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Research Article

Applications of OHAM and MOHAM for Fractional Seventh-Order SKI Equations

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In this article, a comparative study between optimal homotopy asymptotic method and multistage optimal homotopy asymptotic method is presented. These methods will be applied to obtain an approximate solution to the seventh-order Sawada-Kotera Ito equation. The results of optimal homotopy asymptotic method are compared with those of multistage optimal homotopy asymptotic method as well as with the exact solutions. The multistage optimal homotopy asymptotic method relies on optimal homotopy asymptotic method to obtain an analytic approximate solution. It actually applies optimal homotopy asymptotic method in each subinterval, and we show that it achieves better results than optimal homotopy asymptotic method over a large interval; this is one of the advantages of this method that can be used for long intervals and leads to more accurate results. As far as the authors are aware that multistage optimal homotopy asymptotic method has not been yet used to solve fractional partial differential equations of high order, we have shown that this method can be used to solve these problems. The convergence of the method is also addressed. The fractional derivatives are described in the Caputo sense.

1. Introduction

Some phenomena in many disciplines are usually modeled by fractional differential equations or fractional integrodifferential equations. Fractional differential equations have been solved by some series methods, homotopy analysis transform [1], Adomian decomposition [2–4], fractional Jacobi collocation [5], variational iteration [6, 7], differential transform [8, 9], homotopy perturbation (HP) [10–12], homotopy analysis (HA) [13–15], least squares [16], and others [17–21].

Perturbation methods are used for solving problems in sciences and engineering [10–15]; however, most perturbation techniques require a small parameter in the equation. An improper choice of the parameter may leads to very bad results. Solutions obtained through perturbation methods can only be valid when a small amount of the parameter is used. Therefore, it is necessary to check the validity of the approximations through numerical processes. In 2008, Marinca and Herisanu introduced a new analytic method known as the optimal homotopy asymptotic method or for short (OHAM) [22]. An advantage of OHAM, in comparison with HAM, is that there is no need of h-curves study. This method provides us with a convenient way to control the convergence of the solution series and allows the adjustment of the convergence region, wherever it is needed. Several authors have demonstrated the effectiveness, generalizability, and reliability of this method. Another advantage of OHAM is built in convergence a criterion that is controllable. OHAM results in to satisfactory solutions on short domains, but when the interval becomes longer, the accuracy of the method decreases, so a new approach was proposed by Anakira et al., which is called multistage optimal homotopy asymptotic method (MOHAM) that is suitable for analytic approximate solutions for any long interval [23, 24]. Finally, the approximate solutions obtained from both methods are compared with the exact solution.

2. Basic Definitions

In this section, some basic definitions of fractional calculus are explained briefly [25].

Definition 1. A real-valued function $f(t)$ with $t > 0$ can be defined on the space C_μ, $\mu \in \mathbb{R}$, if there is a real number $\rho > \mu$ such that $f(t) = t^\rho f_1(t)$, where $f_1(t) \in (0,+\infty)$ and it is defined on the space C_μ^n, if and only if $f^{(n)}(t) \in C\mu$, for $n \in \mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$, for a continuous function $f \in C_\mu$, $\mu > -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \alpha > 0, \quad J^0 f(t) = f(t). \tag{1}$$

112121Some main properties of the operator J^α are listed below.

For $f \in C_\mu$, $\mu > -1$, $\alpha, \beta \geq 0$, and $\gamma \geq -1$,

$$\begin{aligned} J^\alpha J^\beta f(t) &= J^{\alpha+\beta} f(t), \\ J^\alpha J^\beta f(t) &= J^\beta J^\alpha f(t), \\ J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned} \tag{2}$$

Definition 3. The fractional derivative of $f(t)$ in Caputo sense is defined as the following:

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} f^m(s)ds, \tag{3}$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}, t > 0, f \in C_{-1}^m$.

Lemma 4. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}, t > 0, f \in C_{-1}^m$, $\mu \geq -1$, then the following two properties will be set:*

$$\begin{aligned} D^\alpha J^\alpha f(t) &= f(t), \\ (J^\alpha D^\alpha) f(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}. \end{aligned} \tag{4}$$

3. The Proposed Techniques

In this section, first the basic idea of OHAM is discussed; then, MOHAM is introduced with its convergence analysis will be stated.

3.1. OHAM. Let us consider the following fractional equation with the boundary condition:

$$L(u(\xi)) + N(u(\xi)) + f(\xi) = 0. \; B\left(u(\xi), \frac{du}{d\xi}\right) = 0, \tag{5}$$

where ξ is an independent variable L, N, f, and $u(\xi)$ represent a linear operator, a nonlinear operator, a known function, and an unknown function, respectively. Also, $B(u(\xi))$ stands for a boundary operator.

