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Research Article

Periodic Oscillations in MEMS under Squeeze Film Damping Force

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We provide sufficient conditions for the existence of periodic solutions for an idealized electrostatic actuator modeled by the Liénard-type equation $\ddot{x} + F_D(x, \dot{x}) + x = \beta \mathscr{V}^2(t)/(1-x)^2$, $x \in]-\infty, 1[$ with $\beta \in \mathbb{R}^+$, $\mathscr{V} \in C(\mathbb{R}/T\mathbb{Z})$, and $F_D(x, \dot{x}) = \kappa \dot{x}/(1-x)^3$, $\kappa \in \mathbb{R}^+$ (called squeeze film damping force), or $F_D(x, \dot{x}) = c\dot{x}$, $c \in \mathbb{R}^+$ (called linear damping force). If F_D is of squeeze film type, we have proven that there exists at least two positive periodic solutions, one of them locally asymptotically stable. Meanwhile, if F_D is a linear damping force, we have proven that there are only two positive periodic solutions. One is unstable, and the other is locally exponentially asymptotically stable with rate of decay of $c/2$. Our technique can be applied to a class of Liénard equations that model several microelectromechanical system devices, including the comb-drive finger model and torsional actuators.

1. Introduction

This paper presents a mathematical study of the existence and stability of periodic solutions of a Liénard-type equation that describes the motions of the movable plate (movable electrode) in an idealized parallel-plate electrostatic actuator, nowadays known as the *Nathanson model*. This actuator is an example of a large set of devices composed of microscale (one or more of their dimensions are in the micrometers range) mechanical and electronic elements integrated in a common silicon substrate. This recent technology, known in the literature as microsystem technology (MST) or microelectromechanical systems (MEMS), has become very successful in the commercial front (a complete and recent survey of MEMS literature with applications can be found in [1]). Due to their size, they can fit in several devices such as TVs, microwaves circuits, cardiac pacemakers, pressure sensors, accelerometers and gyroscopes for automobiles, and wearable electronic devices [2]. Examples of MEMS are the acceleration sensor and scanner developed and produced by Bosch [3, 4]. The mathematical formulation of the Nathanson model was initially presented in 1967 by the American electrical engineer H. C. Nathanson et al. [5]. The study focuses on the structural instability phenomenon which results from the variations in voltage load and leads to a saddle-node bifurcation, called *pull-in*. After more than 50 years, the Nathanson model continues to draw a lot of attention. Many researchers have been devoted to its analytical and numerical study, mainly to understanding and characterizing the pull-in phenomenon through different techniques and mathematical formulations (see for instance [1, 2, 6–10]).

The fundamental configuration of the Nathanson model assumes rectangular electrodes. One stationary and the other are allowed to move. The electrodes can have any shape, but for simplicity, rectangular electrodes are more commonly used. If d is the initial distance between the electrodes and both are biased by a voltage V, then an electrostatic force will be generated which pulls the movable electrode.

If the schematic diagram of the Nathanson model is like the one shown in Figure 1, the electrostatic force $F_E(\tau, \hat{x})$ acting on the movable electrode is expressed as

$$F_E(\tau, \hat{x}) = \frac{\varepsilon A V^2(\tau)}{2(d-\hat{x})^2}, \quad \hat{x} \in]-\infty, d[, \tag{1}$$

