Research Article

Irreversible $k$-Threshold Conversion Number of Circulant Graphs

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An irreversible conversion process is a dynamic process on a graph where a one-way change of state (from state 0 to state 1) is applied on the vertices if they satisfy a conversion rule that is determined at the beginning of the study. The irreversible $k$-threshold conversion process on a graph $G = (V, E)$ is an iterative process which begins by choosing a set $S_0 \subseteq V$, and for each step $t (t = 1, 2, \ldots)$, $S_t$ is obtained from $S_{t-1}$ by adjoining all vertices that have at least $k$ neighbors in $S_{t-1}$. $S_t$ is called the seed set of the $k$-threshold conversion process, and if $S_t = V(G)$ for some $t \geq 0$, then $S_t$ is an irreversible $k$-threshold conversion set (IkCS) of $G$. The $k$-threshold conversion number of $G$ (denoted by $(C_k(G))$ is the minimum cardinality of all the IkCSs of $G$. In this paper, we determine $C_2(G)$ for the circulant graph $C_n([1, r])$ when $r$ is arbitrary; we also find $C_2(C_n([1, r]))$ when $r = 2, 3$. We also introduce an upper bound for $C_2(C_n([1, 4]))$. Finally, we suggest an upper bound for $C_2(C_n([1, r]))$ if $n \geq 2(r + 1)$ and $n \equiv 0 (\text{mod } 2(r + 1))$.

1. Introduction

As usual, $n = |V|$ and $m = |E|$ denote the numbers of vertices and edges at a graph $G(V, E)$, respectively. Let $\deg(v)$ be the degree of a vertex $v$; a graph is $t$-regular if all of its vertices are of degree $t$. The open neighborhood of a vertex $v$ is $N(v) = \{ u \in V : uv \in E \}$ while the closed neighborhood of $v$ is $N[v] = N(v) \cup \{ v \}$. For any undefined term in the paper, we refer to Harary [1]. An irreversible $k$-threshold conversion process on a graph $G = (V, E)$ is the process of finding the least number of vertices we need to initially convert in step $t = 0$ in order to spread the conversion to all the remaining vertices of the graph according to a conversion rule. This iterative process starts by choosing a seed set $S_0 \subseteq V$, and for each step $t (t = 1, 2, \ldots)$, $S_t$ is obtained from $S_{t-1}$ by adjoining all vertices that have at least $k$ neighbors in $S_{t-1}$. $S_0$ is called the seed set of the $k$-threshold conversion process, and if $S_t = V(G)$ for some $t \geq 0$, then $S_t$ is an irreversible $k$-threshold conversion set (IkCS) of $G$. The $k$-threshold conversion number of $G$ (denoted by $(C_k(G))$ is the minimum cardinality of all the IkCSs of $G$. Therefore, $1 \leq k \leq \Delta(G)$ and $C_1(G) = 1$ for connected graphs. The first graph model of the irreversible $k$-threshold conversion problem was presented by Dreyer and Roberts in [2] where they determined the value of $C_2$ for paths and cycles. For further information on the irreversible $k$-threshold conversion problem on graphs, see [2–6]. The circulant graph $C_n(S)$ with the connection set $S \subseteq \{1, 2, \ldots, n\}$ is an undirected graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ where two vertices $v_i, v_j$ are adjacent if $|i - j| \in S$. Therefore, the circulant graph $C_n(1)$ is a cycle, and the circulant graph $C_n(\{1, 2, \ldots, [n/2]\})$ forms the complete graph $K_n$. It is obvious that the circulant graph $C_n(1)$ is 4-regular when $1 < r \leq \lfloor (n - 1)/2 \rfloor$. Through this paper, we will denote the vertex set by $V = \{v_i : i = 1, \ldots, n\}$ taking into consideration that we exchange the subscript of the vertex $v_0$ by $v_n$. For further information on the circulant graph, see [7].

**Proposition 1** (see [2]).

$$C_2(P_n) = \left\lfloor \frac{n + 1}{2} \right\rfloor.$$  \hspace{1cm} (1)

**Proposition 2** (see [2]).

$$C_2(C_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$  \hspace{1cm} (2)
Proposition 3 (see [2]). If $G$ is a $k$-regular graph, then $S$ is a $k$-conversion set of $G$ if and only if $V - S$ is independent.

Note 1: in every figure of this article, we assign the black color to the converted vertices and the white color to unconverted ones.

2. Main Results

In this paper, we determine $C_2(G)$ for the circulant graph $C_n(1, r)$ for arbitrary $r$; we also find $C_2(C_n(1, r))$ when $r = 2, 3$. We also introduce an upper bound for $C_2(C_n(1, 4))$. Then we suggest an upper bound for $C_2(C_n(1, r))$ if $n \geq 2(r + 1)$ and $n \equiv 0 \mod (2r + 1)$. Let $C_n(1, r)$ be a circulant graph on which an irreversible $k$-threshold conversion process is being studied. Since $C_n(1, r)$ is 4-regular, we define the $k$-unconvertible set of $G$ (denoted by $U$) as follows:

$$U \subseteq V - S_0$$

which means each vertex of $U$ is unconverted and is adjacent to at least $4 - k + 1$ vertices of $V$ at $t = 0$; therefore, the conversion cannot reach any vertex of $U$ during any step of the process unless at least one of its vertices is converted at $t = 0$. Figure 1 shows a 3-unconvertible set on $C_4(1, 2)$.

Note 2: let $C_n(1, r)$ be a circulant graph, and let the conversion threshold be $k = 2$. We will define a conversion generating path (CGP) as a series of $r$ consecutive vertices (a path) on $C_n(1, r)$ such as $(v_1, v_1+1, \cdots, v_{r+1})$, so that when all of these vertices are converted in a step $t = l$, they can spread the conversion to the entire graph by converting two new (unconverted) vertices at every following step. The process goes as follows:

$$t = l:$$ the conversion reaches all vertices of the CGP which are $v_1, v_{r+1}, \cdots, v_{r+1-l}$

$$t = l + 1:$$ since $v_1, v_{r+1-l}$ are adjacent to both $v_{r+1}, v_{r+1-l}$, then conversion spreads to $v_{r+1}, v_{r+1-l}$, and the converted vertices are $v_1, v_{r+1}, \cdots, v_{r+1-l}, v_{r+1}$

$$t = l + 2:$$ since $v_{r+1}, v_{r+1-l}$ are adjacent to both $v_{r+1}, v_{r+1-l}$, then conversion spreads to $v_{r+1}, v_{r+1-l}$

The conversion process continues until all vertices of $V$ are converted. This goal is achieved on step $t = l + [(n - r - m)/2]$ where $m$ represents the number of converted vertices of $V - V(CGP)$ at $t = l$.

If $n - r - m$ is even, then the last two unconverted vertices are converted at the last step, which is $t = l + [(n - r - m)/2]$. If $n - r - m$ is odd, then the last two unconverted vertices are converted at the last step, which is $t = l + [(n - r - m)/2] - 1$; two unconverted vertices are converted, and then only one unconverted vertex remains to be converted in the last step which is $t = l + [(n - r - m)/2]$. Figure 2 illustrates a 5-vertex CGP on $C_{12}(1, 5)$.

$$C_2(C_n(1, r)).$$

In this subsection, we determine $C_2(C_n(1, r))$ for arbitrary $r$ when $1 < r \leq [(n - 1)/2]$.

Theorem 4. For $n \geq 5, C_2(C_n(1, 2)) = 2$.

