Research Article

Relation between Borweins' Cubic Theta Functions and Ramanujan's Eisenstein Series

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Received 25 November 2020; Revised 24 February 2021; Accepted 15 April 2021; Published 10 May 2021

Academic Editor: Bruno Carpentieri

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Two-dimensional theta functions were found by the Borwein brothers to work on Gauss and Legendre's arithmetic-geometric mean iteration. In this paper, some new Eisenstein series identities are obtained by using (p, k) -parametrization in terms of Borweins' theta functions.

1. Introduction

The series $L(q)$, $M(q)$, and $N(q)$ by Eisenstein are defined below, primarily mentioned by Ramanujan in his second notebook [1].

$$L(q) := 1 - 24 \sum_{m=1}^{\infty} \frac{mq^m}{1-q^m}, \quad (1)$$

$$M(q) := 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1-q^m}, \quad (2)$$

$$N(q) := 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1-q^m}. \quad (3)$$

Borwein brothers [2, 3] established the consecutive three two-dimensional theta functions on cubic; as parallel to Jacobi's theta function identity and the arithmetic-geometric mean iteration of Gauss and Legendre, they are stated as follows for $\omega = \exp(2\pi i/3)$:

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

One can easily verify that $a(0) = b(0) = 1$ and $c(0) = 0$. By using Euler binomial theorem [2, 3], Borwein brothers found an expression for $b(q)$ and $c(q)$ in terms of infinite products, namely,

$$b(q) = \frac{f_1^3}{f_3}, \quad (5)$$

$$c(q) = 3q^{1/3} \frac{f_3^3}{f_1}, \quad (6)$$

$$a^3(q) = b^3(q) + c^3(q), \quad (7)$$

where f_k is defined as

$$f_k := (q^k; q^k)_\infty = \prod_{m=1}^{\infty} (1 - q^{mk}), \quad |q| < 1.$$

Proofs of (5)–(7) can be found in [2, 3]. Berndt et al. [4] demonstrated the following identities by using Ramanujan's elliptic functions in the theory of signature 3, namely,

$$M(q) := a(q)(a^3(q) + 8c^3(q)), \quad (9)$$

$$N(q) := a^6(q) - 20a^3(q)c^3(q) - 8c^3(q), \quad (10)$$

$$M(q^3) := \frac{1}{9}a(q)(9a^3(q) - 8c^3(q)), \quad (11)$$

$$N(q^3) := a^6(q) - \frac{4}{3}a^3(q)c^3(q) + \frac{8}{27}c^3(q). \quad (12)$$

In the current work, using (p, k) -parameters introduced by Alaca et al. [5–7], the new Eisenstein series identities are obtained which connects $a(q)$, $b(q)$, and $c(q)$; these are the examples of sum-to-product identities. Using the classical theory of elliptic functions and modular equations of degree 3, Chan [8] proved (9) and (12). Moreover, Liu [9, 10] provided alternate proofs of (9)–(12) by using the simple elliptic functions. The identities of Ramanujan's Eisenstein series P_n of weight 2 found in [10, 11] have been proved by Bhuvan [12]. Xia and Yao [11] also obtained several Eisenstein series identities that includes Borweins' theta functions containing $b(q)$, $b(q^2)$, $b(q^4)$, $c(q)$, $c(q^2)$, and $c(q^4)$ by using (p, k) -parametrization. Further, in 2020, Shruthi and Srivatsa Kumar [13] obtained new Eisenstein series involving Borweins' cubic theta functions. This paper is classified as follows. Section 2 is devoted for preliminary results which are required to prove our main results in Section 3.

