Hindawi
Journal of Applied Mathematics
Volume 2021, Article ID 6627272, 10 pages
https://doi.org/10.1155/2021/6627272

Research Article

Eternal Domination of Generalized Petersen Graph

Ramy Shaheen and Ali Kassem

Department of Mathematics, Faculty of Science, Tishreen University, Latakia, Syria

Correspondence should be addressed to Ramy Shaheen; shaheenramy2010@hotmail.com

Received 4 October 2020; Accepted 2 January 2021; Published 27 January 2021

Academic Editor: Jian G. Zhou

An eternal dominating set of a graph G is a set of guards distributed on the vertices of a dominating set so that each vertex can be occupied by one guard only. These guards can defend any infinite series of attacks; an attack is defended by moving one guard along an edge from its position to the attacked vertex. We consider the "all guards move" of the eternal dominating set problem, in which one guard has to move to the attacked vertex and all the remaining guards are allowed to move to an adjacent vertex or stay in their current positions after each attack in order to form a dominating set on the graph and at each step can be moved after each attack. The "all guards move model" is called the m-eternal domination model. The size of the smallest m-eternal dominating set is called the m-eternal domination number and is denoted by $\gamma_m^\infty(G)$. In this paper, we find $\gamma_m^\infty(P(n,1))$ and $\gamma_m^\infty(P(n,3))$ for $n \equiv 0 \,(\mathrm{mod}\ 4)$. We also find upper bounds for $\gamma_m^\infty(P(n,2))$ and $\gamma_m^\infty(P(n,3))$ when n is arbitrary.

1. Introduction

The term graph protection refers to the process of placing guards or mobile agents in order to defend against a sequence of attacks on a network. Go to [1–5] for more background of the graph protection problem. Burger et al. introduced the concept of eternal domination in 2004 [2]. Goddard et al. introduced the "all guards move model" in [3]. General bounds of $\gamma(G) \le \gamma_m^\infty(G) \le \alpha(G)$ were determined in [3], where $\alpha(G)$ denotes the independence number of G and $\gamma(G)$ denotes the domination number of G. The m-eternal domination numbers for cycles C_n and paths P_n were found by Goddard et al. [3] as follows: $\gamma_m^\infty(C_n) = \lceil n/3 \rceil$ and $\gamma_m^\infty(P_n) = \lceil n/2 \rceil$. For further information on eternal domination, see survey [1]. A generalized Petersen graph $P(n,k)$ is a graph with vertex set $V \cup U = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and edge set $E = \cup\{v_i v_{i+1}, v_i u_1, u_i u_{i+k}\}$ with $v_{n+1} = v_1$, $u_{n+1} = u_1$, and $1 \le i \le n$ and $1 \le k \le \lfloor n/2 \rfloor$; see [6] for more information on generalized Petersen graph. The k-dominating graph $H(G,k)$ was defined by Goldwasser et al. in [7] as follows. Let G be a graph with a dominating set of cardinality k. The vertex set of the k-dominating graph $H(G,k)$, denoted $V(H)$, is the set of all subsets of $V(G)$ of size k which are dominating sets and two vertices of H are adjacent if and only if the k guards occupying the vertices of G of one can move (at most distance one each) to the vertices of the other, $\gamma_m^\infty(G) \le k$ if and only if $H(G,k)$ has an induced subgraph $S(G,k)$ such that for each vertex x of $S(G,k)$ the union of the vertices in the closed neighborhood of x in $S(G,k)$ is equal to $V(G)$. A generalized Jahangir graph $J_{s,m}$ for $m \ge 2$ is a graph on $sm+1$ vertices, i.e., a graph consisting of a cycle C_{sm} with one additional vertex which is adjacent to m vertices of C_{sm} at distance s from each other on C_{sm}; see [8] for more information on the Jahangir graph. In [9], we found $\gamma_m^\infty(J_{s,m})$ for $s \equiv 2, 3$. In [10], we found $\gamma_m^\infty(J_{s,m})$ for s, m are arbitraries.

Proposition 1 [11]. *For $n \ge 3$, we have* $\gamma(P(n,1)) =$

$$\begin{cases} \lceil n/2 \rceil \text{ for } n \equiv 0, 1, 3 (\mathrm{mod}\ 4), \\ (n/2) + 1 \text{ for } n \equiv 2 (\mathrm{mod}\ 4). \end{cases}$$

Proposition 2 [11]. *For $n \geq 5$, we have $\gamma(P(n,2)) = \lceil 3n/5 \rceil$.*

Proposition 3 [11]. *For $n \geq 7$, we have $\gamma(P(n,3)) =$*

$$\begin{cases} (n/2)+1 \text{ for } n \equiv 2 (\bmod\ 4), \\ \lceil n/2 \rceil \text{ for } n \equiv 0, 1 (\bmod\ 4) \text{ or } n = 11, \\ \lceil n/2 \rceil + 1 \text{ for } n \equiv 3 (\bmod\ 4), n \neq 11. \end{cases}$$

Proposition 4 [12]. *For $n \geq 5$, we have $\alpha(P(n,2)) = \lfloor 4n/5 \rfloor$.*

Proposition 5 [12]. *For $n \geq 7$, we have $\alpha(P(n,3)) =$*

$$\begin{cases} n \text{ if } n \text{ is even}, \\ n-2 \text{ if } n \text{ is odd}. \end{cases}$$

Proposition 6 [3]. *For any graph $G : \gamma(G) \leq \gamma_m^{\infty}(G) \leq \alpha(G)$.*

2. Main Results

2.1. Eternal Domination Number of $P(n,1)$. In this section, we find the m-eternal domination number of $P(n,1)$ for n is arbitrary.

Theorem 1. $\gamma_m^{\infty}(P(n,1)) = \begin{cases} \lceil n/2 \rceil \text{ for } n \equiv 0, 1, 3 (\bmod\ 4), \\ (n/2)+1 \text{ for } n \equiv 2 (\bmod\ 4). \end{cases}$

Proof. We consider all four cases of n.

Case 1 ($n \equiv 0 (\bmod\ 4)$). Let $n = 4l : l \in N$. As it was found in [11] that $\gamma(P(n,1)) = n/2$ for this specific case and the γ-dominating set S is the union of two sets A and B which are $A = \{v_{4i+1} ; 0 \leq i \leq l-1\}$, $B = \{u_{4i+3} ; 0 \leq i \leq l-1\}$.

We form the k-dominating graph $H(G,k)$ where $G = P(n,1)$, $k = n/2$, and with sets

$$\begin{aligned} D_1 &= A_1 \cup B_1 = \{v_{4i+1} ; 0 \leq i \leq l-1\} \cup \{u_{4i+3} ; 0 \leq i \leq l-1\}, \\ D_2 &= A_2 \cup B_2 = \{v_{4i+2} ; 0 \leq i \leq l-1\} \cup \{u_{4i+4} ; 0 \leq i \leq l-1\}, \\ D_3 &= A_3 \cup B_3 = \{v_{4i+3} ; 0 \leq i \leq l-1\} \cup \{u_{4i+1} ; 0 \leq i \leq l-1\}, \\ D_4 &= A_4 \cup B_4 = \{v_{4i+4} ; 0 \leq i \leq l-1\} \cup \{u_{4i+2} ; 0 \leq i \leq l-1\}. \end{aligned} \tag{1}$$

Each of these sets D_1, D_2, D_3, D_4 has a cardinality of $n/2$ and they are adjacent in $H(P(n,1), n/2)$ for the following reasons.