Proof. We know by definition that for any graph $G, C_2(G)$ $\geq k$, which means that $C_2(C_n(1, 2)) \geq 2$. Let $S_0 = \{v_1, v_2\}$ be the seed set of the conversion process. $S_0$ forms a CGP on $C_n(1, 2)$ with $l = 0$, and the process goes as follows:

$t = 0$: we convert $v_1, v_2$

$t = 1$: the conversion spreads to $v_3, v_4$

$t = 2$: the conversion spreads to $v_5, v_{n-1}$

The process continues, spreading the conversion to two new (unconverted) vertices each step. If $n$ is even, the process ends in step $t = n/2 - 1$ when the last two unconverted vertices $(v_{n/2+1}, v_{n/2+2})$ are converted.

If $n$ is odd, at the next to last step $t = [(n - 2)/2] = (n - 3)/2$, there are three unconverted vertices left which are $(v_{n/2+1}, v_{n/2+2}, v_{n/2+3})$. Two of them $(v_{n/2+1}, v_{n/2+2})$ are converted in step $t = (n - 3)/2$; while the last unconverted vertex $(v_{n/2+3})$ is converted in the last step $t = [(n - 2)/2] = (n - 1)/2$.

We conclude that $S_0$ is an I2CS of $C_n(1, 2)$, which means $C_2(C_n(1, 2)) \leq 2$; therefore, $C_2(C_n(1, 2)) = 2$, and we prove the requested.

Theorem 5. For $n \geq 7, C_2(C_n(1, 3)) = 2$.

Proof. Since $C_2(C_n(1, 3)) \geq 2$, we need to prove that $C_2(C_n(1, 3)) \leq 2$. Let $S_0 = \{v_1, v_3\}$ be the seed set. The process goes as follows:

$t = 0$: we convert $S_0 = \{v_1, v_3\}$

$t = 1$: since $v_2$ adj $v_1, v_3$, the conversion spreads to $v_2$.

Similarly, $v_4$ adj $v_1, v_3$, and we also have $v_6$ adj $v_1, v_3$, which means $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Each of the sets $\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_4\}$ forms a CGP on $C_n(1, 3)$ with $l = 1$.

$t = 2$: the conversion spreads to $v_5, v_{n-1}$

$t = 3$: the conversion spreads to $v_6, v_{n-2}$

The process continues similarly to Theorem 4 until the graph is successfully converted at step $t = (n - 5)/2 + 1$ if $n$ is odd, or at step $t = [(n - 5)/2] + 1$ if $n$ is even.

We conclude that $S_0 = \{v_1, v_3\}$ is an I2CS of $C_n(1, 3)$ which means $C_2(C_n(1, 3)) \leq 2$; then $C_2(C_n(1, 3)) = 2$, and the requested is proven.

Theorem 6. $C_2(C_n(1, 4)) = \begin{cases} 2 & \text{for } 9 \leq n \leq 11; \\ 3 & \text{otherwise.} \end{cases}$
Similarly, Case 2.

**Case 1.** $n = 9$. Let $S_0 = \{v_1, v_6\}$ be the seed set. The process goes as follows:
- $t = 0$: we convert $S_0 = \{v_1, v_6\}$
- $t = 1$: since $v_2 \text{adj } v_1, v_6$, the conversion spreads to $v_2$.

Similarly, $v_5 \text{adj } v_1, v_6$, and we also have $S_1 = \{v_1, v_2, v_5, v_6\}$

**Case 2.** $n = 10$. Let $S_0 = \{v_1, v_6\}$ be the seed set. The process goes as follows:
- $t = 0$: we convert $S_0 = \{v_1, v_6\}$
- $t = 1$: the conversion spreads to $v_2, v_5, v_7, v_{10}$ which means $S_1 = \{v_1, v_2, v_5, v_6, v_7, v_{10}\}$
- $t = 2$: the conversion spreads to $v_3, v_4, v_9$ which means $S_2 = V(C_{10}(\{1, 4\}))$

We conclude that $C_2(C_{10}(\{1, 4\})) = 2$.

**Case 3.** $n = 11$. Let $S_0 = \{v_1, v_6\}$ be the seed set. The process goes as follows:
- $t = 0$: we convert $S_0 = \{v_1, v_6\}$
- $t = 1$: the conversion spreads to $v_2, v_5$; therefore, $S_1 = \{v_1, v_2, v_5, v_6\}$
- $t = 2$: the conversion spreads to $v_9$ which means $S_2 = \{v_1, v_2, v_5, v_6, v_9\}$
- $t = 3$: the conversion spreads to $v_8, v_{10}$; therefore, $S_3 = \{v_1, v_2, v_5, v_6, v_8, v_9, v_{10}\}$
- $t = 4$: the conversion spreads to $v_3, v_4, v_7, v_{11}$ which means $S_4 = V(C_{11}(\{1, 4\}))$

We conclude that $C_2(C_{11}(\{1, 4\})) = 2$.

**Case 4.** $n \geq 12$. We start by proving that $C_2(C_n(\{1, 4\})) > 2$ for $n \geq 12$. We consider the following subcases:

**Case 4.a.** $S_0 = \{v_1, v_2\}$; since $N(v_1) \cap N(v_2) = \emptyset$, then $S_1 = S_0$, and the conversion does not spread after the initial step $t = 0$, which means the process fails. Without loss of generality, the same argument can be applied for any $S_0 = \{v_i, v_{i+1} : 1 \leq i \leq n\}$.

**Case 4.b.** $S_0 = \{v_1, v_3\}$; since $N(v_1) \cap N(v_3) = \emptyset$, then at step $t = 1$, we get $S_1 = \{v_1, v_2, v_3\}$. However, $N(v_1) \cap N(v_2) \cap N(v_3) = \emptyset$ which means $S_2 = S_1$ and the spread stops at the end of step $t = 1$. Without loss of generality, this applies to any $S_0 = \{v_i, v_{i+2} : 1 \leq i \leq n\}$.

**Case 4.c.** $S_0 = \{v_1, v_4\}$. In a similar way to the previous two cases, $S_1 = \{v_1, v_4, v_5, v_6\}$, $S_2 = S_1$ which means the spread stops at the end of step $t = 1$, and without loss of generality; this applies to any $S_0 = \{v_i, v_{i+3} : 1 \leq i \leq n\}$.

**Case 4.d.** $S_0 = \{v_1, v_5\}$; then $S_1 = S_0$, and the process fails at the end of step $t = 0$. Without loss of generality, this applies to any $S_0 = \{v_i, v_{i+4} : 1 \leq i \leq n\}$.

**Case 4.e.** $S_0 = \{v_1, v_6\}$; then $S_1 = \{v_1, v_2, v_5, v_6\}$, but $S_2 = S_1$ and the spread stops at the end of step $t = 1$. Without loss of generality, this applies to any $S_0 = \{v_i, v_{i+5} : 1 \leq i \leq n\}$.

**Case 4.f.** $S_0 = \{v_1, v_7 : l \geq 7\}$. Since $N(v_1) \cap N(v_7) = \emptyset$, then $S_1 = S_0$, and the conversion does not spread after the initial step $t = 0$, which means the process fails. Without loss of generality, the same argument can be applied for any $S_0 = \{v_i, v_{i+l} : 1 \leq i \leq n \text{ and } l \geq 7\}$.