2. Preliminaries

Alaca and Williams [14] ascertained the parametric representations for $mL(q^m) - L(q)$, $M(q^n)$, $b(q^i)$, and $c(q^i)$ for $m \in \{2, 3, 4, 6, 12\}$, $n \in \{1, 2, 3, 6\}$, and $i \in \{1, 2, 4\}$ in terms of parameters p and k , namely,

$$p = p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)},$$

$$k = k(q) = \frac{\varphi^3(q^3)}{\varphi(q)},$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty.$$

Since $\varphi(0) = 1$, one can verify $p(0) = 0$ and $k(0) = 1$. The sequential parametric representations in terms of p and k given by Alaca et al. [6, 7] and Alaca and Williams [14] for $M(q^n)$ for $n \in \{1, 2, 3, 6, 12\}$ and $mL(q^m) - L(q)$ for $m \in \{2, 3, 4, 6, 12\}$ are listed below.

$$M(q) = k^4(1 + 124p + 964p^2 + 2788p^3 + 3910p^4 + 2788p^5 + 964p^6 + 124p^7 + p^8),$$

$$M(q^2) = k^4(1 + 4p + 64p^2 + 178p^3 + 235p^4 + 178p^5 + 64p^6 + 4p^7 + p^8),$$

$$M(q^3) = k^4(1 + 4p + 4p^2 + 28p^3 + 70p^4 + 28p^5 + 4p^6 + 4p^7 + p^8),$$

$$M(q^6) = k^4(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + 4p^6 + 4p^7 + p^8),$$

$$M(q^{12}) = k^4(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 + 1/4p^6 + 1/4p^7 + 1/16p^8),$$

$$2L(q^2) - L(q) = k^2(1 + 4p + 24p^2 + 14p^3 + p^4),$$

$$3L(q^3) - L(q) = 2k^2(1 + 8p + 18p^2 + 8p^3 + p^4),$$

$$4L(q^4) - L(q) = 3k^2(1 + 6p + 12p^2 + 8p^3),$$

$$6L(q^6) - L(q) = k^2(5 + 22p + 36p^2 + 22p^3 + 5p^4),$$

$$12L(q^{12}) - L(q) = k^2(11 + 34p + 36p^2 + 16p^3 + 2p^4).$$

The following parametric representations in terms of the parameters p and k are given by Alaca et al. [5].

$$b(q) = \frac{k(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}}{2^{1/3}},$$

$$b(-q) = \frac{k(1-p)^{1/3}(1+2p)^{4/3}(2+p)^{1/3}}{2^{1/3}},$$

$$b(q^2) = \frac{k(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}}{2^{2/3}},$$

$$b(q^4) = \frac{k(2+p)^{4/3}(1-p)^{1/3}(1+2p)^{1/3}}{2^{4/3}},$$

$$c(q) = \frac{3k(1+p)^{4/3}p^{1/3}}{2^{1/3}},$$

$$c(-q) = -2^{1/3}3k(p(1+p))^{1/3},$$

$$c(q^2) = \frac{3kp^{2/3}(1+p)^{2/3}}{2^{2/3}},$$

$$c(q^4) = \frac{3kp^{4/3}(1+p)^{1/3}}{2^{4/3}}.$$

Now, we explain the concept of (p, k) -parametrization as explained in [11]. Using the identities listed above, we derive the representations $\Omega(p, k)$ for $\Omega(\omega(q), \omega(-q), \omega(q^2), \omega(q^4))$ in terms of p and k . The said representation in terms of (p, k) is a polynomial in p and k , and we want to show that

$$\Omega(p, k) = \sum_{i=1}^n a_i T_i, \quad (17)$$

where every a_i is a rational number and T_i is the product involving $M(q^n)$ or $mL(q^m) - L(q)$. Substituting the representations for T_i in terms of p and k in (17), then both sides of (17) are functions of p and k . Further on equating the

coefficients of $p^i k^j$ on both sides of (17), a set of linear equations in a_i are deduced. If these equations have a solution, then by using Maple, we solve the equations and evaluate the values of a_i , which gives the Eisenstein series identities involving Borweins' cubic theta functions.

