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Research Article

Approximate Analytical Solution of One-Dimensional Beam Equations by Using Time-Fractional Reduced Differential Transform Method

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In this paper, a recent and reliable method, named the fractional reduced differential transform method (FRDTM) is employed to solve one-dimensional time-fractional Beam equation subject to the appropriate initial conditions. This method provides the solutions very accurately and efficiently in convergent series form with easily computable coefficients. The efficacy and accuracy of this method are verified by means of three illustrative examples which indicate that the present method is very effective, simple, and easy to implement. Finally, it is observed that the FRDTM is the prevailing and convergent method for the solutions of linear and nonlinear fractional-order partial differential equations.

1. Introduction

In last few decades, fractional calculus has been attracted much attention due to its enormous numbers of applications in almost all disciplines of applied sciences and engineering. The fractional calculus became an aspirant to find out the solution of complex systems that exist in numerous fields of sciences (for detail see [1–4]). This branch of mathematical analysis, extensively investigated in the recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. One of the key factors for the popularity of the subject is the nonlocal nature of fractional-order operators. In the field of mathematical modeling, having partial derivatives of fractional order naturally seems to be dealing with the generality of the current traditional models [5]. In the field of modern science and engineering, the fourth-order parabolic time-fractional beam equation plays an important role in modern science and engineering. For example, airplane wings and transverse vibrations of sustained tensile beams can be modeled as plates with initial/different boundary supports which are successfully governed by superdiffusion fourth-order differential equations [6].

The fourth-order parabolic PDEs are of great importance. These PDEs describe various physical phenomena, including deformation of beams, viscoelastic and inelastic flows, transverse vibrations of a homogeneous beam, plate deflection theory, engineering, and applied sciences [7–13].

In this work, we concentrate our discussion on the following classes of time-fractional nonlinear Beam equation (fourth-order time-fractional nonlinear parabolic PDEs) of the form [14].

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + A(x,t)\frac{\partial^4 u}{\partial x^4} = f(x,t,u,u_t,u_x,u_{xx},u_{xxx}), (x,t) \in \Omega, \tag{1}$$

with initial conditions

$$\begin{aligned} u(\mathrm{x},0) &= f(x) \\ u_t(\mathrm{x},0) &= g(x), a \le x \le b, \end{aligned} \tag{2}$$

finding the solution of this equation has been the subject of many investigators in the recent years.

Before the nineteenth century, there was no scheme available for the analytical solutions of the fractional differential equations. In the beginning of the twentieth century, researchers started to pay attention to find the robust and stable analytical approaches for the exact (approximate) solution of the fractional differential equations [15]. Subsequently, several schemes such as the Adomian decomposition methods [16–18], differential transform method [19–21], Homotopy perturbation method [22–25], Local fractional variation iteration method [26], Variation iteration method [27, 28], and Shifted Chebyshev polynomials based method [29] have been developed for the analytical solutions of fractional differential equations.

Most of these methods sometimes require complex and huge calculation in order to obtain approximate solutions. To overcome such difficulties and drawbacks, an alternative method, the so-called fractional reduced differential transform method (FRDTM), has been developed by Keskin and Oturanc [30]. FRDTM plays a vital role among all the listed methods because it takes small size computation, easy to implement as compared to other techniques [31].

The basic motivation of this paper is to propose FRDTM to find an approximate analytical solution of the time-fractional Beam equation (the governing equation) given in (1). Using this method, it is possible to find both exact and approximate solutions in a rapidly convergent power series form.

It is worth mentioning that the FRDTM is applied without any linearization or discretization or restrictive assumptions. FRDTM is a very reliable, efficient, and effective powerful computational technique for solving physical problems [32–37].

The rest of this paper is organized as follows: in Section 2, we give some fundamental definitions and lemmas associated with fractional calculus. In Section 3, some basic definitions and properties related to one-dimensional fractional reduced differential transformation in the Caputo sense are presented; some lemmas are proved. In Section 4, we present the formulation of the method. Section 6 is devoted to apply the method to solve linear and nonlinear time-fractional beam equation in one dimension and present graphs to show the effectiveness, validity, and performance of the FRDTM for some values of α. Finally, the conclusion is presented in Section 6.

2. Fractional Calculus

In this section, some basic definitions and lemmas of FRDTM associated with fractional calculus are presented. Some of these definitions are due to Riemann Liouville and Caputo sense; for details, see [34, 36, 37].

Definition 1. The Gamma function. The Gamma function $\Gamma(z)$ is simply a generalization of the factorial real arguments. The Gamma function can be defined as [3]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, z > 0. \tag{3}$$

Definition 2. Let $\mu \in \mathbb{R}, m \in \mathbb{N}$. A function $f : \mathbb{R}^+ \to \mathbb{R}$ belongs to the space C_μ if there exists a real number $k > \mu$ such that

$$f(t) = t^p g(t), \mathrm{t} > 0, \tag{4}$$

where

$$g(t) \in C[0,\infty) \text{ and } f \in \mathbb{C}_\mu^m \text{ if } f^{(m)} \in C_\mu. \tag{5}$$

Definition 3. Let J_x^α be Riemann-Liouville fractional integral operator of order $\alpha > 0$ and let $f \in \mathbb{C}_\mu$, $\mu \geq -1$, then

$$\begin{aligned} J_t{}^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \\ J_t{}^0 f(t) &= f(t), t > 0. \end{aligned} \tag{6}$$

Definition 4. If $m-1 < \alpha \leq m, m \in \mathbb{N}, t > 0$, then the Caputo's fractional derivative of $f \in C_\mu$ is defined as

$$D_x{}^\alpha f(x) = J_x{}^{m-\alpha} D_x{}^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt. \tag{7}$$

The fundamental properties of the Caputo fractional derivative are given in the following lemma.

Lemma 5. *If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f(x) \in C_\mu^m$, $\mu \geq -1$, then*

$$\begin{gathered} (i) D_t^\alpha D_t^\beta f(t) = D_t^{\alpha+\beta} f(t) = D_t^\beta D_t^\alpha f(t) \\ (ii) D_t^\alpha J_t^\alpha f(t) = f(t), t > 0 \\ (iii) J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, t > 0. \end{gathered} \tag{8}$$

3. Fractional Reduced Differential Transform Method (FRDTM)

Definition 6 (see [34, 36, 37]). If $u(x,t)$ is analytic and continuously differentiable with respect to space variable x and time variable t in the domain of interest, then time-fractional reduced differential transform (or the spectrum function) is

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x,t) \right]_{t=t_0}, \tag{9}$$

where α is a parameter which describes the order of time-fractional derivative in the Caputo sense, and k is an integer $(k \geq 0)$.