An optimal convex homotopy $\mathbf{H}(\varphi(\xi; p, c_i))$: $\Omega \times [0,1] \longrightarrow \mathbb{R}$ can be constructed as follows:

$$\begin{aligned} \mathbf{H}(\varphi(\xi; p, c_i)) &= (1-p)\left[L(\varphi(\xi; p, c_i)) + f(\xi)\right] \\ &= H(\xi, p, c_i)\left[L(\varphi(\xi; p, c_i)) + f(\xi)\right. \\ &\quad \left. + N(\varphi(\xi; p, c_i))\right], \; B(\varphi(\xi; p.,c_i)) = 0, \end{aligned} \tag{6}$$

Let us consider the following zeroth-order deformation equation

$$\begin{aligned} &(1-p)\left[L(\varphi(\xi, p)) + f(\xi)\right] \\ &\quad = H(p)\left[L(\varphi(\xi, p)) + f(\xi) + N(\varphi(\xi, p))\right], \\ &\quad B\left(\varphi(\xi, p), \frac{d\varphi(\xi.p)}{d\xi}\right) = 0, \end{aligned} \tag{7}$$

where p is an embedding parameter in the interval $[0, 1]$, $H(\xi.p)$ is an auxiliary function with nonzero and zero outputs for $p \neq 0$ and $p = 0$, respectively, $u_0(\xi)$ represents the initial condition of $u(\xi)$, and $\varphi(\xi, p)$ is an unknown function. By inserting $p = 0$ and 1 into Equation (7), one can obtain the following boundary conditions:

$$\varphi(\xi, 0) = u_0(\xi), \tag{8}$$

$$\varphi(\xi, 1) = u(\xi). \tag{9}$$

Therefore, $\varphi(\xi, p)$ will change continuously from the initial guess to the solution, $u_0(\xi)$ to $u(\xi)$ when p increases from 0 to 1.

Putting $p = 0$ into Equation (7), the initial solution $u_0(\xi)$ is determined as a solution for the problem

$$L(u_0(\xi)) + f(\xi) = 0, \quad B\left(u_0, \frac{du_0}{d\xi}\right) = 0. \tag{10}$$

Next, choose an auxiliary function $H(p)$ as the following form:

$$H(p) = pc_1 + p^2 c_2 + p^3 c_3 + \cdots, \tag{11}$$

where $c_1, c_2, c_3, \cdots$ are called the convergence control parameter that will be determined shortly. The auxiliary function $H(p)$ can be expressed in other forms, see, for exampl,e Herisanu and Marinca [7].

Expansion of $\varphi(\xi; p, c_i)$, in Taylor series about p, reads

$$\varphi(\xi; p, c_i) = u_0(\xi) + \sum_{m=1}^{\infty} u_m(\xi, c_1, c_2, \cdots, c_m)p^m. \tag{12}$$

Substituting following Equations (14)–(18) into (7), and equating the coefficients of the terms with identical powers of p, one will obtain the governing equation of the initial approximation $u_0(\xi)$, given by Equation (8); then, the governing equation of the first order problem is defined as follows:

$$L(u_1(\xi)) + f(\xi) = c_1 N_0(u_0(\xi)), B\left(u_1, \frac{du_1}{d\xi}\right) = 0. \quad (13)$$

The governing equation of the m^{th}-order is defined as

$$\begin{aligned} &L(u_m(\xi)) - L(u_{m-1}(\xi)) \\ &\quad = c_m N_0(u_0(\xi)) + \sum_{j=1}^{m-1} c_j\left[L\left(u_{m-j}(\xi)\right)\right. \\ &\qquad \left. + N_{m-j}(u_0(\xi), u_1(\xi), \cdots, u_{m-1}(\xi))\right] \ B\left(u_m, \frac{du_m}{d\xi}\right) \\ &\quad = 0, \quad m = 2, 3, \cdots, \end{aligned} \quad (14)$$

where $N_m(u_0(\xi), u_1(\xi), \cdots, u_m(\xi))$ is the coefficient of p^m in the expansion of $N(\varphi(\xi, p))$ about the embedding parameter p

$$N(\varphi(\xi, p)) = N_0(u_0(\xi)) + \sum_{m=1}^{\infty} N_m(u_0(\xi), u_1(\xi), \cdots, u_m(\xi))p^m. \quad (15)$$

Solving Equation (14) gives various approximate solutions $u_m(\xi, c_1, c_2, c_3, \cdots, c_m)$, but there exist still m unknowns, auxiliary parameters, $c_1, c_2, c_3, \cdots, c_m$ in the obtained solutions. It is assumed that the auxiliary parameters $c_1, c_2, c_3, \cdots, c_m$, the linear operator L, and the deformation equation of the zeroth order, (10), are appropriately determined in order to ensure the convergence of series (12), at $p = 1$. Hence, substituting Equations (8) and (9) into Equation (12) for $p = 1$ gives the solution $u(\xi)$ as follows:

$$u(\xi, c_1, c_2, c_3, \cdots) = u_0(\xi) + \sum_{m=1}^{\infty} u_m(\xi, c_1, c_2, c_3, \cdots, c_m). \quad (16)$$