3. Main Results

Theorem 1. *One has*

$$\begin{aligned}\frac{b^4(q^4)}{b^2(q^2)} &= 16 - 12 \sum_{m=1}^{\infty} \left(\frac{3mq^{12m}}{1-q^{12m}} + \frac{mq^{4m}}{1-q^{4m}} - \frac{4mq^{2m}}{1-q^{2m}} \right), \\ \frac{b^4(q^2)}{b^2(q^4)} &= -1 - 6 \sum_{m=1}^{\infty} \left(\frac{3mq^{6m}}{1-q^{6m}} - \frac{16mq^{4m}}{1-q^{4m}} + \frac{mq^{2m}}{1-q^{2m}} \right), \\ \frac{b^3(-q)b(q^2)}{b^2(q)} &= 1 + \frac{1}{3} \sum_{m=1}^{\infty} \left(\frac{195mq^{12m}}{1-q^{12m}} - \frac{619mq^{6m}}{1-q^{6m}} + \frac{159mq^{4m}}{1-q^{4m}} \right. \\ &\quad \left. + \frac{175mq^{3m}}{1-q^{3m}} + \frac{51mq^{2m}}{1-q^{2m}} + \frac{39mq^m}{1-q^m} \right), \\ \frac{c^4(q^2)c(q)}{c^3(-q)} &= \frac{225}{128} + \frac{1}{32} \sum_{m=1}^{\infty} \left(-\frac{135mq^{12m}}{1-q^{12m}} + \frac{81mq^{6m}}{1-q^{6m}} - \frac{189mq^{4m}}{1-q^{4m}} \right. \\ &\quad \left. - \frac{171mq^{3m}}{1-q^{3m}} + \frac{639mq^{2m}}{1-q^{2m}} + \frac{351mq^m}{1-q^m} \right), \\ \frac{c^4(q)c(q^2)}{c^3(-q)} &= 1 - \frac{1}{4} \sum_{m=1}^{\infty} \left(-\frac{660mq^{12m}}{1-q^{12m}} + \frac{108mq^{6m}}{1-q^{6m}} + \frac{2228mq^{4m}}{1-q^{4m}} \right. \\ &\quad \left. - \frac{1545mq^{3m}}{1-q^{3m}} - \frac{60mq^{2m}}{1-q^{2m}} + \frac{25mq^m}{1-q^m} \right), \\ \frac{c^4(q^2)c(q^4)}{c^3(-q)} &= \frac{255}{256} - \frac{1}{64} \sum_{m=1}^{\infty} \left(\frac{1620mq^{12m}}{1-q^{12m}} - \frac{702mq^{6m}}{1-q^{6m}} + \frac{1332mq^{4m}}{1-q^{4m}} \right. \\ &\quad \left. + \frac{81mq^{3m}}{1-q^{3m}} - \frac{1206mq^{2m}}{1-q^{2m}} + \frac{225mq^m}{1-q^m} \right), \\ \frac{c^3(q)c(q^2)}{2^{2/3}c^2(-q)} &= \frac{9}{4} \sum_{m=1}^{\infty} \left(\frac{5220mq^{12m}}{1-q^{12m}} - \frac{156mq^{6m}}{1-q^{6m}} + \frac{12mq^{4m}}{1-q^{4m}} \right. \\ &\quad \left. + \frac{11mq^{3m}}{1-q^{3m}} + \frac{2mq^{2m}}{1-q^{2m}} + \frac{mq^m}{1-q^m} \right), \\ \frac{b^4(q^2)b^2(-q)}{b^2(q)b^2(q^4)} &= 1 + 12 \sum_{m=1}^{\infty} \left(\frac{36mq^{12m}}{1-q^{12m}} - \frac{54mq^{6m}}{1-q^{6m}} + \frac{4mq^{4m}}{1-q^{4m}} \right. \\ &\quad \left. + \frac{9mq^{3m}}{1-q^{3m}} + \frac{2mq^{2m}}{1-q^{2m}} + \frac{mq^m}{1-q^m} \right), \\ \frac{b^4(q^2)}{b(q)b(-q)} &= 1 - 3 \sum_{m=1}^{\infty} \left(\frac{18mq^{12m}}{1-q^{12m}} - \frac{9mq^{6m}}{1-q^{6m}} - \frac{2mq^{4m}}{1-q^{4m}} + \frac{mq^{2m}}{1-q^{2m}} \right), \\ \frac{b^3(q)b(q^2)}{b(-q)b(q^4)} &= 1 - 12 \sum_{m=1}^{\infty} \left(\frac{9mq^{3m}}{1-q^{3m}} - \frac{8mq^{2m}}{1-q^{2m}} + \frac{mq^m}{1-q^m} \right), \\ \frac{b^3(q^4)b(q^2)}{b(q)b(-q)} &= 1 + 3 \sum_{m=1}^{\infty} \left(-\frac{9mq^{12m}}{1-q^{12m}} - \frac{mq^{4m}}{1-q^{4m}} + \frac{2mq^{2m}}{1-q^{2m}} \right),\end{aligned}$$

