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Research Article

Irreversible k-Threshold Conversion Number of Circulant Graphs

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An irreversible conversion process is a dynamic process on a graph where a one-way change of state (from state 0 to state 1) is applied on the vertices if they satisfy a conversion rule that is determined at the beginning of the study. The irreversible k-threshold conversion process on a graph $G=(V,E)$ is an iterative process which begins by choosing a set $S_0 \subseteq V$, and for each step $t(t=1,2,\cdots,)$,S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. S_0 is called the seed set of the k-threshold conversion process, and if $S_t = V(G)$ for some $t \geq 0$, then S_0 is an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $(C_k(G))$ is the minimum cardinality of all the IkCSs of G. In this paper, we determine $C_2(G)$ for the circulant graph $C_n(\{1,\mathrm{r}\})$ when r is arbitrary; we also find $C_3(C_n(\{1,\mathrm{r}\}))$ when $r=2,3$. We also introduce an upper bound for $C_3(C_n(\{1,4\}))$. Finally, we suggest an upper bound for $C_3(C_n(\{1,\mathrm{r}\}))$ if $n \geq 2(r+1)$ and $n \equiv 0 (\mathrm{mod}\, 2(r+1))$.

1. Introduction

As usual, $n=|V|$ and $m=|E|$ denote the numbers of vertices and edges at a graph $G(V,E)$, respectively. Let deg (v) be the degree of a vertex v; a graph is t-regular if all of its vertices are of degree t. The open neighborhood of a vertex v is $N(v)=\{u \in V : uv \in E\}$ while the closed neighborhood of v is $N[v]=N(v) \cup \{v\}$. For any undefined term in the paper, we refer to Harary [1]. An irreversible k-threshold conversion process on a graph $G=(V,E)$ is the process of finding the least number of vertices we need to initially convert in step $t=0$ in order to spread the conversion to all the remaining vertices of the graph according to a conversion rule. This iterative process starts by choosing a seed set $S_0 \subseteq V$, and for each step $t(t=1,2,\cdots,)$,S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. S_0 is called the seed set of the k-threshold conversion process, and if $S_t=V(G)$ for some $t \geq 0$, then S_0 is an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $(C_k(G))$ is the minimum cardinality of all the IkCSs of G. Therefore, $1 \leq k \leq \Delta(G)$ and $C_1(G)=1$ for connected graphs. The first graph model of the irreversible k-threshold conversion problem was presented by Dreyer and Roberts in [2] where they determined the value of C_2 for paths and cycles. For further information on the irreversible k-threshold conversion problem on graphs, see [2–6]. The circulant graph $C_n(S)$ with the connection set $S \subseteq \{1,2,\cdots,n\}$ is an undirected graph with the vertex set $V=\{v_1,v_2,\cdots,v_n\}$ where two vertices v_i, v_j are adjacent if $|i-j| \in S$. Therefore, the circulant graph $C_n(1)$ is a cycle, and the circulant graph $C_n(\{1,2,\cdots,\lfloor n/2 \rfloor\})$ forms the complete graph K_n. It is obvious that the circulant graph $C_n(\{1,r\})$ is 4-regular when $1<r \leq \lfloor (n-1)/2 \rfloor$. Through this paper, we will denote the vertex set by $V=\{v_i | i=1,\cdots,n\}$ taking into consideration that we exchange the subscript of the vertex v_0 by v_n. For further information on the circulant graph, see [7].

Proposition 1 (see [2]).

$$C_2(P_n)=\left\lceil \frac{n+1}{2} \right\rceil. \tag{1}$$

Proposition 2 (see [2]).

$$C_2(C_n)=\left\lceil \frac{n}{2} \right\rceil. \tag{2}$$

Proposition 3 (see [2]). *If G is a k-regular graph, then S is a k-conversion set of G if and only if V − S is independent.*

Note 1: in every figure of this article, we assign the black color to the converted vertices and the white color to unconverted ones.

2. Main Results

In this paper, we determine $C_2(G)$ for the circulant graph $C_n(\{1, r\})$ for arbitrary r; we also find $C_3(C_n(\{1, r\}))$ when $r = 2, 3$. We also introduce an upper bound for $C_3(C_n(\{1, 4\}))$. Then we suggest an upper bound for $C_3(C_n(\{1, r\}))$ if $n \geq 2(r+1)$ and $n \equiv 0 (\text{mod}\, 2(r+1))$. Let $C_n(\{1, r\})$ be a circulant graph on which an irreversible k-threshold conversion process is being studied. Since $C_n(\{1, r\})$ is 4-regular, we define the k-unconvertable set of G (denoted by U) as follows:

$$U \subseteq V - S_0 \text{ and for every } x \in U \text{ then } |N(x) \cap U| \geq 4 - k + 1. \tag{3}$$

which means each vertex of U is unconverted and is adjacent to at least $4 - k + 1$ vertices of U at $t = 0$; therefore, the conversion cannot reach any vertex of U during any step of the process unless at least one of its vertices is converted at $t = 0$. Figure 1 shows a 3-unconvertable set on $C_n(\{1, 2\})$.

Note 2: let $C_n(\{1, r\})$ be a circulant graph, and let the conversion threshold be $k = 2$. We will define a conversion generating path (CGP) as a series of r consecutive vertices (a path) on $C_n(\{1, r\})$ such as $(v_i, v_{i+1}, \cdots, v_{i+r-1})$, so that when all of these vertices are converted in a step $t = l$, they can spread the conversion to the entire graph by converting two new (unconverted) vertices at every following step. The process goes as follows:

$t = l$: the conversion reaches all vertices of the CGP which are $v_i, v_{i+1}, \cdots, v_{i+r-1}$

$t = l + 1$: since v_i, v_{i+r-1} are adjacent to both v_{i-1}, v_{i+r}, then conversion spreads to v_{i-1}, v_{i+r}, and the converted vertices are $v_{i-1}, v_{i+1}, \cdots, v_{i+r-1}, v_{i+r}$

$t = l + 2$: since v_{i-1}, v_{i+r} are adjacent to both v_{i-2}, v_{i+r+1}, then conversion spreads to v_{i-2}, v_{i+r+1}

The conversion process continues until all vertices of V are converted. This goal is achieved on step $t = l + \lceil (n - r - m)/2 \rceil$ where m represents the number of converted vertices of $V - V(CGP)$ at $t = l$.

If $n - r - m$ is even, then the last two unconverted vertices are converted at the last step, which is $t = l + ((n - r - m)/2)$.

If $n - r - m$ is odd, then at the next to last step $t = l + \lceil (n - r - m)/2 \rceil - 1$; two unconverted vertices are converted, and then only one unconverted vertex remains to be converted in the last step which is $t = l + \lceil (n - r - m)/2 \rceil$. Figure 2 illustrates a 5-vertex CGP on $C_{12}(\{1, 5\})$.

$$C_2(C_n(\{1, r\})). \tag{4}$$

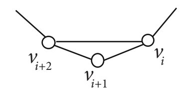

Figure 1: $X = \{v_i, v_{i+1}, v_{i+2}\}$ is 3-unconvertable on $C_n(\{1, 2\})$. $C_{12}(\{1, 5\})$. $C_{12}(\{1, 5\}).J_{2,m}$.

In this subsection, we determine $C_2(C_n(\{1, r\}))$ for arbitrary r when $1 < r \leq \lfloor (n-1)/2 \rfloor$.

Theorem 4. *For* $n \geq 5, C_2(C_n(\{1, 2\})) = 2$.

Proof. We know by definition that for any graph G, $C_k(G) \geq k$, which means that $C_2(C_n(\{1, 2\})) \geq 2$. Let $S_0 = \{v_1, v_2\}$ be the seed set of the conversion process. S_0 forms a CGP on $C_n(\{1, 2\})$ with $l = 0$, and the process goes as follows:

$t = 0$: we convert v_1, v_2

$t = 1$: the conversion spreads to v_3, v_n

$t = 2$: the conversion spreads to v_4, v_{n-1} □

The process continues, spreading the conversion to two new (unconverted) vertices each step. If n is even, the process ends in step $t = n/2 - 1$ when the last two unconverted vertices $(v_{(n/2)+1}, v_{(n/2)+2})$ are converted.

If n is odd, at the next to last step $t = \lceil (n-2)/2 \rceil - 1 = (n-3)/2$, there are three unconverted vertices left which are $(v_{\lceil n/2 \rceil}, v_{\lceil n/2 \rceil + 1}, v_{\lceil n/2 \rceil + 2})$. Two of them $(v_{\lceil n/2 \rceil}, v_{\lceil n/2 \rceil + 2})$ are converted in $t = (n-3)/2$ while the last unconverted vertex $(v_{\lceil n/2 \rceil + 1})$ is converted in the last step $t = \lceil (n-2)/2 \rceil = (n-1)/2$.

We conclude that S_0 is an I2CS of $C_n(\{1, 2\})$ which means $C_2(C_n(\{1, 2\})) \leq 2$; therefore, $C_2(C_n(\{1, 2\})) = 2$, and we prove the requested.

Theorem 5. *For* $n \geq 7, C_2(C_n(\{1, 3\})) = 2$.

Proof. Since $C_2(C_n(\{1, 3\})) \geq 2$, we need to prove that $C_2(C_n(\{1, 3\})) \leq 2$. Let $S_0 = \{v_1, v_3\}$ be the seed set. The process goes as follows:

$t = 0$: we convert $S_0 = \{v_1, v_3\}$

$t = 1$: since $v_2 \, adj \, v_1, v_3$ the conversion spreads to v_2. Similarly, $v_4 \, adj \, v_1, v_3$, and we also have $v_n \, adj \, v_1, v_3$ which means $S_1 = \{v_1, v_2, v_3, v_4, v_n\}$. Each of the sets $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, and $\{v_1, v_2, v_n\}$ forms a CGP on $C_n(\{1, 3\})$ with $l = 1$.

$t = 2$: the conversion spreads to v_5, v_{n-1}

$t = 3$: the conversion spreads to v_6, v_{n-2} □

The process continues similarly to Theorem 4 until the graph is successfully converted at step $t = (n-5)/2 + 1$ if n is odd, or at step $t = \lceil (n-5)/2 \rceil + 1$ if n is even.

We conclude that $S_0 = \{v_1, v_3\}$ is an I2CS of $C_n(\{1, 3\})$ which means $C_2(C_n(\{1, 3\})) \leq 2$; then $C_2(C_n(\{1, 3\})) = 2$, and the requested is proven.