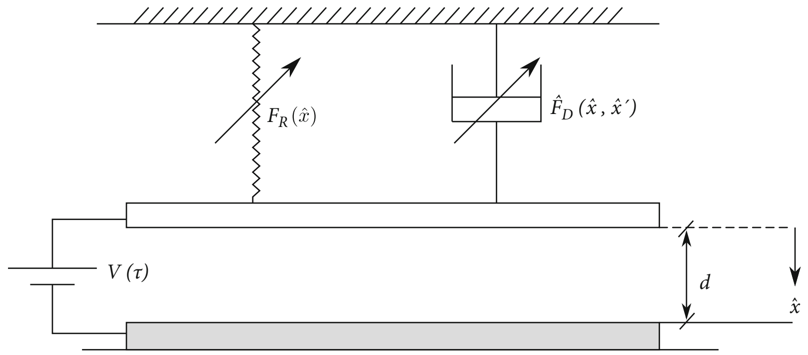
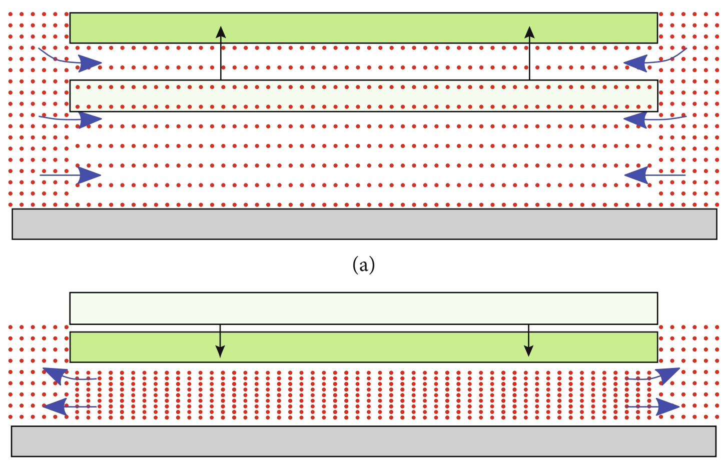


FIGURE 1: Parallel plate capacitor with one movable plate.



(a)

(b)

FIGURE 2: Squeeze film damping on parallel plate capacitors.

where τ is an independent variable related to time, $V(\tau)$ is the potential difference between the two plates, A is the area of the plate that is facing the other, ε is the dielectric constant of the medium in between the plates, and $\hat{x}$ is the position of the movable electrode with respect to d. The other forces involved are the restoring $F_R(\hat{x})$ and the damping force $\hat{F}_D(\hat{x}, \hat{x}')$, where $\hat{x}'$ represents the induced velocity of the movable electrode. For the restoring force, we have considered a linear stiffness force, in such case $F_R(\hat{x}) = k\hat{x}$ with $k > 0$, and for the damping force, two types were considered:

$$\hat{F}_D\left(\hat{x}, \hat{x}'\right) = \tilde{c}\hat{x}', \quad \text{(linear damping)}, \tag{2}$$

$$\hat{F}_D\left(\hat{x}, \hat{x}'\right) = \frac{\gamma}{(d-\hat{x})^3}\hat{x}', \quad \text{(squeeze film damping)}, \tag{3}$$

with $\tilde{c}, \gamma \in \mathbb{R}^+$. The first comes from simplifying the problem to a moving sphere in fluid at a velocity $\hat{x}'$. The second corresponds to the most common and dominant dissipation mechanism in MEMS, which is related to the study of the damping force on a microstructure with a big surface that traps a fluid in a small space. When the plates separate, the inner pressure is smaller than the outer pressure as shown in Figure 2(a). When the plates get closer, the opposite occurs, as seen in Figure 2(b). This effect is called *squeeze film* damping, and it is significantly present in parallel plate actuators, which have a proportionally bigger surface area in comparison with the distance in between the plates. For more details on damping in MEMS, see [2] and the references therein.

Under the previous considerations and from Newton's second law, the equation of motion of the movable electrode is given by the following second order nonlinear differential equation:

$$m\hat{x}'' + \hat{F}_D\left(\hat{x}, \hat{x}'\right) + k\hat{x} = \frac{\epsilon A V^2(\tau)}{2(d-\hat{x})^2}, \tag{4}$$

where $\hat{x}'' = \hat{x}''(\tau)$, represents the induced acceleration of the position $\hat{x} = \hat{x}(\tau)$. The gravitational force is not considered because it is too small compared to the electrostatic force in microstructures. In order to reduce the number of parameters, we can normalize $\hat{x}$ with respect to d and τ with respect to $\mathcal{T} = \sqrt{m/k}$. Therefore, our nondimensional variables x and t satisfies