$$\frac{b^4(q^2)b(q)}{b^2(-q)b(q^4)} = 1 - 3 \sum_{m=1}^{\infty} \left(\frac{16mq^{4m}}{1-q^{4m}} + \frac{9mq^{3m}}{1-q^{3m}} + \frac{20mq^{2m}}{1-q^{2m}} + \frac{3mq^m}{1-q^m} \right),$$

$$\frac{b^4(q^2)b(q)}{b(-q)b^2(q^4)} = 1 - 6 \sum_{m=1}^{\infty} \left(-\frac{16mq^{4m}}{1-q^{4m}} + \frac{9mq^{3m}}{1-q^{3m}} + \frac{4mq^{2m}}{1-q^{2m}} + \frac{mq^m}{1-q^m} \right),$$

$$\frac{b^4(q^2)b(q^4)}{b^2(q)b(-q)} = 1 + 3 \sum_{m=1}^{\infty} \left(-\frac{9mq^{12m}}{1-q^{12m}} + \frac{mq^{4m}}{1-q^{4m}} - \frac{mq^{2m}}{1-q^{2m}} + \frac{mq^m}{1-q^m} \right),$$

$$\frac{b^4(q^2)b(q^4)}{b(q)b^2(-q)} = 1 - 3 \sum_{m=1}^{\infty} \left(\frac{9mq^{12m}}{1-q^{12m}} + \frac{3mq^{4m}}{1-q^{4m}} - \frac{5mq^{2m}}{1-q^{2m}} + \frac{mq^m}{1-q^m} \right).$$

Proof. If suppose to be the case

$$\begin{aligned}a_5(12L(q^{12}) - L(q)) + a_4(6L(q^6) - L(q)) + a_3(4L(q^4) - L(q)) \\ + a_2(3L(q^3) - L(q)) + a_1(2L(q^2) - L(q)) = \frac{b^4(q^4)}{b^2(q^2)}.\end{aligned}\quad (19)$$

On equating the various powers of $p^4 k^2, p^3 k^2, p^2 k^2, pk^2, k^2$ on both sides of the above, it is observed that

$$\begin{aligned}11a_5 + 5a_4 + 3a_3 + 2a_2 + a_1 &= 16, \\ 34a_5 + 22a_4 + 18a_3 + 16a_2 + 14a_1 &= 32, \\ 36a_5 + 36a_4 + 36a_3 + 36a_2 + 24a_1 &= 24, \\ 16a_5 + 22a_4 + 24a_3 + 16a_2 + 14a_1 &= 8, \\ 2a_5 + 5a_4 + 2a_2 + a_1 &= 1.\end{aligned}$$