Remark 7. In this study, the lowercase $u(x,t)$ represents the original function, while the uppercase $U_k(x)$ stands for the transformed function.

Definition 8 (see [34, 36, 37]). The fractional reduced differential inverse transform of $U_k(x)$ is defined as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)(t-t_o)^{k\alpha}. \tag{10}$$

Substituting Eq. (9) into Eq. (10), we obtain

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_o}(t-t_o)^{k\alpha}, \tag{11}$$

which in practical application can be approximated by a finite series

$$u_n(x,t)=\sum_{k=0}^{n}U_k(x)(t-t_o)^{k\alpha}, \tag{12}$$

where n is the order of this approximate solution. Therefore, the exact solution can be obtained as

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\sum_{k=0}^{\infty}U_k(x)(t-t_o)^{k\alpha}. \tag{13}$$

If $t_0=0$, then Eq. (13) reduces to the form

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\sum_{k=0}^{\infty}U_k(x)(t)^{k\alpha}. \tag{14}$$

Some of the basic properties of one-dimensional fractional reduced differential transform function that are constructed based on Definitions 6 and 8 are given below.

Lemma 9. *If $f(x,t)=x^m\sin(\eta x+\theta t)$, then the fractional reduced differential transform of f is $F_k(x)=(\theta^k/k!)x^m\sin(\eta x+(\pi k/2))$, where η and θ are constants.*

Proof. From Definition 6 and FRDTM properties, we have

$$\begin{aligned}
F_k(x)&=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}f(x,t)\right]_{t=t_0}\\
F_k(x)&=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}x^m\sin(\eta x+\theta t)\right]_{t=t_0}\\
F_k(x)&=x^m\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\sin(\eta x+\theta t)\right]_{t=t_0}\\
F_k(x)&=\frac{\theta^k}{k!}x^m\sin\left(\eta x+\frac{\pi k}{2}\right),\ fork=0,1,2,3,\cdots
\end{aligned} \tag{15}$$

Lemma 10. *If $f(x,t)=x^m\cos(\eta x+\theta t)$, then the fractional reduced differential transform of f is*

$F_k(x)=x^m(\theta^k/k!)\cos(\eta x+(\pi k/2))$, where η and θ are constants.

Proof. From Definition 6 and FRDTM properties, we have.

$$\begin{aligned}
F_k(x)&=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}f(x,t)\right]_{t=t_0}\\
F_k(x)&=\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}x^m\cos(\eta x+\theta t)\right]_{t=t_0}\\
F_k(x)&=x^m\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\cos(\eta x+\theta t)\right]_{t=t_0}\\
F_k(x)&=x^m\frac{\theta^{\theta k}}{k!}\cos\left(\eta x+\frac{\pi k}{2}\right),\text{ for }k=0,1,2,3,\cdots
\end{aligned} \tag{16}$$

4. Solution of the Problem by FRDTM

Applying properties of FRDTM to Eq. (1), we obtain the following recurrence relation:

$$U_{k+2}(x)=\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left(-\sum_{r=0}^{k}A_r(x)\frac{\partial^4}{\partial x^4}U_{k-r}(x)+F(U_k(x))\right), \tag{17}$$

and using FRDTM to the initial conditions (2), we get

$$U_0(x)=f(x),\ U_1(x)=g(x). \tag{18}$$

Using Eqs. (17) and (18) and $k=0,1,2,3$, and by iterative calculation, the following results are obtained:

$$\begin{aligned}
U_2(x)&=\frac{1}{\Gamma(2\alpha+1)}\left(-A_0(x)\frac{\partial^4}{\partial x^4}U_0(x)+F(U_0(x))\right),\\
U_3(x)&=\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-\left[A_0(x)\frac{\partial^4}{\partial x^4}U_1(x)+A_1(x)\frac{\partial^4}{\partial x^4}U_0(x)\right]+F(U_1(x))\right),\\
U_4(x)&=\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left[-\left[\begin{matrix}A_0(x)\frac{\partial^4}{\partial x^4}U_2(x)+A_1(x)\frac{\partial^4}{\partial x^4}U_1(x)+\\A_2(x)\frac{\partial^4}{\partial x^4}U_0(x)\end{matrix}\right]+F(U_2(x))\right],\\
U_5(x)&=\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-\left[\begin{matrix}A_0(x)\frac{\partial^4}{\partial x^4}U_3(x)+A_1(x)\frac{\partial^4}{\partial x^4}U_2(x)+\\A_2(x)\frac{\partial^4}{\partial x^4}U_1(x)+A_3(x)\frac{\partial^4}{\partial x^4}U_0(x)\end{matrix}\right]+F(U_3(x))\right).
\end{aligned} \tag{19}$$

Continuing in a similar fashion the remaining successive terms of the FRDTM can be obtained.