$D_1 \longrightarrow D_2$, $D_2 \longrightarrow D_3$, $D_3 \longrightarrow D_4$, $D_4 \longrightarrow D_1$ by rotating the guards one step clockwise.

$D_2 \longrightarrow D_1$, $D_3 \longrightarrow D_2$, $D_4 \longrightarrow D_3$, $D_1 \longrightarrow D_4$ by rotating the guards one step counterclockwise.

$D_1 \longrightarrow D_3$, $D_3 \longrightarrow D_1$, $D_2 \longrightarrow D_4$, $D_4 \longrightarrow D_2$ by reflection which means moving the guards one step to occupy the corresponding unoccupied vertex on the other cycle.

Therefore, D_1, D_2, D_3, D_4 are adjacent in $S(P(n,1), n/2)$ (the induced subgraph of $H(P(n,1), n/2)$) on D_1, D_2, D_3, D_4 and $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,1))$; therefore, we conclude that $\gamma_m^{\infty}(P(n,1)) \leq n/2$ for $n \equiv 0 (\bmod\ 4)$ but $\gamma_m^{\infty}(P(n,1)) \geq \gamma(P(n,1))$ from the definition of eternal domination; therefore, $\gamma_m^{\infty}(P(n,1)) = \gamma(P(n,1)) = n/2$ for $n \equiv 0 (\bmod\ 4)$; see Figure 1.

Case 2 ($n \equiv 1 (\bmod\ 4)$). Let $n = 4l+1 : l \in N$. It was found in [11] that $\gamma(P(n,1)) = \lceil n/2 \rceil$ for this case and it was found that the γ-dominating set S is $S = A \cup B \cup \{x\}$: $x \in \{v_n, u_n\}$.

We form the k-dominating graph $H(P(n,1), \lceil n/2 \rceil)$ with the following sets of cardinality $\lceil n/2 \rceil$:

$$\begin{aligned} D_1 &= \{v_{4i+1} ; 0 \leq i \leq l-1\} \cup \{u_{4i+3} ; 0 \leq i \leq l-1\} \cup \{v_n\}, \\ D_2 &= \{v_{4i+2} ; 0 \leq i \leq l-1\} \cup \{u_{4i+4} ; 0 \leq i \leq l-1\} \cup \{u_n\}, \\ D_3 &= \{v_{4i+3} ; 0 \leq i \leq l-1\} \cup \{u_{4i+1} ; 0 \leq i \leq l-1\} \cup \{u_n\}, \\ D_4 &= \{v_{4i+4} ; 0 \leq i \leq l-1\} \cup \{u_{4i+2} ; 0 \leq i \leq l-1\} \cup \{v_n\}. \end{aligned} \tag{2}$$

It is easy to notice that D_1, D_2, D_3, D_4 are all adjacent in $H(P(n,1), \lceil n/2 \rceil)$ and the union $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,1))$; therefore, $\gamma_m^{\infty}(P(n,1)) \leq \lceil n/2 \rceil$ for $n \equiv 1 (\bmod\ 4)$ but we have $\gamma_m^{\infty}(P(n,1)) \geq \gamma(P(n,1))$ from the definition of eternal domination; therefore, $\gamma_m^{\infty}(P(n,1)) = \gamma(P(n,1)) = \lceil n/2 \rceil$ for $n \equiv 1 (\bmod\ 4)$; see Figure 2.

Case 3 ($n \equiv 2 (\bmod\ 4)$). Let $n = 4l+2 : l \in N$. As it was found in [11], $\gamma(P(n,1)) = (n/2)+1$ for this specific case and the γ-dominating set S is $S = A \cup B \cup \{x, y\}$: $x, y \in \{v_{n-1}, v_n, u_{n-1}, u_n\}$.

We form the k-dominating graph $H(P(n,1), (n/2)+1)$ with the following sets of the cardinality $(n/2)+1$:

$$\begin{aligned} D_1 &= \{v_{4i+1} ; 0 \leq i \leq l-1\} \cup \{u_{4i+3} ; 0 \leq i \leq l-1\} \cup \{v_{n-1}, u_n\}, \\ D_2 &= \{v_{4i+2} ; 0 \leq i \leq l-1\} \cup \{u_{4i+4} ; 0 \leq i \leq l-1\} \cup \{v_n, u_n\}, \\ D_3 &= \{v_{4i+3} ; 0 \leq i \leq l-1\} \cup \{u_{4i+1} ; 0 \leq i \leq l-1\} \cup \{v_{n-1}, u_{n-1}\}, \\ D_4 &= \{v_{4i+4} ; 0 \leq i \leq l-1\} \cup \{u_{4i+2} ; 0 \leq i \leq l-1\} \cup \{v_n, u_{n-1}\}. \end{aligned} \tag{3}$$

It is easy to notice that D_1, D_2, D_3, D_4 are all adjacent in $H(P(n,1), (n/2)+1)$ and the union of sets $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,1))$; therefore, $\gamma_m^{\infty}(P(n,1)) \leq n/2+1$ for $n \equiv 1 (\bmod\ 4)$, $\gamma_m^{\infty}(P(n,1)) \geq \gamma(P(n,1))$; therefore, $\gamma_m^{\infty}(P(n,1)) = \gamma(P(n,1)) = (n/2)+1$; see Figure 3.

Case 4 ($n \equiv 3 (\bmod\ 4)$). Let $n = 4l+3 : l \in N$. Ebrahimi et al. found in [11] that $\gamma(P(n,1)) = \lceil n/2 \rceil$ for this case and the γ-dominating set S is $S = A \cup B \cup \{x, y\}$: $x, y \in \{v_{n-2}, v_{n-1}, v_n, u_{n-2}, u_{n-1}, u_n\}$.

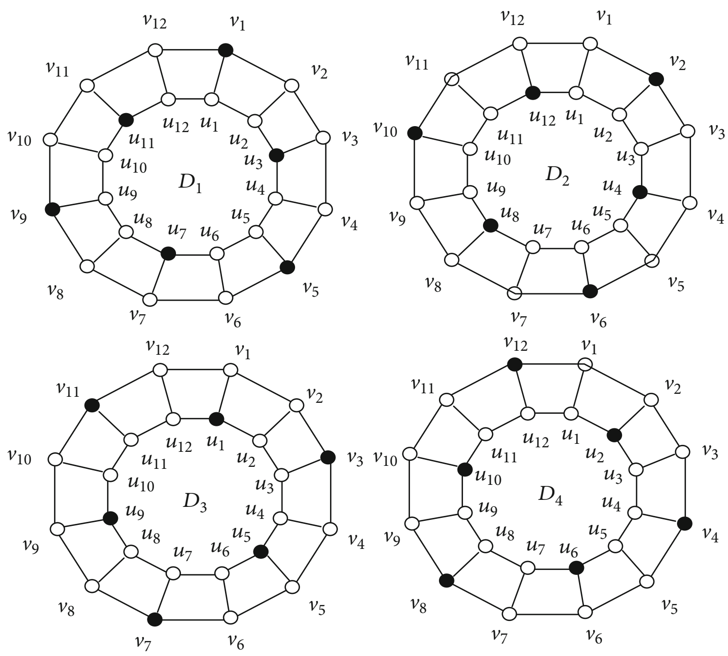

Figure 1: $\gamma_m^\infty(P(12,1)) = \gamma(P(12,1)) = 6$.