On solving the above five equations, one can see that

$$a_5 = \frac{3}{2}, a_4 = 0, a_3 = \frac{1}{2}, a_2 = 0, a_1 = -2. \quad (21)$$

Substituting (21) in (19) and using (1), the desired result is obtained. In the same way, applying this technique, the following identities are also derived:

$$\begin{aligned}\frac{b^4(q^2)}{b^2(q^4)} &= -\frac{3}{4}(L(q) - 6L(q^6)) + \frac{1}{4}(L(q) - 4L(q^4)) \\ &\quad - \frac{1}{4}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{b^3(-q)b(q^2)}{b^2(q)} &= \frac{65}{24}(L(q) - 12L(q^{12})) - \frac{619}{72}(L(q) - 6L(q^6)) \\ &\quad + \frac{53}{24}(L(q) - 4L(q^4)) + \frac{17}{24}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{c^4(q^2)c(q)}{c^3(-q)} &= -\frac{45}{256}(L(q) - 12L(q^{12})) + \frac{27}{256}(L(q) - 6L(q^6)) \\ &\quad - \frac{63}{256}(L(q) - 4L(q^4)) - \frac{57}{256}(L(q) - 3L(q^3)) \\ &\quad - \frac{213}{256}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{c^4(q)c(q^2)}{c^3(-q)} &= \frac{55}{96}(L(q) - 12L(q^{12})) - \frac{3}{16}(L(q) - 6L(q^6)) \\ &\quad - \frac{557}{96}(L(q) - 4L(q^4)) + \frac{515}{96}(L(q) - 3L(q^3)) \\ &\quad + \frac{5}{16}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{c^4(q^2)c(q^4)}{c^3(-q)} &= -\frac{45}{512}(L(q) - 12L(q^{12})) + \frac{39}{512}(L(q) - 6L(q^6)) \\ &\quad - \frac{111}{512}(L(q) - 4L(q^4)) - \frac{9}{512}(L(q) - 3L(q^3)) \\ &\quad + \frac{201}{512}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{c^3(q)c(q^2)}{2^{2/3}c^2(-q)} &= \frac{13}{32}(L(q) - 12L(q^{12})) - \frac{39}{32}(L(q) - 6L(q^6)) \\ &\quad + \frac{9}{32}(L(q) - 4L(q^4)) + \frac{11}{32}(L(q) - 3L(q^3)) \\ &\quad + \frac{3}{32}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{b^4(q^2)b^2(-q)}{b^2(q)b^2(q^4)} &= \frac{3}{2}(L(q) - 12L(q^{12})) - \frac{9}{2}(L(q) - 6L(q^6)) \\ &\quad + \frac{1}{2}(L(q) - 4L(q^4)) + \frac{3}{2}(L(q) - 3L(q^3)) \\ &\quad + \frac{1}{2}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{b^4(q^2)}{b(q)b(-q)} &= -\frac{3}{16}(L(q) - 12L(q^{12})) + \frac{3}{16}(L(q) - 6L(q^6)) \\ &\quad + \frac{1}{16}(L(q) - 4L(q^4)) - \frac{1}{16}(L(q) - 2L(q^2)),\end{aligned}$$

$$\frac{b^3(q)b(q^2)}{b(-q)b(q^4)} = -\frac{3}{2}(L(q) - 3L(q^3)) + 2(L(q) - 2L(q^2)),$$

$$\begin{aligned}\frac{b^3(q^4)b(q^2)}{b(q)b(-q)} &= -\frac{3}{32}(L(q) - 12L(q^{12})) - \frac{1}{32}(L(q) - 4L(q^4)) \\ &\quad + \frac{1}{8}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{b^4(q^2)b(q)}{b^2(-q)b(q^4)} &= -\frac{1}{2}(L(q) - 4L(q^4)) - \frac{3}{8}(L(q) - 3L(q^3)) \\ &\quad + \frac{5}{4}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{b^4(q^2)b(q)}{b(-q)b^2(q^4)} &= -(L(q) - 4L(q^4)) + \frac{3}{4}(L(q) - 3L(q^3)) \\ &\quad + \frac{1}{2}(L(q) - 2L(q^2)),\end{aligned}$$

$$\begin{aligned}\frac{b^4(q^2)b(q^4)}{b^2(q)b(-q)} &= -\frac{3}{32}(L(q) - 12L(q^{12})) - \frac{1}{16}(L(q) - 2L(q^2)) \\ &\quad + \frac{1}{32}(L(q) - 4L(q^4)),\end{aligned}$$

$$\begin{aligned}\frac{b^4(q^2)b(q^4)}{b(q)b^2(-q)} &= -\frac{3}{32}(L(q) - 12L(q^{12})) + \frac{5}{16}(L(q) - 2L(q^2)) \\ &\quad - \frac{3}{32}(L(q) - 4L(q^4)).\end{aligned}$$