Theorem 6. $C_2(C_n(\{1, 4\})) = \begin{cases} 2 \text{ for } 9 \leq n \leq 11; \\ 3 \text{ otherwise.} \end{cases}$

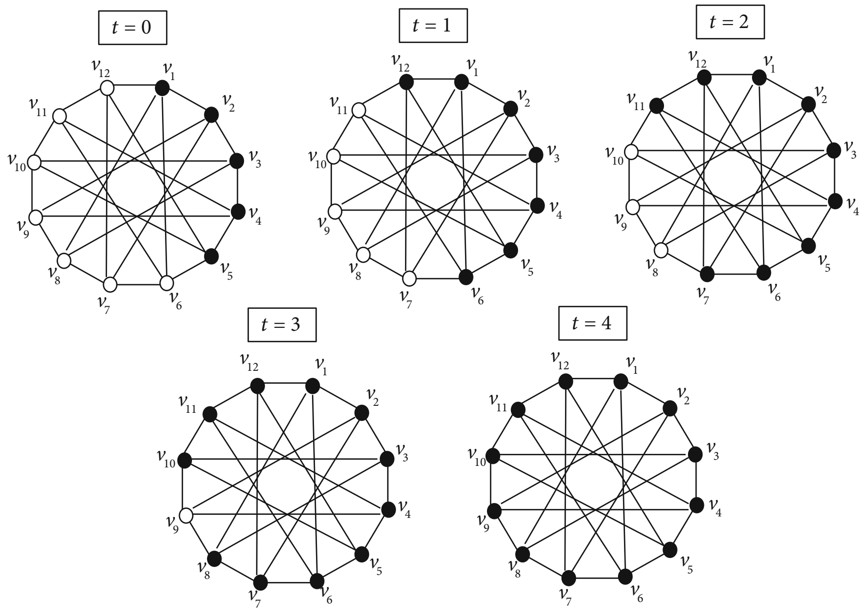

Figure 2: $(v_1, v_2, v_3, v_4, v_5)$ form a CGP on $C_{12}(\{1,5\})$.

Proof. We consider the following cases: □

Case 1. $n = 9$. Let $S_0 = \{v_1, v_6\}$ be the seed set. The process goes as follows:

t = 0: we convert $S_0 = \{v_1, v_6\}$

t = 1: since $v_2\, adj\, v_1, v_6$ the conversion spreads to v_2. Similarly, $v_5\, adj\, v_1, v_6$, and we also have $S_1 = \{v_1, v_2, v_5, v_6\}$

t = 2: the conversion spreads to v_7, v_9; therefore, $S_2 = \{v_1, v_2, v_5, v_6, v_7, v_9\}$

t = 3: the conversion spreads to v_3, v_4, v_8; therefore, $S_3 = V(C_9(\{1,4\}))$

We conclude that $C_2(C_9(\{1,4\})) = 2$.

Case 2. $n = 10$. Let $S_0 = \{v_1, v_6\}$ be the seed set. The process goes as follows:

t = 0: we convert $S_0 = \{v_1, v_6\}$

t = 1: the conversion spreads to $v_2.v_5, v_7, v_{10}$ which means $S_1 = \{v_1, v_2, v_5, v_6, v_7, v_{10}\}$

t = 2: the conversion spreads to $v_3.v_4, v_8, v_9$ which means $S_2 = V(C_{10}(\{1,4\}))$

We conclude that $C_2(C_{10}(\{1,4\})) = 2$.

Case 3. $n = 11$. Let $S_0 = \{v_1, v_6\}$ be the seed set. The process goes as follows:

t = 0: we convert $S_0 = \{v_1, v_6\}$

t = 1: the conversion spreads to v_2, v_5; therefore, $S_1 = \{v_1, v_2, v_5, v_6\}$

t = 2: the conversion spreads to v_9 which means $S_2 = \{v_1, v_2, v_5, v_6, v_9\}$

t = 3: the conversion spreads to v_8, v_{10}; therefore, $S_3 = \{v_1, v_2, v_5, v_6, v_8, v_9, v_{10}\}$

t = 4: the conversion spreads to $v_3.v_4, v_7, v_{11}$ which means $S_4 = V(C_{11}(\{1,4\}))$

We conclude that $C_2(C_{11}(\{1,4\})) = 2$.

Case 4. $n \geq 12$. We start by proving that $C_2(C_n(\{1,4\})) > 2$ for $n \geq 12$. We consider the following subcases:

Case 4.a. $S_0 = \{v_1, v_2\}$; since $N(v_1) \cap N(v_2) = \varnothing$, then $S_1 = S_0$, and the conversion does not spread after the initial step t = 0, which means the process fails. Without loss of generality, the same argument can be applied for any $S_0 = \{v_i, v_{i+1} : 1 \leq i \leq n\}$.

Case 4.b. $S_0 = \{v_1, v_3\}$; since $N(v_1) \cap N(v_3) = \{v_2\}$, then at step t = 1, we get $S_1 = \{v_1, v_2, v_3\}$. However, $N(v_1) \cap N(v_2) \cap N(v_3) = \varnothing$ which means $S_2 = S_1$ and the spread stops at the end of step t = 1. Without loss of generality, this applies to any $S_0 = \{v_i, v_{i+2} : 1 \leq i \leq n\}$.

Case 4.c. $S_0 = \{v_1, v_4\}$. In a similar way to the previous two cases, $S_1 = \{v_1, v_4, v_5, v_n\}$, $S_2 = S_1$ which means the spread stops at the end of step t = 1, and without loss of generality; this applies to any $S_0 = \{v_i, v_{i+3} : 1 \leq i \leq n\}$.

Case 4.d. $S_0 = \{v_1, v_5\}$; then $S_1 = S_0$, and the process fails at the end of step t = 0. Without loss of generality, this applies to any $S_0 = \{v_i, v_{i+4} : 1 \leq i \leq n\}$.

Case 4.e. $S_0 = \{v_1, v_6\}$; then $S_1 = \{v_1, v_2, v_5, v_6\}$, but $S_2 = S_1$ and the spread stops at the end of step t = 1. Without loss of generality, this applies to any $S_0 = \{v_i, v_{i+5} : 1 \leq i \leq n\}$.

Case 4.f. $S_0 = \{v_1, v_l : l \geq 7\}$. Since $N(v_1) \cap N(v_l) = \varnothing$, then $S_1 = S_0$, and the conversion does not spread after the initial step t = 0, which means the process fails. Without loss of generality, the same argument can be applied for any $S_0 = \{v_i, v_{i+l} : 1 \leq i \leq n \text{ and } l \geq 7\}$.

From subcases 4.a to 4.f, we conclude that

$$C_2(C_n(\{1,4\})) > 2. \quad (5)$$

Now, let $S_0 = \{v_1, v_3, v_4\}$ be the seed set; then at t = 1, the conversion spreads to v_2, v_5, v_n which makes $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_n\}$. We notice that each set of four consecutive

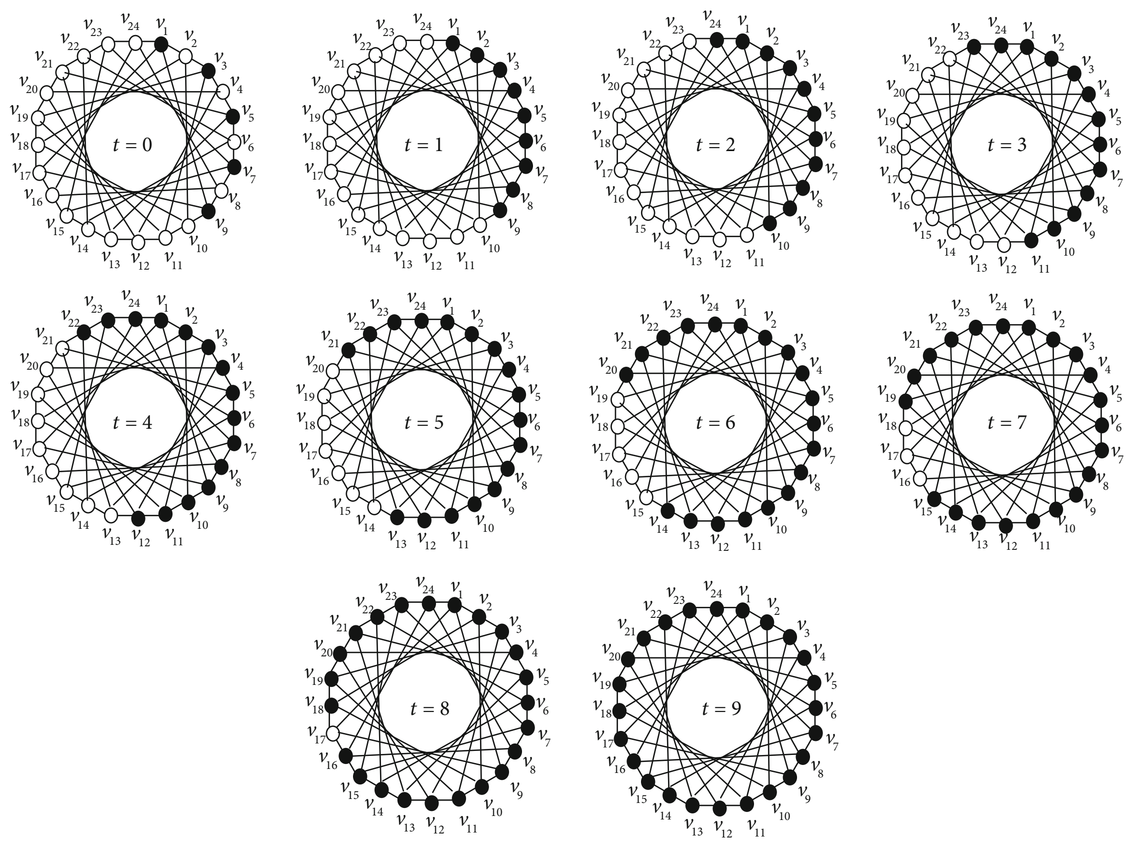

Figure 3: An I2CS of 5 vertices on $C_{24}(\{1,8\})$.

vertices of S_1 forms a CGP on $C_n(\{1,4\})$ with $l=1$ and $m=2$. The process goes as follows:

t = 0: we convert $S_0=\{v_1,v_3,v_4\}$

t = l =1: $S_1=\{v_1,v_2,v_3,v_4,v_5,v_n\}$

t = 2 :$S_2=\{v_1,v_2,v_3,v_4,v_5,v_6,v_{n-1},v_n\}$

The process continues converting two new vertices each step until it ends at t = $(n-4)/2$ if n is even, or at t = $(n-3)/2$ if n is odd. We conclude that $S_0=\{v_1,v_3,v_4\}$ is an I2CS of $C_n(\{1,4\})$. Therefore

$$C_2(C_n(\{1,4\}))\le 3. \tag{6}$$

From (5) and (6), we conclude that $C_2(C_n(\{1,4\}))=3$ for $n\ge 12$.

