Research Article

Frequency Assignment Model of Zero Divisor Graph

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Given a frequency assignment network model is a zero divisor graph $\Gamma = (V, E)$ of commutative ring R_q , in this model, each node is considered to be a channel and their labelings are said to be the frequencies, which are assigned by the $L(2, 1)$ and $L(3, 2, 1)$ labeling constraints. For a graph Γ , $L(2, 1)$ labeling is a nonnegative real valued function $f : V(G) \rightarrow [0, \infty)$ such that $|f(x) - f(y)| \geq 2d$ if $d = 1$ and $|f(x) - f(y)| \geq d$ if $d = 2$ where x and y are any two vertices in V and $d > 0$ is a distance between x and y . Similarly, one can extend this distance labeling terminology up to the diameter of a graph in order to enhance the channel clarity and to prevent the overlapping of signal produced with the minimum resource (frequency) provided. In general, this terminology is known as the $L(h, k)$ labeling where h is the difference of any two vertex frequencies connected by a two length path. In this paper, our aim is to find the minimum spanning sharp upper frequency bound $\lambda_{(2,1)}$ and $\lambda_{(3,2,1)}$, within Δ^2 , in terms of maximum and minimum degree of Γ by the distance labeling $L(2, 1)$ and $L(3, 2, 1)$, respectively, for some order $\eta = p^n q, pqr, p^n$ where p, q, r are distinct prime and n is any positive integer.

1. Introduction

Frequency assignment problem is one of the mathematical optimization technique used in wireless communication. The frequency assignment problem (FAP) initially emerged in the 1960s. Today, the wireless telecommunication networks such as mobile network, radio, television broadcasting, satellite, and radar string our world in our palm. The development and innovation on this network become natural, essential, and unavoidable. It is clear that a number of waves produced in one second is known as frequency and is denoted as λ . For various frequencies, we can access different channels. The proximity between the channel frequencies may affect the channel clarity of one another, and at the same time, our objective is to limit the use of resources. Therefore, distance between the channels is a main determinable factor to overcome the interference as well as for a feasible solution. FCA (fixed channel assignment), DCA (dynamic channel assignment), and HCA (hybrid channel assignment) are the

different types of FAP models; for further more classification, one can read the literature source [1].

In 1980, Hale introduced this as a vertex colouring problem [2]. Shao et al. proposed “efficient assignment of radio channels to transmitters” in 1982 [3]. The $L(2, 1)$ analogue was first proposed by Griggs and Yeh in 1992 [4], and they discussed $L(2, 1)$ labeling bound for some basic graphs in terms of maximum degree Δ . According to Griggs, for a graph $G(V, E)$, an $L(2, 1)$ labeling is a nonnegative real valued function $f : V(G) \rightarrow [0, \infty)$ such that $|f(x) - f(y)| \geq 2d$ if $d = 1$ and $|f(x) - f(y)| \geq d$ if $d = 2$ where x and y are any two vertices in V and $d > 0$ is the distance between x and y . The $L(2, 1)$ labeling number of G is the smallest number such that no other label does not exceed that and is denoted by $\lambda = \lambda(G)$. The canonical generalization of $L(2, 1)$ labeling is $L(h, k)$ labeling where h is the difference of any two vertex frequencies connected by a two length path and $\lambda_{(h,k)}$ is the minimum spanning labeling number.

At this instance, our FAP input model is zero divisor graph $\Gamma = (V, E)$ of a commutative ring R_η [5]. This problem was also first introduced as a colouring problem like FAP. The notion of this graph was initiated by Beck in 1988; after one decade, it was even more fined by Anderson and Livingston as an algebraic graph problem which gives more exposure to this graph. Let ZD be a set of all nonzero zero divisors of R_η , which is the vertex set $V(\Gamma)$. An edge between two distinct vertices u and v if and only if $uv = 0$. Our aim of output is to attain minimum spanning frequency $\lambda = \lambda(\Gamma)$ such that no other frequency is greater than λ under the constraint of distance. Suppose, for a (frequency model) graph, Γ has a diameter two then $L(2, 1)$ labeling (assignment) as same as $L(h, k)$ labeling (assignment) of Γ . Let f be a frequency function and d be a distance of v from u , on the vertex set $V(\Gamma)$ to the frequency domain $[0, \infty)$ where $f(v)$ be a frequency for any vertex (channel) $v \in V(\Gamma)$. Suppose $f \in L_d(2, 1)(\Gamma)$, then f is a $L(2, 1)$ labeling function of Γ if for all $u, v \in V$

$$|f(u) - f(v)| \geq \begin{cases} 2 & \text{if } d(u, v) = 1, \\ 1 & \text{if } d(u, v) = 2. \end{cases} \quad (1)$$

Suppose $f \in L_d(3, 2, 1)(\Gamma)$, then f is a $L(3, 2, 1)$ labeling function of Γ if for all $u, v \in V$

$$|f(u) - f(v)| \geq \begin{cases} 3 & \text{if } d(u, v) = 1, \\ 2 & \text{if } d(u, v) = 2, \\ 1 & \text{if } d(u, v) = 3. \end{cases} \quad (2)$$