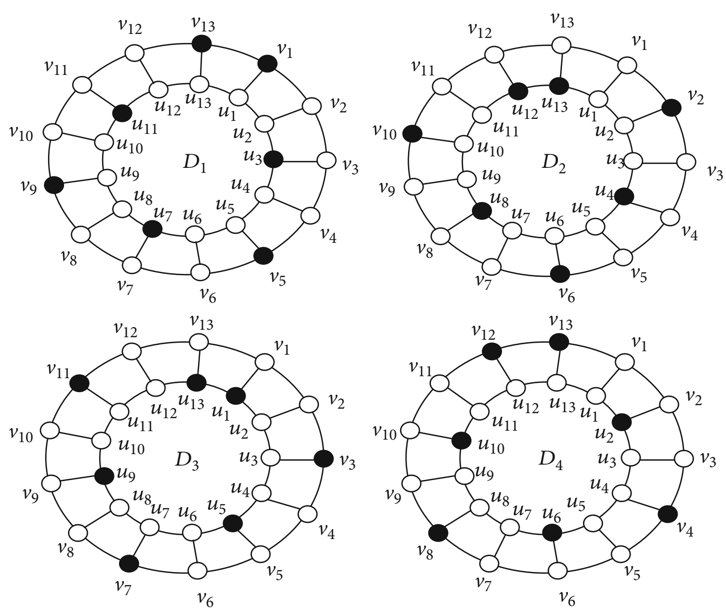

Figure 2: $\gamma_m^\infty(P(13,1)) = \gamma(P(13,1)) = 7$.

We form the k-dominating graph $H(P(n,1), \lceil n/2 \rceil)$ with the following sets of cardinality $\lceil n/2 \rceil$:

$$D_1 = \{v_{4i+1}\,; 0 \le i \le l-1\} \cup \{u_{4i+3}\,; 0 \le i \le l-1\} \cup \{v_{n-2}, u_n\},$$
$$D_2 = \{v_{4i+2}\,; 0 \le i \le l-1\} \cup \{u_{4i+4}\,; 0 \le i \le l-1\} \cup \{v_{n-1}, u_n\},$$
$$D_3 = \{v_{4i+3}\,; 0 \le i \le l-1\} \cup \{u_{4i+1}\,; 0 \le i \le l-1\} \cup \{u_{n-2}, v_n\},$$
$$D_4 = \{v_{4i+4}\,; 0 \le i \le l-1\} \cup \{u_{4i+2}\,; 0 \le i \le l-1\} \cup \{v_{n-1}, v_n\}. \tag{4}$$

It is easy to notice that D_1, D_2, D_3, D_4 are all adjacent in $H(P(n,1), \lceil n/2 \rceil)$ and the union $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,1))$; therefore, $\gamma_m^\infty(P(n,1)) \le \lceil n/2 \rceil$ for $n \equiv 3 \pmod 4$ but $\gamma_m^\infty(P(n,1)) \ge \gamma(P(n,1))$; therefore, $\gamma_m^\infty(P(n,1)) = \gamma(P(n,1)) = \lceil n/2 \rceil$ for $n \equiv 3 \pmod 4$; see Figure 4. From Cases 1–4, we conclude that $\gamma_m^\infty(P(n,1)) = \gamma(P(n,1))$ for $n \ge 3$.

2.2. Eternal Domination Number of $P(n,2)$. In this section, we give an upper bound for the eternal domination number of generalized Petersen graph $P(n,2)$.

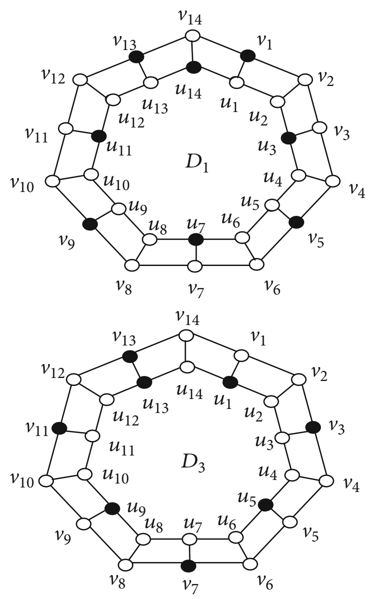

Figure 3: $\gamma_m^\infty(P(14,1)) = \gamma(P(14,1)) = 8$.

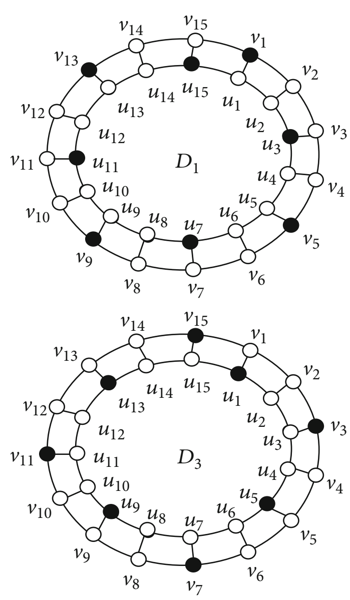

Figure 4: $\gamma_m^\infty(P(15,1)) = \gamma(P(15,1)) = 8$.

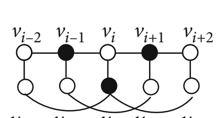

Figure 5: Dominating B_j with 3 vertices.

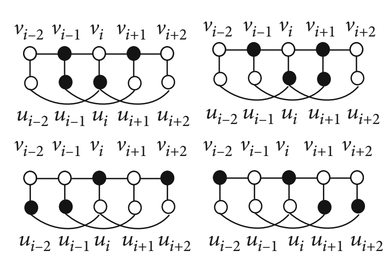

Figure 6: Eternally dominating B_j with 4 vertices.

Lemma 1. *If we divide $P(n,2)$ into $\lfloor n/5 \rfloor$ blocks where each block consists of 10 vertices (5 vertices from the outer cycle and their 5 adjacent vertices from the inner cycle), then 4 guards are enough to eternally dominate each block.*

Proof. We denote these blocks by $B_j : 1 \le j \le \lfloor n/5 \rfloor$; we also denote the subblock that remains in case $n \not\equiv 0 \pmod 5$ by SB keeping in mind that SB consists of 2, 4, 6, and 8 vertices when we have $n \equiv 1, 2, 3, 4 \pmod 5$, respectively. It was found in [11] that three vertices are enough to dominate each block B_j if the dominating vertices were distributed as shown in Figure 5.

To prove that four guards can eternally protect any block B_j it is enough to find a set of configurations for the guards so that these configurations cover all the vertices of B_j (every vertex of B_j belongs to at least one configuration) and every two configurations are reachable from each other in one step; we notice that the configurations presented in Figure 6 can achieve that; therefore, according to the definition of the k-dominating graph, we conclude that $\gamma_m^\infty(B_j) \le 4$.