The approximate solution of Equation (5) will be as follows:

$$\widehat{u}(\xi, c_i) = u_0(\xi) + \sum_{k=1}^{m} u_i(\xi, c_1, c_2, c_3, \cdots, c_k). \quad (17)$$

Substituting Equation (17) in Equation (5), results in the following residual:

$$R(\xi, c_i) = L(\widehat{u}(\xi, c_i)) + N(\widehat{u}(\xi., c_i)) + f(\xi), i = 1, 2, 3, \cdots, m. \quad (18)$$

By considering $R(\xi, c_i) = 0$, the exact solution is $\widehat{u}(\xi, c_i)$. However, such a case could not be true for a non-linear equation. By least square technique, the functional $J(c_i)$ should be minimized

$$J(c_i) = \int_a^b R^2(\xi, c_i) d\xi, \quad (19)$$

where a and b are the bounds of the interval in hand. The optimal values of the unknown coefficients c_i $(i = 1, 2, \cdots, m)$ can be determined based on the following conditions:

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \frac{\partial J}{\partial c_3} = \cdots = \frac{\partial J}{\partial c_m} = 0. \quad (20)$$

In order to get an analytic approximate solution at the level m, the obtained optimal coefficients will be substituted in Equation (17).

3.2. MOHAM. Although the OHAM is used to obtain approximate solutions of problems, it has some disadvantage in nonlinear problems with long domains. To control this drawback, let us introduce the multistage OHAM to obtain the nonlinear problem in some small intervals. A simple way to confirm the validity of the approximate solutions of a large domain, T is by dividing the interval $[0, T]$ in to subinterval as $[t_0, t_1), \ldots, [t_{j-1}, t_j]$, where $t_j = T$ and utilizing OHAM on each subinterval. The initial approximation in each interval is given from the solution in the last interval. First, by considering the following initial condition:

$$y_i(t_j) = \alpha_i. \quad (21)$$

Then, we can obtain the initial approximation $y_{i,0}(t) = \alpha$ and the following zero-order equation:

$$\begin{aligned} (1-p)\left[L(\varphi_i(t, p)) - y_{i,0}(t)\right] = H_i(p)[L(\varphi_i(t, p)) \\ + f_i(t) + N_i(\varphi_i(t, p))]. \end{aligned} \quad (22)$$

Next, we pick out the auxiliary function $H_i(p)$ in the form

$$H_i(p) = C_{1,j}p + C_{2,j}p^2 + C_{3,j}p^3 + \cdots, \quad (23)$$

or

$$H_i(p) = \left(C_{1,j} + C_{2,j}t + C_{3,j}t^2 + \cdots\right)p. \quad (24)$$

Then, the first, second and m^{th}-order-approximate solutions can be generated subject to initial conditions $y_{i,1}(t_j) = y_{i,2}(t_j) = \cdots = y_{i,m}(t_j) = 0$ and the approximate solutions as follows:

$$\tilde{y}_i(t, C_{1,j}, C_{2,j}, \cdots, C_{m,j}) = y_{i,0}(t) + \sum_{k=1}^{m} y_{i,k}(t, C_{1,j}, C_{2,j}, \cdots, C_{k,j}). \quad (25)$$

Substituting Equation (25) into Equation (5) yields the following residual:

$$R_i\left(t, C_{1,j}, C_{2,j}, \cdots, C_{m,j}\right) = L\left(\tilde{y}_i\left(t, C_{1,j}, C_{2,j}, \cdots, C_{m,j}\right)\right) + f_i(t) + N_i\left(\tilde{y}_i\left(t, C_{1,j}, C_{2,j}, \cdots, C_{m,j}\right)\right). \quad (26)$$

If $R_i = 0$, then $\tilde{y}_i$ will be the exact solution. Generally, such a case will not arise for nonlinear problems, but we can minimize the functional

$$J_i\left(C_{1,j}, C_{2,j}, \cdots, C_{m,j}\right) = \int_{t_j}^{t_j+h} R_i^2\left(t, C_{1,j}, C_{2,j}, \cdots, C_{m,j}\right) dt, \quad (27)$$

where h is the length of the subinterval $[t_j, t_{j+1})$ and $N = [T/h]$ is the number of subinterval. Now, we can solve Equtaion (27) for $j = 0, 1, \cdots, N$ with changing the initial approximation α_i in each subinterval from the solution in the last point of the prior interval. The unknown convergence control parameters $C_{i,j} (i = 1, 2, 3, \cdots, m, \ j = 1, 2, 3, \cdots, N)$ can be obtained by solving the following system of equations:

$$\frac{\partial J_i}{\partial C_{1.j}} = \frac{\partial J_i}{\partial C_{2.j}} = \cdots = \frac{\partial J_i}{\partial C_{m.j}} = 0. \quad (28)$$

Thus, the analytic approximate solution will be as follows:

$$\tilde{y}(t) = \begin{cases} \tilde{y}_1(t), & t_0 \le t < t_1, \\ \tilde{y}_2(t), & t_1 \le t < t_2, \\ \vdots & \\ \tilde{y}_N(t), & t_{N-1} \le t \le T. \end{cases} \quad (29)$$

In such a way, we successfully gain the solution of the initial value problem for a large interval T. It should be noted that if $j = 0$, then the MOHAM reduces to OHAM. One of the benefits of MOHAM is that it provides a simple way to control the convergence, regulate convergence region, and adjust the convergence region though the auxiliary function $H_i(p)$, involving several convergent control parameters $C_{i,j}{}'$. In general, this method eliminates the difficulty of finding approximate solutions in large ranges.