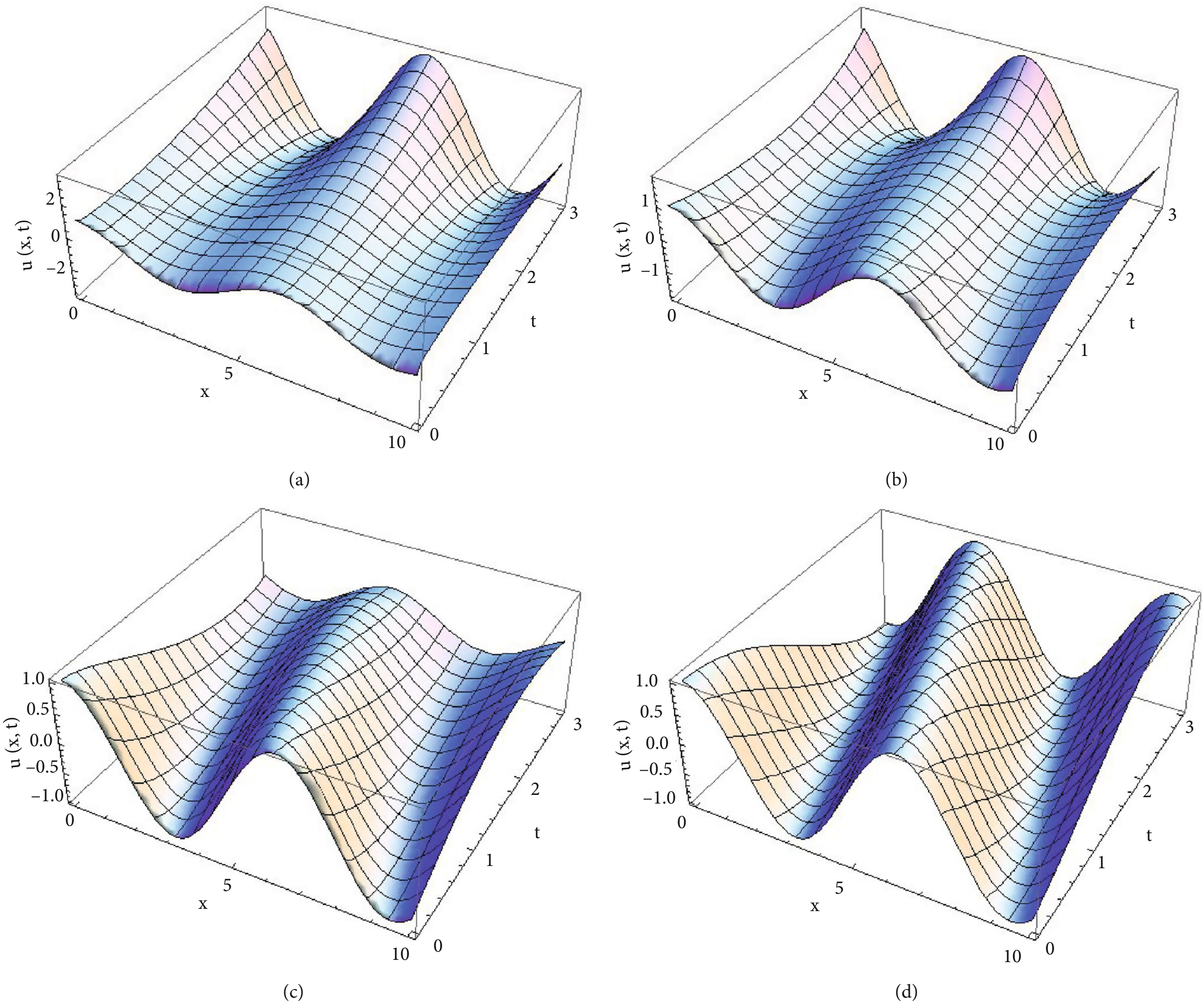

FIGURE 1: The physical behavior of FRDTM solution of Eq. (21) for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$, and (d) $\alpha = 1$.

Then, the fractional reduced differential inverse transform of the set of values of $[U_k(x)]_{k=0}^{\infty}$ giving the series solution of Eq. (1) as

$$u(x,t) = f(x) + g(x)t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)}\left(-A_0(x)\frac{\partial^4}{\partial x^4}U_0(x) + F(U_0(x))\right)t^{2\alpha} + \cdots. \tag{20}$$

If $\alpha = 1$, the FRDTM solution (20) gives the exact solution of Eq. (1).

5. Illustrative examples

Example 11. Consider one-dimensional homogeneous time-fractional beam equation:

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) + \frac{\partial^4}{\partial x^4}u(x,t) = 0, x \in \mathbb{R},\ t > 0,\ 0 < \alpha \le 1. \tag{21}$$

Subjected to the initial conditions:

$$u(x,0) = \cos x,\ u_t(x,0) = -\sin x. \tag{22}$$

Applying properties of FRDTM to Eq. (21), we obtain the following recurrence relation:

$$U_{k+2}(x) = -\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\frac{\partial^4}{\partial x^4}U_k(x). \tag{23}$$

and using FRDTM to the initial conditions (22), we get

$$U_0(x) = \cos x, U_1(x) = -\sin x. \tag{24}$$

When $k = 0, 1, 2, 3, \cdots$, by iterative calculation we obtain

$$\begin{aligned} U_2(x) &= \frac{-\cos x}{\Gamma(2\alpha+1)}, \\ U_3(x) &= \frac{-\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} \sin x, \\ U_4(x) &= \frac{1}{\Gamma(4\alpha+1)} \cos x, \\ U_5(x) &= -\frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)} \sin x. \end{aligned} \tag{25}$$

Continuing in this way, the remaining successive terms of the FRDTM can be obtained. Then, the fractional reduced differential inverse transform of the set of values of $[U_k(x)]_{k=0}^{\infty}$ gives the following approximate analytic solution

$$\begin{aligned} u(x,t) = \Bigg[\cos x - \frac{\sin x}{\Gamma(\alpha+1)} t^{\alpha} - \frac{\cos x}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{\sin x\,\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha} \\ + \frac{\cos x}{\Gamma(4\alpha+1)} t^{4\alpha} - \frac{\sin x\,\Gamma(\alpha+1)}{\Gamma(5\alpha+1)} t^{5\alpha} + \cdots \end{aligned} \tag{26}$$

Finally, for $\alpha = 1$, Eq. (26) reduces to the form:

$$u(x,t) = \cos x\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots\right) - \sin x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right), \tag{27}$$

and whose exact solution of the problem is $u(x, t) = \cos x \cos t - \sin x \sin t = \cos(x+t)$ [38].

The approximate numerical solutions corresponding to Example 11 are given in Figures 1 and 2 and Table 1.

Example 12. Consider the following fourth-order parabolic time-fractional beam equation with variable coefficient:

$$\begin{aligned} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} u(x,t) = -(x+1)\frac{\partial^4}{\partial x^4} u(x,t) + \left(x^4 + x^3 - \frac{6}{7!}x^7\right) \cos t, 0 \\ < x < 1, t > 0, 0 < \alpha \le 1, \end{aligned} \tag{28}$$

subject to the initial conditions:

$$u(x,0) = \frac{6}{7!}x^7 \text{ and } u_t(x,0) = 0,\ 0 < x < 1,\ t > 0. \tag{29}$$

Applying properties of FRDTM to Eq. (28), we obtain the following recurrence relation:

$$U_{k+2}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left(-(x+1)\frac{\partial^4}{\partial x^4}U_k(x) + \left(x^4 + x^3 - \frac{6}{7!}x^7\right)\frac{1}{k!}\cos\left(\frac{\pi k}{2}\right)\right), \tag{30}$$

and using FRDTM to the initial conditions (29), we get

$$U_0(x) = \frac{6}{7!}x^7 \text{ and } U_1(x) = 0,\ 0 < x < 1. \tag{31}$$

Iterative calculations for $k = 0, 1, 2, 3, \cdots$ gives the following successive values.