Theorem 2. *For $n \ge 5$,*

$$\begin{aligned}
&\gamma_m^\infty(P(n,2)) = 4\,;\, n \in \{5, 6\},\\
&\gamma_m^\infty(P(7,2)) = 5,\\
&\gamma_m^\infty(P(8,2)) \in \{5, 6\},\\
&\gamma_m^\infty(P(n,2)) \le \left\lfloor \frac{3n}{5} \right\rfloor + \left\lceil \frac{n}{10} \right\rceil ;\, n \ge 9 \text{ and } n \equiv 0 \pmod{10},\\
&\gamma_m^\infty(P(n,2)) \le \left\lfloor \frac{3n}{5} \right\rfloor + \left\lceil \frac{n}{10} \right\rceil ;\, n \ge 9 \text{ and } n \equiv 0 \pmod 5 \text{ and } n \not\equiv 0 \pmod{10},\\
&\gamma_m^\infty(P(n,2)) \le \left\lfloor \frac{3n}{5} \right\rfloor + \left\lceil \frac{n}{10} \right\rceil + l\,;\, n \equiv l \pmod{10} \text{ and } l = 1, 2, 3,\\
&\gamma_m^\infty(P(n,2)) \le \left\lfloor \frac{3n}{5} \right\rfloor + \left\lceil \frac{n}{10} \right\rceil + l\,;\, n \equiv l \pmod 5 \text{ and } n \not\equiv l \pmod{10} \text{ and } l = 1, 2, 3,\\
&\gamma_m^\infty(P(n,2)) \le \left\lfloor \frac{3n}{5} \right\rfloor + \left\lceil \frac{n}{10} \right\rceil + 2\,;\, n \equiv 4 \pmod{10},\\
&\gamma_m^\infty(P(n,2)) \le \left\lfloor \frac{3n}{5} \right\rfloor + \left\lceil \frac{n}{10} \right\rceil + 3\,;\, n \equiv 4 \pmod 5 \text{ and } n \not\equiv 4 \pmod{10}.
\end{aligned} \tag{5}$$

Proof. We discuss all the cases as follows.

Case i.1 $(n = 5)$.

From Propositions 2, 4, and 6, we conclude that

$$3 \le \gamma_m^\infty(P(5,2)) \le 4. \tag{6}$$

By forming $H(P(n,2),3)$ (the 3-dominating graph for $P(n,2)$), we notice that $V(H(P(n,2),3))$ consists of the following sets:

$$\begin{gathered}
D_1 = \{v_1, u_3, u_4\}, D_2 = \{v_2, u_4, u_5\}, D_3 = \{v_3, u_1, u_5\}, D_4 = \{v_4, u_1, u_2\},\\
D_5 = \{v_5, u_2, u_3\}, D_6 = \{v_1, v_3, u_2\}, D_7 = \{v_2, v_4, u_3\}, D_8 = \{v_3, v_5, u_4\},\\
D_9 = \{v_1, v_4, u_5\}, D_{10} = \{v_2, v_5, u_1\}.
\end{gathered} \tag{7}$$

Figure 7 illustrates $H(P(n,2),3)$.

We denote the closed neighborhood of vertex D_i in $H(P(n,2),3)$ by X_i, and we notice the following:

$$\begin{gathered}
u_1 \notin X_1, u_2 \notin X_2, u_3 \notin X_3, u_4 \notin X_4, u_5 \notin X_5, v_2 \notin X_6, v_3\\
\notin X_7, v_4 \notin X_8, v_5 \notin X_9, v_1 \notin X_{10}.
\end{gathered} \tag{8}$$

Therefore, by the definition of k-dominating graph, we conclude that $\gamma_m^\infty(P(5,2)) \ne 3$ which means $\gamma_m^\infty(P(5,2)) = 4$ from (6).

We can easily prove that the following family is a 4-vertex eternal dominating family of $(P(5,2))$:

$$D_1 = \{v_1, v_2, u_4, u_5\}, D_2 = \{v_4, v_5, u_1, u_2\}, D_3 = \{v_1, v_3, u_3, u_4\}. \tag{9}$$

Case i.2 $(n = 6)$.

From Propositions 2, 4, and 6, we conclude that $4 \le \gamma_m^\infty(P(6,2)) \le 4$ which means $\gamma_m^\infty(P(6,2)) = 4$ and the following family is a 4-vertex eternal dominating family of $P(6,2)$:

$$D_1 = \{v_1, v_4, u_3, u_6\}, D_2 = \{v_2, v_5, u_1, u_4\}, D_3 = \{v_3, v_6, u_2, u_5\}. \tag{10}$$

Case ii $(n = 7)$.

From Propositions 2, 4, and 6, we conclude that $\gamma_m^\infty(P(7,2)) = 5$. The following family is a 5-vertex eternal dominating family of $P(7,2)$:

$$D_1 = \{v_1, v_4, v_6, u_3, u_7\}, D_2 = \{v_2, v_5, v_7, u_1, u_2\}, D_3 = \{v_3, v_7, u_4, u_5, u_6\}. \quad (11)$$

Case iii ($n = 8$).
From Propositions 2, 4, and 6, it can be implied that

$$5 \le \gamma_m^\infty(P(8,2)) \le 6. \quad (12)$$

Case iv ($n \equiv 0 \pmod{10}$).
In this case, we have an even number of blocks that were defined in Lemma 1. For every pair of adjacent blocks, let us assign 3 guards to one of them and 4 guards to the other; we start by studying the first two blocks B_1, B_2; to prove that the 7 guards assigned to these two blocks are enough to eternally protect them, it is enough to find a set of guard configurations for which every vertex of $V(B_1) \cup V(B_2)$ belongs to at least one of the configurations and every two configurations are reachable from one another in one step. We imply that the configurations shown in Figure 8 satisfy this purpose.

Therefore, the number of guards we assigned to B_1, B_2 is enough to eternally protect these two blocks. Let us now consider the next attack to occur on a vertex that does not belong to any of these two blocks; for example, the vertex denoted by v_{18} which is the middle "outer" vertex of B_4 and then the guards of B_1, B_2 return to configuration 1 of Figure 7 and at the same time the guards of B_3, B_4, B_5, B_6 move to configuration 3 of Figure 7. It is obvious that these configurations can be applied to all sets of blocks in $P(n,2)$ which means without loss of generality the theory holds and we conclude that $\gamma_m^\infty(P(n,2)) \le (3n/5) + \lceil n/10 \rceil$ in this case.

Case v ($n > 9$ and $n \equiv 0 \pmod 5$ and $n \not\equiv 0 \pmod{10}$).
In this case, we have an odd number of blocks that were defined in Lemma 1. This proof is equivalent to proving that the same distribution in Figure 7 of Case iv can eternally protect $P(n,2)$ against an endless series of attacks taking into consideration the need to assign 4 guards to $B_{\lfloor n/5 \rfloor}$ in order to eternally protect it without the need to summon a guard from another block and that is because $B_{\lfloor n/5 \rfloor}$ does not belong to a pair (we have an odd number of blocks), which explains the need to have one additional guard in this case over Case iv.