Theorem 2. One has

$$\frac{c^3(q)c^2(-q)}{2^{1/3}c(q^2)} = -17 + 162 \sum_{m=1}^{\infty} \left(\frac{m^3 q^m}{1-q^m} - \frac{8m^3 q^{2m}}{1-q^{2m}} - \frac{21m^3 q^{3m}}{1-q^{3m}} + \frac{28m^3 q^{6m}}{1-q^{6m}} \right),$$

$$\frac{c^3(q^4)c^2(-q)}{2^{1/3}c(q^2)} = 1296 \sum_{m=1}^{\infty} \left(\frac{m^3 q^{2m}}{1-q^{2m}} - \frac{6m^3 q^{3m}}{1-q^{3m}} + \frac{5m^3 q^{6m}}{1-q^{6m}} \right),$$

$$\frac{c^3(-q)c^2(q)}{c(q^2)} = 324 \sum_{m=1}^{\infty} \left(-\frac{m^3 q^m}{1-q^m} - \frac{10m^3 q^{2m}}{1-q^{2m}} - \frac{21m^3 q^{3m}}{1-q^{3m}} + \frac{30m^3 q^{6m}}{1-q^{6m}} \right),$$

$$\frac{c^3(-q)c^2(q^2)}{c(q^4)} = 324 \sum_{m=1}^{\infty} \left(-\frac{m^3 q^m}{1-q^m} + \frac{12m^3 q^{2m}}{1-q^{2m}} + \frac{17m^3 q^{3m}}{1-q^{3m}} - \frac{28m^3 q^{6m}}{1-q^{6m}} \right),$$

$$\begin{aligned}b^2(q^2)b^2(q^4) &= \frac{5}{24} - \frac{1}{12} \sum_{m=1}^{\infty} \left(\frac{95m^3 q^m}{1-q^m} + \frac{270m^3 q^{2m}}{1-q^{2m}} \right. \\ &\quad \left. + \frac{912m^3 q^{3m}}{1-q^{3m}} - \frac{582m^3 q^{6m}}{1-q^{6m}} + \frac{95m^3 q^{12m}}{1-q^{12m}} \right),\end{aligned}$$

$$\begin{aligned}\frac{b^3(q)b^2(q^2)}{b(q^4)} &= \frac{25}{3} + \frac{1}{3} \sum_{m=1}^{\infty} \left(\frac{193m^3 q^m}{1-q^m} - \frac{486m^3 q^{2m}}{1-q^{2m}} \right. \\ &\quad \left. - \frac{1221m^3 q^{3m}}{1-q^{3m}} + \frac{7734m^3 q^{6m}}{1-q^{6m}} - \frac{220m^3 q^{12m}}{1-q^{12m}} \right),\end{aligned}$$

$$\begin{aligned}\frac{b^3(q)b^2(q^4)}{b(q^2)} &= \frac{685}{72} + \frac{1}{36} \sum_{m=1}^{\infty} \left(\frac{2741m^3 q^m}{1-q^m} - \frac{7650m^3 q^{2m}}{1-q^{2m}} \right. \\ &\quad \left. - \frac{17976m^3 q^{3m}}{1-q^{3m}} + \frac{108150m^3 q^{6m}}{1-q^{6m}} - \frac{3065m^3 q^{12m}}{1-q^{12m}} \right),\end{aligned}$$