Note that our FAP follows the static model [1] FCA, and then, the connections remain the same ever. For a graph G , the general upper bound on $\lambda(2, 1)$ is never much larger than Δ^2 said by Griggs [4] and also he proved $\lambda(G) \leq \Delta^2 + 2\Delta$. Over time, this general bound was again restricted as $\lambda(G) \leq \Delta^2 + \Delta$ by Chang and Kuo [6]. In this manuscript, we determined the sharp upper bound on $\lambda_{(2,1)}$ and $\lambda_{(3,2,1)}$ of a zero divisor graph of a commutative ring, in terms of Δ (maximum degree of Γ) and δ (minimum degree of Γ) which also satisfies the general upper bound of Griggs and others [3, 4]. The algorithmic approach is an optimized way for solving these types of polynomial time, NP complete and network problems. Many methods have been found for constructing the algorithm for creating a zero divisor graph. Labeling scheme points the correct labeling block or a label in localized networks associated with some defined conditions. This was introduced by Breuer in 1996 [7]. We the authors mention the labeling scheme before executing every labeling function, which points the correct labeling block or a subblock of graph Γ .

2. Upper Bounds in terms of Δ and δ

In this section, we have obtained the exact bound on $L(2, 1)$ and $L(3, 2, 1)$ labeling number for some classes of zero divisor graph of finite commutative ring whose order is denoted by η . Throughout this paper, we used f as a nonnegative real valued function for labeling of Γ for some order η is $p^n q, p^n$

and pqr . It is very essential to analyze the structure of the graph, and it is clear that $|\text{ZD}(R_\eta)| = (\eta - \phi(\eta)) - 1$. These zero divisors are contributed by each nontrivial divisor of η . Let $p^{n-1}q, p^{n-2}q, p^{n-3}q, \dots, p^{n-(n-1)}q, q$ and $p^n, p^{n-1}, \dots, p^2, p$ be the nontrivial divisors of the ring R of order $\eta = p^n q$. Consider the set $s_2 = \sum_{i=1}^{n-1} |m_u(p^{n-i}q)|$, where $m_u(p^{n-i}q)$ is the unique multiple of $p^{n-i}q$ and $|m_u(p^{n-i}q)|$ is the contribution of vertices (zero divisors) by nontrivial divisors of η , including itself such that, for any i

$$|m_u(p^{n-i}q)| = \frac{\eta - (p^{n-i}q)}{p^{n-i}}. \quad (3)$$

Therefore, $s_2 = |m_u(p^{n-1}q)| + |m_u(p^{n-2}q)| + \dots + |m_u(p^{n-(n-1)}q)|$. Now, the problem is to climb the cardinality of $s_2 = p^{n-1} - 1$ by inductive steps

$$\begin{aligned} |m_u(p^{n-1}q)| &= |m(p^{n-1}q)| = p - 1, \text{ if } i = 1 \text{ (canonical),} \\ |m_u(p^{n-2}q)| &= |m(p^{n-2}q)| - |m(p^{n-1}q)|, \text{ if } i = 2, \\ &\vdots \\ |m_u(p^{n-(n-1)}q)| &= |m(p^{n-(n-1)}q)| - |m(p^{n-(n-2)}q)|, \text{ if } i = n - 1. \end{aligned} \quad (4)$$

By observation, we get

$$\begin{aligned} \sum_{i=1}^{n-1} |m_u(p^{n-i}q)| &= p - 1 + p(p - 1) \\ &\quad + p^2(p - 1) + \dots + p^{n-2}(p - 1) = p^{n-1} - 1. \end{aligned} \quad (5)$$

In particular,

$$|m_u(q)| = |m_u(p^{n-n}q)| = p^n - p^{n-1}. \quad (6)$$

From equation (6), we can construct the set s_4 and $|s_4| = p^n - p^{n-1}$. Let s_3 be a set of all vertices of $\Gamma(R_\eta)$ such that $|s_3| = \sum_{i=2}^n |m_u(p^i)|$, where $|m_u(p^i)|$ is the number of unique multiple of p^i , and by the previous discussion, for any i , $|m(p^i)| = (\eta - p^i)/p^i$.

$$\begin{aligned} |m_u(p^n)| &= |m(p^n)| = q - 1, \text{ if } i = n, \\ |m_u(p^{n-1})| &= |m(p^{n-1})| - [|m(p^n)| + |m_u(p^{n-1}q)|], \text{ if } i = n - 1, \\ |m_u(p^{n-2})| &= |m(p^{n-2})| - [|m(p^{n-1})| + |m_u(p^{n-2}q)|], \text{ if } i = n - 2, \\ &\vdots \\ |m_u(p^{n-(n-2)})| &= |m(p^2)| - [|m(p^3)| + |m_u(p^2q)|], \text{ if } i = 2. \end{aligned} \quad (7)$$

Hence,

$$\sum_{i=2}^n |m_u(p^i)| = q-1 + p^0(q-1) \cdot (p-1) + \dots + p^{n-3}(q-1) \cdot (p-1) = (q-1)p^{n-2}. \quad (8)$$

When the summation is taken from $i=1$, we get

$$|m_u(p^{n-(n-1)})| = |m_u(p)| = (q-1) \cdot (p^{n-1} - p^{n-2}) = |s_1|(\text{let}). \quad (9)$$