Case vi ($n \equiv l \pmod{10}$ and $l = 1, 2, 3$).
In this case, we have an even number of blocks that were defined in Lemma 1. This proof is equivalent to proving that the same distribution in Figure 7 of Case iv can eternally protect $P(n,2)$ against an endless series of attacks with one difference which is placing 1,2,3 additional guards on the additional 2,4,6-vertex subblock when $n \equiv 1, 2, 3 \pmod 5$, respectively. In each of Figures 9–11 and in a similar way to Case iv, we find a set of configurations of the 7 guards placed on B_1, B_2 and the additional guards on the subblock SB so that these configurations can cover all the vertices of B_1, B_2, SB and they are all reachable from each other in one step.

As for the remaining pairs of blocks, we can simply apply the same configurations shown in Figure 8 of Case iv. Therefore, the theory holds for all the blocks and we prove the need for $\lfloor 3n/5 \rfloor + \lceil n/10 \rceil + l - 1$ guards when $l = 1, 2, 3$.

Case vii ($n \equiv l \pmod 5$ and $n \not\equiv l \pmod{10}$ and $l = 1, 2, 3$).

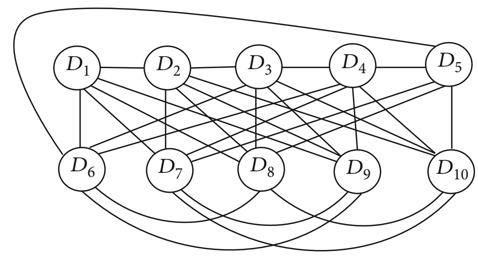

Figure 7: $H(P(5,2),3)$.

This case is similar to Case vi with the exception that we need one additional guard on $B_{\lfloor n/5 \rfloor}$ because we have an odd number of blocks (similar to Case v).

Case viii ($n \equiv 4 \pmod{10}$).
This case is similar to Case vi, with the exception that we can eternally protect the 8-vertex subblock SB with l-2 = 2 guards instead of l-1 guards like in Case vi. Figure 12 shows the distributions of guards in this case.

Case ix ($n \equiv 4 \pmod 5$ and $n \not\equiv 4 \pmod{10}$).
This case is similar to Case viii with the exception that we need one additional guard on $B_{\lfloor n/5 \rfloor}$ because we have an odd number of blocks (similar to Cases v and vii).

We conclude that the theorem holds for all cases of n.

2.3. Eternal Domination Number of $P(n, 3)$. In this section, we find the eternal domination number of generalized Petersen graph $P(n,3)$ when $n \equiv 0 \pmod 4$ and we give an upper bound for $\gamma_m^\infty(P(n,3))$ in the remaining cases: $n \equiv 1, 2, 3 \pmod 4$.

Theorem 3. $\gamma_m^\infty(P(n,3)) = n/2$ *for* $n \equiv 0 \pmod 4$.

Proof. Let $n = 4l : l \in N$. From Proposition 3, we know that $\gamma(P(n,3)) = n/2$ for this case and the γ-dominating set S is the union of two sets A and B which are $A = \{v_{4i+1}; 0 \le i \le l-1\}$, $B = \{u_{4i+3}; 0 \le i \le l-1\}$. Therefore, according to Propositions 3 and 5, the trivial bounds for $\gamma_m^\infty(P(n,3))$ are $n/2 \le \gamma_m^\infty(P(n,3)) \le n$.

We form the k-dominating graph $H(G,k)$ where $G = P(n,3), k = n/2$, and with sets

$$\begin{aligned}
D_1 &= A_1 \cup B_1 = \{v_{4i+1}; 0 \le i \le l-1\} \cup \{u_{4i+3}; 0 \le i \le l-1\},\\
D_2 &= A_2 \cup B_2 = \{v_{4i+2}; 0 \le i \le l-1\} \cup \{u_{4i+4}; 0 \le i \le l-1\},\\
D_3 &= A_3 \cup B_3 = \{v_{4i+3}; 0 \le i \le l-1\} \cup \{u_{4i+1}; 0 \le i \le l-1\},\\
D_4 &= A_4 \cup B_4 = \{v_{4i+4}; 0 \le i \le l-1\} \cup \{u_{4i+2}; 0 \le i \le l-1\}.
\end{aligned} \quad (13)$$

Each of these sets D_1, D_2, D_3, D_4 has a cardinality of $n/2$ and they are all adjacent in $H(P(n,3), n/2)$ for the following reasons.

$D_1 \longrightarrow D_2$, $D_2 \longrightarrow D_3$, $D_3 \longrightarrow D_4$, $D_4 \longrightarrow D_1$ by rotating the guards of the outer cycle one step clockwise and rotating the guards of the inner cycle one step counterclockwise.

$D_2 \longrightarrow D_1$, $D_3 \longrightarrow D_2$, $D_4 \longrightarrow D_3$, $D_1 \longrightarrow D_4$ by rotating the guards on the outer cycle one step

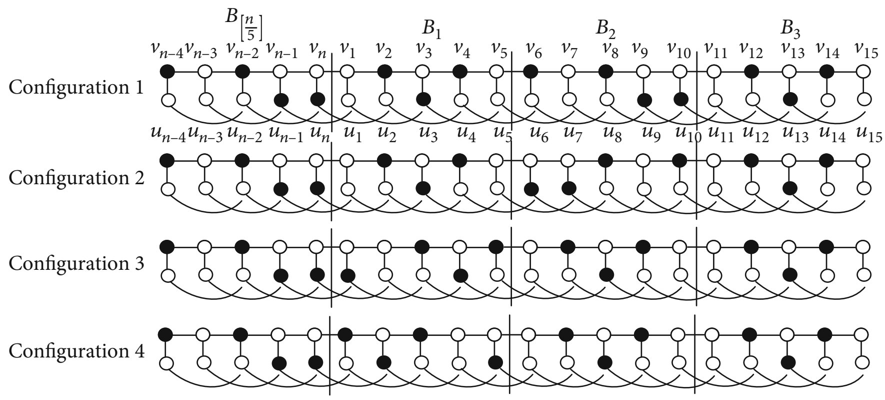

Figure 8: Eternally protecting B_1, B_2 with 7 guards.

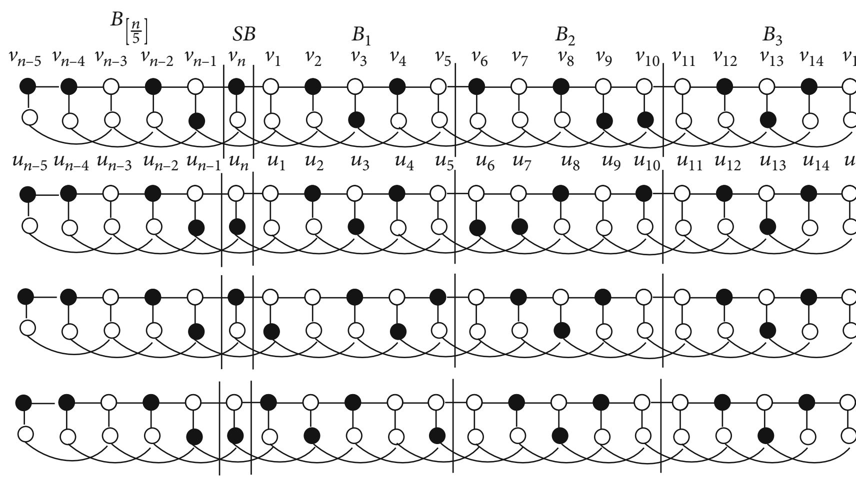

Figure 9: Eternally protecting B_1, B_2, SB with 8 guards.