$$\begin{aligned} U_2(x) &= \frac{1}{\Gamma(2\alpha+1)}\left(-\frac{6}{7!}x^7\right), \\ U_3(x) &= 0, \end{aligned} \tag{32}$$

$$\begin{aligned} U_4(x) &= \frac{x^4+x^3}{\Gamma(4\alpha+1)} - \frac{\Gamma(2\alpha+1)(x^4+x^3)}{2!\Gamma(4\alpha+1)} + \frac{6x^7\Gamma(2\alpha+1)}{7!2!\Gamma(4\alpha+1)}, \\ U_5(x) &= 0, \\ U_6(x) &= \Bigg[\frac{-24(x+1)}{\Gamma(6\alpha+1)} + \frac{12(x+1)\Gamma(2\alpha+1)}{\Gamma(6\alpha+1)} - \frac{(x^4+x^3)\Gamma(2\alpha+1)}{2\Gamma(6\alpha+1)} \\ &\quad + \frac{(x^4+x^3)\Gamma(4\alpha+1)}{4\Gamma(6\alpha+1)} - \frac{6x^7\Gamma(4\alpha+1)}{7!4!\Gamma(6\alpha+1)}, \\ U_7(x) &= 0, \end{aligned} \tag{33}$$

$$\begin{aligned} U_8(x) = \Bigg[\frac{12(x+1)\Gamma(2\alpha+1)}{\Gamma(8\alpha+1)} - \frac{(x+1)\Gamma(4\alpha+1)}{\Gamma(8\alpha+1)} \\ + \frac{(x^4+x^3)\Gamma(4\alpha+1)}{4!\Gamma(8\alpha+1)} - \frac{(x^4+x^3)\Gamma(6\alpha+1)}{6!\Gamma(8\alpha+1)} + \frac{6x^7\Gamma(6\alpha+1)}{7!6!\Gamma(8\alpha+1)}, \end{aligned} \tag{34}$$

and so on. Continuing in this manner, the remaining iterative values can be obtained. Then, the fractional reduced differential inverse transform of the set of values of $[U_k(x)]_{k=0}^{\infty}$ gives the following approximate analytic solution

$$\begin{aligned} u(x,t) = \Bigg[&\frac{6x^7}{7!} - \frac{6x^7}{7!\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{x^4+x^3}{\Gamma(2\alpha+1)} - \frac{x^4+x^3-(6/7!)x^7}{2!}\right)t^{4\alpha} \\ &- \frac{\Gamma(4\alpha+1)}{\Gamma(6\alpha+1)}\left(\left(-(x+1)\frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(\frac{24}{\Gamma(2\alpha+1)} - 12 + \frac{x^3}{2!}\right)\right) + \frac{x^4+x^3}{4!} + \frac{6x^7}{7!4!}\right)t^{6\alpha} \\ &+ \left(\begin{array}{c} \dfrac{12(x+1)\Gamma(2\alpha+1)}{\Gamma(8\alpha+1)} - \dfrac{(x+1)\Gamma(4\alpha+1)}{\Gamma(8\alpha+1)} + \dfrac{(x^4+x^3)\Gamma(4\alpha+1)}{4!\Gamma(8\alpha+1)} \\ - \dfrac{(x^4+x^3)\Gamma(6\alpha+1)}{6!\Gamma(8\alpha+1)} + \dfrac{6x^7\Gamma(6\alpha+1)}{7!6!\Gamma(8\alpha+1)} \end{array}\right)t^{8\alpha} + \cdots. \end{aligned} \tag{35}$$

If $\alpha = 1$, then Eq. (35) gives

$$u(x,t) = \frac{6x^7}{7!} - \frac{6x^7}{7!2!}t^2 + \frac{6}{7!4!}x^7t^4 - \frac{6x^7}{7!6!}t^6 + \frac{6x^7}{7!8!}t^8 + \cdots = \frac{6x^7}{7!}\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots\right). \tag{36}$$

The exact solution of the classical form of Eq. (28) is $u(x,t) = (6x^7/7!)\cos t$ [14, 39–42].

The approximate numerical solutions corresponding to Example 12 are given in Figures 3 and 4 and Table 2.

Example 13. Consider the following one-dimensional nonhomogeneous nonlinear Beam equation

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) + xu^2(x,t)\frac{\partial^4}{\partial x^4}u(x,t) = 24x^9t^3, 0 < x < 1, t > 0, 0 < \alpha \le 1. \tag{37}$$

with initial conditions:

$$u(x,0) = 0 \text{ and } u_t(x,0) = x^4. \tag{38}$$

Applying properties of FRDTM to (37), we obtain the following recurrence relation

$$U_{k+2}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2\alpha+1)}\left(-x\sum_{r=0}^{k}\sum_{i=0}^{r}U_i(x)U_{r-i}(x)\frac{\partial^4}{\partial x^4}U_{k-r}(x) + 24x^9\delta(k\alpha-3)\right), \tag{39}$$

and using FRDTM to the initial conditions (38), we get

$$U_0(x) = 0 \text{ and } U_1(x) = x^4. \tag{40}$$

By iterative calculations for $k = 0, 1, 2, 3, \cdots$ equations (39) and (40) give the following successive values

$$\begin{aligned}
U_2(x) &= \frac{1}{\Gamma(2\alpha+1)}\left(-xU_0{}^2(x)\frac{\partial^4}{\partial x^4}U_0(x) + \delta(-3)\right) = \frac{1}{\Gamma(2\alpha+1)}\delta(-3),\\
U_3(x) &= \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(-x\left(U_0{}^2(x)\frac{\partial^4}{\partial x^4}U_1(x) + 2U_0(x)U_1(x)\frac{\partial^4}{\partial x^4}U_0(x)\right) + 24x^9\delta(\alpha-3)\right),\\
U_3(x) &= \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(24x^9\delta(\alpha-3)\right),\\
U_4(x) &= \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(-x\begin{pmatrix} U_0{}^2(x)\frac{\partial^4}{\partial x^4}U_2(x) + 2U_0(x)U_1(x)\frac{\partial^4}{\partial x^4}U_1(x) + \\ 2U_0(x)U_2(x)\frac{\partial^4}{\partial x^4}U_0(x) + U_1{}^2(x)\frac{\partial^4}{\partial x^4}U_0(x)\end{pmatrix} + 24x^9\delta(2\alpha-3)\right),\\
U_4(x) &= \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(24x^9\delta(2\alpha-3)\right),\\
U_5(x) &= \frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-x\begin{pmatrix} U_0{}^2(x)\frac{\partial^4}{\partial x^4}U_3(x) + 2U_0(x)U_1(x)\frac{\partial^4}{\partial x^4}U_2(x) + \\ 2U_0(x)U_2(x)\frac{\partial^4}{\partial x^4}U_1(x) + U_1{}^2(x)\frac{\partial^4}{\partial x^4}U_1(x) + \\ 2U_0(x)U_3(x)\frac{\partial^4}{\partial x^4}U_0(x) + 2U_1(x)U_2(x)\frac{\partial^4}{\partial x^4}U_0(x)\end{pmatrix} + 24x^9\delta(3\alpha-3)\right),\\
U_5(x) &= \frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-24x^9 + 24x^9\delta(3\alpha-3)\right),
\end{aligned} \tag{41}$$

and so on. Then, the fractional reduced differential inverse transform of the set of values of $[U_k(x)]_{k=0}^{\infty}$ gives the following approximate analytic solution

$$u(x,t) = x^4t^\alpha + \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}\left(24x^9\delta(\alpha-3)\right)t^{3\alpha} + \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left(24x^9\delta(2\alpha-3)\right)t^{4\alpha} + \left(\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left(-24x^9 + 24x^9\delta(3\alpha-3)\right)\right)t^{5\alpha} + \cdots. \tag{42}$$