From Cases 1–4, we conclude the requested.

Theorem 7. *For* $r\ge 4$,$C_2(C_n(\{1,r\}))=\lceil (r+1)/2\rceil$.

Proof. Since a CGP is a path of length r, and by Proposition 1, $C_2(P_n)=\lceil (n+1)/2\rceil$; therefore, $Y=\{v_1,v_3,\cdots,v_m : m = r \text{ if } r \text{ is odd}, m=r+1 \text{ if } r \text{ is even}\}$ can be an I($\lceil (r+1)/2\rceil$)CS of $C_n(\{1,\mathrm{r}\})$. This means

$$C_2(C_n(\{1,\mathrm{r}\}))\le \left\lceil \frac{r+1}{2}\right\rceil. \tag{7}$$

Figure 3 illustrates an I2CS of 5 vertices on $C_{24}(\{1,8\})$. □

We now consider the following cases for r:

Case 1. r is odd. Let $S_0=\{v_1,v_3,\cdots,v_{r-2}\}$ be a conversion seed set of cardinality $\lceil (r+1)/2\rceil-1=(r-1)/2$. The process goes as follows:

t = 0: we convert $S_0=\{v_1,v_3,\cdots,v_{r-4},v_{r-2}\}$

t = 1: $S_1=\{v_1,v_2,v_3,v_4\cdots,v_{r-3},v_{r-2}\}$

t = 2: $S_2=\{v_1,v_2,v_3,v_4\cdots,v_{r-3},v_{r-2}\}=S_1$. The process stops at the end of step t = 1. Without loss of generality, the process applies to all configurations of S_0 on the vertices of $\{v_1,v_2,v_3,v_4\cdots,v_{r-1},v_r\}$. Therefore, S_0 cannot produce a CGP on $C_n(\{1,\mathrm{r}\})$, and since we need to convert at least $\lceil (r+1)/2\rceil=(r+1)/2$ vertices of the path $\{v_1,v_2,v_3,v_4\cdots,v_{r-1},v_r\}$ in order to convert it entirely, it is impossible to convert $C_n(\{1,\mathrm{r}\})$ if we initially convert less than $(r+1)/2$ vertices at t = 0, which means $C_2(C_n(\{1,\mathrm{r}\}))>\lceil (r+1)/2\rceil-1$.

Case 2. r is even. Let $S_0=\{v_1,v_3,\cdots,v_{r-1}\}$ be a conversion seed set of cardinality $\lceil (r+1)/2\rceil-1=r/2$. The process goes as follows:

t = 0: we convert $S_0=\{v_1,v_3,\cdots,v_{r-3},v_{r-1}\}$

t = 1: $S_1=\{v_1,v_2,v_3,v_4\cdots,v_{r-2},v_{r-1}\}$

t = 2: $S_2=\{v_1,v_2,v_3,v_4\cdots,v_{r-2},v_{r-1}\}=S_1$

The process stops at the end of step t = 1. Without loss of generality, the process applies to all configurations of S_0 on the vertices of $\{v_1, v_2, v_3, v_4 \cdots, v_{r-1}, v_r\}$. Therefore, S_0 cannot produce a CGP on $C_n(\{1, r\})$, and since we need to convert at least $\lceil (r+1)/2 \rceil = (r+1)/2$ vertices of the path $\{v_1, v_2, v_3, v_4 \cdots, v_{r-1}, v_r\}$ in order to convert it entirely, it is impossible to convert $C_n(\{1, r\})$ if we initially convert less than $(r+1)/2$ vertices at t = 0, which means $C_2(C_n(\{1, r\})) > \lceil (r+1)/2 \rceil - 1$.

From Case 1 and Case 2, we conclude that

$$C_2(C_n(\{1, r\})) > \left\lceil \frac{r+1}{2} \right\rceil - 1. \quad (8)$$

From (7) and (8), we conclude that $C_2(C_n(\{1, r\})) = \lceil (r+1)/2 \rceil$.

$$C_3(C_n(\{1, r\})). \quad (9)$$

In this subsection, we determine $C_3(C_n(\{1, r\}))$ for $r = 2, 3$. Then, we introduce an upper bound for $C_3(C_n(\{1, 4\}))$. Finally, we introduce an upper bound for $C_3(C_n(\{1, r\}))$ if $n \geq 2(r+1)$ and $n \equiv 0(\bmod 2(r+1))$.

Theorem 8. *For $n \geq 5$, $C_3(C_n(\{1, 2\})) = \begin{cases} \lfloor n/3 \rfloor + 1 \text{ if } n \equiv 0, 1(\bmod 3), \\ \lceil n/3 \rceil + 1 \text{ if } n \equiv 2(\bmod 3). \end{cases}$*

Proof. We have $C_3(C_n(\{1, 2\})) \geq 3$, and we consider the following cases: □

Case 1. $n \equiv 0(\bmod 3)$.

Let $S_0 = \{v_{3l} : 1 \leq l \leq n/3\} \cup \{v_1\}$ be the seed set of the conversion process. It is obvious that S_0 is of cardinality $(n/3) + 1$. The process goes as follows:

t = 0: we convert $S_0 = \{v_{3l} : 1 \leq l \leq n/3\} \cup \{v_1\}$

t = 1: we notice that $|N(v_2) \cap S_0| = |N(v_{n-1}) \cap S_0| = 3$ which means that the conversion spreads to $\{v_2, v_{n-1}\}$ making $S_1 = \{v_{3l} : 1 \leq l \leq n/3\} \cup \{v_1, v_2, v_{n-1}\}$

t = 2: we notice that $|N(v_4) \cap S_1| = |N(v_{n-2}) \cap S_1| = 3$; therefore, the conversion spreads to $\{v_4, v_{n-2}\}$, which makes $S_2 = \{v_{3l} : 1 \leq l \leq n/3\} \cup \{v_1, v_2, v_4, v_{n-2}, v_{n-1}\}$

The process goes on, and with each step, two new (unconverted) vertices $\{v_m, v_{n-d}\}$ are converted, where $5 \leq m \leq \lceil n/2 \rceil$ and $m \not\equiv 0(\bmod\ 3)$. Similarly, $4 \leq d \leq \lceil n/2 \rceil$ and $m \not\equiv 0(\bmod\ 3)$. Which means, the next to last step is $t = (n/3) - 1$, and $S_{(n/3)-1}$ is defined as:

$$t = \frac{n}{3} - 1 : S_{n/3-1} = \begin{cases} V - \{v_{n/2+1}\} \text{if } n \text{ is even}; \\ V - \left\{v_{\lceil \frac{n}{2} \rceil}\right\} \text{if } n \text{ is odd}. \end{cases} \quad (10)$$

$t = n/3$: the conversion spreads to the last remaining vertex $v_{(n/2)+1}$ (or $v_{\lceil n/2 \rceil}$) if n is odd (or even), respectively, and then the conversion is spread to all vertices of $C_n(\{1, 2\})$ which makes S_0 an I3CS of $C_n(\{1, 2\})$. Therefore

$$C_3(C_n(\{1, 2\})) \leq \frac{n}{3} + 1 \text{ if } n \equiv 0(\bmod 3). \quad (11)$$

Let v_i, v_{i+1}, v_{i+2} be three consecutive unconverted vertices of $C_n(\{1, 2\})$. At any step of the process, in order to convert any of these three vertices, it needs to be adjacent to three converted vertices. However, this is impossible since each one of them is adjacent to the other two and they are all unconverted, which means $X = \{v_i, v_{i+1}, v_{i+2}\}$ is 3-unconvertable if $X \cap S_0 = \varnothing$. We imply that any seed set of cardinality $n/3$ on $C_n(\{1, 2\})$ should be distributed as one of the following:

$$D_0 = \left\{v_{3l} : 1 \leq l \leq \frac{n}{3}\right\},$$
$$B_0 = \left\{v_{3l+1} : 0 \leq l \leq \frac{n}{3} - 1\right\}, \quad (12)$$
$$E_0 = \left\{v_{3l+2} : 0 \leq l \leq \frac{n}{3} - 1\right\}.$$

Let us assume that D_0 is the seed set. The conversion process goes as follows:

t = 0: we convert $D_0 = \{v_{3l} : 1 \leq l \leq n/3\}$

t = 1 : with the absence of any unconverted vertex that is adjacent to three vertices of D_0, then $D_1 = D_0$ which means the spread stops at the end of step t = 0. Therefore, the process fails. Without loss of generality, the same result can be obtained if the seed set was B_0 or E_0, then:

$$C_3(C_n(\{1, 2\})) > n/3 \text{ if } n \equiv 0(\bmod 3) \quad (13)$$

From (11) and (13), we conclude that $C_3(C_n(\{1, 2\})) = n/3 + 1$ if $n \equiv 0(\bmod 3)$.

Case 2. $n \equiv 1(\bmod 3)$.