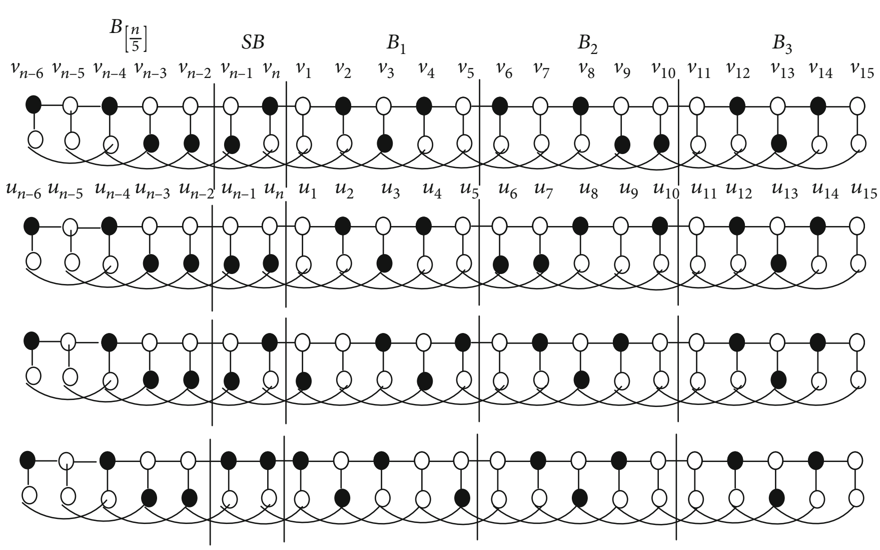

Figure 10: Eternally protecting B_1, B_2, SB with 9 guards.

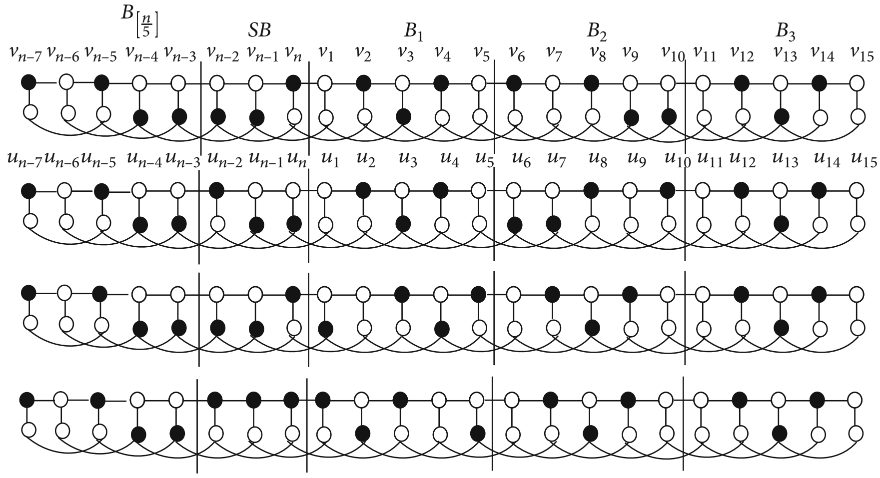

Figure 11: Eternally protecting B_1, B_2, SB with 10 guards.

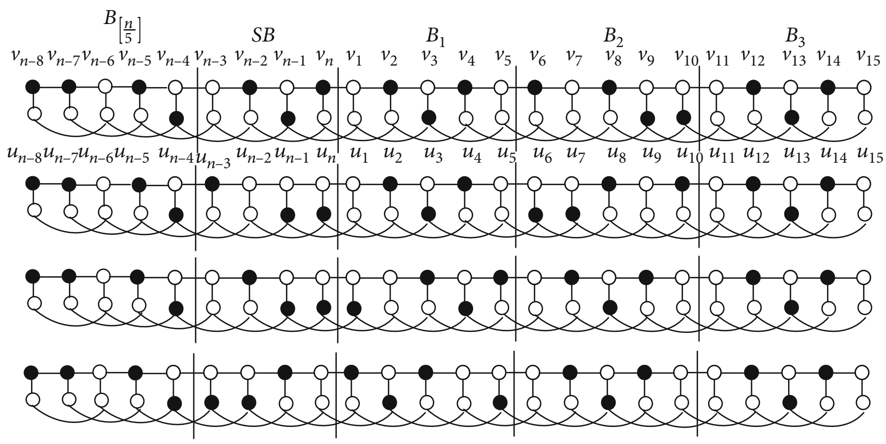

Figure 12: Eternally protecting B_1, B_2, SB with 10 guards.

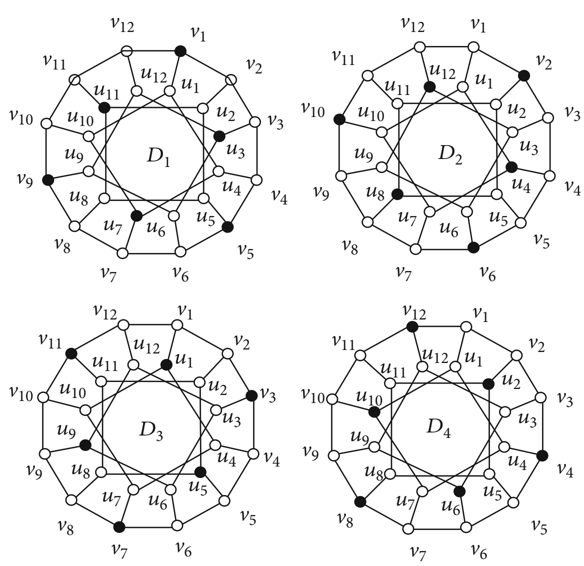

Figure 13: $\gamma_m^\infty(P(12,3)) = \gamma(P(12,3)) = 6$.

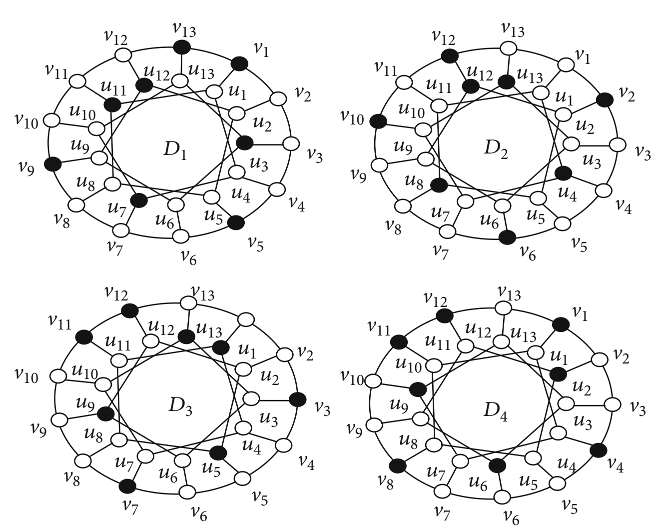

Figure 14: Eternally protecting $P(13,3)$ with 8 guards.

counterclockwise and rotating the guards of the inner cycle one step clockwise.