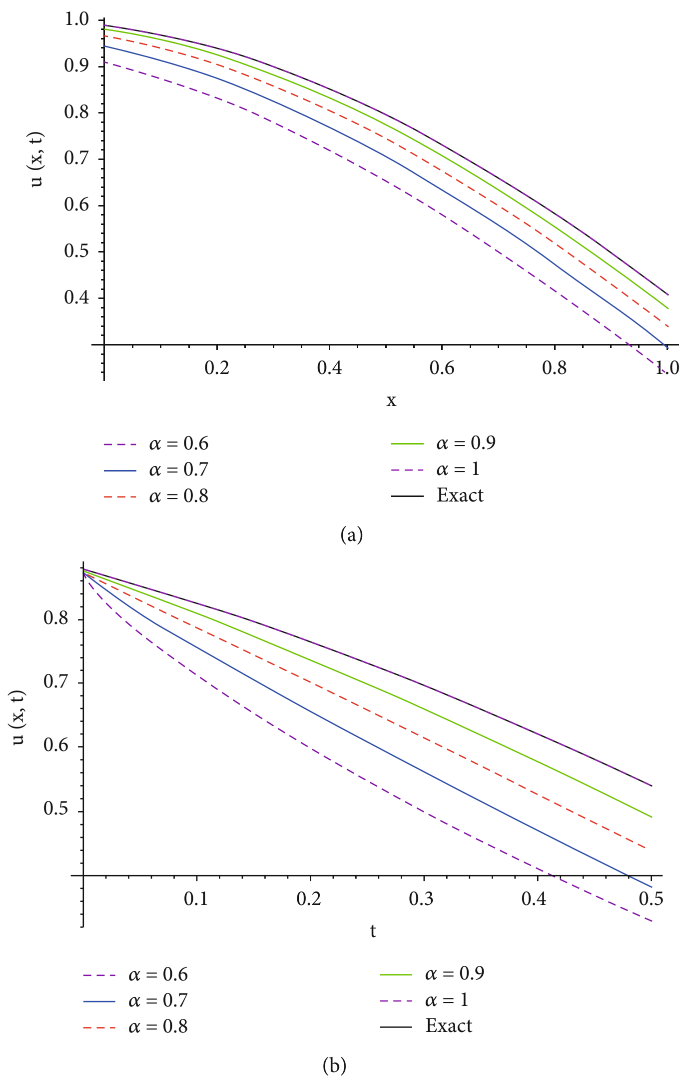

Figure 2: FRDTM solution profile of Eq. (21): (a) $u(x, t)$ vs. x for different values α and $t = 0.15$ and (b) $u(x, t)$ vs. time t for different values α and $x = 0.5$.

Using the definition $\delta(k)$ given in Table 3, Eq. (42) reduces to the form

$$u(x, t) = x^4 t^\alpha. \tag{43}$$

If $\alpha = 1$, then Eq. (42) gives the exact solution $u(x, t) = x^4 t$ of the classical form of Eq. (37) see Ref. [42].

The approximate numerical solutions corresponding to Example 13 are given in Figures 5 and 6 and Table 4.

Figures 1–6 exhibit the physical behavior of the FRDTM solutions $u(x, t)$ of Example 11, Example 12, and Example 13 for different values of time-fractional order α and time t. It is evident from the figures that, as the values of time-fractional order α approaches to 1, the graph of the FRDTM solutions $u(x, t)$ of the illustrated examples resembles to the graph of the exact solutions $u(x, t)$ of their corresponding classical (nonfractional) one-dimensional beam equations. Furthermore, Figures 6(c) and 6(d) depict the long time range physical behavior of the solution of Eq. (37).

Table 1 shows the ninth-order approximate numerical solutions $u(x, t)$ of Eq. (21) for different values of α and the absolute error of FRDTM solution for $\alpha = 1$. Table 2 illustrates the eighth-order approximate numerical solution $u(x, t)$ of Eq. (28) for different values of α and the absolute error of FRDTM solution for $\alpha = 0.8$. Table 4 reveals the first-order approximate numerical solution $u(x, t)$ of Eq. (37) for different values of α and the absolute error of FRDTM solution for $\alpha = 0.9$. Generally, from Tables 1–4, it is distinguished that the approximate solutions obtained by FRDTM are close to the exact solution of the classical form of Examples 11–13 as the values of α are close to 1 for any values of t.

Table 1: Ninth-order approximate numerical solution by FRDTM of Eq. (21) at different values of α and comparison of absolute error at $\alpha = 1$.

x	t		u_{FRDTM}		u_{Exact}	Absolute error ($\lvert u_{\text{exact}} - u_{\alpha=1} \rvert$)
		$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$		
4	1	0.209769	0.274293	0.283662	0.283662	0
	2	0.312931	0.589007	0.960029	0.960170	$1.410e-04$
	3	0.117991	0.365617	0.747129	0.753902	$6.773e-03$
6	1	0.515730	0.603985	0.753903	0.753902	$1.000e-06$
	2	0.241086	-0.030377	-0.145223	-0.145500	$2.770e-04$
	3	0.849243	-0.183085	-0.895349	-0.911130	$1.578e-03$
8	1	-0.639008	-0.776986	-0.911130	-0.911130	0
	2	-0.513585	-0.563724	-0.839161	-0.839072	$8.900e-05$
	3	-0.824811	-0.213237	-0.001936	0.004423	$6.359e-03$
10	1	0.016113	0.042696	0.004425	0.004426	$2.200e-07$
	2	0.186367	0.499561	0.843651	0.843854	$2.040e-04$
	3	-0.162758	0.360561	0.896961	0.907447	$1.048e-02$