From subcases 4.a to 4.f, we conclude that

$$C_2(C_n(\{1, 4\})) > 2.$$  \hfill (5)
vertices of $S_1$ forms a CGP on $C_n\{1, 4\}$ with $l = 1$ and $m = 2$. The process goes as follows:

- $t = 0$: we convert $S_0 = \{v_1, v_3, v_4\}$
- $t = 1$: $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$
- $t = 2$: $S_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{n-1}, v_n\}$

The process continues converting two new vertices each step until it ends at $t = (n - 4)/2$ if $n$ is even, or at $t = (n - 3)/2$ if $n$ is odd. We conclude that $S_0 = \{v_1, v_3, v_4\}$ is an I2CS of $C_n\{1, 4\}$. Therefore

$$C_2(C_n\{1, 4\}) \leq 3.$$  

From (5) and (6), we conclude that $C_2(C_n\{1, 4\}) = 3$ for $n \geq 12$.

From Cases 1–4, we conclude the requested.

**Theorem 7.** For $r \geq 4$, $C_2(C_n\{1, r\}) = \lceil (r+1)/2 \rceil$.

**Proof.** Since a CGP is a path of length $r$, and by Proposition 1, $C_2(P_r) = \lceil (n+1)/2 \rceil$; therefore, $Y = \{v_1, v_2, \ldots, v_m : m = r$ if $r$ is odd, $m = r + 1$ if $r$ is even$\}$ can be an I((r+1)/2)CS of $C_n\{1, r\}$. This means

$$C_2(C_n\{1, r\}) \leq \left\lceil \frac{r+1}{2} \right\rceil.$$  

Figure 3 illustrates an I2CS of 5 vertices on $C_{24}\{1, 8\}$.

We now consider the following cases for $r$:

**Case 1.** $r$ is odd. Let $S_0 = \{v_1, v_3, \ldots, v_{r-2}\}$ be a conversion seed set of cardinality $\lceil (r+1)/2 \rceil - 1 = (r-1)/2$. The process goes as follows:

- $t = 0$: we convert $S_0 = \{v_1, v_3, \ldots, v_{r-2}\}$
- $t = 1$: $S_1 = \{v_1, v_2, v_3, v_4, \ldots, v_{r-3}, v_{r-2}\}$
- $t = 2$: $S_2 = \{v_1, v_2, v_3, v_4, \ldots, v_{r-3}, v_{r-2}\}$

The process stops at the end of step $t = 1$. Without loss of generality, the process applies to all configurations of $S_0$ on the vertices of $\{v_1, v_2, v_3, v_4, \ldots, v_{r-3}, v_{r-2}\}$. Therefore, $S_0$ cannot produce a CGP on $C_n\{1, r\}$, and since we need to convert at least $\lceil (r+1)/2 \rceil = (r+1)/2$ vertices of the path $\{v_1, v_2, v_3, v_4, \ldots, v_{r-3}, v_{r-2}\}$ in order to convert it entirely, it is impossible to convert $C_n\{1, r\}$ if we initially convert less than $(r+1)/2$ vertices at $t = 0$, which means $C_2(C_n\{1, r\}) > \lceil (r+1)/2 \rceil - 1$.

**Case 2.** $r$ is even. Let $S_0 = \{v_1, v_3, \ldots, v_{r-1}\}$ be a conversion seed set of cardinality $\lceil (r+1)/2 \rceil - 1 = r/2$. The process goes as follows:

- $t = 0$: we convert $S_0 = \{v_1, v_3, \ldots, v_{r-3}, v_{r-1}\}$
- $t = 1$: $S_1 = \{v_1, v_2, v_3, v_4, \ldots, v_{r-2}, v_{r-1}\}$
- $t = 2$: $S_2 = \{v_1, v_2, v_3, v_4, \ldots, v_{r-2}, v_{r-1}\}$

Figure 3: An I2CS of 5 vertices on $C_{24}\{1, 8\}$.
The process stops at the end of step $t = 1$. Without loss of generality, the process applies to all configurations of $S_0$ on the vertices of $\{v_1, v_2, v_3, \ldots, v_n, v_r\}$. Therefore, $S_0$ cannot produce a CGP on $C_n(1, r)$, and since we need to convert at least $\lceil (r + 1)/2 \rceil = (r + 1)/2$ vertices of the path $\{v_1, v_2, v_3, \ldots, v_{r-1}, v_r\}$ in order to convert it entirely, it is impossible to convert $C_n(\{1, r\})$ if we initially convert less than $(r + 1)/2$ vertices at $t = 0$, which means $C_2(C_n(\{1, r\})) > \lceil (r + 1)/2 \rceil - 1$.

From Case 1 and Case 2, we conclude that
\[
C_2(C_n(\{1, r\})) > \left\lceil \frac{r + 1}{2} \right\rceil - 1. \quad (8)
\]

From (7) and (8), we conclude that $C_3(C_n(\{1, r\})) = \lceil (r + 1)/2 \rceil$.

\[
C_3(C_n(\{1, r\})) = \lceil (r + 1)/2 \rceil. \quad (9)
\]

In this subsection, we determine $C_3(C_n(\{1, r\}))$ for $r = 2, 3$. Then, we introduce an upper bound for $C_3(C_n(\{1, 4\}))$. Finally, we introduce an upper bound for $C_3(C_n(\{1, r\}))$ if $n \geq 2(r + 1)$ and $n \equiv 0(\mod 2(r + 1))$.

**Theorem 8.** For $n \geq 5$, $C_3(C_n(\{1, 2\})) = \begin{cases} \lfloor n/3 \rfloor + 1 & \text{if } n \equiv 0, 1(\mod 3), \\ \lfloor n/3 \rfloor + 1 & \text{if } n \equiv 2(\mod 3). \end{cases}$

*Proof.* We have $C_3(C_n(\{1, 2\})) \geq 3$, and we consider the following cases:

**Case 1.** $n \equiv 0(\mod 3)$.

Let $S_0 = \{v_{3i} : 1 \leq i \leq n/3\} \cup \{v_1\}$ be the seed set of the conversion process. It is obvious that $S_0$ is of cardinality $(n/3) + 1$. The process goes as follows:

$t = 0$: we convert $S_0 = \{v_{3i} : 1 \leq i \leq n/3\} \cup \{v_1\}$

$t = 1$: we notice that $|N(v_2) \cap S_0| = |N(v_{n-1}) \cap S_0| = 3$, which means that the conversion spreads to $\{v_2, v_{n-1}\}$ making $S_1 = \{v_{3i} : 1 \leq i \leq n/3\} \cup \{v_1, v_2, v_{n-1}\}$

$t = 2$: we notice that $|N(v_3) \cap S_1| = |N(v_{n-2}) \cap S_1| = 3$; therefore, the conversion spreads to $\{v_3, v_{n-2}\}$, which makes $S_2 = \{v_{3i} : 1 \leq i \leq n/3\} \cup \{v_1, v_2, v_3, v_{n-2}, v_{n-1}\}$.