$D_1 \longrightarrow D_3, D_3 \longrightarrow D_1, D_2 \longrightarrow D_4, D_4 \longrightarrow D_2$ by reflection which means moving the guards from each cycle one step to occupy the corresponding unoccupied vertex on the other cycle.

Therefore, D_1, D_2, D_3, D_4 are adjacent in $S(P(n,3), n/2)$ (the induced subgraph of $H(P(n,3), n/2)$) on D_1, D_2, D_3, D_4 and $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,3))$; therefore, we conclude that $\gamma_m^\infty(P(n,3)) \leq n/2$ for $n \equiv 0 \pmod 4$ but $\gamma_m^\infty(P(n,1)) \geq \gamma(P(n,1))$ from the definition of eternal domination, so $\gamma_m^\infty(P(n,3)) = \gamma(P(n,3)) = n/2$ for $n \equiv 0 \pmod 4$; see Figure 13.

This proof proposes the following idea.

For $n \equiv 1, 2, 3 \pmod 4$, we can divide $V(P(n,3))$ into two sets X and Y defined as

$$X = \{v_i, u_i ; 1 \leq i \leq 4l\} ; n \equiv 1,2,3 \pmod 4,$$
$$Y = \begin{cases} \{v_{4l+1}, u_{4l+1}\} ; n \equiv 1 \pmod 4, \\ \{v_{4l+1}, v_{4l+2}, u_{4l+1}, u_{4l+2}\} ; n \equiv 2 \pmod 4, \\ \{v_{4l+1}, v_{4l+2}, v_{4l+3}, u_{4l+1}, u_{4l+2}, u_{4l+3}\} ; n \equiv 3 \pmod 4. \end{cases} \tag{14}$$

We notice that any change of n equals a change in the number of the vertices of X; meanwhile, the number of the vertices of Y does not change. We can use this observation to study the eternal domination problem on $P(n,3)$ when $n \equiv 1,2,3 \pmod 4$, because this observation allows us to take the relationships between the guards situated on the vertices of Y and the end vertices of X of one example "one value of n" then generalize these relationships for any value of n, taking into consideration that the relationships between the guards situated on the inner vertices of X will always be similar to the ones in Theorem 3.

Theorem 4. *For $n \geq 7$, we have* $\gamma_m^\infty(P(n,3)) \leq$

$$\begin{cases} \lceil n/2 \rceil + 1 ; n \equiv 1 \pmod 4, \\ (n/2) + 2 ; n \equiv 2 \pmod 4, \\ \lceil n/2 \rceil + 2 ; n \equiv 3 \pmod 4. \end{cases}$$

Case 1 ($n \geq 9$ and $n \equiv 1 \pmod 4$). Let $n = 4l + 1 : l \in N$. From Proposition 3, we know that $\gamma(P(n,3)) = \lceil n/2 \rceil$ for this case and the γ-dominating set S is $S = A \cup B \cup \{u_{n-1}\}$. Therefore, according to Propositions 3 and 5, the trivial bounds for $\gamma_m^\infty(P(n,3))$ are $\lceil n/2 \rceil \leq \gamma_m^\infty(P(n,3)) \leq n$.

We form the k-dominating graph $H(P(n,3), \lceil n/2 \rceil + 1)$ with these sets of cardinality $\lceil n/2 \rceil + 1$:

$$\begin{aligned} D_1 &= \{v_{4i+1} ; 0 \leq i \leq l-1\} \cup \{u_{4i+3} ; 0 \leq i \leq l-1\} \cup \{v_n, u_{n-1}\}, \\ D_2 &= \{v_{4i+2} ; 0 \leq i \leq l-1\} \cup \{u_{4i+4} ; 0 \leq i \leq l-1\} \cup \{v_{n-1}, u_n\}, \\ D_3 &= \{v_{4i+3} ; 0 \leq i \leq l-1\} \cup \{u_{4i+1} ; 0 \leq i \leq l-1\} \cup \{v_{n-1}, u_n\}, \\ D_4 &= \{v_{4i+4} ; 0 \leq i \leq l-1\} \cup \{u_{4i+2} ; 0 \leq i \leq l-1\} \cup \{v_1, u_{n-2}\}. \end{aligned} \tag{15}$$

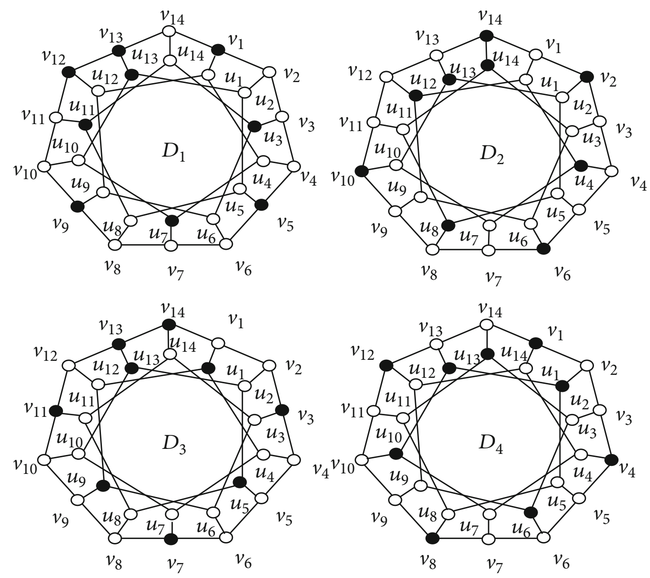

FIGURE 15: Eternally protecting $P(14,3)$ with 9 guards.

It is easy to notice that D_1, D_2, D_3, D_4 are all adjacent in $H(P(n,3), \lceil n/2 \rceil + 1)$ and the union $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,3))$; therefore, $\gamma_m^\infty(P(n,3)) \leq \lceil n/2 \rceil + 1$ for $n \equiv 1 \pmod 4$. See Figure 14.

Case 2 ($n \geq 10$ and $n \equiv 2 \pmod 4$). Let $n = 4l + 2 : l \in N$. From Proposition 3, we know that $\gamma(P(n,3)) = (n/2) + 1$ for this case and the γ-dominating set S is $S = A \cup B \cup \{v_{n-1}, u_{n-1}\}$. Then, according to Propositions 3 and 5, the trivial bounds for $\gamma_m^\infty(P(n,3))$ in this case are $(n/2) + 1 \leq \gamma_m^\infty(P(n,3)) \leq n$.