$$\begin{aligned}\frac{b^5(q)}{b^2(q^2)} &= 10 + 3 \sum_{m=1}^{\infty} \left(\frac{25m^3 q^m}{1-q^m} - \frac{32m^3 q^{2m}}{1-q^{2m}} - \frac{243m^3 q^{3m}}{1-q^{3m}} \right. \\ &\quad \left. + \frac{960m^3 q^{6m}}{1-q^{6m}} - \frac{30m^3 q^{12m}}{1-q^{12m}} \right),\end{aligned}$$

$$\begin{aligned}\frac{b^5(q^2)}{b(q^4)} &= -\frac{13}{6} - \frac{1}{3} \sum_{m=1}^{\infty} \left(\frac{95m^3 q^m}{1-q^m} + \frac{297m^3 q^{2m}}{1-q^{2m}} + \frac{912m^3 q^{3m}}{1-q^{3m}} \right. \\ &\quad \left. - \frac{2769m^3 q^{6m}}{1-q^{6m}} + \frac{95m^3 q^{12m}}{1-q^{12m}} \right),\end{aligned}$$

$$\frac{b^5(q^2)}{b(-q)} = \frac{13}{24} - \frac{1}{12} \sum_{m=1}^{\infty} \left(\frac{91m^3 q^m}{1-q^m} + \frac{450m^3 q^{2m}}{1-q^{2m}} + \frac{1824m^3 q^{3m}}{1-q^{3m}} - \frac{678m^3 q^{6m}}{1-q^{6m}} + \frac{55m^3 q^{12m}}{1-q^{12m}} \right),$$

$$\frac{b^5(q^4)}{b(q^2)} = \frac{59}{40} + \frac{1}{12} \sum_{m=1}^{\infty} \left(\frac{55m^3 q^m}{1-q^m} - \frac{162m^3 q^{2m}}{1-q^{2m}} - \frac{528m^3 q^{3m}}{1-q^{3m}} + \frac{4890m^3 q^{6m}}{1-q^{6m}} - \frac{55m^3 q^{12m}}{1-q^{12m}} \right),$$

$$\frac{b^4(q)b(q^4)}{b^2(q^2)} = \frac{55}{3} + \frac{1}{3} \sum_{m=1}^{\infty} \left(\frac{418m^3 q^m}{1-q^m} - \frac{774m^3 q^{2m}}{1-q^{2m}} - \frac{3408m^3 q^{3m}}{1-q^{3m}} + \frac{17454m^3 q^{6m}}{1-q^{6m}} - \frac{490m^3 q^{12m}}{1-q^{12m}} \right),$$

$$\frac{b^4(q^4)b(q)}{b^2(q^2)} = \frac{43}{24} + \frac{1}{12} \sum_{m=1}^{\infty} \left(\frac{59m^3 q^m}{1-q^m} + \frac{18m^3 q^{2m}}{1-q^{2m}} + \frac{60m^3 q^{3m}}{1-q^{3m}} + \frac{5118m^3 q^{6m}}{1-q^{6m}} - \frac{95m^3 q^{12m}}{1-q^{12m}} \right).$$

Proof. If suppose to be the case

$$a_1 M(q) + a_2 M(q^2) + a_3 M(q^3) + a_4 M(q^6) = \frac{c^3(q)c^2(-q)}{2^{1/3}c(q^2)}. \quad (24)$$

On equating the various powers of $p^4 k^2, p^3 k^2, p^2 k^2, p k^2, k^2$ on both sides of the above, it is observed that

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0, \\ 124a_1 + 4a_2 + 4a_3 + 4a_4 &= 81, \\ 964a_1 + 64a_2 + 4a_3 + 4a_4 &= 324, \\ 2788a_1 + 178a_2 + 28a_3 - 2a_4 &= 48. \end{aligned}$$

On solving the above, it is easy to see that

$$a_1 = \frac{27}{40}, a_2 = -\frac{27}{5}, a_3 = -\frac{567}{40}, a_4 = \frac{189}{10}. \quad (26)$$