The process goes on, and with each step, two new (unconverted) vertices $\{v_{3i}, v_{n-d}\}$ are converted, where $5 \leq m \leq \lfloor n/2 \rfloor$ and $m \not\equiv 0(\mod 3)$. Similarly, $4 \leq d \leq \lfloor n/2 \rfloor$ and $m \not\equiv 0(\mod 3)$. Where $t = n/3 - 1$, and $S_{(n/3)-1}$ is defined as:

\[
t = n/3 - 1 : S_{(n/3)-1} = \begin{cases} V \setminus \{v_{n+1}\} & \text{if } n \text{ is even}; \\ V \setminus \{v_{n/2}\} & \text{if } n \text{ is odd}. \end{cases} \quad (10)
\]

$t = n/3$: the conversion spreads to the last remaining vertex $v_{(n/3)+1}$ (or $v_{(n/3)}$) if $n$ is odd (or even), respectively, and then the conversion is spread to all vertices of $C_n(\{1, 2\})$ which makes $S_n$ an ICS of $C_n(\{1, 2\})$.

\[
C_3(C_n(\{1, 2\})) \leq n/3 + 1 \text{ if } n \equiv 0(\mod 3). \quad (11)
\]

Let $v_1, v_{n+1}, v_{n+2}$ be three consecutive unconverted vertices of $C_n(\{1, 2\})$. At any step of the process, in order to convert any of these three vertices, it needs to be adjacent to three converted vertices. However, this is impossible since each one of them is adjacent to the other two and they are all unconverted, which means $X = \{v_1, v_{n+1}, v_{n+2}\}$ is 3-unconvertible if $X \cap S_0 = \emptyset$. We imply that any seed set of cardinality $n/3$ on $C_n(\{1, 2\})$ should be distributed as one of the following:

\[
D_0 = \{v_{3i} : 1 \leq i \leq \lfloor n/3 \rfloor\},
\]
\[
B_0 = \{v_{3i+1} : 0 \leq i \leq \lfloor n/3 - 1 \rfloor\},
\]
\[
E_0 = \{v_{3i+2} : 0 \leq i \leq \lfloor n/3 - 1 \rfloor\}.
\]

Let us assume that $D_0$ is the seed set. The conversion process goes as follows:

$t = 0$: we convert $D_0 = \{v_{3i} : 1 \leq i \leq \lfloor n/3 \rfloor\}$

$t = 1$: with the absence of any unconverted vertex that is adjacent to three vertices of $D_0$, then $D_1 = D_0$ which means the spread stops at the end of step $t = 0$. Therefore, the process fails. Without loss of generality, the same result can be obtained if the seed set was $B_0$ or $E_0$, then:

\[
C_3(C_n(\{1, 2\})) > n/3 \text{ if } n \equiv 0(\mod 3). \quad (13)
\]

From (11) and (13), we conclude that $C_3(C_n(\{1, 2\})) = n/3 + 1 \text{ if } n \equiv 0(\mod 3)$.

**Case 2.** $n \equiv 1(\mod 3)$.

Similarly to Case 1, a seed set of cardinality $\lfloor n/3 \rfloor$ is not enough to convert $C_n(\{1, 2\})$. Let the seed set be $S_0 = \{v_{3i} : 1 \leq i \leq (n-1)/3\} \cup \{v_1\}$. It is obvious that $|S_0| = (n-1)/3 + 1 = \lfloor n/3 \rfloor + 1$. The process goes (similarly to Case 1) as follows:

$t = 0$: we convert $S_0 = \{v_{3i} : 1 \leq i \leq (n-1)/3\} \cup \{v_1\}$

$t = 1$: since $|N(v_1) \cap S_0| = |N(v_{n-2}) \cap S_0| = 3$, the conversion spreads to $\{v_2, v_{n-2}\}$, which makes $S_1 = \{v_{3i} : 1 \leq i \leq (n-1)/3\} \cup \{v_1, v_{n-2}, v_n\}$

$t = 2$: since $|N(v_{3i}) \cap S_1| = |N(v_{n-3}) \cap S_1| = 3$, the conversion spreads to $\{v_{3i}, v_{n-3}\}$, which makes $S_2 = \{v_{3i} : 1 \leq i \leq (n-1)/3\} \cup \{v_1, v_2, v_{n-3}, v_{n-2}, v_n\}$

The process goes on, and with each step, two new (unconverted) vertices $\{v_m, v_{n-d}\}$ are converted, where $4 \leq
Proof. Let \( S_0 \) be the seed set of the conversion process. We implied in Proposition 9 that the process fails if \( V - S_0 \) contains four (or more) consecutive vertices, which means \( S_0 \) contains at least \( \lceil n/4 \rceil \) vertices. Let \( D_0, E_0, L_0, B_0 \) be the following sets:

\[
\begin{align*}
D_0 &= \left\{ v_i : 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil \right\}, \\
E_0 &= \left\{ v_{i+1} : 0 \leq i \leq \left\lceil \frac{n}{4} - 1 \right\rceil \right\}, \\
L_0 &= \left\{ v_{4i+2} : 0 \leq i \leq \left\lceil \frac{n}{4} - 1 \right\rceil \right\}, \\
B_0 &= \left\{ v_{4i+3} : 0 \leq i \leq \left\lceil \frac{n}{4} - 1 \right\rceil \right\}.
\end{align*}
\]

In order to avoid having four consecutive unconverted vertices, \( S_0 \) must contain either \( D_0, E_0, L_0, \) or \( B_0 \). We assume that \( D_0 \subseteq S_0 \); we notice that the vertices of \( D_0 \) divide \( C_n(1, 1) \), or \( E_0 \); therefore

\[
C_3(C_n(1, 2)) > \left\lceil \frac{n}{3} \right\rceil + 1 \text{ if } n \equiv 2(\text{mod } 3).
\]

(16)

Let \( S_0 = \{ v_{2i} : 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil \} \cup \{ v_{n-1}, v_n \} \) be seed set of cardinality \( \lceil n/3 \rceil + 1 = \left\lceil \frac{n}{3} \right\rceil + 1 \). The process goes as follows:

\( t = 0 \) we convert \( S_0 = \{ v_{2i} : 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil \} \cup \{ v_{n-1}, v_n \} \)

\( t = 1 \) the conversion spreads to \( v_1, v_{n-1} \)

\( t = 2 \) the conversion spreads to \( v_2, v_{n-4} \).

Similarly to Case 1 and Case 2, by the end of each step, two new (unconverted) vertices are converted, and the process continues until the last step \( t = \left\lceil \frac{n}{3} \right\rceil \) when the last two unconverted vertices which are \( \{ v_{2n-2}, v_{2n-2} \} \) (in case \( n \) is even), and \( \{ v_{2n-4}, v_{2n-4} \} \) (in case \( n \) is odd) get converted and then the conversion process reaches the entire graph. We conclude that

\[
C_3(C_n(1, 2)) > \left\lceil \frac{n}{3} \right\rceil + 1 \text{ if } n \equiv 2(\text{mod } 3).
\]

(17)

From (16) and (17), we conclude that \( C_3(C_n(1, 2)) = \left\lceil \frac{n}{3} \right\rceil + 1 \) if \( n \equiv 2(\text{mod } 3) \).

From all the previous cases, we conclude the requested.

**Proposition 9.** A set of \( r + 1 \) consecutive unconverted vertices in \( C_n(1, r) \) is 3-unconvertable.

**Proof.** Let there be a conversion process on \( C_n(1, r) \), and at the initial step \( t = 0 \), let the set \( X = \{ v_i, v_{i+1}, v_{i+r-1}, v_{i+r} \} \) be a set of \( r + 1 \) unconverted vertices. Every vertex of \( X \) is of degree 4 and is adjacent to two other vertices of \( X \). Since \( k = 3 \), it is impossible for any of these vertices to satisfy the conversion condition at any step of the conversion process even if all the vertices of \( V - X \) get converted, which means \( X \) is 3-unconvertable.

**Theorem 10.** For \( n \geq 7 \) we have:

\[
C_3(C_n(1, 3)) = \begin{cases} 
\frac{3n/8 ; n \equiv 0(\text{mod } 8) + \left\lfloor n/4 \right\rfloor + \left\lceil n/8 \right\rceil + \left\lfloor n/4 \right\rfloor + \left\lceil n/8 \right\rceil + \left\lfloor n/4 \right\rfloor + \left\lceil n/8 \right\rceil + \left\lfloor (n/4) - 1 \right\rfloor + \left\lceil (n/4) - 1 \right\rceil + \left\lceil (n/4) - 1 \right\rceil + \left\lfloor (n/4) - 1 \right\rfloor. 
\end{cases}
\]

(18)
We consider the following cases for $n$:

Case 1. $n \equiv 0 \pmod{4}$. We consider two subcases:

Case 1.a. $n \equiv 0 \pmod{4}$. In this subcase, we have an even number of subgraphs on $C_n(\{1, 3\})$ and $m = 0$. Let $M_0$ be the following set:

$$M_0 = \{v_{2n} : 0 \leq l \leq (n/8) - 1\}.$$

Let $S_0 = D_0 \cup M_0$. We consider the conversion process as follows:

$t = 0$: We convert $S_0 = \{v_{2n} : 0 \leq l \leq (n/8) - 1\}$.