$$\hat{x} = xd, \quad \tau = t\mathcal{T}, \tag{5}$$

and the corresponding nondimensional equation from (4) is

$$\ddot{x} + F_D(x, \dot{x}) + x = \frac{\beta \mathscr{V}^2(t)}{(1-x)^2}, \quad x \in]-\infty, 1[, \tag{6}$$

with

$$F_D(x, \dot{x}) = \frac{\mathscr{T}^2}{md} \widehat{F}_D\left(xd, \frac{\dot{x}d}{\mathscr{T}}\right), \quad \mathscr{V}(t) = V(t\mathscr{T}), \quad \text{and} \quad \beta = \frac{\varepsilon A}{2kd^3}. \tag{7}$$

Throughout this document, we consider a DC-AC voltage source $\mathscr{V}(t)$ of the form

$$\mathscr{V}(t) = v_0 + \delta v(t), \tag{8}$$

with $v_0 \in \mathbb{R}^+$ (*DC*-voltage source) and $v(t) \in C(\mathbb{R}/T\mathbb{Z})$ with zero average. Voltage $\mathscr{V}$ will also be nonnegative; therefore, from now on we assume $\delta \in [0, -v_0/v_{\min}[$ where

$$v_{\max} := \max_{t\in[0,T]} v(t), \quad v_{\min} := \min_{t\in[0,T]} v(t). \tag{9}$$

When the damping force $\widehat{F}_D$ is given by (2) (namely, the linear damping force), the authors in [7, 9] present a rigorous analysis of the existence and stability of exactly two positive T-periodic solutions of (6) for the nonconservative ($\tilde{c} \neq 0$) and for the conservative case ($\tilde{c} = 0$), respectively. In both papers, classical functional and topological techniques were employed such as the upper and lower solution method, Leray-Schauder degree, and the topological index of a periodic solution. As far as we are aware, no papers have been published regarding the study of periodic solutions of (6) when the damping force is given by (3). Hence, this paper pursues two goals: firstly, to provide an alternative and accurate stability criteria for the two periodic solutions of (6) with linear damping force, and secondly, to present sufficient conditions for the existence and linear stability of periodic solutions of (6) with under squeeze film damping force. We remark that the techniques and ideas in this document can be applied to study periodic motions in other MEMS devices and microstructures, for example, torsional actuators, comb-drive devices, atomic force microscope microcantilevers (see [2]), and the recent graphene-based Nathanson model (see [11]).

We have divided the document into four sections: following the introduction, in Section 2, we developed the main tools for the proofs. Sections 3 and 4 are devoted to state and prove the main results. In addition, numerical validations are provided to illustrate the results applied to (4) using explicit values of the parameters taken from the specialized literature [2]. Finally, to provide a self-contained manuscript, we included an appendix in which we established well-known results about the method of lower and upper solutions for second order differential equations and multiplicity and stability of periodic solutions of Liénard equations.

2. Preliminary Results

Different approaches can be used to study the existence of solutions of the boundary value problem:

$$\begin{gathered} \ddot{x} + f(t, x, \dot{x}) = 0, \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \end{gathered} \tag{10}$$

for $f : D \longrightarrow \mathbb{R}$ continuous function, where $D \subseteq \mathbb{R} \times]l_1, l_2[\times \mathbb{R}$ is an open connected set with $-\infty \leq l_1 \leq l_2 \leq \infty$. Topological degree, averaging method, and lower and upper solutions are perhaps the most common used tools from nonlinear analysis to address this problem. In this section, we use the lower and upper solution method to obtain existence results for Liénard type families of (10). It is worth to mention that if f is a T-periodic function in the variable t, then all the solutions of (10) would be also T-periodic.