Similarly to Case 1, a seed set of cardinality $\lfloor n/3 \rfloor$ is not enough to convert $C_n(\{1, 2\})$. Let the seed set be $S_0 = \{v_{3l} : 1 \leq l \leq (n-1)/3\} \cup \{v_n\}$. It is obvious that $|S_0| = (n-1)/3 + 1 = \lfloor n/3 \rfloor + 1$. The process goes (similarly to Case 1) as follows:

t = 0: we convert $S_0 = \{v_{3l} : 1 \leq l \leq (n-1)/3\} \cup \{v_n\}$

t = 1: since $|N(v_1) \cap S_0| = |N(v_{n-2}) \cap S_0| = 3$, the conversion spreads to $\{v_2, v_{n-2}\}$, which makes $S_1 = \{v_{3l} : 1 \leq l \leq (n-1)/3\} \cup \{v_1, v_{n-2}, v_n\}$

t = 2: since $|N(v_2) \cap S_1| = |N(v_{n-3}) \cap S_1| = 3$, the conversion spreads to $\{v_2, v_{n-3}\}$, which makes $S_2 = \{v_{3l} : 1 \leq l \leq (n-1)/3\} \cup \{v_1, v_2, v_{n-3}, v_{n-2}, v_n\}$

The process goes on, and with each step, two new (unconverted) vertices $\{v_m, v_{n-d}\}$ are converted, where $4 \leq$

$m \leq \lceil n/2 \rceil$ and $m \not\equiv 0 (\bmod\ 3)$. Similarly, $5 \leq d \leq \lceil n/2 \rceil$ and $m \not\equiv 0 (\bmod\ 3)$, which means the next to last step is $t = \lfloor n/3 \rfloor - 1$, and $S_{\lfloor n/3 \rfloor - 1}$ is defined as:

$$t = \left\lfloor \frac{n}{3} \right\rfloor - 1 : S_{\left\lfloor \frac{n}{3} \right\rfloor - 1} = \begin{cases} V - \left\{ v_{\left\lfloor \frac{n}{2} \right\rfloor - 1}, v_{\left\lfloor \frac{n}{2} \right\rfloor + 1} \right\} \text{if } n \text{ is odd}; \\ V - \{ v_{n/2-1}, v_{n/2} \} \text{if } n \text{ is even}. \end{cases} \tag{14}$$

$t = \lfloor n/3 \rfloor$: the conversion spreads to the last two unconverted vertices $v_{\lceil n/2 \rceil}, v_{\lceil n/2 \rceil + 1}$ if n is odd, or $v_{(n/2)-1}, v_{(n/2)}$ if n is even. Either way, the process ends successfully at the end of $t = \lfloor n/3 \rfloor$ which means S_0 is an I3CS of $C_n(\{1,2\})$ and $C_3(C_n(\{1,2\})) \leq \lfloor n/3 \rfloor + 1$. However, since $C_3(C_n(\{1,2\})) > \lfloor n/3 \rfloor$, we conclude that $C_3(C_n(\{1,2\})) = \lfloor n/3 \rfloor + 1$ if $n \equiv 1 (\bmod\ 3)$.

Case 3. $n \equiv 2 (\bmod\ 3)$.

Similarly to Case 1 and Case 2, $C_3(C_n(\{1,2\})) > \lfloor n/3 \rfloor$ and in order to avoid getting any version of X on $C_n(\{1, 2\})$, the seed set must take one of the following forms:

$$\begin{aligned} D_0 &= \{ v_{3l} : 1 \leq l \leq \lfloor n/3 \rfloor \} \cup \{ x : x \in \{ v_1, v_n \} \}, \\ B_0 &= \left\{ v_{3l+1} : 0 \leq l \leq \left\lfloor \frac{n}{3} \right\rfloor - 1 \right\} \cup \{ x : x \in \{ v_{n-2}, v_{n-1} \} \}, \\ E_0 &= \left\{ v_{3l+2} : 0 \leq l \leq \left\lfloor \frac{n}{3} \right\rfloor - 1 \right\} \cup \{ x : x \in \{ v_{n-1}, v_n \} \}. \end{aligned} \tag{15}$$

Let D_0 be the seed set (of cardinality $\lfloor n/3 \rfloor + 1 = \lceil n/3 \rceil$). The process goes as follows:

$t = 0$: we convert $D_0 = \{ v_{3l} : 1 \leq l \leq \lfloor n/3 \rfloor \} \cup \{ x : x \in \{ v_1, v_n \} \}$. Whether $x = v_1$ or $x = v_n$, the conversion does not spread to any vertex from $V - D_0$ because no vertex of $V - D_0$ is adjacent to three vertices of D_0

$t = 1$: $D_1 = D_0$, and the process fails. Without loss of generality, the same result is obtained if the seed set was B_0 or E_0; therefore

$$C_3(C_n(\{1,2\})) > \left\lfloor \frac{n}{3} \right\rfloor + 1 \text{ if } n \equiv 2 (\bmod\ 3). \tag{16}$$

Let $S_0 = \{ v_{3l} : 1 \leq l \leq \lfloor n/3 \rfloor \} \cup \{ v_{n-1}, v_n \}$ be seed set of cardinality $\lfloor n/3 \rfloor + 1 = \lceil n/3 \rceil + 1$. The process goes as follows:

$t = 0$: we convert $S_0 = \{ v_{3l} : 1 \leq l \leq \lfloor n/3 \rfloor \} \cup \{ v_{n-1}, v_n \}$

$t = 1$: the conversion spreads to v_1, v_{n-3}

$t = 2$: the conversion spreads to v_2, v_{n-4}

Similarly to Case 1 and Case 2, by the end of each step, two new (unconverted) vertices are converted, and the process continues until the last step $t = \lfloor n/3 \rfloor$ when the last two unconverted vertices which are $\{ v_{n/2}, v_{(n/2)-2} \}$ (in case n is even), and $\{ v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor - 1} \}$ (in case n is odd) get converted and then the conversion process reaches the entire graph. We conclude that:

$$C_3(C_n(\{1,2\})) \leq \left\lceil \frac{n}{3} \right\rceil + 1 \text{ if } n \equiv 2 (\bmod\ 3). \tag{17}$$

From (16) and (17), we conclude that $C_3(C_n(\{1,2\})) = \lceil n/3 \rceil + 1$ if $n \equiv 2 (\bmod\ 3)$.

From all the previous cases, we conclude the requested.

Proposition 9. *A set of $r+1$ consecutive unconverted vertices in $C_n(\{1,r\})$ is 3-unconvertable.*

Proof. Let there be a conversion process on $C_n(\{1, \mathrm{r}\})$, and at the initial step $t = 0$, let the set $X = \{ v_i, v_{i+1}, v_{i+r-1}, v_{i+r} \}$ be a set of $r+1$ unconverted vertices. Every vertex of X is of degree 4 and is adjacent to two other vertices of X. Since $k = 3$, it is impossible for any of these vertices to satisfy the conversion condition at any step of the conversion process even if all the vertices of $V - X$ get converted, which means X is 3-unconvertble. □

Theorem 10. *For $n \geq 7$ we have:*

$$C_3(C_n(\{1,3\})) = \left\{ \frac{3n/8 ; n \equiv 0 (\bmod 8), / \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1 ; n \equiv 0,1,2 (\bmod 4) \text{ and } n \not\equiv 0,1,2 (\bmod 8);}{./(3 \lfloor n/8 \rfloor + 1 ; n \equiv 1,2,3 (\bmod 8), / \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 2 ; n \equiv 3 (\bmod 4) \text{ and } n \not\equiv 3 (\bmod 8).)} \right. \tag{18}$$

Proof.

Let S_0 be the seed set of the conversion process. We implied in Proposition 9 that the process fails if $V - S_0$ contains four (or more) consecutive vertices, which means S_0 contains at least $\lfloor n/4 \rfloor$ vertices. Let D_0, E_0, L_0, B_0 be the following sets:

$$\begin{aligned} D_0 &= \left\{ v_{4l} : 1 \leq l \leq \left\lfloor \frac{n}{4} \right\rfloor \right\}, \\ E_0 &= \left\{ v_{4l+1} : 0 \leq l \leq \left\lfloor \frac{n}{4} \right\rfloor - 1 \right\}, \\ L_0 &= \left\{ v_{4l+2} : 0 \leq l \leq \left\lfloor \frac{n}{4} \right\rfloor - 1 \right\}, \\ B_0 &= \left\{ v_{4l+3} : 0 \leq l \leq \left\lfloor \frac{n}{4} \right\rfloor - 1 \right\}. \end{aligned} \tag{19}$$

□

In order to avoid having four consecutive unconverted vertices, S_0 must contain either D_0, E_0, L_0, or B_0. We assume that $D_0 \subseteq S_0$; we notice that the vertices of D_0 divide $C_n(\{1,$

3}) into $\lfloor n/4 \rfloor$ subgraph, each of which consists of four consecutive vertices, and only one of these vertices is converted at t = 0. We denote these subgraphs by $SG_i : 1 \le i \le \lfloor n/4 \rfloor$, when:

$$\begin{aligned} SG_1 &= \{v_1, v_2, v_3, v_4\}, \\ SG_2 &= \{v_5, v_6, v_7, v_8\}, \\ &\vdots \\ SG_{\lfloor \frac{n}{2} \rfloor} &= \{v_{n-m-3}, v_{n-m-2}, v_{n-m-1}, v_{n-m}\}. \end{aligned} \tag{20}$$

We take into consideration that $m \in \{0, 1, 2, 3\}$ if $n \equiv 0, 1, 2, 3 (\bmod 4)$, respectively. The last m vertices form a minisubgraph of 1,2, and 3 vertices if $n \equiv 1, 2, 3 (\bmod 4)$, respectively. We imply that for $S_0 = D_0$, the conversion does not spread to any vertex of $V - S_0$ because none of them is adjacent to three vertices of S_0 which means $S_1 = S_0 = D_0$, and the process fails. This means $D_0 \subset S_0$. Let $Y = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$ be a set of five consecutive vertices where the middle vertex v_{i+2} is the only converted vertex of Y; then the four unconverted vertices of Y form a 3-unconvertable set on $C_n(\{1, 3\})$ and that is because each of them is adjacent to two other unconverted vertices of Y, which means there cannot be a step t when any of these four vertices satisfies the conversion rule (being adjacent to three or more converted vertices). We conclude that there cannot be two consecutive subgraphs SG_i, SG_{i+1}, of the subgraphs identified previously, or else, a version of Y will be created consisting of the last two vertices of SG_i and the first three vertices of SG_{i+1}; this means that we need to add at least one converted vertex to one of every two consecutive subgraphs. Therefore

$$C_3(C_n(\{1, 3\})) \ge \begin{cases} 3\left\lfloor \dfrac{n}{8} \right\rfloor & \text{if } n \equiv 0, 1, 2, 3 (\bmod 8); \\ \left\lfloor \dfrac{n}{4} \right\rfloor + \left\lfloor \dfrac{n}{8} \right\rfloor + 1 & \text{if } n \equiv 0, 1, 2, 3 (\bmod 4) \text{ and } n \not\equiv 0, 1, 2, 3 (\bmod 8). \end{cases} \tag{21}$$

We consider the following cases for n:

Case 1. $n \equiv 0 (\bmod 4)$. We consider two subcases:

Case 1.a. $n \equiv 0 (\bmod 8)$.