$t = 1$: The conversion spreads to $S_0$.

$t = 2$: The conversion spreads to $v_{6n+1}$.

By the end of step $t = 2$, the conversion reaches all vertices of $C_n(\{1, 3\})$.

We consider the case when $n \equiv 0 \pmod{4}$.

Case 1.b. $n \equiv 0 \pmod{4}$ and $n \neq 0 \pmod{8}$.

In this subcase, we have an odd number of subgraphs on $C_n(\{1, 3\})$ and $m = 0$. Let $S_0 = D_0 \cup M_0$. We consider the conversion process as follows:

$t = 0$: We convert $S_0$.

$t = 1$: The conversion spreads to $S_0$.

$t = 2$: The conversion spreads to $v_{6n+1}$.

By the end of step $t = 2$, the conversion reaches all vertices of $C_n(\{1, 3\})$.

We consider the case when $n \equiv 1 \pmod{4}$.

Case 2. $n \equiv 1 \pmod{4}$. We consider two subcases:

Case 2.a. $n \equiv 1 \pmod{8}$. In this subcase, we have an even number of subgraphs on $C_n(\{1, 3\})$ and $m = 0$. Let $S_0 = D_0 \cup M_0$. We consider the conversion process as follows:

$t = 0$: We convert $S_0$.

$t = 1$: The conversion spreads to $S_0$.

$t = 2$: The conversion spreads to $v_{6n+1}$.

By the end of step $t = 2$, the conversion reaches all vertices of $C_n(\{1, 3\})$.

Case 2.b. $n \equiv 1 \pmod{4}$ and $n \neq 0 \pmod{8}$.
This subcase is similar to subcase 2.a with the only difference of having an odd number of SGs. This means that similarly to subcase 1.b, we need to convert one additional vertex from this last subgraph \((v_{n-3})\); the process goes as follows:

\[
\begin{align*}
t & = 0: \text{we convert } S_0 = \{v_{d_l} : 1 \leq l \leq \lfloor n/4 \rfloor \} \cup \{v_{2+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_1\} \\
t & = 1: \text{the conversion spreads to } \{v_{1+8l}, v_{3+8l} : 1 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-2}\} \\
t & = 2: \text{the conversion spreads to } \{v_{6+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \\
t & = 3: \text{the conversion spreads to } \{v_{3}, v_{n-4}\} \\
t & = 4: \text{the conversion spreads to } \{v_{n-3}, v_{n}\} \\
\end{align*}
\]

By the end of step \(t = 4\), the conversion reaches all vertices of \(C_n(\{1, 3\})\). Therefore, \(S_0\) is an I3CS of \(C_n(\{1, 3\})\), and since \(|S_0| = |n/4 + |n/8| + 1\), this means \(C_3(C_n(\{1, 3\})) = |n/4| + |n/8| + 1\) if \(n \equiv 1(\text{mod } 4)\) and \(n \not\equiv 1(\text{mod } 8)\).

\[\text{Case 3.a. } n \equiv 2(\text{mod } 4)\]. We consider two subcases:

\[\text{Case 3.a. } n \equiv 2(\text{mod } 8)\].

In this subcase, we have an even number of subgraphs on \(C_n(\{1, 3\})\) and \(m = 2\). Let \(D_0\) and \(M_0\) be the same sets identified in subcase 1.a. Let the seed set be \(S_0 = D_0 \cup M_0\). In a similar process to the one in subcase 1.a, all vertices of \(V - \{v_{n-4}, v_{n-3}, v_{n-1}, v_1\}\) by the end of step \(t = 2\). However, the five consecutive vertices \(v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\) form a version of \(Y\), and since taking out any vertex from \(D_0 \cup M_0\) results in a version of either \(X\) or \(Y\), we conclude that \(C_3(C_n(\{1, 3\})) \geq 3|n/8|\) in this subcase as well. Let \(S_0 = D_0 \cup M_0 \cup \{v_{n}\}\) be the seed set of cardinality \(3|n/8| + 1\); the process goes as follows:

\[
\begin{align*}
t & = 0: \text{we convert } S_0 = \{v_{d_l} : 1 \leq l \leq \lfloor n/4 \rfloor \} \cup \{v_{2+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_1\} \\
t & = 1: \text{the conversion spreads to } \{v_{1+8l}, v_{3+8l}, v_{5+8l}, v_{7+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-1}\} \text{ which means } v_{1}, v_{n-3}, v_{n-1} \text{ are converted in this step} \\
t & = 2: \text{the conversion spreads to } \{v_{6+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \text{ which means } v_{n-4} \text{ is converted at this step} \\
\end{align*}
\]

By the end of step \(t = 2\), the conversion reaches all vertices of \(C_n(\{1, 3\})\); therefore, \(S_0\) is an I3CS of \(C_n(\{1, 3\})\), and since \(|S_0| = 3|n/8| + 1\), we conclude that \(C_3(C_n(\{1, 3\})) = 3|n/8| + 1\) if \(n \equiv 2(\text{mod } 8)\).

\[\text{Case 3.b. } n \equiv 2(\text{mod } 4)\] and \(n \not\equiv 2(\text{mod } 8)\).

By following the same argument in subcase 1.b and subcase 3.a, let \(S_0 = D_0 \cup M_0 \cup \{v_{n-4}\} \) be the seed set of cardinality \(|n/4| + |n/8| + 1\); the process goes as follows:

\[
\begin{align*}
t & = 0: \text{we convert } S_0 = \{v_{d_l} : 1 \leq l \leq \lfloor n/4 \rfloor \} \cup \{v_{2+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-4}\} \\
t & = 1: \text{the conversion spreads to the vertices } \{v_{1+8l}, v_{3+8l}, v_{5+8l}, v_{7+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-5}, v_{n-3}, v_{n-1}\} \text{ which means } v_{1}, v_{n-5}, v_{n-3}, v_{n-1} \text{ are converted in this step} \\
t & = 2: \text{the conversion spreads to the vertices } \{v_{6+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-4}\} \\
t & = 3: \text{the conversion spreads to the last unconverted vertex } v_{1} \text{ and the entire graph is converted} \\
\end{align*}
\]

We conclude that \(C_3(C_n(\{1, 3\})) = |n/4| + |n/8| + 1\) if \(n \equiv 2(\text{mod } 4)\) and \(n \not\equiv 2(\text{mod } 8)\).

\[\text{Case 4. } n \equiv 3(\text{mod } 4)\]. We consider two subcases:

\[\text{Case 4.a. } n \equiv 3(\text{mod } 8)\].