From equations (9), (8), (6), and (5), one can find the cardinality of vertices contributed by the nontrivial divisors of η , which are mentioned initially. By adding those equations, we get $p^n + (q-1)p^{n-1} - 1$ which is the number of zero divisors of a ring as well as the vertex set $V(\Gamma)$; in addition, the author will express the maximum degree Δ and minimum degree δ of Γ in terms of p and q such that

$$\Delta = \begin{cases} p^{n-1}q - 2 & \text{if } p < q, \\ p^n - 1 & \text{if } p > q, \end{cases} \quad (10)$$

$$\delta = \begin{cases} p - 1 & \text{if } p < q, \\ q - 1 & \text{if } p > q, \end{cases}$$

of Γ for a ring with order $p^n q$ where p and q are distinct prime and n is any positive integer.

Theorem 1. Let Γ be a zero divisor graph of order $\eta = p^n q$ where p and q are distinct prime and n is any positive integer. Then, $\lambda_{(2,1)}(\Gamma) = \Delta + \delta$ where Δ and δ are maximum and minimum degrees of Γ , respectively.

Proof. Type 1. Suppose $p < q$.

Case 1. If $\lfloor n/2 \rfloor > 1$, then the labeling scheme $s_2^{K_i} \longrightarrow s_1 \longrightarrow s_3 \longrightarrow s_2^{d_i}$.

Here, K_i 's denote the vertices of complete subblock whereas d_i 's denote the other vertices of s_2 .

$$f(v_{s_2}^{K_i}) = 2i - 2, \quad 1 \leq i \leq p^{\lfloor n/2 \rfloor} - 1,$$

$$f(v_{s_1}^{d_i}) = \begin{cases} 2i + 2p - 3, & 1 \leq i \leq p^{\lfloor n/2 \rfloor} - p, \\ i + p^{\lfloor n/2 \rfloor} + p - 3, & p^{\lfloor n/2 \rfloor} - p + 1 \leq i \leq (q-1)(p^{n-1} - p^{n-2}), \end{cases}$$

$$f(v_{s_3}^{d_i}) = i + p^{\lfloor n/2 \rfloor} + p - 3 + (q-1)(p^{n-1} - p^{n-2}), \quad 1 \leq i \leq (q-1)p^{n-2},$$

$$f(v_{s_2}^{d_i}) = i + p^{\lfloor n/2 \rfloor} + p - 3 + (q-1)p^{n-1}, \quad 1 \leq i \leq p^{n-1} - p^{\lfloor n/2 \rfloor}. \quad (11)$$

Case 2. If $\lfloor n/2 \rfloor = 1$, then the labeling scheme $s_2 \longrightarrow s_1 \longrightarrow s_3$.

$$f(v_{s_2}^{K_i}) = 2i - 2, \quad 1 \leq i \leq p^{\lfloor n/2 \rfloor} - 1,$$

$$f(v_{s_2}^{d_i}) = i + 2p^{\lfloor n/2 \rfloor} - 3, \quad 1 \leq i \leq p^{n-1} - p^{\lfloor n/2 \rfloor} - 1.$$

$$f(v_{s_1}^{d_i}) = i + p^{\lfloor n/2 \rfloor} + p^{n-1} - 3, \quad 1 \leq i \leq (q-1)p^{n-1} - p^{n-2},$$

$$f(v_{s_3}^{d_i}) = i + qp^{n-1} - p^{n-2}(q-1) + p^{\lfloor n/2 \rfloor} - 3, \quad 1 \leq i \leq (q-1)p^{n-2}. \quad (12)$$

As per requirement, we can assign the distinct labels for all $v \in s_4$ from the same labeling domain of s_1 . Since $|s_1| > |s_4|$ in addition the distance from s_1 to s_4 and s_4 to s_3 is $-(q-1)$, vertices of s_3 are three.

Subcase 1. If $n=2$, then such graphs have no v^{d_i} 's in s_2 .

$$f(v_{s_2}^{K_i}) = 2i - 2, \quad 1 \leq i \leq p^{\lfloor n/2 \rfloor} - 1,$$

$$f(v_{s_1}^{d_i}) = i + 2p^{\lfloor n/2 \rfloor} - 3, \quad 1 \leq i \leq (q-1)p^{n-1} - p^{n-2},$$

$$f(v_{s_3}^{d_i}) = i + (q-1)(p^{n-1} - p^{n-2}) + (q-1) + 2p^{\lfloor n/2 \rfloor} - 3, \quad 1 \leq i \leq (q-1)p^{n-2}. \quad (13)$$

In this subcase, the set s_4 vertices are at the distance of three from s_1 . The analogue of this labeling allows to assign the same labeling of s_1 to the vertices of s_4 .

Type 2. Suppose $p > q$.