In this subcase, we have an even number of subgraphs on $C_n(\{1, 3\})$ and $m = 0$. Let M_0 be the following set: $M_0 = \{v_{2+8l} : 0 \le l \le (n/8) - 1\}$. Let $S_0 = D_0 \cup M_0$ be the seed set. It is obvious that $|S_0| = 3n/8$. The conversion process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le n/4\} \cup \{v_{2+8l} : 0 \le l \le (n/8) - 1\}$

t = 1: the conversion spreads to $\{v_{1+8l}, v_{3+8l}, v_{5+8l}, v_{7+8l} : 0 \le l \le (n - 8) - 1\}$

t = 2: the conversion spreads to $\{v_{6+8l} : 0 \le l \le (n/8) - 1\}$

By the end of step t = 2, the conversion reaches all vertices of $C_n(\{1, 3\})$; therefore, S_0 is an I3CS of $C_n(\{1, 3\})$ and $C_3(C_n(\{1, 3\})) \le 3n/8$, and from (21), we conclude that $C_3(C_n(\{1, 3\})) = 3n/8$ when $n \equiv 0 (\bmod 8)$.

Case 1.b. $n \equiv 0 (\bmod 4)$ and $n \not\equiv 0 (\bmod 8)$.

This subcase is similar to subcase 1.a with the only difference of having an odd number of subgraphs. This means we need to convert one additional vertex (v_{n-2}). The conversion process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-2}\}$

t = 1: the conversion spreads to $\{v_{1+8l}, v_{3+8l}, v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}\{v_{n-1}\}$

t = 2: the conversion spreads to the remaining vertices which are $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$

Therefore, S_0 is an I3CS of $C_n(\{1, 3\})$ and since $|S_0| = \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$, according to (21), we conclude that $C_3(C_n(\{1, 3\})) = \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$ if $n \equiv 0 (\bmod 4)$ and $n \not\equiv 0 (\bmod 8)$. Figure 4 illustrates an I3CS of 8 vertices on $C_{20}(\{1, 3\})$.

Case 2. $n \equiv 1 (\bmod 4)$. We consider two subcases:

Case 2.a. $n \equiv 1 (\bmod 8)$.

In this subcase, we have an even number of subgraphs on $C_n(\{1, 3\})$ and $m = 1$. Let D_0 and M_0 be the same sets identified in subcase 1.a. Let the seed set be $N_0 = D_0 \cup M_0$. In a similar process to the one in subcase 1.a, all vertices of $V - \{v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_1\}$ by the end of step t = 2. However, the five consecutive vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_1$ form a version of Y which was identified in Theorem 8 as containing a 3-unconvertable set. In addition to that, since taking out any vertex from $D_0 \cup M_0$ results in a version of either X or Y on $C_n(\{1, 3\})$, we conclude that $C_3(C_n(\{1, 3\})) > 3\lfloor n/8 \rfloor$ in this subcase. Let $S_0 = D_0 \cup M_0 \cup \{v_1\}$ be the seed set of cardinality $3\lfloor n/8 \rfloor + 1$; the process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_1\}$

t = 1: the conversion spreads to $\{v_{1+8l}, v_{3+8l} : 1 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-2}\}$

t = 2: the conversion spreads to $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$ which means v_{n-3} is converted at this step as well

t = 3: the conversion spreads to $\{v_3, v_n\}$

By the end of step t = 3, the conversion reaches all vertices of $C_n(\{1, 3\})$; therefore, S_0 is an I3CS of $C_n(\{1, 3\})$, and since $|S_0| = 3\lfloor n/8 \rfloor + 1$, we conclude that $C_3(C_n(\{1, 3\})) = 3\lfloor n/8 \rfloor + 1$ if $n \equiv 1 (\bmod 8)$.

Case 2.b. $n \equiv 1 (\bmod 4)$ and $n \not\equiv 1 (\bmod 8)$.

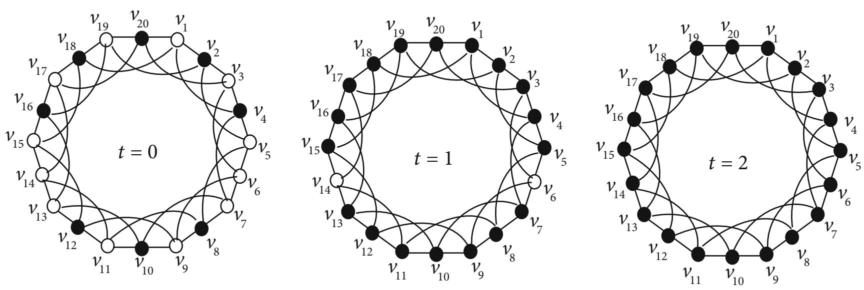

Figure 4: An I3CS of 8 vertices on $C_{20}(\{1,3\})$.

This subcase is similar to subcase 2.a with the only difference of having an odd number of SGs. This means that similarly to subcase 1.b, we need to convert one additional vertex from this last subgraph (v_{n-3}); the process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_1\}$

t = 1: the conversion spreads to $\{v_{1+8l}, v_{3+8l} : 1 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-2}\}$

t = 2: the conversion spreads to $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$

t = 3: the conversion spreads to $\{v_3, v_{n-4}\}$

t = 4: the conversion spreads to $\{v_{n-3}, v_n\}$

By the end of step t = 4, the conversion reaches all vertices of $C_n(\{1,3\})$. Therefore, S_0 is an I3CS of $C_n(\{1,3\})$, and since $|S_0| = \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$, this means $C_3(C_n(\{1,3\})) = \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$ if $n \equiv 1 (\mathrm{mod}\ 4)$ and $n \not\equiv 1 (\mathrm{mod}\ 8)$.

Case 3. $n \equiv 2 (\mathrm{mod}\ 4)$. We consider two subcases:

Case 3.a. $n \equiv 2 (\mathrm{mod}\ 8)$.

In this subcase, we have an even number of subgraphs on $C_n(\{1,3\})$ and $m = 2$. Let D_0 and M_0 be the same sets identified in subcase 1.a. Let the seed set be $N_0 = D_0 \cup M_0$. In a similar process to the one in subcase 1.a, all vertices of $V - \{v_{n-4}, v_{n-3}, v_{n-1}, v_n, v_1\}$ by the end of step t = 2. However, the five consecutive vertices $v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n$ form a version of Y, and since taking out any vertex from $D_0 \cup M_0$ results in a version of either X or Y, we conclude that $C_3(C_n(\{1,3\})) > 3\lfloor n/8 \rfloor$ in this subcase as well. Let $S_0 = D_0 \cup M_0 \cup \{v_n\}$ be the seed set of cardinality $3\lfloor n/8 \rfloor + 1$; the process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_n\}$

t = 1: the conversion spreads to $\{v_{1+8l}, v_{3+8l}, v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-1}\}$ which means v_1, v_{n-3}, v_{n-1} are converted in this step

t = 2: the conversion spreads to $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$ which means v_{n-4} is converted at this step

By the end of step t = 2, the conversion reaches all vertices of $C_n(\{1,3\})$; therefore, S_0 is an I3CS of $C_n(\{1,3\})$, and since $|S_0| = 3\lfloor n/8 \rfloor + 1$, we conclude that $C_3(C_n(\{1,3\})) = 3\lfloor n/8 \rfloor + 1$ if $n \equiv 2 (\mathrm{mod}\ 8)$.

Case 3.b. $n \equiv 2 (mod\ 4)$ and $n \not\equiv 2 (mod\ 8)$.

By following the same argument in subcase 1.b and subcase 3.a, let $S_0 = D_0 \cup M_0 \cup \{v_{n-4}\}$ be the seed set of cardinality $\lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$; the process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-4}\}$

t = 1: the conversion spreads to the vertices $\{v_{1+8l}, v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{3+8l} : 1 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-5}, v_{n-3}, v_{n-1}\}$ which means $v_1, v_{n-5}, v_{n-3}, v_{n-1}$ are converted in this step

t = 2: the conversion spreads to the vertices $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_n\}$

t = 3: the conversion spreads to the last unconverted vertex v_3, and the entire graph is converted

We conclude that $C_3(C_n(\{1,3\})) = \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$ if $n \equiv 2 (mod\ 4)$ and $n \not\equiv 2 (mod\ 8)$.

Case 4. $n \equiv 3 (mod\ 4)$.

We consider two subcases:

Case 4.a. $n \equiv 3 (mod\ 8)$.

In this subcase, we have an even number of subgraphs on $C_n(\{1,3\})$ and $m = 3$. Let D_0 and M_0 be the same sets identified in subcase 1.a. Let the seed set be $N_0 = D_0 \cup M_0$. In a similar process to the one in subcase 1.a, all vertices of $V - \{v_1, v_3, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$ by the end of step t = 2. However, the five consecutive vertices $v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$ form a version of Y, and since taking out any vertex from $D_0 \cup M_0$ results in a version of either X or Y, we conclude that $C_3(C_n(\{1,3\})) > 3\lfloor n/8 \rfloor$ in this subcase as well. Let $S_0 = D_0 \cup M_0 \cup \{v_{n-1}\}$ be the seed set of cardinality $3\lfloor n/8 \rfloor + 1$; the process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-1}\}$

t = 1: vertices $\{v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{1+8l}, v_{3+8l} : 1 \le l \le \lfloor n/8 \rfloor\}$ are converted, which means v_{n-4} is converted at this step

t = 2: the conversion spreads to $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$ which means v_{n-5} is converted at this step

t = 3: v_3, v_{n-2} are converted

t = 4: v_1, v_n are converted

By the end of step t = 4, the conversion reaches all vertices of $C_n(\{1,3\})$; therefore, S_0 is an I3CS of $C_n(\{1,3\})$, and since $|S_0| = 3\lfloor n/8 \rfloor + 1$, we conclude that $C_3(C_n(\{1,3\})) = 3\lfloor n/8 \rfloor + 1$ if $n \equiv 3 (mod\ 8)$.

Case 4.b. $n \equiv 3 (mod\ 4)$ and $n \not\equiv 3 (mod\ 8)$. We consider two subcases:

Case 4.b.1. $n = 7$.