In this subcase, we have an even number of subgraphs on \(C_n(\{1, 3\})\) and \(m = 2\). Let \(D_0\) and \(M_0\) be the same sets identified in subcase 1.a. Let the seed set be \(S_0 = D_0 \cup M_0\). In a similar process to the one in subcase 1.a, all vertices of \(V - \{v_{n-4}, v_{n-3}, v_{n-1}, v_1\}\) by the end of step \(t = 2\). However, the five consecutive vertices \(v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\) form a version of \(Y\), and since taking out any vertex from \(D_0 \cup M_0\) results in a version of either \(X\) or \(Y\), we conclude that \(C_3(C_n(\{1, 3\})) > 3|n/8|\) in this subcase as well. Let \(S_0 = D_0 \cup M_0 \cup \{v_{n-1}\}\) be the seed set of cardinality \(3|n/8| + 1\); the process goes as follows:

\[
\begin{align*}
t & = 0: \text{we convert } S_0 = \{v_{d_l} : 1 \leq l \leq \lfloor n/4 \rfloor \} \cup \{v_{2+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-4}\} \\
t & = 1: \text{the conversion spreads to } \{v_{1+8l}, v_{3+8l}, v_{5+8l}, v_{7+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-1}\} \text{ which means } v_{1}, v_{n-3}, v_{n-1} \text{ are converted in this step} \\
t & = 2: \text{the conversion spreads to } \{v_{6+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \text{ which means } v_{n-4} \text{ is converted at this step} \\
\end{align*}
\]

By the end of step \(t = 2\), the conversion reaches all vertices of \(C_n(\{1, 3\})\); therefore, \(S_0\) is an I3CS of \(C_n(\{1, 3\})\), and since \(|S_0| = 3|n/8| + 1\), we conclude that \(C_3(C_n(\{1, 3\})) = 3|n/8| + 1\) if \(n \equiv 3(\text{mod } 8)\).

\[\text{Case 4.b. } n \equiv 3(\text{mod } 4)\] and \(n \not\equiv 3(\text{mod } 8)\). We consider two subcases:

\[\text{Case 4.b.1. } n = 7\].

This proof is equivalent to proving that \(C_3(C_7(\{1, 3\})) = 3\). It is obvious by definition that \(C_3(C_7(\{1, 3\})) \geq 3\). Let
$S_0$ be a seed set of cardinality 3 and defined as $S_0 = \{v_1, v_3, v_6\}$. The process goes as:

$$
t = 0 : S_0 = \{v_1, v_3, v_6\}. \tag{22}
$$

$$
t = 1 : S_1 = S_0 \cup \{v_2, v_7\}. \tag{23}
$$

$$
t = 2 : S_2 = S_1 \cup \{v_4, v_5\} = V(C_2(\{1, 3\}))
$$

which means $C_3(C_2(\{1, 3\})) \leq 3$; therefore, $C_3(C_2(\{1, 3\})) = 3$.

Case 4.b.2. $n \geq 11$.

Let $S_0 = D_0 \cup M_0 = \{v_{4k} : 1 \leq l \leq \lfloor n/4 \rfloor \} \cup \{v_{2k+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\}$; then that would make the following nine vertices $v_{n-9}, v_{n-8}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n$ unconverted which means creating several versions of $A_1$, and the process fails. Now let $S_0 = D_0 \cup M_0 \cup \{v_2\}$ be the seed set. We consider the following options for $v_4$:

(i) If $v_4 \in \{v_{n-9}, v_{n-8}, v_{n-6}, v_{n-5}\}$, then $v_{n-2}, v_{n-1}, v_n, v_1$ are four consecutive unconverted vertices; therefore, they form a version of $A_1$, and the process fails. However, we notice that

(ii) If $v_4 \in \{v_{n-4}, v_{n-3}, v_{n-1}, v_n, v_1\}$, then $v_{n-9}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5}$ form a version of $Y$, and since $Y$ is 3-unconvertable, then the process fails

(iii) If $v_4 \notin \{v_{n-9}, v_{n-8}, v_{n-6}, v_{n-5}\}$, then $v_{n-2}, v_{n-1}, v_n, v_1$ form a version of $A_1$, $(v_{n-9}, v_{n-8}, v_{n-7}, v_{n-6}, v_{n-5})$ form a version of $Y$, and the process fails

We conclude $D_0 \cup M_0 \cup \{v_2\}$ cannot be a 13CS of $C_n(\{1, 3\})$ when $n \equiv 3 \mod 4$ and $n \not\equiv 3 \mod 8$ which means that $C_3(C_n(\{1, 3\})) > \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 1$ if $n \equiv 2 \mod 4$ and $n \not\equiv 2 \mod 8$.

Now let $S_0 = \{v_{4k} : 1 \leq l \leq 7\} \cup \{v_{2k+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-5}, v_{n-1}\}$ be the seed set of cardinality $\lfloor n/4 \rfloor + \lfloor n/8 \rfloor + 2$; the process goes as follows:

$t = 0$: we convert $S_0 = \{v_{4k} : 1 \leq l \leq \lfloor n/4 \rfloor \} \cup \{v_{2k+8l} : 0 \leq l \leq \lfloor n/8 \rfloor - 1\} \cup \{v_{n-5}, v_{n-1}\}$

$t = 1$: the conversion spreads to $\{v_{1+10}, v_{4+10}, v_{6+10}, v_{9+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$

$t = 2$: the remaining unconverted vertices which are $\{v_{2+10}, v_{5+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ get converted

By the end of step $t = 2$, the entire graph’s vertex set is converted. We conclude that $S_0$ is an 13CS of cardinality 2 $n/5$, which means $C_3(C_n(\{1, 4\})) \leq 2n/5$ if $n \equiv 0 \mod (10)$. Figure 6 illustrates that $C_{20}(\{1, 4\}) \leq 8$.

**Case 2.** $n \equiv 1 \mod (10)$.

Let $S_0 = \{v_{3+10}, v_{5+10}, v_{7+10}, v_{9+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

$$
t = 0 : S_0 = \{v_{3+10}, v_{5+10}, v_{7+10}, v_{9+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \cup \{v_1\}, \tag{25}
$$

$$
t = 1 : S_1 = S_0 \cup \{v_{1+10}, v_{4+10}, v_{6+10}, v_{9+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \cup \{v_{n-5}, v_{n-1}\}, \tag{26}
$$

$$
t = 2 : S_2 = S_1 \cup \{v_{2+10}, v_{5+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1, 1 \leq d \leq \lfloor n/10 \rfloor - 1\}, \tag{27}
$$

$$
t = 3 : S_3 = S_2 \cup \{v_{8}, v_{n-2}\} = V(C_n(\{1, 4\})). \tag{28}
$$

**Theorem 11.** For $n \geq 9$.

$$C_3(C_n(\{1, 4\})) \leq \begin{cases} \frac{n}{5} + 2 & \text{if } n \equiv 0, 5, 6, 7 \mod (10) \text{ and } n \not\equiv \{16, 17\}; \\ \frac{n}{5} + 1 & \text{otherwise.} \end{cases} \tag{29}
$$
Therefore, \( S_0 \) is I3CS of \( C_n(\{1,4\}) \) which means \( C_3(\{1,4\}) \) ≤ 2 \( |n/5| + 1 \) if \( n \equiv 1 \) \((mod 10)\).

**Case 3.** \( n \equiv 2 \) \((mod 10)\).

Let \( S_0 = \{v_3+10, v_5+10, v_7+10, v_{10}+10 : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \)
\( U\{v_n\} \) be the seed set. The process goes as follows:

\[
\begin{align*}
t & = 0 : S_0 = \{v_3+10, v_5+10, v_7+10, v_{10}+10 \} \cup \{v_n\} \\
& = 1 : S_1 = S_0 \cup \{v_3+10, v_4+10, v_6+10, v_{9}+10 \} : 1 \leq j \leq \lfloor n/10 \rfloor, 0 \leq l \\
& \leq \lfloor n/10 \rfloor - 1, 0 \leq l \leq \lfloor n/10 \rfloor - 2, \\
& = 2 : S_2 = S_1 \cup \{v_{2+10}, v_{8+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\}, \\
& = 3 : S_3 = S_2 \cup \{v_1, v_{n-3}\} = V(C_n(\{1,4\})).
\end{align*}
\]

(25)

Therefore, \( S_0 \) is I3CS of \( C_n(\{1,4\}) \) which means \( C_3(\{1,4\}) \) ≤ 2 \( |n/5| + 1 \) if \( n \equiv 2 \) \((mod 10)\).