Theorem 5. *If the series (17) converges to* $u(\xi)$, *where* $u_n(\xi) \in L(R^+)$ *(linear and continuous functions on real numbers whose absolute value integral is finite), is produced by Equations (10)–(13) and satisfies the* n^{th}*-order deformation equation (14), then* $u(\xi)$ *is the exact solution of (5).*

Proof. This theorem is proved by Liao, in his valuable book [26]; here, we prove it with more details. Since the series $\sum_{n=1}^{\infty} u_n(\xi, c_1, c_2, \cdots, c_n)$ is convergent, it can be written as follows:

$$S(\xi) = \sum_{n=1}^{\infty} u_n(\xi, c_1, c_2, \cdots, c_n). \quad (30)$$

Therefore, we have

$$\lim_{n \longrightarrow \infty} u_n(\xi, c_1, c_2, \cdots, c_n) = 0. \quad (31)$$

The left hand-side of (14) satisfies

$$u_1(\xi, c_1) + \sum_{j=2}^{n} u_j\left(\xi, \vec{c}_j\right) - \sum_{j=2}^{n} u_{j-1}\left(\xi, \vec{c}_{j-1}\right) = u_n\left(\xi, \vec{c}_n\right). \quad (32)$$

According to (32) and the limit, we have

$$u_1(\xi, c_1) + \sum_{j=2}^{\infty} u_j\left(\xi, \vec{c}_j\right) - \sum_{j=2}^{\infty} u_{j-1}\left(\xi, \vec{c}_{j-1}\right) = \lim_{n \longrightarrow \infty} u_n\left(\xi, \vec{c}_n\right) = 0. \quad (33)$$

Applying the linear operator,

$$L(u_1(\xi, c_1)) + L\left(\sum_{j=2}^{\infty} u_j\left(\xi, \vec{c}_j\right)\right) - L\left(\sum_{j=2}^{\infty} u_{j-1}\left(\xi, \vec{c}_{j-1}\right)\right) = 0. \quad (34)$$

Equation (34) can be written as the following:

$$\begin{aligned} &L(u_1(\xi, c_1)) + \sum_{j=2}^{\infty}\left(L\left(u_j\left(\xi, \vec{c}_j\right)\right) - L\left(u_{j-1}\left(\xi, \vec{c}_{j-1}\right)\right)\right) \\ &= c_1 N_0(u_0) + f(\xi) + \sum_{j=2}^{\infty}\left(c_j N_0(u_0) + \sum_{k=1}^{j-1} c_k\left[L\left(u_{j-k}\right) + N_{j-k}\left(\vec{u}_{j-1}\right)\right]\right) \\ &= c_1 N_0(u_0) + f(\xi) + \sum_{j=2}^{\infty} c_j N_0(u_0) + \sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\left(c_k\left[L\left(u_{j-k}\right) + N_{j-k}\left(\vec{u}_{j-1}\right)\right]\right) \\ &= \sum_{j=2}^{\infty}\sum_{k=1}^{j-1} c_k\left[L\left(u_{j-k}\right)\right] + \sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\left(c_j N_0(u_0) + c_k\left[N_{j-k}\left(\vec{u}_{j-1}\right)\right]\right) + f(\xi) = 0. \end{aligned} \quad (35)$$

So by choosing the optimal $c_k, k = 1, 2, \cdots,$ Equation (35) is converted to the following:

$$L(u(\xi)) + N(u(\xi)) + f(\xi) = 0, \quad (36)$$

which is the exact solution to the problem. □

4. Numerical Result

Consider time-fractional SKI equation as [27]

$$D_t^{\alpha} u + 252u^3 u_x + 63u_x^3 + 378uu_x u_{xx} + 126u^2 u_{xxx} + 63u_{xx}u_{xxx} + 42u_x u_{xxxx} + 21u_{xxxxx} + u_{xxxxxxx} = 0, t > 0, \ 0 < \alpha \le 1. \quad (37)$$

with initial condition

Table 1: The values of the control parameters c_{ij}.