Case 1. If $\lfloor n/2 \rfloor > 1$, then the labeling scheme $s_2^{K_i} \longrightarrow s_4 \longrightarrow s_2^{d_i} \longrightarrow s_3(q-1)$ elements.

$$f(v_{s_2}^{K_i}) = 2i - 2, \quad 1 \leq i \leq p^{\lfloor n/2 \rfloor} - 1. \quad (14)$$

At this instance, $p^{\lfloor n/2 \rfloor} - 2$ vertices of s_4 can be labeled by the numbers between the two existing labels that are assigned to the vertices of $s_2^{K_i}$

$$f(v_{s_4}^{d_i}) = \begin{cases} i + 2p - 4, & 1 \leq i \leq p^n - p^{n-1} - p^{\lfloor n/2 \rfloor} + 2, \\ i - \left((2(p^n - p^{n-1} - p^{\lfloor n/2 \rfloor} + 2) + 1) - i \right), & p^n - p^{n-1} - p^{\lfloor n/2 \rfloor} + 3 \leq p^n - p^{n-1}, \end{cases}$$

$$f(v_{s_2}^{d_i}) = i + p^{\lfloor n/2 \rfloor} + p^n - p^{n-1} - 2, \quad 1 \leq i \leq p^{n-1} - p^{\lfloor n/2 \rfloor},$$

$$f(v_{s_3}^{d_i}) = i + p^n - 1, \quad 1 \leq i \leq q - 1. \quad (15)$$

Case 2. If $n=3$, then the labeling scheme $s_2 \longrightarrow s_4 \longrightarrow s_3$.

$$f(v_{s_2}^{K_i}) = 2i - 2, \quad 1 \leq i \leq p - 1,$$

$$f(v_{s_2}^{d_i}) = i + 2p - 9, \quad p \leq i \leq p^{n-1} - 1. \quad (16)$$

At the end of this execution, there are $p^{\lfloor n/2 \rfloor} - 1$ missed numbers between the existing labels of s_2 which can be

accessed and assigned as the labels, for some $v \in s_4$, with the next step of scheme execution as follows:

$$\begin{aligned} f(v_{s_4}^i) &= \begin{cases} i + 2p + p^{n-1} - 10, & 1 \leq i \leq p^n - p^{n-1} - p^{\lfloor n/2 \rfloor} + 1, \\ i - \left(\left(2(p^n - p^{n-1} - p^{\lfloor n/2 \rfloor} + 1) + 1 \right) - i \right), & p^n - p^{n-1} - p^{\lfloor n/2 \rfloor} + 2 \leq i \leq p^n - p^{n-1}, \end{cases} \\ f(v_{s_3}^i) &= i + 2p + p^n - p^{\lfloor n/2 \rfloor} - 8, & 1 \leq i \leq q - 1. \end{aligned} \quad (17)$$

This frequency assignment system can be classified into two types along with their labeling scheme. By applying the above defined labeling function step by step as per the labeling scheme, the channel frequency least upper bound is attained.

Theorem 2. Let Γ be a zero divisor graph of a ring of order $\eta = p^n q$ where p and q are distinct prime and n is any positive integer. Then, for a maximum degree Δ of Γ ,

$$\lambda_{(3,2,1)}(\Gamma) = \begin{cases} 2\Delta + p - 1, & \text{if } \lfloor \frac{n}{2} \rfloor = 1 \text{ when } p < q, \\ 2\Delta + p^{\lfloor n/2 \rfloor} - 2, & \text{if } \lfloor \frac{n}{2} \rfloor \geq 2 \text{ when } p < q, \\ 2\Delta + p^{\lfloor n/2 \rfloor} + 2q - 5, & \text{when } p > q. \end{cases} \quad (18)$$

Proof. Since Γ is a zero divisor graph of a finite commutative ring of order $\eta = p^n q$, the set of all nonzero zero divisors $\text{ZD}(R_\eta)$ is the vertex set $V(\Gamma)$ of cardinality $p^n + (q - 1)p^{n-1}$

– 1. Any two vertices x and y of $V(\Gamma)$ are said to be adjacent if $xy = 0$ to η where zero is the additive identity. The vertex set $V(\Gamma)$ has the partition such that $V(\Gamma) = s_1 \cup s_2 \cup s_3 \cup s_4$, which are the unique multiples of $p, p^{n-1}q \cup p^{n-2}q \cup \dots \cup p^{n-(n-1)}q, p^n q \cup p^{n-1}q \cup \dots \cup p^2, q$, respectively. An element x belongs to any one of the nontrivial divisor collection ℓ such that $(x, \eta) = \ell$; this will ensure the unique existence of the element in each set. An edge xy belongs to the different partition or in the same partition; the corresponding vertices are zero to at least prime factor of η . In s_2 , $\lfloor n/2 \rfloor$ multiples of nontrivial divisor collections are adjacent to all other element of the same collection and they individually form the $\lfloor n/2 \rfloor$ complete graphs; at the same time, they together form $K_{p^{\lfloor n/2 \rfloor - 1}}$ in s_2 , which is denoted by $v_{s_2}^{K_i}$. The remaining $p^{n-1} - p^{\lfloor n/2 \rfloor}$ elements are not adjacent to one another, which is denoted by $v_{s_2}^{d_i}$.