**Case 4.** \( n \equiv 3 \) \((mod 10)\).

Let \( S_0 = \{v_3+10, v_5+10, v_7+10, v_{10}+10 : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \)
\( U\{v_n\} \) be the seed set. The process goes as follows:

\[
\begin{align*}
t & = 0 : S_0 = \{v_3+10, v_5+10, v_7+10, v_{10}+10 \} \cup \{v_n\} \\
& = 1 : S_1 = S_0 \cup \{v_3+10, v_4+10, v_6+10, v_{9}+10 \} : 0 \leq l \leq \lfloor n/10 \rfloor - 1, \\
& = 2 : S_2 = S_1 \cup \{v_{2+10}, v_{8+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\}, \\
& = 3 : S_3 = S_2 \cup \{v_{n-2}, v_{n-1}\} = V(C_n(\{1,4\})).
\end{align*}
\]

(26)

Therefore, \( C_3(\{1,4\}) \) ≤ 2 \( |n/5| + 1 \) if \( n \equiv 3 \) \((mod 10)\).

**Case 5.** \( n \equiv 4 \) \((mod 10)\).

We consider the following subcases:

**Case 5.a.** \( n \equiv 14 \).

Let the seed set be \( S_0 = \{v_1, v_4, v_7, v_9, v_{11}, v_{13}\} \) which is of cardinality \( 2 \lfloor n/5 \rfloor + 2 \). Then

\[
\begin{align*}
t & = 0 : S_0 = \{v_1, v_4, v_7, v_9, v_{11}, v_{13}\}, \\
& = 1 : S_1 = S_0 \cup \{v_3, v_5, v_9, v_{14}\}, \\
& = 2 : S_2 = S_1 \cup \{v_{10}, v_{12}\}, \\
& = 3 : S_3 = S_2 \cup \{v_2, v_6\} = V(C_{14}(\{1,4\})).
\end{align*}
\]

(27)

Therefore, \( C_3(\{1,4\}) \) ≤ 2 \( |n/5| + 2 \).

**Case 5.b.** \( n \geq 24 \) and \( n \equiv 4 \) \((mod 10)\).

Let the seed set be \( S_0 = \{v_3+10, v_5+10, v_7+10, v_{10}+10 : 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \cup \{v_{n-2}\} \). The process goes as follows:

\[
\begin{align*}
t & = 0 : S_0 = \{v_3+10, v_5+10, v_7+10, v_{10}+10 \} \cup \{v_n\} \\
& = 1 : S_1 = S_0 \cup \{v_3+10, v_4+10, v_6+10, v_{9}+10 \} : 1 \leq j \leq \lfloor n/10 \rfloor, 1 \leq d \\
& \leq \lfloor n/10 \rfloor - 1, 0 \leq l \leq \lfloor n/10 \rfloor - 2, \lfloor n/10 \rfloor - 1, 1 \leq d \leq \lfloor n/10 \rfloor - 1, \\
& = 2 : S_2 = S_1 \cup \{v_{2+10}, v_{8+10} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\}, \\
& = 3 : S_3 = S_2 \cup \{v_1, v_{n-3}\}, \\
& = 4 : S_4 = S_3 \cup \{v_{n-1}\}, \\
& = 5 : S_5 = S_4 \cup \{v_n\} = V(C_n(\{1,4\})).
\end{align*}
\]

(28)
Case 6. $n \equiv 5 \pmod{10}$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

\[
t = 0 : S_0 = \left\{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\},
\]
\[
t = 1 : S_1 = S_0 \cup \left\{v_{1+10p}, v_{4+10p}, v_{6+10p}, v_{9+10p} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\},
\]
\[
t = 2 : S_2 = S_1 \cup \left\{v_{2+10p}, v_{5+10p}, v_{8+10p} : 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\}.
\]

Therefore, $C_3(C_n(\{1, 4\})) \leq 2\lfloor n/5 \rfloor + 1$.

Case 7. $n \equiv 6 \pmod{10}$.

We consider the following subcases:

Case 7.a. $n = 16$.

Let the seed set be $S_0 = \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\}$ which is of cardinality $2\lfloor n/5 \rfloor + 1$. The process involves the following steps:

\[
t = 0 : S_0 = \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\},
\]
\[
t = 1 : S_1 = S_0 \cup \{v_1, v_4, v_6, v_9, v_{11}, v_{14}\},
\]
\[
t = 2 : S_2 = S_1 \cup \{v_2, v_3, v_8, v_{12}\} = V(C_6(\{1, 4\})).
\]

Therefore, $C_3(C_6(\{1, 4\})) \leq 2\lfloor n/5 \rfloor + 1$.

Case 7.b. $n \geq 26$ and $n \equiv 6 \pmod{10}$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

\[
t = 0 : S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 1 : S_1 = S_0 \cup \{v_{1+10p}, v_{4+10p}, v_{6+10p}, v_{9+10p} : 1 \leq j \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 2 : S_2 = S_1 \cup \{v_{2+10p}, v_{5+10p}, v_{8+10p} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\},
\]
\[
\text{We conclude that } C_3(C_n(\{1, 4\})) \leq 2\lfloor n/5 \rfloor \text{ if } n \geq 26 \text{ and } n \equiv 6 \pmod{10}.
\]

Case 8. $n \equiv 7 \pmod{10}$.

We consider the following subcases:

Case 8.a. $n = 17$.

Let the seed set be $S_0 = \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\}$ which is of cardinality $2\lfloor n/5 \rfloor + 1$. The process involves the following steps:

\[
t = 0 : S_0 = \{v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{16}\},
\]
\[
t = 1 : S_1 = S_0 \cup \{v_4, v_6, v_9, v_{11}, v_{14}, v_{16}\},
\]
\[
t = 2 : S_2 = S_1 \cup \{v_1, v_2, v_8, v_{12}\} = V(C_7(\{1, 4\})).
\]

Therefore, $C_3(C_7(\{1, 4\})) \leq 2\lfloor n/5 \rfloor + 1$.

Case 8.b. $n \geq 27$ and $n \equiv 7 \pmod{10}$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\}$ be the seed set. The process goes as follows:

\[
t = 0 : S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 1 : S_1 = S_0 \cup \{v_{1+10p}, v_{4+10p}, v_{6+10p}, v_{9+10p} : 1 \leq j \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 2 : S_2 = S_1 \cup \{v_{2+10p}, v_{5+10p}, v_{8+10p} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 3 : S_3 = S_2 \cup \{v_1, v_3, v_8, v_{12}\} = V(C_n(\{1, 4\})).
\]

Therefore, $C_3(C_n(\{1, 4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \geq 26$ and $n \equiv 7 \pmod{10}$.

Case 9. $n \equiv 8 \pmod{10}$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \cup \{v_n\}$ be the seed set. The process goes as follows:

\[
t = 0 : S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \cup \{v_n\},
\]
\[
t = 1 : S_1 = S_0 \cup \{v_{1+10p}, v_{4+10p}, v_{6+10p}, v_{9+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 2 : S_2 = S_1 \cup \{v_{2+10p}, v_{5+10p}, v_{8+10p} : 0 \leq l \leq \lfloor n/10 \rfloor - 1\},
\]
\[
t = 3 : S_3 = S_2 \cup \{v_{n-6}, v_{n-2}\} = V(C_n(\{1, 4\})).
\]

Therefore, $C_3(C_n(\{1, 4\})) \leq 2\lfloor n/5 \rfloor + 1$ if $n \equiv 8 \pmod{10}$.