α	j	$c_{1,j}$	$c_{2,j}$
$\alpha = 1$	$j = 1$	-4.656060764540397481 e -16	-20.370432288203407083
	$j = 2$	-3.6335619254749103832 e-14	-5.9545639368933960096

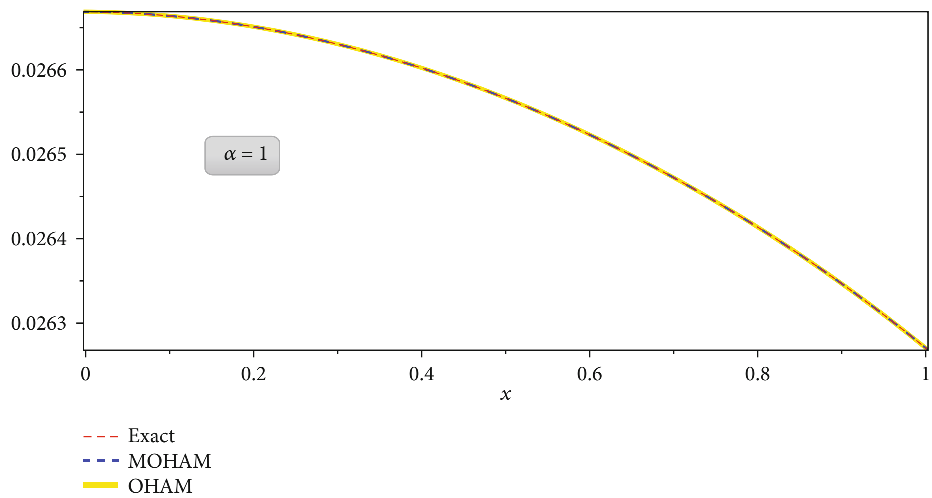

Figure 1: The results of OHAM, MOHAM, and exact solution for $\alpha = 1$ at $t = 0.2$.

$$u(x,0) = \frac{1}{75}\left(2 - 3\tan h^2(0.1x)\right). \tag{38}$$

The exact solution for $\alpha = 1$ is

$$u(x,t) = \frac{1}{75}\left(2 - 3\tan h^2\left(0.1\left(x - \frac{64}{46875}t\right)\right)\right). \tag{39}$$

4.1. Approximate Solution by OHAM. Choosing the linear operator $L = D_t^\alpha$, and applying the algorithm based on OHAM, as presented in Section 3, the initial guess $u_0(x,t)$ will be obtained as $D_t^\alpha(u_0(x,t)) = 0$, with $u(x,o) = (1/75)(2 - 3\tan h^2(0.1x))$.

We obtain

$$u_0(x,t) = \frac{1}{75}\left(2 - 3\tan h^2(0.1x)\right). \tag{40}$$

Substituting $m = 1, 2, 3, \cdots$ successively, the problems of various orders are as follows:

$$\begin{aligned} u_1(x,t) = c_1 \Bigg(& \frac{43690666665 \tanh(x/10)t}{63999999997558594} - \frac{6990506665 \tan h^3(x/10)t}{10239999997558594} \\ & + \frac{\tan h^5(x/10)t}{5000000000000000000000000} - \frac{\tan h^7(x/10)t}{33333333333333333333333} \\ & - \frac{\tan h^9(x/10)t}{100000000000000000000000} \Bigg). \end{aligned} \tag{41}$$

In this paper, three terms approximation of $u(x,t)$ is considered. By substituting zeroth-, first-, and second-order solutions in (7), and by using technique in Section 3, we find the following values for c_1 and c_2:

$$\begin{aligned} c_1 &= -7.9895512250076505116\, 10^{-15}, \; c_2 \\ &= -7.7704946584917721471. \end{aligned} \tag{42}$$

4.2. Approximate Solution by MOHAM. In this section, we utilize MOHAM for Equation (37). We will consider the auxiliary function $H(p,t)$ as the following:

$$H(p,t) = \left(c_{1,j} + c_{2,j}t\right)p. \tag{43}$$

As in OHAM, we obtain two approximate solutions to Equation (37):

$$\tilde{u}_j(x,t) = u_{0j}(x,t) + u_{1j}(x,t). \tag{44}$$

This approach leads to the following sequence of equations:

$$p^0 : D_t^\alpha\left(u_{0j}(x,t)\right) = \alpha_j,$$

$$\begin{aligned} p^1 : D_t^\alpha\left(u_{1j}(x,t)\right) = & -c_{1j}\Big[252{u_{0j}}^3(x,t)\, D_x\left(u_{0j}(x,t)\right) \\ & + 63{D_x}^3\left(u_{0j}(x,t)\right) + 378u_{0j}(x,t)\, D_x\left(u_{0j}(x,t)\right) D_{xx} \\ & \cdot \left(u_{0j}(x,t)\right) + 126{u_{0j}}^2(x,t) D_{xxx}\left(u_{0j}(x,t)\right) + 63D_{xx} \\ & \cdot \left(u_{0j}(x,t)\right) D_{xxx}\left(u_{0j}(x,t)\right) + 42D_x\left(u_{0j}(x,t)\right) D_{xxxx} \\ & \cdot \left(u_{0j}(x,t)\right) + 21D_{xxxxx}\left(u_{0j}(x,t)\right) + D_{xxxxxxx}\left(u_{0j}(x,t)\right)\Big], \end{aligned} \tag{45}$$