Theorem 1. *Let $\zeta, \eta \in C^2([0,T])$ be a lower and an upper solution of the boundary value problem (10) such that $\eta \leq \zeta$. Define*

$$E := \{(t, x, y) \in D \mid t \in [0, T], \eta(t) \leq x \leq \zeta(t)\}. \tag{11}$$

($\dagger$) *Assume that there exists $N \geq 0$ such that $|\partial_y f(t, x, y)| \leq N$ for all (t, x, y) in E. Then for any solution $u(t)$ of (10) such that $\eta(t) \leq u(t) \leq \zeta(t)$ on $[0, T]$, there exists $R := R(N) > 0$ such that $-R \leq \dot{u}(t) \leq R$ for all t in $[0, T]$.*

($\dagger\dagger$) *If the assumption in ($\dagger$) holds. Let*

$$W := \{(t, x, y) \in E \mid t \in [0, T], -R \leq y \leq R\}, \tag{12}$$

and assume that there exists $M \geq 0$ such that for all (t, x, y) in W

$$\partial_x f(t, x, y) \leq M \leq \left(\frac{\pi}{T}\right)^2, \quad \sigma(t) \leq L, \quad N \leq H(L), \tag{13}$$

for some $L \in [M, (\pi/T)^2]$ where

$$\begin{aligned} \sigma(t) &:= \frac{f\left(t, \zeta(t), \dot{\zeta}(t)\right) - f(t, \eta(t), \dot{\eta}(t))}{\zeta(t) - \eta(t)}, \\ H(L) &= \frac{(L-M)}{\sqrt{L}} \cot\left(\frac{T\sqrt{L}}{2}\right). \end{aligned} \tag{14}$$

Then, the boundary value problem (10) has at least one solution ψ such that

$$\eta(t) \leq \psi(t) \leq \zeta(t), \quad \forall t \in [0, T]. \tag{15}$$

Proof. The existence of $N \geq 0$ such that $|\partial_y f(t, x, y)| \leq N$ and Theorem 13 (see Appendix) lead us to the conclusion that there exists $R > 0$ such that for any solution u_1 of (10) and any solution u_2, u_3 of

$$\ddot{x} \geq f(t, x, \dot{x}), \quad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \tag{16}$$

$$\ddot{x} \le f(t,x,\dot{x}), \quad x(0)=x(T), \quad \dot{x}(0)=\dot{x}(T), \tag{17}$$

respectively, with $(t, u_i(t), \dot{u}_i(t))$ in E also satisfies $|\dot{u}_i(t)| < R$ for all t in $[0,T]$ and $i=1,2,3$. This proves the first statement (†). The second statement (††) follows the same lines of the proof of Theorem 3.2 of Chapter 5 in [12] relative to W. □

Now, we consider the family of boundary value problems

$$\ddot{x} + c(t,x)\dot{x} + K(x) = \frac{F(t)}{G(x)}, \quad x(0)=x(T), \quad \dot{x}(0)=\dot{x}(T). \tag{18}$$

where $c:[0,T]\times I \longrightarrow R$, $K: I \longrightarrow R$, $G: I \longrightarrow R$, $F:[0,T] \longrightarrow R$ be continuous functions, $I \subset R$. Define

$$\begin{aligned} &\phi(x) = K(x)G(x), \quad \text{and} \\ &F_{\max} = \max_{t\in[0,T]} F(t), \quad F_{\min} = \min_{t\in[0,T]} F(t). \end{aligned} \tag{19}$$

Following the notation of (10), we have

$$f(t,x,\dot{x}) = c(t,x)\dot{x} + K(x) - F(t)/G(x). \tag{20}$$

Regarding the existence of periodic solutions of (18), we lead to the following results.