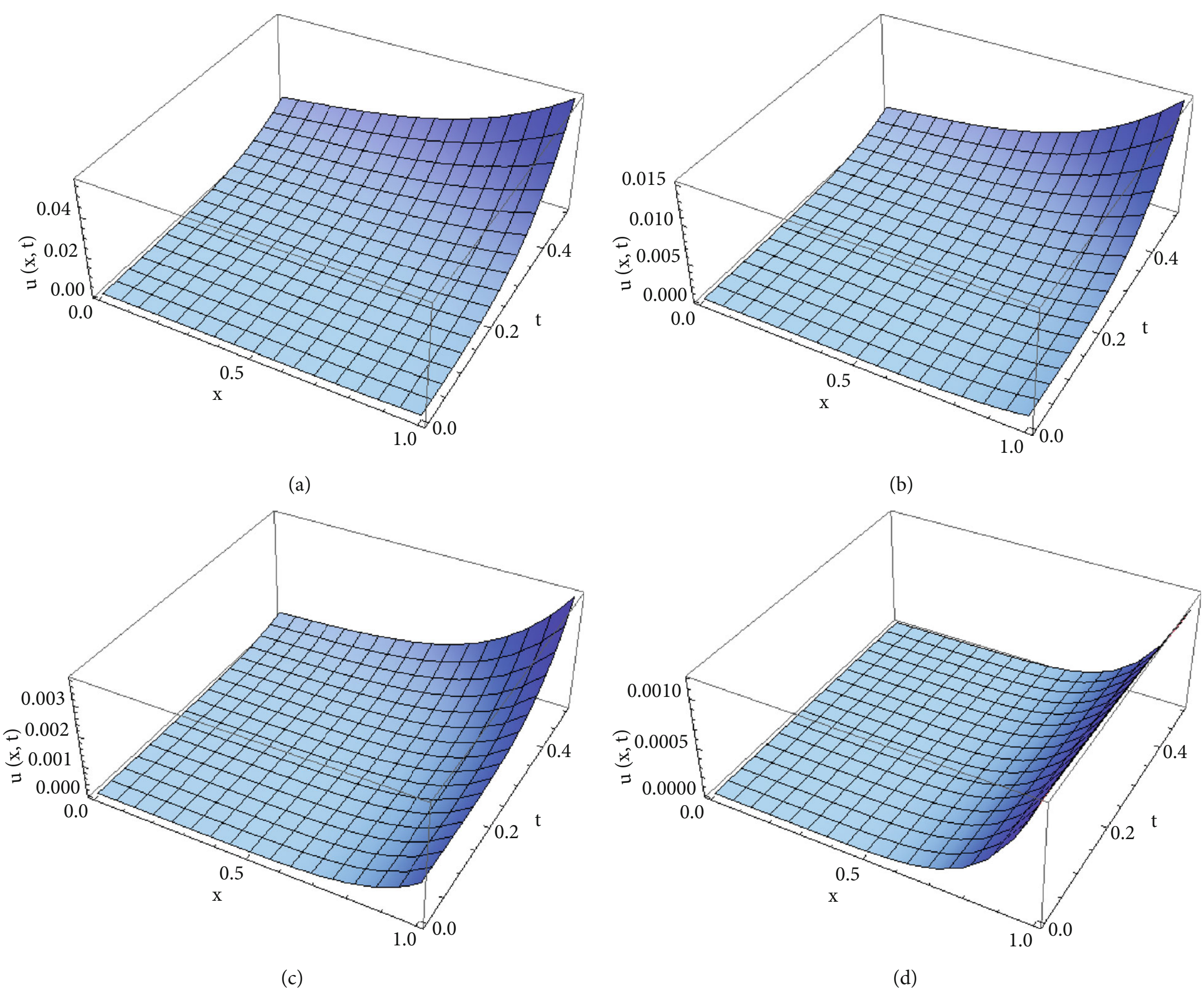

Figure 3: The physical behavior of FRDTM solution of Eq. (28) for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$, and (d) $\alpha = 1$.

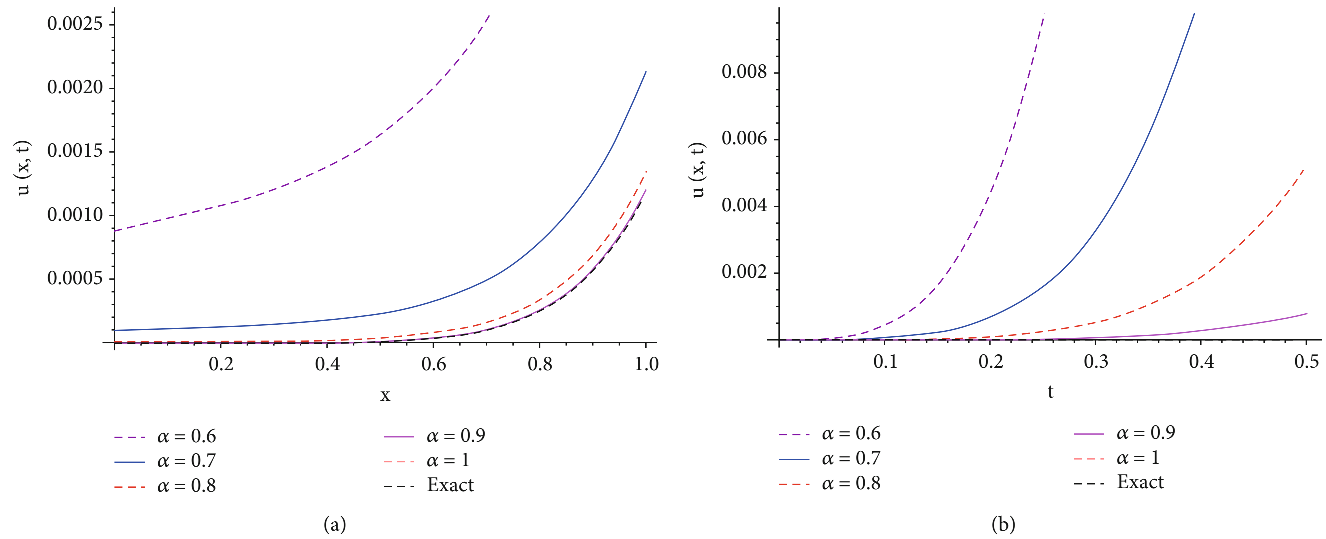

FIGURE 4: FRDTM solution profile of Eq. (28). (a) $u(x,t)$ vs. x for different values α and $t = 0.15$. (b) $u(x,t)$ vs. time t for different values α and $x = 0.5$.