Type 1. Suppose $p < q$, then the labeling scheme $s_2^{K_i} \longrightarrow s_1 \longrightarrow s_3 \longrightarrow s_2^{d_i} \longrightarrow s_4$.

$$\begin{aligned} f(v_{s_2}^{K_i}) &= 3i - 31 \leq i \leq p^{\lfloor n/2 \rfloor} - 1, \\ f(v_{s_1}^i) &= \begin{cases} 3p - 3, & i = 1, \\ 2i + 3p - 5, & 2 \leq i \leq (q - 1)(p^{n-1} - p^{n-2}), \end{cases} \quad \begin{cases} \text{if } \lfloor \frac{n}{2} \rfloor = 1, \\ \text{if } \lfloor \frac{n}{2} \rfloor \geq 1, \end{cases} \\ f(v_{s_1}^i) &= 2i + 3(p^{\lfloor n/2 \rfloor} - 2), \quad 1 \leq i \leq (q - 1)(p^{n-1} - p^{n-2}), \quad \text{if } \lfloor \frac{n}{2} \rfloor \geq 2, \\ f(v_{s_3}^i) &= \begin{cases} 2i + 2(q - 1)(p^{n-1} - p^{n-2}) + 3(p^{\lfloor n/2 \rfloor} - 2), & 1 \leq i \leq (q - 1)p^{n-2}, \text{ if } \lfloor \frac{n}{2} \rfloor \geq 2, \\ 2i + 2(q - 1)(p^{n-1} - p^{n-2}) + 3p - 5, & 1 \leq i \leq (q - 1)p^{n-2}, \text{ if } \lfloor \frac{n}{2} \rfloor = 1, \end{cases} \\ f(v_{s_2}^{d_i}) &= 2i + 2(q - 1)p^{n-1} + 3p^{\lfloor n/2 \rfloor} - 5, \quad 1 \leq i \leq p^{n-1} - p^{\lfloor n/2 \rfloor}. \end{aligned} \quad (19)$$

FIGURE 1: $L(3, 2, 1)$ labeling of Γ of order $7^4 \times 5$.

It is clear that the minimum distance from s_4 is three to s_1 and $|s_3| - (q - 1)$ elements of s_3 . For assigning labels to the elements of s_4 , at this instance, we can expect two possibilities $|s_1| = |s_4|$ and $|s_1| > |s_4|$.

If $|s_1| = |s_4|$, then

$$f(v_{s_4}^i) = \begin{cases} 2i + 3p - 6, & 1 \leq i \leq p^n - p^{n-1} \text{ if } \left\lfloor \frac{n}{2} \right\rfloor = 1, \\ 2i + 3p^{\lfloor n/2 \rfloor} - 5, & 1 \leq i \leq p^n - p^{n-1} - 1 \text{ if } \left\lfloor \frac{n}{2} \right\rfloor \geq 2, \\ 2i + 3p^{\lfloor n/2 \rfloor} + q^2 - 6, & i = p^n - p^{n-1} \text{ if } \left\lfloor \frac{n}{2} \right\rfloor \geq 2. \end{cases} \quad (20)$$

If $|s_1| > |s_4|$, then

$$f(v_{s_4}^i) = 2i + 3p^{\lfloor n/2 \rfloor} - 5 \quad \text{if } 1 \leq i \leq p^n - p^{n-1}. \quad (21)$$

Type 2. Suppose $p > q$, then

$$\begin{aligned} f(v_{s_2}^{k_i}) &= 3i - 31 \leq i \leq p^{\lfloor n/2 \rfloor} - 1, \\ f(v_{s_4}^i) &= 2i + 3(p^{\lfloor n/2 \rfloor} - 2), \\ f(v_{s_1}^i) &= 2i + 3(p^{\lfloor n/2 \rfloor}) - 5, \quad 1 \leq i \leq (q-1)(p^{n-1} - p^{n-2}). \end{aligned} \quad (22)$$

Case 1. If $n = 2$, then

$$f(v_{s_3}^i) = \begin{cases} 3i + 2(p^n - p^{n-1}) + 3(p^{\lfloor n/2 \rfloor} - 2) \\ 2i + 2(p^n - p^{n-1}) + 3(p^{\lfloor n/2 \rfloor}) - 5 \end{cases} \quad \text{if } 2 \leq i \leq q-1. \quad (23)$$

Case 2. If $n > 2$, then

$$\begin{aligned} f(v_{s_2}^{d_i}) &= 2i + 2(p^n - p^{n-1}) + 3(p^{\lfloor n/2 \rfloor} - 2) \quad \text{if } 1 \leq i \leq p^{n-1} - p^{\lfloor n/2 \rfloor}, \\ f(v_{s_3}^i) &= \begin{cases} 2p^n + p^{\lfloor n/2 \rfloor} - 3 & \text{if } i = 1, \\ 2i + 2p^n + p^{\lfloor n/2 \rfloor} - 5 & \text{if } 2 \leq i \leq q-1, \\ 2i + 2(q-1)(p^{n-1} - p^{n-2}) + 3p^{\lfloor n/2 \rfloor} - 5 & \text{if } 1 \leq i \leq (q-1)(p^{n-2} - 1). \end{cases} \end{aligned} \quad (24)$$

Thus, we have attained the $L(3, 2, 1)$ number $\lambda_{(3,2,1)}$ of Γ which is minimum.