Figure 7 illustrates that $C_2(\{1, 4\}) \leq 11$.

Case 10. $n \equiv 9 \pmod{10}$.

We consider the following subcases:

Case 10.a. $n = 9$. 


Let the seed set be $S_0 = \{v_1, v_3, v_6, v_9\}$ which is of cardinality $2 \lfloor n/5 \rfloor + 2$. The process involves the following steps:

$t = 0 : S_0 = \{v_1, v_3, v_6, v_9\}$,
$t = 1 : S_1 = S_0 \cup \{v_2, v_7\}$,
$t = 2 : S_2 = S_1 \cup \{v_4, v_9\} = V(C_9(\{1, 4\}))$. \hspace{1cm} (35)

Therefore, $C_9(C_9(\{1, 4\})) \leq 2 \lfloor n/5 \rfloor + 2$.

Case 10.b. $n = 19$.

Let the seed set be $S_0 = \{v_1, v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{17}\}$ which is of cardinality $2 \lfloor n/5 \rfloor + 2$. The process goes as follows.

$t = 0 : S_0 = \{v_1, v_3, v_5, v_7, v_{10}, v_{13}, v_{15}, v_{17}\}$,
$t = 1 : S_1 = S_0 \cup \{v_2, v_6, v_9, v_{11}, v_{14}, v_{16}\}$,
$t = 2 : S_2 = S_1 \cup \{v_{12}, v_{18}\}$,
$t = 3 : S_3 = S_2 \cup \{v_8, v_{19}\}$,
$t = 4 : S_4 = S_3 \cup \{v_4\} = V(C_{19}(\{1, 4\}))$. \hspace{1cm} (36)

Therefore, $C_9(C_{19}(\{1, 4\})) \leq 2 \lfloor n/5 \rfloor + 2$.

Case 10.c. $n \geq 29$ and $n \equiv 9 \pmod{10}$.

Let $S_0 = \{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \lfloor n/10 \rfloor, 0 \leq l \leq \lfloor n/10 \rfloor - 1\} \cup \{v_1\}$ be the seed set. The process goes as follows:

$t = 0 : S_0 = \left\{v_{3+10p}, v_{5+10p}, v_{7+10p}, v_{10+10p} : 0 \leq p \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\} \cup \{v_1\}$,
$t = 1 : S_1 = S_0 \cup \left\{v_{1+10r}, v_{4+10r}, v_{6+10r}, v_{9+10r} : 1 \leq j \leq \left\lfloor \frac{n}{10} \right\rfloor, 0 \leq l \leq \left\lfloor \frac{n}{10} \right\rfloor - 1 \right\}$. \hspace{1cm} (37)

Therefore, $C_9(C_n(\{1, 4\})) \leq 2 \lfloor n/5 \rfloor + 1$ if $n \geq 29$ and $n \equiv 9 \pmod{10}$.

From all the previous cases and subcases, we conclude the requested.

Theorem 12. For $n \geq 2(r+1)$ and $n \equiv 0 \pmod{2(r+1)}$, $C_9(C_n(\{1, r\})) \leq nr/(2(r+1))$.

Proof. Proposition 9 implies that the conversion process fails if there are $r+1$ consecutive unconverted vertices on $C_n(\{1, 4\})$ at $t = 0$. We divide the vertices of $V(C_n(\{1, r\}))$ into $n/(2(r+1))$ subgraphs denoted by $SG_i : 1 \leq i \leq n/2(r+1)$. Now we try to find a configuration of converted vertices of a random subgraph $SG_i$ at $t = 0$ so that when applied to all the subgraphs, it results in converting all of $V(C_n(\{1, r\}))$. We consider the following cases for $r$: \hspace{1cm} $\Box$

Case 1. $r$ is even. Let the configuration of converted vertices we apply to $SG_i$ at $t = 0$ be $\{v_{i+2l}, v_{i+2l+2m} : 0 \leq l \leq r/2, 0 \leq m \leq (r-4)/2\}$. This means we convert $r$ vertices from each subgraph. As shown in Figure 8, in step $t = 1$, the conversion spreads to $\{v_{i+2l} : 0 \leq l \leq r/2, 0 \leq m \leq (r-4)/2\}$. In the following step $t = 2$, the conversion spreads to $v_{i+2l+2m}$. In step $t = 3$, the conversion spreads to $v_{i+2l+4m}$. In step $t = 4$, the configuration converts $SG_i$ entirely.

Without loss of generality, by applying the same configuration to all subgraphs, we form an I3CS of cardinality $nr/(2(r+1))$. We denote it by $S_0$, and the process goes as follows:

$t = 0 : S_0 = \left\{v_{(1+2l)} + \left\{i+(2r+2)\right\}, v_{(r+4+2m)+i+(2r+2)} : 0 \leq l \leq \frac{n}{2(r+1)} \right\}$,
$t = 1 : S_1 = S_0 \cup \left\{v_{i+2l} \cup v_{i+2l+2m} : 0 \leq l \leq \frac{n}{2(r+1)} \right\}$,
$t = 2 : S_2 = S_1 \cup \left\{v_{i+2l+2m} : 1 \leq i \leq \frac{n}{2(r+1)} \right\}$,
$t = 3 : S_3 = S_2 \cup \left\{v_{(2i+2l+2m)} : 0 \leq i \leq \frac{n}{2(r+1)} - 1 \right\}$,
$t = 4 : S_4 = S_3 \cup \left\{v_{2i+2l+2m} : 0 \leq i \leq \frac{n}{2(r+1)} - 1 \right\} = V(C_n(\{1, r\}))$. \hspace{1cm} (38)
Therefore, $S_0$ is an I3CS which means $C_3(C_n(\{1, r\})) \leq nr/2(r + 1)$ if $n \equiv 0(\text{mod 2}(r + 1))$ and $r$ is even.

Case 2. $r$ is odd. Let the seed set be $S_0 = \{v_{2i+(2r+2)} : 0 \leq i \leq n/2(r + 1) - 1, 1 \leq l \leq r\}$. The process goes as follows:

$$t = 0: S_0 = \{v_{2i+(2r+2)} : 0 \leq i \leq \frac{n}{2(r + 1)} - 1, 1 \leq l \leq r\},$$

$$t = 1: S_1 = S_0 \cup \{v_{(1+2i)+(2r+2)} : 0 \leq i \leq \frac{n}{2(r + 1)} - 1, 0 \leq l \leq r\},$$

$$t = 2: S_2 = S_1 \cup \{v_{2i+2i+(2r+2)} : 0 \leq i \leq \frac{n}{2(r + 1)}\} = V(C_n(\{1, r\})).$$

(39)

Therefore, $S_0$ is an I3CS which means $C_3(C_n(\{1, r\})) \leq nr/2(r + 1)$ if $n \equiv 0(\text{mod 2}(r + 1))$ and $r$ is odd. Figure 9 illustrates how converting $S_0$ at $t = 0$ results in converting $S_{G_1}$ entirely at the end of step $t = 2$, taking into consideration that $v_{n-r+1} \in V(S_{G_n(2r+1)}) \cap S_0$ and $v_{1+3r} \in V(S_{G_1}) \cap S_0$.

Without loss of generality, the same argument applies to all subgraphs. From Case 1 and Case 2, we conclude the requested.

Data Availability

No data was used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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