TABLE 2: Eighth-order approximate numerical solution by FRDTM of Eq. (28) at different values of α and comparison of absolute error at $\alpha = 0.8$.

x	t	u_{FRDTM} $\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	u_{Exact}	Absolute error ($\lvert u_{\text{exact}} - u_{\alpha=0.8} \rvert$)
0.1	0.2	0.00276038	1.19048×10^{-10}	1.16675×10^{-10}	1.16675×10^{-10}	$2.37e-12$
	0.4	0.03436160	1.19048×10^{-10}	1.09650×10^{-10}	1.09650×10^{-10}	$9.40e-12$
	0.6	0.15224600	1.19048×10^{-10}	9.82542×10^{-11}	9.82542×10^{-11}	$2.08e-11$
0.4	0.2	0.00380271	1.95048×10^{-6}	1.91160×10^{-6}	1.91160×10^{-6}	$1.91e-06$
	0.4	0.04531310	1.95048×10^{-6}	1.79651×10^{-6}	1.79651×10^{-6}	$1.95e-06$
	0.6	0.19812500	1.95048×10^{-6}	1.60980×10^{-6}	1.60980×10^{-6}	$3.41e-07$
0.7	0.2	0.00624869	0.0000980408	0.0000960865	0.0000960865	$1.95e-06$
	0.4	0.06358160	0.0000980408	0.0000903016	0.0000903016	$7.74e-06$
	0.6	0.26405700	0.0000980408	0.0000809166	0.0000809166	$1.71e-05$
1	0.2	0.01257680	0.0011904800	0.0011667500	0.0011667500	$2.37e-05$
	0.4	0.09909390	0.0011904800	0.0010965000	0.0010965000	$9.40e-05$
	0.6	0.37615100	0.0011904800	0.0009825420	0.0009825420	$2.08e-04$

6. Conclusions

In this study, the FRDTM is effectively implemented to find approximate analytics solutions of time-fractional Beam equation subject to appropriate initial conditions. The fractional derivative used in this article is in the sense of Caputo. This method requires only initial conditions and represents the solution in an infinite power series. The main advantage of this scheme is that it can be used in a direct way without applying techniques such as restricting conditions, convincing suppositions, and perturbations. This shows that the FRDTM is very simple to utilize and needs brevity of calculation. To check the validity and effectiveness of the method, three illustrative examples are carried out. The infinite power series solutions of Examples 11–13 obtained by FRDTM for $\alpha = 1$ are in excellent agreement with the exact solutions as in [39–42]. The computed results reveal that the FRDTM is accurate, swiftly convergent, and efficient. Consequently, our goal in the further is to apply the FRDTM to nonlinear PDEs which arises in other areas of science, such as Physics, Biology, Medicine, and Engineering.

Table 3: Basic operations of the fractional reduced differential transform method [3, 34, 36, 37].

Original function $w(x,t)$	Transformed function $W_k(x)$
$w(x,t) = \alpha u(x,t) \pm \beta v(x,t)$	$W_k(x) = \alpha U_k(x) \pm \beta V_k(x)$, where α and β are constants
$w(x,t) = (\partial^r/\partial t^r) u(x,t)$	$W_k(x) = (k+1)(k+2)\cdots(k+r)U_{k+1}(x) = ((k+r)!/k!)U_{k+r}(x)$
$w(x,t) = (\partial^r/\partial x^r) u(x,t)$	$W_k(x) = (\partial^r/\partial x^r) U_k(x)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = \begin{cases} x^m U_{k\alpha - n}(x), & \text{if } k\alpha \geq n \\ 0 & \text{otherwise} \end{cases}$
$w(x,t) = \left(\partial^{N\alpha}/\partial t^{N\alpha}\right) u(x,t)$	$W_k(x) = (\Gamma(k\alpha + N\alpha + 1)/\Gamma(k\alpha + 1)) U_{k+N}(x)$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k\alpha - n)$, where $\delta(k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x)$
$w(x,t) = u(x,t)v(x,t)z(x,t)$	$W_k(x) = \sum_{r=0}^{k} \sum_{i=0}^{r} U_i(x) V_{r-i}(x) Z_{k-r}(x)$
$w(x,t) = \sin(\eta x + \theta t)$	$W_k(x) = \left(\theta^k/k!\right) \sin(\eta x + (\pi k/2))$, where η and θ are constants.
$w(x,t) = \cos(\eta x + \theta t)$	$W_k(x) = \left(\theta^k/k!\right) \cos(\eta x + (\pi k/2))$, where η and θ are constants.