Example 3. Figure 1 describes the FAP model which resembles a zero divisor graph of a finite commutative ring of order $p^n q = 7^4 \times 5 = 12,005$ along with their frequency assignments. For this ring, the cardinality of zero divisors acts as a number of channels (vertices) such that $|V(\Gamma)| = |\text{ZD}| = 3772$. The vertices are partitioned into four separate blocks, namely, s_1 , s_2 , s_3 , and s_4 , respectively. Note that $|s_1| = 1176$, $|s_2| = 6 + 42 + 294$, $|s_3| = 4 + 24 + 168$, and $|s_4| = 2058$. The dotted subblock and the darkened subblock of s_2 together form the complete graph which is denoted by $s_2^{K_i}$ where the dotted subblock alone is adjacent with block s_1 . The other subblock of s_2 is adjacent with both darkened and dotted subblock but not within itself. In s_3 , the dotted subblock alone is adjacent with s_4 block. Adjacency relation between s_2 and s_3 blocks is shown in the figure. According to Theorem 2, it comes under the Case 2 of Type 2. By applying the respective frequencies to the vertices, one can attain the least upper frequency $\lambda_{(3,2,1)}(\Gamma)$ which is $2\Delta + p^{\lfloor n/2 \rfloor} + 2q - 5$ where $\Delta = p^n - 1$.

If $i = 1$, the graph can be decomposed into a complete graph, and an independent set is known as a split graph. Our next frequency model is also a zero divisor graph Γ of a finite commutative ring of order $\eta = p^n$, which is isomorphic to the split graph. Let s_1 be a set consisting the vertices of the

Input: zero divisor graph Γ of a finite commutative ring of order $\eta = pqr$ where p, q, r are distinct prime.

Step 1: calculate the cardinality of the nontrivial divisor sets.

$$|s_1| = qr - (q + r) + 1$$

$$|s_2| = pr - (p + r) + 1$$

$$|s_3| = pq - (p + q) + 1$$

$$|s_4| = p - 1$$

$$|s_5| = q - 1$$

$$|s_6| = r - 1.$$

Step 2: compare the cardinalities of s_1, s_2 , and s_3 and find the greatest set.

Step 3: if $|s_1| > |s_2|$ and $|s_1| > |s_3|$

Assign labeling scheme as $s_4 \longrightarrow s_5 \longrightarrow s_6 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3$.

Else if $|s_2| > |s_1|$ and $|s_2| > |s_3|$

Assign labeling scheme as $s_5 \longrightarrow s_4 \longrightarrow s_6 \longrightarrow s_2 \longrightarrow s_1 \longrightarrow s_3$.

Else

Assign labeling scheme as $s_6 \longrightarrow s_5 \longrightarrow s_4 \longrightarrow s_3 \longrightarrow s_2 \longrightarrow s_1$.

Output: determine the suitable labeling scheme of Γ .

ALGORITHM 1

complete graph and the collection of independent vertices is said to be set s_2 . Both sets are having the cardinalities $p - 1$ and $p^{n-1} - p$, respectively. It is clear that $|V(\Gamma)| = |s_1| + |s_2| = p^n - 1$ are the set of zero divisors of ring R_η . Therefore, the maximum degree is $p^{n-1} - 2$ and the minimum degree is $p - 1$.

Theorem 4. Let Γ be a zero divisor graph of order $\eta = p^n$ where p is prime and n is any positive integer. Then, $\lambda_{(2,1)}(\Gamma) = \Delta + \delta$ and $\lambda_{(3,2,1)}(\Gamma) = 2\Delta + \delta$ where Δ and δ denote the maximum and minimum degrees of Γ , respectively.

Proof. For this model, we discussed two different frequency distributions. Let f be a frequency distribution function which is executed in two distributions with the same labeling scheme $s_1 \longrightarrow s_2$ but different distance constraints.

Frequency Assignment 1

$$\begin{aligned} f(v_{s_1}^i) &= 2i - 2, \quad 1 \leq i \leq p - 1, \\ f(v_{s_2}^i) &= i + 2p - 3, \quad 1 \leq i \leq p^2 - p. \end{aligned} \quad (25)$$

Frequency Assignment 2

$$\begin{aligned} f(v_{s_1}^i) &= 3i - 3, \quad 1 \leq i \leq p - 1, \\ f(v_{s_2}^i) &= 2i + 3p - 5, \quad 1 \leq i \leq p^2 - p. \end{aligned} \quad (26)$$

By applying the labeling function f in each assignment, we can obtain the $L(2, 1)$ number $\lambda_{(2,1)} = p^2 + p - 3 = \Delta + \delta$ and the $L(3, 2, 1)$ number $\lambda_{(3,2,1)} = 2(p^2 + p) - 5 = 2\Delta + \delta$ of Γ , respectively.

In a complete tripartite graph, each partite element is adjacent with a certain set of elements but no one path of length less than two does occur between those sets of ele-

ments. This structured graph is isomorphic to the zero divisor graph of a finite commutative ring of order $\eta = pqr$ where p, q, r are distinct prime. For that, p, q, r, qr, pr, pq are the nontrivial divisors of the graph. Each divisor and its multiples together form a collection of vertices of Γ that are indexed as the sets, $s_1, s_2, s_3, s_4, s_5, s_6$. Here, s_4, s_5, s_6 are the members of the tripartite graph and s_1, s_2, s_3 are the immediate neighbours of the partite set, respectively. For this model, we have written an algorithm for finding the suitable labeling scheme according to the cardinality of s_1, s_2, s_3 . It will facilitate to execute the labeling function in an effective manner.