We form the k-dominating graph $H(P(n,3), (n/2) + 2)$ with these sets of cardinality $(n/2) + 2$:

$$\begin{aligned} D_1 &= \{v_{4i+1} ; 0 \leq i \leq l-1\} \cup \{u_{4i+3} ; 0 \leq i \leq l-1\} \cup \{v_{n-2}, v_{n-1}, u_{n-1}\}, \\ D_2 &= \{v_{4i+2} ; 0 \leq i \leq l-1\} \cup \{u_{4i+4} ; 0 \leq i \leq l-1\} \cup \{v_n, u_{n-1}, u_n\}, \\ D_3 &= \{v_{4i+3} ; 0 \leq i \leq l-1\} \cup \{u_{4i+1} ; 0 \leq i \leq l-1\} \cup \{v_{n-1}, v_n, u_{n-1}\}, \\ D_4 &= \{v_{4i+4} ; 0 \leq i \leq l-1\} \cup \{u_{4i+2} ; 0 \leq i \leq l-1\} \cup \{v_1, u_{n-1}, u_n\}. \end{aligned} \tag{16}$$

See Figure 15.

Case 3 ($n \geq 7$ and $n \equiv 3 \pmod 4$). Let $n = 4l + 3 : l \in N$. From Proposition 3, we know that $\gamma(P(n,3)) = \lceil n/2 \rceil + 1$ for this case and the γ-dominating set S is $S = A \cup B \cup \{v_{n-2}, u_{n-3}, u_2\}$. Therefore, according to Propositions 3 and 5, the trivial bounds for $\gamma_m^\infty(P(n,3))$ in this specific case are $\lceil n/2 \rceil + 1 \leq \gamma_m^\infty(P(n,3)) \leq n$.

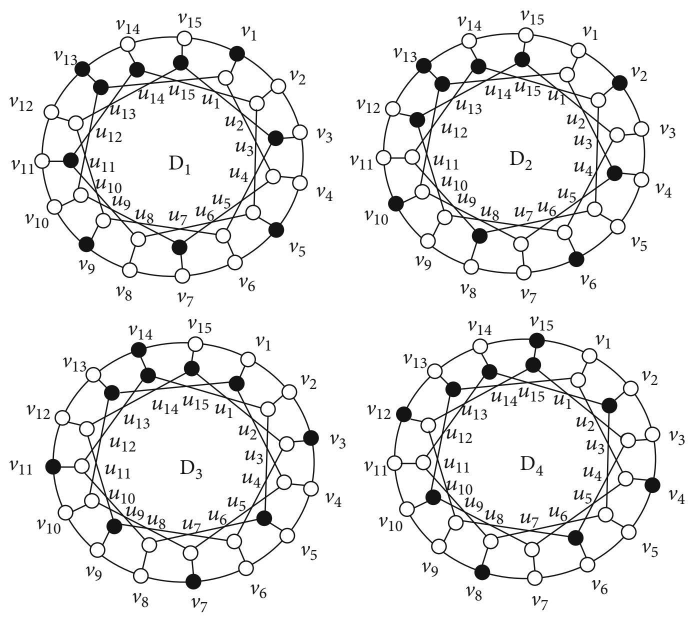

Figure 16: Eternally protecting $P(15, 3)$ with 10 guards.

We form the k-dominating graph $H(P(n,3), \lceil n/2 \rceil + 2)$ with these sets of cardinality $\lceil n/2 \rceil + 2$:

$$
\begin{aligned}
D_1 &= \{v_{4i+1}\,; 0 \le i \le l-1\} \cup \{u_{4i+3}\,; 0 \le i \le l-1\} \cup \{v_{n-2}, u_{n-2}, u_{n-1}, u_n\},\\
D_2 &= \{v_{4i+2}\,; 0 \le i \le l-1\} \cup \{u_{4i+4}\,; 0 \le i \le l-1\} \cup \{v_{n-2}, u_{n-2}, u_{n-1}, u_n\},\\
D_3 &= \{v_{4i+3}\,; 0 \le i \le l-1\} \cup \{u_{4i+1}\,; 0 \le i \le l-1\} \cup \{v_{n-1}, u_{n-2}, u_{n-1}, u_n\},\\
D_4 &= \{v_{4i+4}\,; 0 \le i \le l-1\} \cup \{u_{4i+2}\,; 0 \le i \le l-1\} \cup \{v_n, u_{n-2}, u_{n-1}, u_n\}.
\end{aligned} \tag{17}
$$

It is easy to notice that D_1, D_2, D_3, D_4 are all adjacent in $H(P(n,3), \lceil n/2 \rceil + 2)$ and the union $D_1 \cup D_2 \cup D_3 \cup D_4 = V(P(n,3))$; therefore, $\gamma_m^{\infty}(P(n,3)) \le \lceil n/2 \rceil + 2$ for $n \equiv 3 \pmod 4$. See Figure 16.

From Cases 1–3, we conclude the requested.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Faculty of Science, Tishreen University, Syria.

References

[1] W. F. Klostermeyer and C. M. Mynhardt, "Protecting a graph with mobile guards," *Applicable Analysis and Discrete Mathematics*, vol. 10, no. 1, pp. 1–29, 2016.

[2] A. P. Burger, E. J. Cockayne, W. R. Grundlingh, C. M. Mynhardt, J. H. van Vuuren, and W. Winterbach, "Infinite order domination in graphs," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 50, pp. 179–194, 2004.

[3] W. Goddard, S. M. Hedetniemi, and S. T. Hedetniemi, "Eternal security in graphs," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 52, pp. 169–180, 2005.

[4] S. Finbow, M. E. Messinger, and M. F. Van Bommel, "Eternal domination on 3× n grid graphs," *The Australasian Journal of Combinatorics*, vol. 61, pp. 156–174, 2015.

[5] M. E. Messinger and A. Z. Delaney, "Closing the gap: eternal domination on 3 × N grids," *Contributions to Discrete Mathematics*, vol. 12, no. 1, pp. 47–61, 2017.

[6] Y. Liu, L. Zuo, and C. Shang, "Chemical indices of generalized Petersen graph," *International Journal of Applied Mathematics*, vol. 47, no. 2, pp. 123–129, 2017.

[7] J. Goldwasser, W. Klostermeyer, and C. M. Mynhardt, "Eternal protection in grid graphs," *Utilitas Mathematica*, vol. 91, pp. 47–64, 2013.

[8] D. A. Mojdeh and A. N. Ghameshlou, "Domination in Jahangir graph J_2, m," *International Journal of Contemporary Mathematical Sciences*, vol. 2, no. 24, pp. 1193–1199, 2007.

[9] R. Shaheen, M. Assaad, and A. Kassem, "Domination and eternal domination of Jahangir graph," *Open Journal of Discrete Mathematics*, vol. 9, no. 3, pp. 68–81, 2019.

[10] R. Shaheen, M. Assaad, and A. Kassem, "Domination and eternal domination of generalized Jahangir graph," *Journal of applied Mathematics*, 2019.

[11] B. Javad Ebrahimi, N. Jahanbakht, and E. S. Mahmoodian, "Vertex domination of generalized Petersen graphs," *Discrete Mathematics*, vol. 309, no. 13, pp. 4355–4361, 2009.

[12] N. Besharati, J. B. Ebrahimi, and A. Azadi, "Independence number of generalized Petersen graphs," *Ars Combinatoria*, vol. 124, pp. 239–255, 2016.