This proof is equivalent to proving that $C_3(C_7(\{1,3\})) = 3$. It is obvious by definition that $C_3(C_7(\{1,3\})) \ge 3$. Let

S_0 be a seed set of cardinality 3 and defined as $S_0 = \{v_1, v_3, v_6\}$. The process goes as:

$$\begin{aligned} &t = 0 : S_0 = \{v_1, v_3, v_6\}. \\ &t = 1 : S_1 = S_0 \cup \{v_2, v_7\}. \\ &t = 2 : S_2 = S_1 \cup \{v_4, v_5\} = V(C_7(\{1, 3\})) \end{aligned} \quad (22)$$

which means $C_3(C_7(\{1,3\})) \le 3$; therefore, $C_3(C_7(\{1,3\})) = 3$.

Case 4.b.2. $n \ge 11$.

Let $S_0 = D_0 \cup M_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$; then that would make the following nine vertices $v_{n-9}, v_{n-8}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n, v_1$ unconverted which means creating several versions of A_1, and the process fails. Now let $S_0 = D_0 \cup M_0 \cup \{v_x\}$ be the seed set. We consider the following options for v_x :

(i) If $v_x \in \{v_{n-9}, v_{n-8}, v_{n-6}, v_{n-5}\}$, then $v_{n-2}, v_{n-1}, v_n, v_1$ are four consecutive unconverted vertices; therefore, they form a version of A_1, and the process fails. However, we notice that

(ii) If $v_x \in \{v_{n-4}, v_{n-2}, v_{n-1}, v_n, v_1\}$, then $(v_{n-9}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5})$ form a version of Y, and since Y is 3-unconvertable, then the process fails

(iii) If $v_x \notin \{v_{n-9}, v_{n-8}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n, v_1\}$, then $v_{n-2}, v_{n-1}, v_n, v_1$ form a version of A_1, $(v_{n-9}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5})$ form a version of Y, and the process fails

We conclude $D_0 \cup M_0 \cup \{v_x\}$ cannot be a I3CS of $C_n(\{1, 3\})$ when $n \equiv 3(mod\,4)$ and $n \not\equiv 3(mod\,8)$ which means that $C_3(C_n(\{1, 3\})) > \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$ if $n \equiv 2(mod\,4)$ and $n \not\equiv 2(mod\,8)$.

Now let $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-5}, v_{n-1}\}$ be the seed set of cardinality $\lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 2$; the process goes as follows:

t = 0: we convert $S_0 = \{v_{4l} : 1 \le l \le \lfloor n/4 \rfloor\} \cup \{v_{2+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-5}, v_{n-1}\}$

t = 1: the conversion spreads to $\{v_{1+8l}, v_{3+8l} : 1 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{5+8l}, v_{7+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_{n-6}, v_{n-4}, v_{n-2}\}$

t = 2: the conversion spreads to $\{v_{6+8l} : 0 \le l \le \lfloor n/8 \rfloor - 1\} \cup \{v_1\}$

t = 3 : the last two unconverted vertices v_3, v_n get converted which means the entire graph is converted at the end of this step

We conclude that S_0 is an I3CS, and therefore, $C_3(C_n(\{1, 3\})) = \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 2$ if $n \equiv 3(mod\,4)$ and $n \not\equiv 3(mod\,8)$.

From all cases and subcases, we conclude the requested.

Theorem 11. *For* $n \ge 9$.

$$C_3(C_n(\{1, 4\})) \le \begin{cases} 2\left\lfloor \frac{n}{5} \right\rfloor + 2 \text{ if } n \in \{9, 14, 19\}; \\ 2\left\lfloor \frac{n}{5} \right\rfloor \text{ if } n \equiv 0, 5, 6, 7 (mod\ 10) \text{ and } n \notin \{16, 17\}; \\ 2\left\lfloor \frac{n}{5} \right\rfloor + 1 \text{ otherwise.} \end{cases} \quad (23)$$

Proof. We implied in Proposition 9 that the conversion process fails if there are five consecutive unconverted vertices on $C_n(\{1, 4\})$ at $t = 0$, which means the conversion seed set S_0 cannot consist of less than $\lfloor n/5 \rfloor$ vertices. Let $D_0 = \{v_{5l} : 1 \le l \le \lfloor n/5 \rfloor\}$; we assume that $D_0 \subseteq S_0$, and we notice that the vertices of D_0 divide the first $5\lfloor n/5 \rfloor$ vertices of $V(C_n(\{1, 4\}))$ into $\lfloor n/5 \rfloor$ subgraphs, each of which consists of four consecutive unconverted vertices followed by one converted vertex. We denote them by $SG_i : 1 \le i \le \lfloor n/5 \rfloor$. The two adjacent subgraphs SG_3, SG_4 together have two converted vertices $\{v_{15}, v_{20}\}$ and eight unconverted vertices which are $\{v_{11}, v_{12}, v_{13}, v_{14}, v_{16}, v_{17}, v_{18}, v_{19}\}$. We notice that the set $U = \{v_{12}, v_{13}, v_{16}, v_{17}\}$ consists of four unconverted vertices each of which is adjacent to two vertices of U which means U is 3-unconvertable. Therefore, the process fails if $S_0 = D_0$, Without loss of generality, the same argument applies to any $SG_i, SG_{i+1} : 1 \le i \le \lfloor n/5 \rfloor - 1$. Let us now try to find a configuration of converted vertices that prevents having any unconvertable sets and at the same time guarantees total conversion of SG_3, SG_4. We imply that converting $\{v_{13}, v_{15}, v_{17}, v_{20}\}$ and applying this configuration to the neighboring subgraphs SG_1, SG_2, SG_5, SG_6 achieves the requested for SG_3, SG_4 as shown in Figure 5. □

Therefore, we apply this configuration to every two adjacent subgraphs. As for the remaining $n - 10\lfloor n/10 \rfloor$, we will need to convert additional vertices in order to convert them. In that regard, we consider the following cases of n:

Case 1. $n \equiv 0(mod\,10)$.

Let $S_0 = \{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \le l \le (n/10) - 1\}$ be the seed set. The process goes as follows:

t = 0: we convert $S_0 = \{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \le l \le (n/10) - 1\}$

t = 1: the conversion spreads to $\{v_{1+10l}, v_{4+10l}, v_{6+10l}, v_{9+10l} : 0 \le l \le (n/10) - 1\}$

t = 2: the remaining unconverted vertices which are $\{v_{2+10l}, v_{8+10l} : 0 \le l \le \lfloor n/8 \rfloor - 1\}$ get converted

By the end of step t = 2, the entire graph's vertex set is converted. We conclude that S_0 is an I3CS of cardinality $2n/5$, which means $C_3(C_n(\{1, 4\})) \le 2n/5$ if $n \equiv 0(mod\,10)$. Figure 6 illustrates that $C_{20}(\{1, 4\}) \le 8$.

Case 2. $n \equiv 1(mod\,10)$.

Let $S_0 = \{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \le l \le \lfloor n/10 \rfloor - 1\}$ $U\{v_1\}$ be the seed set. The process goes as follows:

$$\begin{aligned} t = 0 : S_0 &= \left\{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \le l \le \left\lfloor \frac{n}{10} \right\rfloor - 1\right\} U\{v_1\}, \\ t = 1 : S_1 &= S_0 \cup \left\{v_{1+10j}, v_{4+10l}, v_{6+10l}, v_{9+10l} : 1 \le j \le \left\lfloor \frac{n}{10} \right\rfloor, 0 \le l \le \left\lfloor \frac{n}{10} \right\rfloor - 1\right\} U\{v_n\}, \\ t = 2 : S_2 &= S_1 \cup \left\{v_{2+10l}, v_{8+10d} : 0 \le l \le \left\lfloor \frac{n}{10} \right\rfloor - 1,\ 1 \le d \le \left\lfloor \frac{n}{10} \right\rfloor - 1\right\} U\{v_4\}, \\ t = 3 : S_3 &= S_2 \cup \{v_8, v_{n-2}\} = V(C_n(\{1, 4\})). \end{aligned} \quad (24)$$

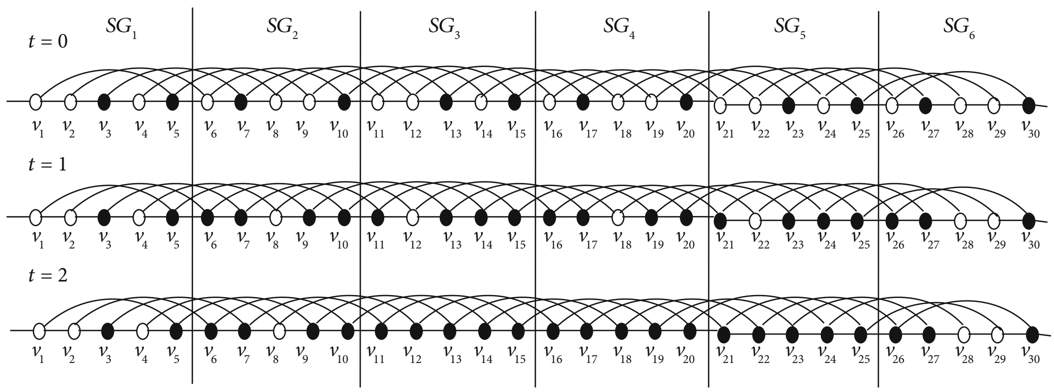

Figure 5: A configuration to convert SG_3, SG_4 starting with 8 converted vertices at $t = 0$.

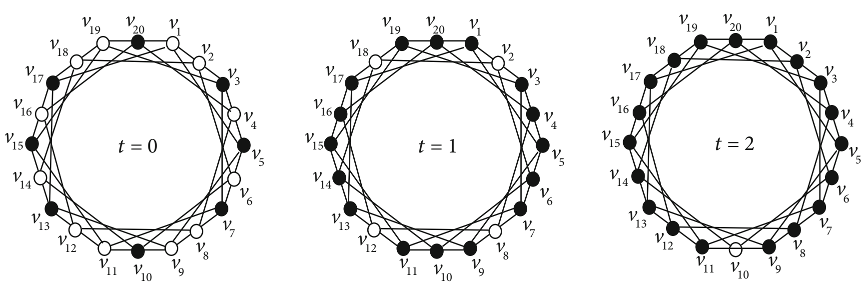

Figure 6: $C_{20}(\{1,4\}) \leq 8$.

Therefore, S_0 is I3CS of $C_n(\{1,4\})$ which means $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \equiv 1(mod\,10)$.