Theorem 5. Let Γ be a zero divisor graph of order $\eta = pqr$ where p, q, r are distinct prime. Then, $\lambda_{(2,1)}(\Gamma) = \Delta + \delta$ (maximum degree + minimum degree).

Proof. Event 1: if the labeling scheme is $s_4 \longrightarrow s_5 \longrightarrow s_6 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3$.

$$\begin{aligned} f(v_{s_4}^i) &= i - 1, \quad 1 \leq i \leq p - 1, \\ f(v_{s_5}^i) &= i + p - 1, \quad 1 \leq i \leq q - 1, \\ f(v_{s_6}^i) &= i + p + q - 1, \quad 1 \leq i \leq r - 1, \\ f(v_{s_1}^i) &= \begin{cases} p + q - 1 & \text{if } i = 1, \\ i + p + q + r - 3 & \text{if } 2 \leq i \leq qr - (q + r) + 1. \end{cases} \end{aligned} \quad (27)$$

The sets s_2 and s_3 channels are at the distance of three from set s_1 , and they will get the same as the frequencies of s_1 .

FIGURE 2: $L(3, 2, 1)$ labeling of Γ of order $\eta = 7 \times 11 \times 13$.

Event 2: if the labeling scheme is $s_5 \longrightarrow s_4 \longrightarrow s_6 \longrightarrow s_2 \longrightarrow s_1 \longrightarrow s_3$.

$$\begin{aligned} f(v_{s_5}^i) &= i - 1, \quad 1 \leq i \leq q - 1, \\ f(v_{s_4}^i) &= i + q - 1, \quad 1 \leq i \leq p - 1, \\ f(v_{s_6}^i) &= i + p + q - 1, \quad 1 \leq i \leq r - 1, \\ f(v_{s_2}^i) &= \begin{cases} p + q - 1 & \text{if } i = 1, \\ i + p + q + r - 3 & \text{if } 2 \leq i \leq pr - (p + r) + 1. \end{cases} \end{aligned} \quad (28)$$

The sets s_1 and s_3 channels are at the distance of three from set s_2 , and they will get the same frequencies of s_2 .

Event 3: if the labeling scheme is $s_6 \longrightarrow s_5 \longrightarrow s_4 \longrightarrow s_3 \longrightarrow s_2 \longrightarrow s_1$.

$$\begin{aligned} f(v_{s_6}^i) &= i - 1, \quad 1 \leq i \leq r - 1, \\ f(v_{s_5}^i) &= i + r - 1, \quad 1 \leq i \leq q - 1, \\ f(v_{s_4}^i) &= i + q + r - 1, \quad 1 \leq i \leq p - 1, \\ f(v_{s_3}^i) &= \begin{cases} q + r - 1 & \text{if } i = 1, \\ i + p + q + r - 3 & \text{if } 2 \leq i \leq pq - (p + q) - 1. \end{cases} \end{aligned} \quad (29)$$

The sets s_1 and s_2 channels are at the distance of three from set s_3 , and they will get the same frequencies of s_3 . The above labeling executions in three different events as per their appropriate labeling scheme will yield $\lambda_{(2,1)}$.

Theorem 6. Let Γ be a zero divisor graph of a finite commutative ring of order $\eta = pqr$ where p, q, r are distinct prime.

Then, for a maximum degree Δ and minimum degree δ , $\lambda_{(3,2,1)}(\Gamma) = 2(\Delta + \delta + d) + 1$, if the cardinality of the smallest prime and its multiple set is less than the sum of the cardinality of the other primes and its multiple sets where d is the difference of two cardinalities. Otherwise, $\lambda_{(3,2,1)}(\Gamma) = 2(\Delta + \delta)$.

Proof. Event 1: if the labeling scheme is $s_4 \longrightarrow s_5 \longrightarrow s_6 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3$, then

$$\begin{aligned} f(v_{s_4}^i) &= 2i - 2, \quad 1 \leq i \leq p - 1, \\ f(v_{s_5}^i) &= 2i + 2p - 3, \quad 1 \leq i \leq q - 1, \\ f(v_{s_6}^i) &= 2i + 2(p + q) - 4, \quad 1 \leq i \leq r - 1, \\ f(v_{s_1}^i) &= \begin{cases} 2i + 2(p + q + r) - 6, & 1 \leq i \leq qr - (q + r) + 1 \text{ if } |s_1| > |s_2| \text{ and } |s_3|, \\ 2i + 2(p + q + r) - 5, & 1 \leq i \leq qr - (q + r) + 1 \text{ if } |s_2| > |s_1| \text{ and } |s_3|, \end{cases} \\ f(v_{s_2}^i) &= \begin{cases} 2i + 2(p + q + r) - 5, & 1 \leq i \leq pr - (p + r) + 1 \text{ if } |s_1| > |s_2| \text{ and } |s_3|, \\ 2i + 2(p + q + r) - 6, & 1 \leq i \leq pr - (p + r) + 1 \text{ if } |s_2| > |s_1| \text{ and } |s_3|, \end{cases} \\ f(v_{s_3}^i) &= \begin{cases} 2i + 2q + 2pr - 3, & 1 \leq i \leq pq - (p + q) + 1 \text{ if } |s_1| > |s_2| \text{ and } |s_3|, \\ 2i + 2p + 2qr - 3, & 1 \leq i \leq pq - (p + q) + 1 \text{ if } |s_2| > |s_1| \text{ and } |s_3|. \end{cases} \end{aligned} \quad (30)$$