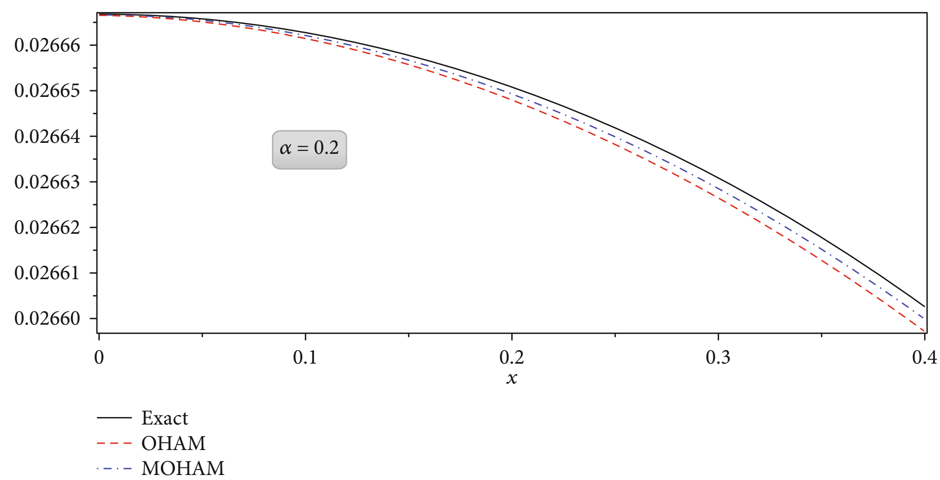

Figure 2: The results of OHAM, MOHAM, and exact solution for $\alpha = 0.2$ at $t = 0.4$.

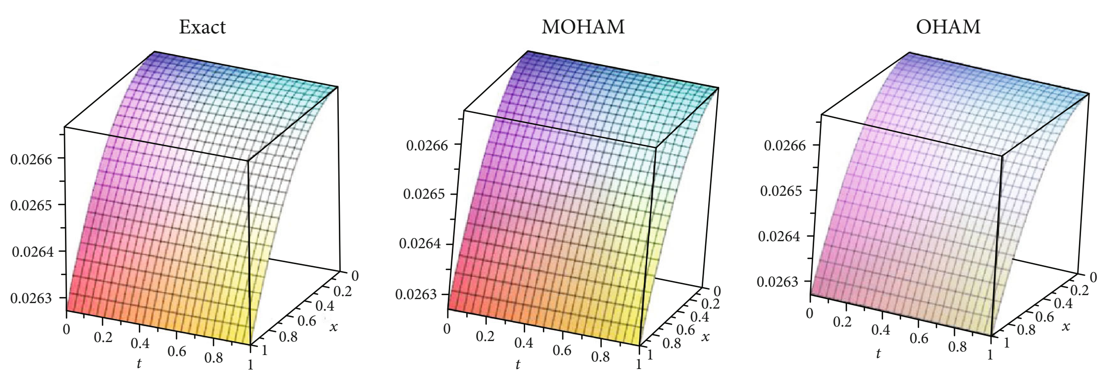

Figure 3: The results of OHAM, MOHAM, and exact solution for $\alpha = 1$.

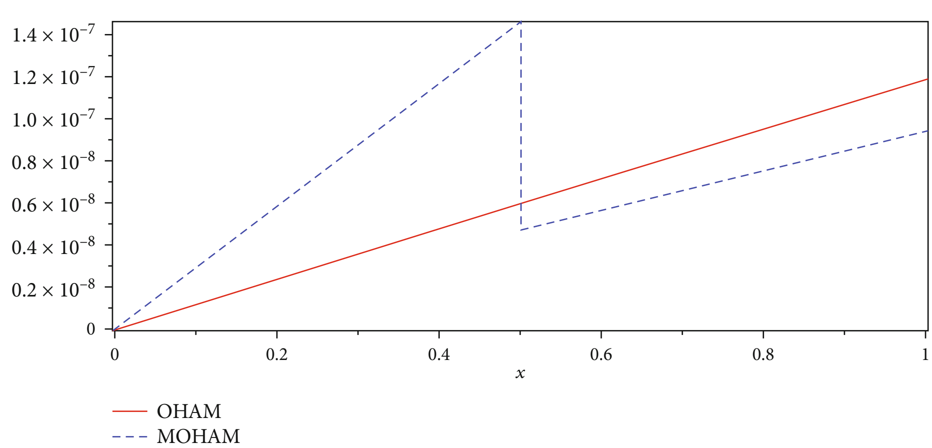

Figure 4: The absolute error of OHAM and MOHAM, for $\alpha = 1$ and $t = 0.2$.