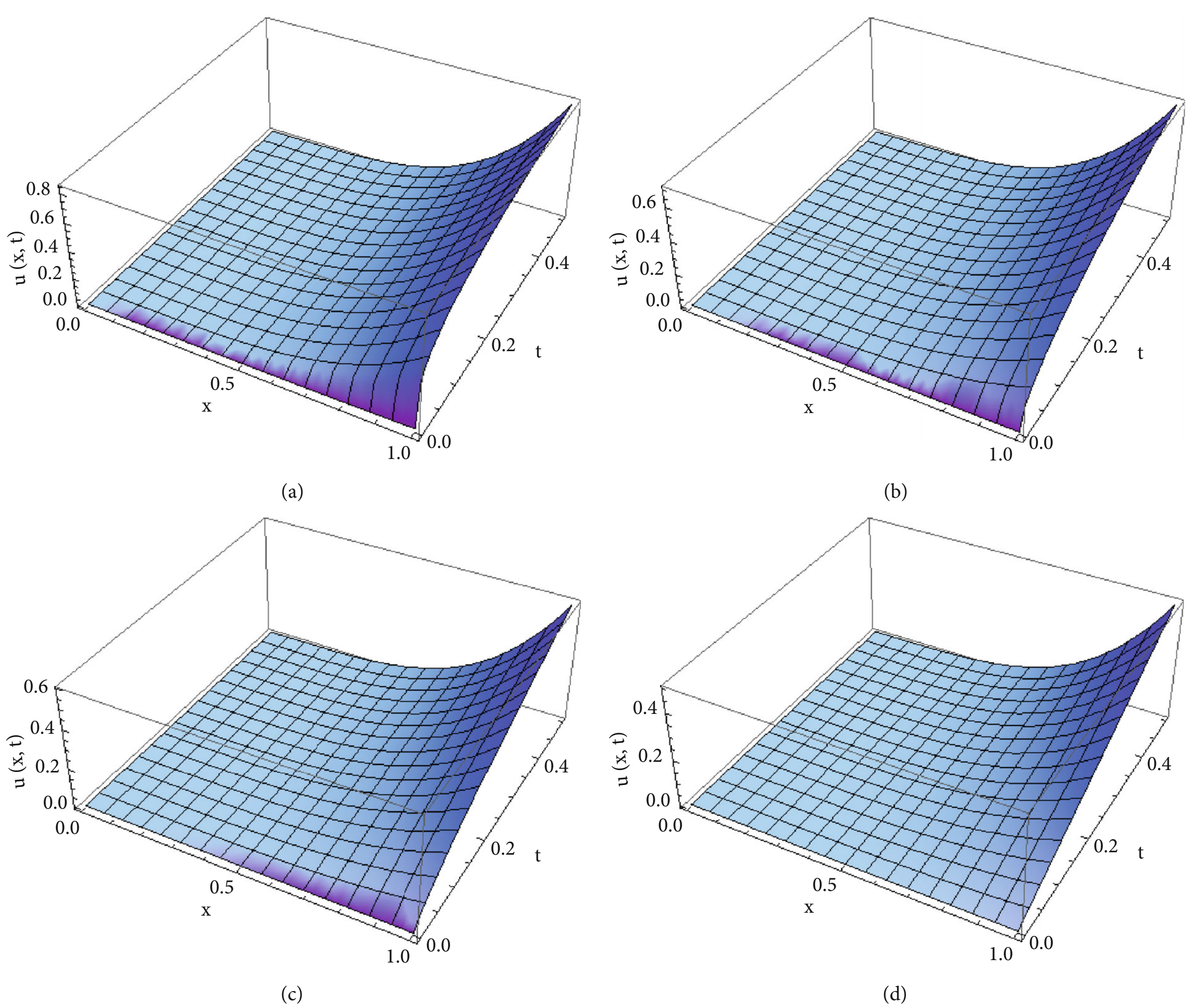

Figure 5: The physical behavior of FRDTM solution of Eq. (37) for (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$, and (d) $\alpha = 1$.

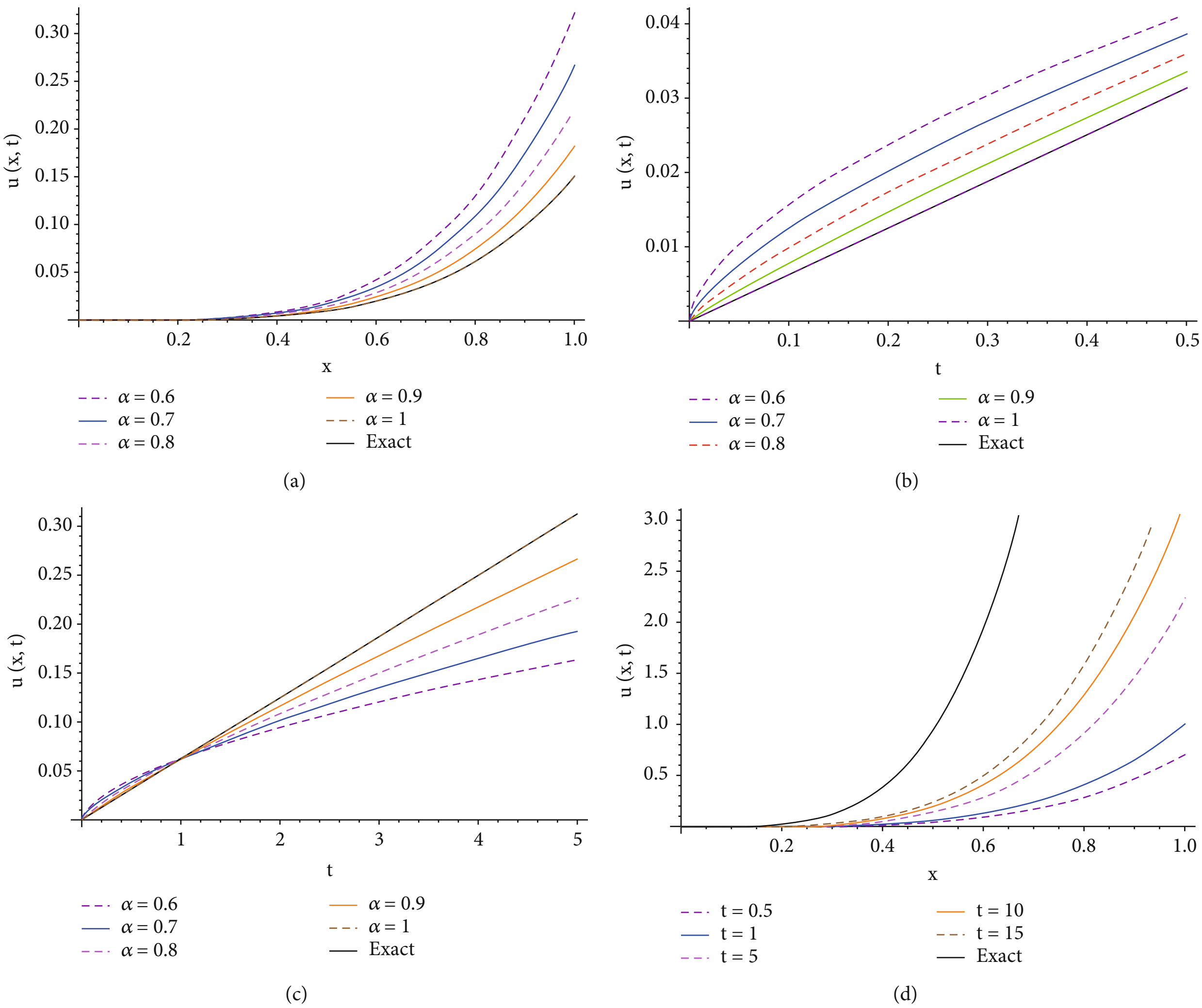

Figure 6: FRDTM solution profile of Eq. (37): (a) $u(x,t)$ vs. x for different values α and $t = 0.15$, (b) $u(x,t)$ vs. time t for different values α and $x = 0.5$, (c) $u(x,t)$ vs. time t for different values α and $x = 0.5$, and (d) $u(x,t)$ vs. x for different values t and $\alpha = 0.5$.

Table 4: First-order approximate numerical solution by FRDTM of Eq. (37) at different values of α and comparison of absolute error at $\alpha = 0.9$.

x	t	u_{FRDTM} $\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	u_{Exact}	Absolute error ($\lvert u_{\text{exact}} - u_{\alpha=0.9} \rvert$)
0.1	0.1	0.0000158489	0.0000125893	0.00001	0.00001	$2.589e-06$
	0.3	0.0000381678	0.0000338383	0.00003	0.00003	$3.838e-06$
	0.5	0.0000574349	0.0000535887	0.00005	0.00005	$3.589e-06$
0.4	0.1	0.00405733	0.00322285	0.00256	0.00256	$6.285e-04$
	0.3	0.00977095	0.00866262	0.00768	0.00768	$9.826e-04$
	0.5	0.01470330	0.01371870	0.01280	0.01280	$9.187e-04$
0.7	0.1	0.03805330	0.0302268	0.02401	0.02401	$6.217e-03$
	0.3	0.09164090	0.0812459	0.07203	0.07203	$9.216e-03$
	0.5	0.13790100	0.1286660	0.12005	0.12005	$8.616e-03$
1	0.1	0.15848900	0.1258930	0.10000	0.10000	$2.589e-02$
	0.3	0.38167800	0.3383830	0.30000	0.30000	$3.838e-02$
	0.5	0.57434900	0.5358870	0.50000	0.50000	$3.589e-02$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests about the publication of this paper.

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