Substituting (26) in (24) and using (2), the desired result is obtained. In the same way, applying this technique, the following identities are deduced:

$$\frac{c^3(q^4)c^2(-q)}{2^{1/3}c(q^2)} = \frac{27}{5} M(q^2) - \frac{162}{5} M(q^3) + 27 M(q^6),$$

$$\frac{c^3(-q)c^2(q)}{c(q^2)} = -\frac{27}{20} M(q) + \frac{27}{2} M(q^2) + \frac{567}{20} M(q^3) - \frac{81}{2} M(q^6),$$

$$\frac{c^3(-q)c^2(q^2)}{c(q^4)} = -\frac{27}{20} M(q) + \frac{81}{5} M(q^2) + \frac{459}{20} M(q^3) - \frac{189}{5} M(q^6),$$

$$\begin{aligned} b^2(q^2)b^2(q^4) &= -\frac{19}{576} M(q) + \frac{3}{32} M(q^2) + \frac{19}{60} M(q^3) \\ &\quad - \frac{97}{480} M(q^6) + \frac{19}{576} M(q^{12}), \end{aligned}$$

$$\begin{aligned} \frac{b^3(q)b^2(q^2)}{b(q^4)} &= \frac{193}{720} M(q) - \frac{27}{40} M(q^2) - \frac{407}{240} M(q^3) \\ &\quad + \frac{1289}{120} M(q^6) - \frac{11}{36} M(q^{12}), \end{aligned}$$

$$\begin{aligned} \frac{b^3(q)b^2(q^4)}{b(q^2)} &= \frac{2741}{8640} M(q) - \frac{85}{96} M(q^2) - \frac{749}{360} M(q^3) \\ &\quad + \frac{3605}{288} M(q^6) - \frac{613}{1728} M(q^{12}), \end{aligned}$$

$$\frac{b^5(q)}{b^2(q^2)} = \frac{5}{16} M(q) - \frac{2}{5} M(q^2) - \frac{243}{80} M(q^3) + \frac{27}{2} M(q^6) - \frac{3}{8} M(q^{12}),$$

$$\begin{aligned} \frac{b^5(q^2)}{b(q^4)} &= -\frac{19}{144} M(q) + \frac{33}{80} M(q^2) + \frac{19}{15} M(q^3) - \frac{923}{240} M(q^6) \\ &\quad + \frac{9}{144} M(q^{12}), \end{aligned}$$

$$\begin{aligned} \frac{b^5(q^2)}{b(-q)} &= -\frac{91}{2880} M(q) + \frac{5}{32} M(q^2) + \frac{19}{30} M(q^3) - \frac{113}{480} M(q^6) \\ &\quad + \frac{11}{576} M(q^{12}), \end{aligned}$$

$$\begin{aligned} \frac{b^5(q^4)}{b(q^2)} &= \frac{11}{576} M(q) - \frac{9}{160} M(q^2) - \frac{11}{60} M(q^3) + \frac{163}{96} M(q^6) \\ &\quad - \frac{11}{576} M(q^{12}), \end{aligned}$$

$$\begin{aligned} \frac{b^4(q)b(q^4)}{b^2(q^2)} &= \frac{209}{360} M(q) - \frac{43}{40} M(q^2) - \frac{71}{15} M(q^3) \\ &\quad + \frac{2909}{120} M(q^6) - \frac{49}{72} M(q^{12}), \end{aligned}$$

$$\begin{aligned} \frac{b^4(q^4)b(q)}{b^2(q^2)} &= \frac{59}{2880} M(q) + \frac{1}{160} M(q^2) + \frac{1}{48} M(q^3) \\ &\quad + \frac{853}{480} M(q^6) - \frac{19}{576} M(q^{12}). \end{aligned}$$

4. Conclusion and Application

Employing the technique developed by Xia and Yao [11], one can derive several relations between Eisenstein Series and Borwien's cubic theta functions using (p, k) -parametrization. As an application, one can evaluate convolution sums involving divisor sum.

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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