Event 2: if the labeling scheme is $s_6 \longrightarrow s_5 \longrightarrow s_4 \longrightarrow s_3 \longrightarrow s_2 \longrightarrow s_1$, then

$$\begin{aligned} f(v_{s_6}^i) &= 2i - 2, \quad 1 \leq i \leq r - 1, \\ f(v_{s_5}^i) &= 2i + 2r - 3, \quad 1 \leq i \leq q - 1, \\ f(v_{s_4}^i) &= 2i + 2(r + q) - 4, \quad 1 \leq i \leq p - 1, \\ f(v_{s_3}^i) &= 2i + 2(p + q + r) - 6, \quad 1 \leq i \leq pq - (p + q) + 1, \\ f(v_{s_1}^i) &= 2i + 2(p + q + r) - 5, \quad 1 \leq i \leq qr - (q + r) + 1, \\ f(v_{s_2}^i) &= 2i + 2p + 2qr - 3, \quad 1 \leq i \leq pr - (p + r) + 1. \end{aligned} \quad (31)$$

Input: zero divisor graph Γ of a finite commutative ring of order $\eta = pqr$ where p, q, r are distinct prime.

Step 1: calculate the cardinality of the nontrivial divisor sets.

$$|s_1| = qr - (q + r) + 1$$

$$|s_2| = pr - (p + r) + 1$$

$$|s_3| = pq - (p + q) + 1$$

$$|s_4| = p - 1$$

$$|s_5| = q - 1$$

$$|s_6| = r - 1$$

Step 2: compare the cardinalities of s_1, s_2 , and s_3 and find the greatest set.

Step 3: if $|s_1| > |s_2|$ and $|s_1| > |s_3|$ OR $|s_2| > |s_1|$ and $|s_2| > |s_3|$

Assign labeling scheme as $s_4 \longrightarrow s_5 \longrightarrow s_6 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3$.

Else

Assign labeling scheme as $s_6 \longrightarrow s_5 \longrightarrow s_4 \longrightarrow s_3 \longrightarrow s_2 \longrightarrow s_1$.

Output: determine the suitable labeling scheme of Γ .

ALGORITHM 2

The execution of the above events will ensure the $L(3, 2, 1)$ number $\lambda_{(3,2,1)}(\Gamma)$ during the appropriate labeling scheme.

Example 7. Figure 2 gives more interpretation on the $L(3, 2, 1)$ frequency assignment to the zero divisor graph Γ of a commutative ring of order $\eta = 7 \times 11 \times 13$ whose $|V(\Gamma)| = 280$. This model comes under Theorem 6 and follows Event 1 labeling scheme by the instruction of Algorithm 2. The least upper frequency of Γ is $2(\Delta + \delta + d) + 1 = 321$ when executing appropriate labeling function on Γ .

3. Conclusion

In this paper, we handled some general classes of zero divisor graphs as frequency assignment networks. We have discussed the characteristics of their vertices and their edges. Algorithm construction for algebraic graphs is the most familiar, and it provides the systematic approach on these types of NP complete problems. Enormous applications are scattered as well behind every growing sciences; because of this reason, we took this graph under frequency labeling. Through this distance labeling, one can restrict a frequency domain along with finding λ of Γ in terms of Δ and δ . Our future research plan is to analyze some other classes of zero divisor graphs to find λ as well as break those networks by using the decycling technique which was introduced by Lowell W. Beineke and Robert C. Vandell in 1996.

Data Availability

The nature of the problem is to find the minimum sharp upper numerical bound for some general classes of zero divisor graph. This bound was obtained by $L(2, 1)$ and $L(3, 2, 1)$ labeling techniques. This is popularly known as channel assignment problem. The data that support the findings of this study are available in the research article, about the graph [<https://scholar.rose-hulman.edu/rhumj/vol17/iss2/6> and <https://rb.gy/oa3nbs>] and labeling [<https://rb.gy/nvyweij>, DOI: 10.1016/j.aml.2006.08.029, 10.1016/j.endm.2017.11.013]. In 1992, Griggs conjectured that for any graph, fre-

quency upper bound $\lambda \leq \Delta^2 + 2\Delta$. This was refined to $\Delta^2 + \Delta$ by Chang, Kuo, and Havet et al. [DOI: 10.7151/dmgt.2086]. But we have claimed this frequency bound for general classes of zero divisor graphs with in Δ^2 , where Δ is the maximum degree of the graph.

Conflicts of Interest

No potential conflict of interest was reported by the authors.

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