Table 2: Absolute error of OHAM and MOHAM for $\alpha = 1$ and $t = 0.2$.

x	0	0.2	0.4	0.6	0.8	1
Abs(exact-OHAM)	$1.17e-13$	$2.4e-8$	$4.78e-8$	$7.15e-8$	$9.5e-8$	$1.18e-7$
Abs(exact-MOHAM)	$1.17e-13$	$5.8e-8$	$1.16e-8$	$5.7e-8$	$7.5e-8$	$9.4e-8$

By applying the least squares technique, the parameters $c_{1,j}$ and $c_{2,j}$ are determined for $\alpha = 1, h = 0.5$, and t from $t_0 = 0$ up to $t_2 = T = 1$. The results are presented in Table 1.

Approximate solution for $\alpha = 1$ is in the following form:

$$\begin{aligned}
&\Big(1.8 \times 10^{-54} \tanh^{16}(0.1x) - 8.9 \times 10^{-54} \tanh^{14}(0.1x) \\
&\quad + 1.9 \times 10^{-53} \tanh^{12}(0.1x) - 2.1 \times 10^{-53} \tanh^{10}(0.1x) \\
&\quad + 1.4 \times 10^{-54} \tanh^{8}(0.1x) - 5.2 \times 10^{-54} \tanh^{6}(0.1x) \\
&\quad - 1.8 \times 10^{-42} \tanh^{4}(0.1x) + 2.5 \times 10^{-42} \tanh^{2}(0.1x) \\
&\quad - 6.3 \times 10^{-43} \tanh^{16}(0.1x)\Big) t^2 + \Big(2.04 \times 10^{-22} \tanh^{9} \\
&\quad \cdot (0.1x) - 2.03 \times 10^{-22} \tanh^{7}(0.1x) \\
&\quad + 1.8 \times 10^{-21} \tanh^{5}(0.1x) + 0.0000 \tanh^{3}(0.1x) \\
&\quad - 0.04 \tanh^{2}(0.1x) - 0.00001 \tanh(0.1x)\Big) t \\
&\quad + 0.03, \quad 0 < x < 0.5, \\
&\Big(1.1 \times 10^{-50} \tanh^{16}(0.1x) - 5.5 \times 10^{-50} \tanh^{14}(0.1x) + 1.1 \\
&\quad \times 10^{-49} \tanh^{12}(0.1x) - 1.3 \times 10^{-49} \tanh^{10}(0.1x) + 8.3 \\
&\quad \times 10^{-50} \tanh^{8}(0.1x) - 3.2 \times 10^{-50} \tanh^{6}(0.1x) - 1.2 \\
&\quad \times 10^{-38} \tanh^{4}(0.1x) + 1.5 \times 10^{-38} \tanh^{2}(0.1x) - 3.8 \\
&\quad \times 10^{-39}\Big) t^2 \Big(6 \times 10^{-23} \tanh^{9}(0.1x) - 6 \times 10^{-23} \tanh^{7}(0.1x) \\
&\quad + 5.4 \times 10^{-22} \tanh^{5}(0.1x) \\
&\quad + 0.000004 \tanh^{3}(0.1x) - 0.000004 \tanh(0.1x)\Big) t \\
&\quad - 0.04 \tanh^{2}(0.1x), \quad 0.5 \le x \le 1.
\end{aligned} \tag{46}$$

From Figures 1–3, one can see that the solutions obtained by OHAM and MOHAM are nearly identical with the exact solution.

5. Conclusion

In this study, the optimal homotopy asymptotic method and the multistage optimal homotopy asymptotic method are used to derive an analytic approximate solution to the time-fractional seventh order Sawada-Kotera-Ito equation. The results obtained from these methods show that multistage optimal homotopy asymptotic method converges better than the optimal homotopy asymptotic method. One observes that the results agree very well with the exact solution. The multistage optimal homotopy asymptotic method by dividing the interval $[0, T]$ into some subintervals can obtain more exact solution than the optimal homotopy asymptotic method. The disadvantage of the multistage optimal homotopy asymptotic method is that it is time consuming and requires more time to solve problems than the optimal homotopy asymptotic method. As far as the authors are aware, the multistage optimal homotopy asymptotic method has not been used to solve fractional partial differential equations yet. In this study, the method has been tested on fractional PDE and yields to satisfactory results. Figure 4 and Table 2 expose that the multistage optimal homotopy asymptotic method results to more accurate solution as compared to the optimal homotopy asymptotic method. The convergence of the method is addressed.

Data Availability

The boundary operator data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. M. Saad, E. H. F. al-Shareef, A. K. Alomari, D. Baleanu, and J. F. Gómez-Aguilar, “On exact solutions for time-fractional Korteweg-de Vries and Korteweg-de Vries-Burger's equations using homotopy analysis transform method,” *Chinese Journal of Physics*, vol. 63, pp. 149–162, 2020.

[2] M. G. Sakar and F. Erdogan, “The homotopy analysis method for solving the time-fractional Fornberg-Whitham equation and comparison with Adomian's decomposition method,” *Applied Mathematical Modelling*, vol. 37, no. 20-21, pp. 8876–8885, 2013.