Case 3. $n \equiv 2(mod\,10)$.

Let $S_0 = \{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} U\{v_n\}$ be the seed set. The process goes as follows:

$$\begin{aligned}
&t = 0 : S_0 = \left\{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\} U\{v_n\},\\
&t = 1 : S_1 = S_0 \cup \left\{v_{1+10j}, v_{4+10l}, v_{6+10l}, v_{9+10t} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \right.\\
&\qquad \left. \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 0 \leq t \leq \left\lfloor \frac{n}{10} \right\rfloor - 2\right\},\\
&t = 2 : S_2 = S_1 \cup \left\{v_{2+10l}, v_{8+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\},\\
&t = 3 : S_3 = S_2 \cup \{v_1, v_{n-3}\} = V(C_n(\{1,4\})).
\end{aligned} \tag{25}$$

Therefore, S_0 is I3CS of $C_n(\{1,4\})$ which means $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \equiv 2(mod\,10)$.

Case 4. $n \equiv 3(mod\,10)$.

Let $S_0 = \{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} U\{v_n\}$ be the seed set. The process goes as follows:

$$\begin{aligned}
&t = 0 : S_0 = \left\{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\} U\{v_n\},\\
&t = 1 : S_1 = S_0 \cup \left\{v_{1+10l}, v_{4+10l}, v_{6+10l}, v_{9+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\},\\
&t = 2 : S_2 = S_1 \cup \left\{v_{2+10l}, v_{8+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\},\\
&t = 3 : S_3 = S_2 \cup \{v_{n-2}, v_{n-1}\} = V(C_n(\{1,4\})).
\end{aligned} \tag{26}$$

Therefore, $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \equiv 3(mod\,10)$.

Case 5. $n \equiv 4(mod\,10)$.

We consider the following subcases:

Case 5.a. $n = 14$.

Let the seed set be $S_0 = \{v_1, v_4, v_7, v_9, v_{11}, v_{13}\}$ which is of cardinality $2\lfloor n/5 \rfloor + 2$. Then

$$\begin{aligned}
&t = 0 : S_0 = \{v_1, v_4, v_7, v_9, v_{11}, v_{13}\},\\
&t = 1 : S_1 = S_0 \cup \{v_3, v_5, v_8, v_{14}\},\\
&t = 2 : S_2 = S_1 \cup \{v_{10}, v_{12}\},\\
&t = 3 : S_3 = S_2 \cup \{v_2, v_6\} = V(C_{14}(\{1,4\})).
\end{aligned} \tag{27}$$

Therefore, $C_3(C_{14}(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 2$.

Case 5.b. $n \geq 24$ and $n \equiv 4(mod\,10)$.

Let the seed set be $S_0 = \{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} U\{v_{n-2}\}$. The process goes as follows:

$$\begin{aligned}
&t = 0 : S_0 = \left\{v_{3+10l}, v_{5+10l}, v_{7+10l}, v_{10+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\} U\{v_{n-2}\},\\
&t = 1 : S_1 = S_0 \cup \left\{v_{1+10j}, v_{4+10d}, v_{6+10l}, v_{9+10t} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 1 \leq d \right.\\
&\qquad \left. \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 0 \leq t \leq \left\lfloor \frac{n}{10} \right\rfloor - 2\right\},\\
&t = 2 : S_2 = S_1 \cup \left\{v_{2+10l}, v_{8+10d} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 1 \leq d \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\right\},\\
&t = 3 : S_3 = S_2 \cup \{v_1, v_8, v_{n-5}\},\\
&t = 4 : S_4 = S_3 \cup \{v_4, v_{n-1}\},\\
&t = 5 : S_5 = S_4 \cup \{v_n\} = V(C_n(\{1,4\})).
\end{aligned} \tag{28}$$

Therefore, $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \geq 24$ and $n \equiv 4(mod\, 10)$.

Case 6. $n \equiv 5(mod\, 10)$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

$$\begin{aligned} t=0 : S_0 &= \Big\{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}. \\ t=1 : S_1 &= S_0 \cup \Big\{v_{1+10j}, v_{4+10p}, v_{6+10l}, v_{9+10l} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}. \\ t=2 : S_2 &= S_1 \cup \Big\{v_{2+10l}, v_{8+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}. \\ t=3 : S_3 &= S_2 \cup \{v_1, v_{n-3}\} = V(C_n(\{1,4\})), \end{aligned} \tag{29}$$

which means $C_3(C_n(\{1,4\})) \leq 2n/5$ if $n \equiv 5(mod\, 10)$.

Case 7. $n \equiv 6(mod\, 10)$.

We consider the following subcases:

Case 7.a. $n = 16$.

Let the seed set be $S_0 = \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\}$ which is of cardinality $2\lfloor n/5 \rfloor + 1$. The process involves the following steps:

$$\begin{aligned} t=0 : S_0 &= \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\}, \\ t=1 : S_1 &= S_0 \cup = \{v_1, v_4, v_6, v_9, v_{11}, v_{14}\}, \\ t=2 : S_2 &= S_1 \cup \{v_2, v_8, v_{12}\} = V(C_{16}(\{1,4\})). \end{aligned} \tag{30}$$

Therefore, $C_3(C_{16}(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$.

Case 7.b. $n \geq 26$ and $n \equiv 6(mod\, 10)$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

$$\begin{aligned} t=0 : S_0 &= \{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}, \\ t=1 : S_1 &= S_0 \cup \Big\{v_{1+10j}, v_{4+10j}, v_{6+10l}, v_{9+10l} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}. \\ t=2 : S_2 &= S_1 \cup \Big\{v_{2+10l}, v_{8+10d} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 1 \leq d \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}. \\ t=3 : S_3 &= S_2 \cup \{v_1, v_8, v_{n-4}\}. \\ t=4 : S_4 &= S_3 \cup \{v_4, v_n\} = V(C_n(\{1,4\})). \end{aligned} \tag{31}$$

We conclude that $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor$ if $n \geq 26$ and $n \equiv 6(mod\, 10)$.

Case 8. $n \equiv 7(mod\, 10)$. We consider the following subcases:

Case 8.a. $n = 17$.

Let the seed set be $S_0 = \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\}$ which is of cardinality $2\lfloor n/5 \rfloor + 1$. The process involves the following steps:

$$\begin{aligned} t=0 : S_0 &= \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{17}\}, \\ t=1 : S_1 &= S_0 \cup = \{v_4, v_6, v_9, v_{11}, v_{14}, v_{16}\}, \\ t=2 : S_2 &= S_1 \cup \{v_1, v_2, v_8, v_{12}\} = V(C_{17}(\{1,4\})). \end{aligned} \tag{32}$$

Therefore, $C_3(C_{17}(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$.

Case 8.b. $n \geq 27$ and $n \equiv 7(mod\, 10)$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

$$\begin{aligned} t=0 : S_0 &= \Big\{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}, \\ t=1 : S_1 &= S_0 \cup \Big\{v_{1+10j}, v_{4+10j}, v_{6+10l}, v_{9+10l} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}, \\ t=2 : S_2 &= S_1 \cup \Big\{v_{2+10l}, v_{8+10d} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 1 \leq d \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}, \\ t=3 : S_3 &= S_2 \cup \{v_1, v_8, v_{n-5}\}, \\ t=4 : S_4 &= S_3 \cup \{v_4, v_{n-1}\}, \\ t=5 : S_5 &= S_4 \cup \{v_n\} = V(C_n(\{1,4\})). \end{aligned} \tag{33}$$

Therefore, $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor$ if $n \geq 27$ and $n \equiv 7$ $(mod\, 10)$.

Case 9. $n \equiv 8(mod\, 10)$. Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\} U\{v_n\}$ be the seed set. The process goes as follows:

$$\begin{aligned} t=0 : S_0 &= \Big\{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\} U\{v_n\}, \\ t=1 : S_1 &= S_0 \cup \Big\{v_{1+10p}, v_{4+10p}, v_{6+10l}, v_{9+10l} : 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\} U\{v_{n-1}\}, \\ t=2 : S_2 &= S_1 \cup \Big\{v_{2+10l}, v_{8+10l} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1\Big\}, \\ t=3 : S_3 &= S_2 \cup \{v_{n-6}, v_{n-2}\} = V(C_n(\{1,4\})). \end{aligned} \tag{34}$$

Therefore, $C_3(C_n(\{1,4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \equiv 8(mod\, 10)$. Figure 7 illustrates that $C_{28}(\{1,4\}) \leq 11$.

Case 10. $n \equiv 9(mod\, 10)$. We consider the following subcases:

Case 10.a. $n = 9$.

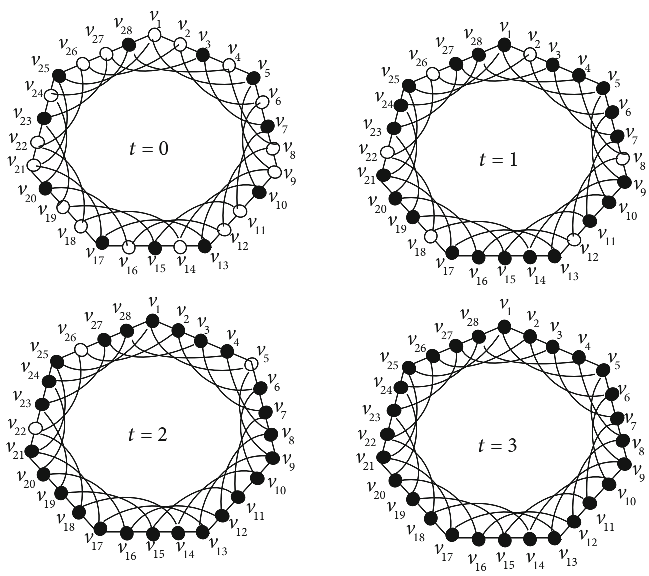

FIGURE 7: An I3CS of 11 vertices on $C_{28}(\{1,4\})$.

Let the seed set be $S_0 = \{v_1, v_3, v_6, v_8\}$ which is of cardinality $2\lfloor n/5 \rfloor + 2$. The process involves the following steps:

$$\begin{aligned} &t = 0 : S_0 = \{v_1, v_3, v_6, v_8\}, \\ &t = 1 : S_1 = S_0 \cup = \{v_2, v_7, v_9\}, \\ &t = 2 : S_2 = S_1 \cup \{v_4, v_5\} = V(C_9(\{1,4\})). \end{aligned} \tag{35}$$

Therefore, $C_3(C_9(\{1,4\})) \leq 2\lfloor \mathrm{n}/5 \rfloor + 2$.

Case 10.b. $n = 19$.

Let the seed set be $S_0 = \{v_1, v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{17}\}$ which is of cardinality $2\lfloor n/5 \rfloor + 2$. The process goes as follows:

$$\begin{aligned} &t = 0 : S_0 = \{v_1, v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{17}\}, \\ &t = 1 : S_1 = S_0 \cup \{v_2, v_6, v_9, v_{11}, v_{14}, v_{16}\}, \\ &t = 2 : S_2 = S_1 \cup \{v_{12}, v_{18}\}, \\ &t = 3 : S_3 = S_2 \cup \{v_8, v_{19}\}, \\ &t = 4 : S_4 = S_3 \cup \{v_4\} = V(C_{19}(\{1,4\})). \end{aligned} \tag{36}$$

Therefore, $C_3(C_{19}(\{1,4\})) \leq 2\lfloor \mathrm{n}/5 \rfloor + 2$.

Case 10.c. $n \geq 29$ and $n \equiv 9(mod\, 10)$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\} U\{v_1\}$ be the seed set. The process goes as follows:

$$\begin{aligned} &t = 0 : S_0 = \left\{v_{3+10p}, v_{5+10p}, v_{7+10l}, v_{10+10l} : 0 \leq p \right. \\ &\qquad \left. \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\} \cup \{v_1\}, \\ &t = 1 : S_1 = S_0 \cup \left\{v_{1+10j}, v_{4+10j}, v_{6+10l}, v_{9+10l} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \right. \\ &\qquad \left. \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\}, \\ &t = 2 : S_2 = S_1 \cup \left\{v_{2+10l}, v_{8+10d} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1, 1 \right. \\ &\qquad \left. \leq d \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\}, \\ &t = 3 : S_3 = S_2 \cup \{v_8, v_{n-7}\}, \\ &t = 4 : S_4 = S_3 \cup \{v_4, v_{n-3}\}, \\ &t = 5 : S_5 = S_4 \cup \{v_{n-2}, v_n\}, \\ &t = 6 : S_6 = S_2 \cup \{v_{n-1}\} = V(C_n(\{1,4\})). \end{aligned} \tag{37}$$

Therefore, $C_3(C_n(\{1,4\})) \leq 2\lfloor \mathrm{n}/5 \rfloor + 1$ if $n \geq 29$ and $n \equiv 9(mod\, 10)$.

From all the previous cases and subcases, we conclude the requested.

Theorem 12. *For $n \geq 2(r+1)$ and $n \equiv 0(mod\, 2(r+1))$: $C_3(C_n(\{1, r\})) \leq nr/(2(r+1))$.*

Proof. Proposition 9 implies that the conversion process fails if there are $r+1$ consecutive unconverted vertices on $C_n(\{1,4\})$ at $t = 0$. We divide the vertices of $V(C_n(\{1, r\}))$ into $n/2(r+1)$ subgraphs denoted by $SG_i : 1 \leq i \leq n/2(r+1)$. Now we try to find a configuration of converted vertices of a random subgraph (SG_i) at $t = 0$ so that when applied to all the subgraphs, it results in converting all of $V(C_n(\{1, r\}))$. We consider the following cases for r: □

Case 1. r is even. Let the configuration of converted vertices we apply to SG_1 at $t = 0$ be $\{v_{(1+2l)}, v_{(r+4+2m)} : 0 \leq l \leq r/2, 0 \leq m \leq (r-4)/2\}$. This means we convert r vertices from each subgraph. As shown in Figure 8, in step $t = 1$, the conversion spreads to $\{v_{2l} : 0 \leq l \leq r/2, 0 \leq m \leq (r-2)/2\}$. In the following step $t = 2$, the conversion spreads to v_{2r+2}. In step $t = 3$, the conversion spreads to v_{r+2}. In step $t = 4$, the configuration converts SG_1 entirely.

Without loss of generality, by applying the same configuration to all subgraphs, we form an I3CS of cardinality $nr(2(r+1))$. We denote it by S_0, and the process goes as follows:

$$\begin{aligned} &t = 0 : S_0 = \left\{v_{(1+2l)+i(2r+2)}, v_{(r+4+2m)+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} \right. \\ &\qquad \left. - 1, 0 \leq l \leq \frac{r}{2}, 0 \leq m \leq \frac{r-4}{2} \right\}, \\ &t = 1 : S_1 = S_0 \cup \left\{v_{2l+i(2r+2)}, v_{r+3+2m+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} \right. \\ &\qquad \left. - 1, 2 \leq l \leq \frac{r}{2}, 0 \leq m \leq \frac{r-2}{2} \right\}, \\ &t = 2 : S_2 = S_1 \cup \left\{v_{i(2r+2)} : 1 \leq i \leq \frac{n}{2(r+1)} \right\}, \\ &t = 3 : S_3 = S_2 \cup \left\{v_{(r+2)+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} - 1 \right\}, \\ &t = 4 : S_4 = S_3 \cup \left\{v_{2+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} - 1 \right\} \\ &\qquad = V(C_n(\{1, \mathrm{r}\})). \end{aligned} \tag{38}$$

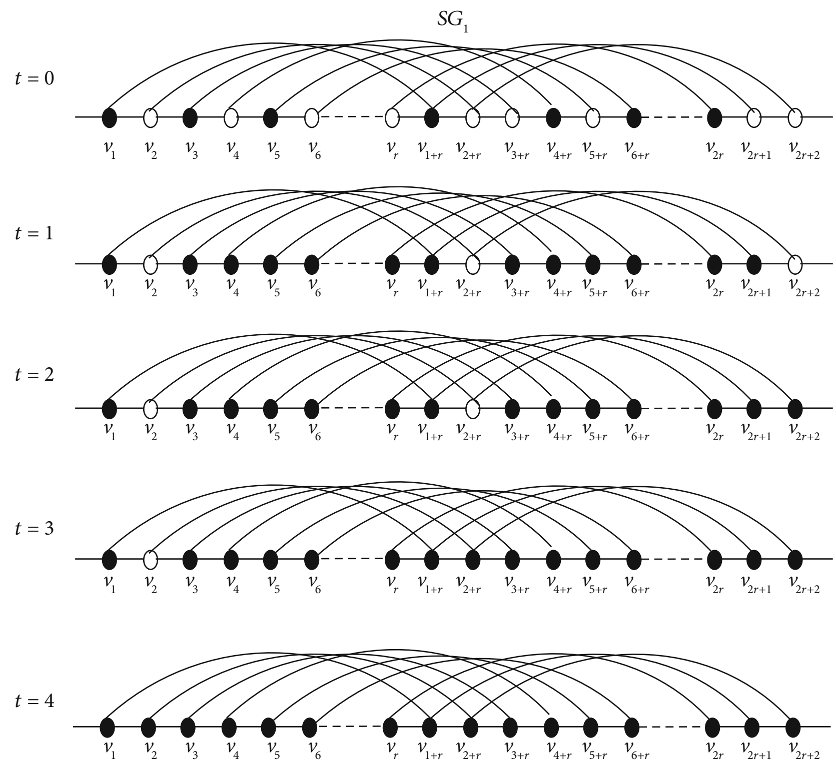

FIGURE 8: A configuration to convert SG_1 starting with r converted vertices when r is even.

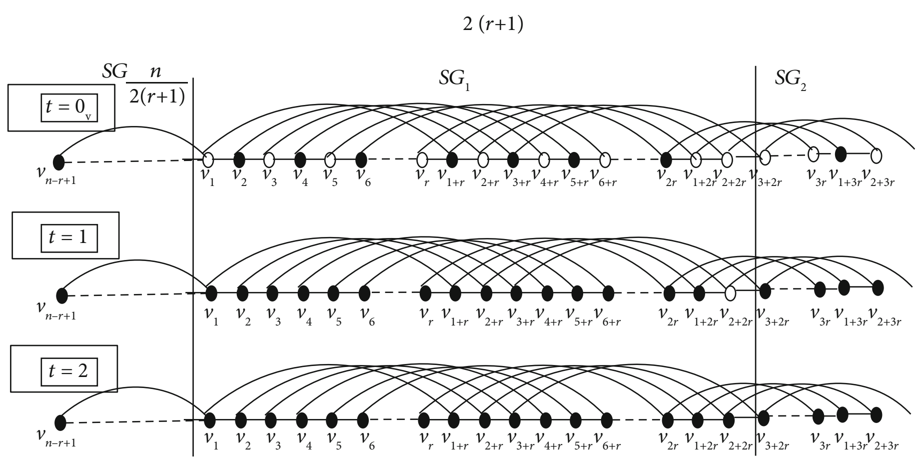

FIGURE 9: A configuration to convert SG_1 starting with r converted vertices when r is odd.

Therefore, S_0 is an I3CS which means $C_3(C_n(\{1,r\})) \leq$ nr$/2(r+1)$ if $n \equiv 0(mod\,2(r+1))$ and r is even.

Case 2. r is odd. Let the seed set be $S_0 = \{v_{2l+i(2r+2)} : 0 \leq i \leq n/2(r+1) - 1,\ 1 \leq l \leq r\}$. The process goes as follows:

$$\begin{aligned}
t=0 : S_0 &= \left\{ v_{2l+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} - 1, 1 \leq l \leq r \right\},\\
t=1 : S_1 &= S_0 \cup \left\{ v_{(1+2l)+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} - 1, 0 \leq l \leq r \right\},\\
t=2 : S_2 &= S_1 \cup \left\{ v_{2r+2+i(2r+2)} : 0 \leq i \leq \frac{n}{2(r+1)} \right\} = V(C_n(\{1,r\})).
\end{aligned} \tag{39}$$

Therefore, S_0 is an I3CS which means $C_3(C_n(\{1,r\})) \leq nr/2(r+1)$ if $n \equiv 0(mod\,2(r+1))$ and r is odd. Figure 9 illustrates how converting S_0 at $t=0$ results in converting SG_1 entirely at the end of step $t=2$, taking into consideration that $v_{n-r+1} \in V(SG_{n/2(r+1)}) \cap S_0$ and $v_{1+3r} \in V(SG_1) \cap S_0$.

Without loss of generality, the same argument applies to all subgraphs. From Case 1 and Case 2, we conclude the requested.

Data Availability

No data was used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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