[3] Y. Hu, Y. Luo, and Z. Lu, “Analytical solution of the linear fractional differential equation by Adomian decomposition method,” *Journal of Computational and Applied Mathematics*, vol. 215, no. 1, pp. 220–229, 2008.

[4] M. Turkyilmazoglu, “Nonlinear problems via a convergence accelerated decomposition method of Adomian,” *Computer Modeling in Engineering & Sciences*, vol. 127, no. 1, pp. 1–22, 2021.

[5] M. A. Abdelkawy, A. M. Lopes, and M. M. Babatin, “Shifted fractional Jacobi collocation method for solving fractional functional differential equations of variable order,” *Chaos, Solitons & Fractals*, vol. 134, article 109721, 2020.

[6] G.-C. Wu, “A fractional variational iteration method for solving fractional nonlinear differential equations,” *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2186–2190, 2011.

[7] Y. Molliq R, M. S. M. Noorani, and I. Hashim, “Variational iteration method for fractional heat- and wave-like equations,”

Nonlinear Analysis: Real World Applications, vol. 10, no. 3, pp. 1854–1869, 2009.

[8] A. Aytac and O. Ibrahim, "Solution of fractional differential equations by using differential transform method," *Chaos, Solitons & Fractals*, vol. 34, pp. 1473–1481, 2007.

[9] B. İbiş, M. Bayram, and A. Agargum, "Applications of fractional differential transform method to fractional differential-algebraic equations," *European Journal of Pure and Applied Mathematics*, vol. 4, pp. 129–141, 2011.

[10] S. Hosseinnia, A. Ranjbar, and S. Momani, "Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3138–3149, 2008.

[11] S. Momani and Z. Odibat, "Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 910–919, 2007.

[12] J. H. He, "Homotopy perturbation method with two expanding parameters," *Indian Journal of Physics*, vol. 88, no. 2, pp. 193–196, 2014.

[13] M. Ganjiani, "Solution of nonlinear fractional differential equations using homotopy analysis method," *Applied Mathematical Modelling*, vol. 34, no. 6, pp. 1634–1641, 2010.

[14] S. Das and D. K. Gupta, "Homotopy analysis method for solving fractional hyperbolic partial differential equations," *International Journal of Computer Mathematics*, vol. 88, pp. 578–588, 2011.

[15] S. Abbasbandy, M. S. Hashemi, and I. Hashim, "On convergence of homotopy analysis method and its application to fractional integro-differential equations," *Quaestiones Mathematicae*, vol. 36, no. 1, pp. 93–105, 2013.

[16] C. Bota and B. Căruntu, "Analytic approximate solutions for a class of variable order fractional differential equations using the polynomial least squares method," *Fractional Calculus and Applied Analysis*, vol. 20, no. 4, pp. 1043–1050, 2017.

[17] H. Khalil, R. A. Khan, and M. M. Rashidi, "Brenstien polynomials and applications to fractional differential equations," *Computational Methods for Differential Equations*, vol. 3, pp. 14–35, 2015.

[18] S. Kumar, D. Kumar, S. Abbasbandy, and M. M. Rashidi, "Analytical solution of fractional Navier-Stokes equation by using modified Laplace decomposition method," *Ain Shams Engineering Journal*, vol. 5, no. 2, pp. 569–574, 2014.

[19] J. Biazar, S. Safaei, and M. Tango, "Fractional modelling for prey and predator problem by using optimal homotopy asymptotic method," *International Journal of Applied Mathematical Research*, vol. 9, pp. 35–40, 2020.

[20] M. S. Hashemi and A. Akgül, "Solitary wave solutions of time-space nonlinear fractional Schrodinger's equation: two analytical approaches," *Journal of Computational and Applied Mathematics*, vol. 339, pp. 147–160, 2018.

[21] M. Turkyilmazoglu, "Parametrized Adomian decomposition method with optimum convergence," *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, vol. 27, no. 4, pp. 1–22, 2017.

[22] V. Marinca and N. Herisanu, "Application of optimal homotopy asymptotic method for solving non-linear equations arising in heat transfer," *International Communications in Heat and Mass Transfer*, vol. 35, pp. 710–715, 2008.

[23] N. R. Anakira, A. K. Alomari, A. F. Jameel, and I. Hashim, "Multistage optimal homotopy asymptotic method for solving initial–value problems," *Journal of Nonlinear Sciences and Applications*, vol. 9, pp. 1826–1843, 2016.

[24] B. Jafar and S. Saghi, "Applications of OHAM and MOHAM for time-fractional Klein-Fock-Gordon equation," *International Journal of Optics and Photonic Engineering*, vol. 5, no. 2, p. 26, 2020.

[25] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering V198*, Academic press, 1999.

[26] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall, CRC Press, Boca Raton, 2003.

[27] D. Ganji, M. Safari, and R. Ghayor, *Application of He's Variational Iteration Method and Adomian's Decomposition Method to Sawada–Kotera–Ito Seventh-Order Equation*, Wiley Online Library, 2009.