Theorem 2. *Assume that $F_{\min}, F_{\max} \in \phi(I)$, $G(x) > 0$ and $\phi(x)$ is decreasing with isolated critical points in I. Then, (18) admits unique constant lower and upper solution ζ and η, respectively, such that $\zeta \le \eta$ satisfying*

$$\phi(\zeta) = F_{\max} \quad \text{and} \quad \phi(\eta) = F_{\min}. \tag{21}$$

Moreover, there exists a solution $\varphi(t)$ of (18) such that

$$\zeta \le \varphi(t) \le \eta, \quad \forall t \in [0,T]. \tag{22}$$

Proof. Since ϕ is decreasing with isolated critical points in I and $F_{\min}, F_{\max} \in \phi(I)$, then there exist unique solutions ζ, η, in I for

$$\phi(\zeta) = F_{\max}, \quad \text{and} \quad \phi(\eta) = F_{\min}, \tag{23}$$

respectively. Moreover, $\zeta \le \eta$. Additionally, notice that

$$K(\eta)G(\eta) = F_{\min} \le F(t) \le F_{\max} = K(\zeta)G(\zeta), \quad \forall t \in [0,T], \tag{24}$$

which implies

$$K(\eta) - F(t)/G(\eta) \le 0 \quad \text{and} \quad K(\zeta) - F(t)/G(\zeta) \ge 0. \tag{25}$$

It is equivalent to

$$f(t,\eta,0) \le 0 \quad \text{and} \quad f(t,\zeta,0) \ge 0. \tag{26}$$

Therefore, $\zeta \le \eta$, and they are lower and upper solutions, respectively.

Let

$$E := [0,T]\times[\zeta,\eta]\times\mathbb{R} \quad \text{and} \quad N = \max_{[0,T]\times[\zeta,\eta]} |c(t,x)|. \tag{27}$$

Notice that for all $(t,x,v_1), (t,x,v_2) \in E$, it follows

$$|f(t,x,v_1) - f(t,x,v_2)| = |c(t,x)||v_1 - v_2| \le N|v_1 - v_2|. \tag{28}$$

Then, the positive function $\rho(s) = Ns + \max_E |f(t,x,0)|$ satisfies

$$\int_0^\infty \frac{s}{\rho(s)} ds = \infty \quad \text{and} \quad |f(t,x,v)| \le \rho(|v|), \quad \forall (t,x,v) \in E. \tag{29}$$

□

Hence, f satisfies the *Nagumo condition* and Theorem 12 (see Appendix); we can conclude that there exists a solution φ of (18) such that

$$\zeta \le \varphi(t) \le \eta, \quad \forall t \in [0,T]. \tag{30}$$

Theorem 3. *Let $\widehat{I} \subset \mathbb{R}$ and $c:[0,T]\times\widehat{I} \longrightarrow \mathbb{R}$, $K:\widehat{I} \longrightarrow \mathbb{R}$, $G:\widehat{I} \longrightarrow \mathbb{R}$ be differentiable functions and $F:[0,T] \longrightarrow \mathbb{R}$ continuous.*

(‡) *Assume that $F_{\min}, F_{\max} \in \phi(\widehat{I})$, $G(x) > 0$, and $\phi'(x) \ge 0$ with isolated critical points in $\widehat{I}$. Then, (18) admits unique constant reversed-ordered upper and lower solutions η, ζ, respectively, such that $\phi(\eta) = F_{\min}$ and $\phi(\zeta) = F_{\max}$.*

(‡‡) *If the assumptions in* (‡) *holds, define*

$$\begin{gathered} \widehat{E} = [0,T]\times[\eta,\zeta]\times\mathbb{R}, \\ N = \max_{\widehat{E}} |c(t,x)|, \quad \widehat{a} = \max\left\{N, \max_{\widehat{E}} |K(x) - F(t)/G(x)|\right\}, \end{gathered} \tag{31}$$

and let R the unique positive solution of

$$R - \ln(R+1) = \widehat{a}(\zeta - \eta). \tag{32}$$

Assume that there exists $M > 0$ such that

$$\max_W \left| \frac{\partial}{\partial x}\left(c(t,x)\dot{x} + K(x) - \frac{F(t)}{G(x)} \right) \right| \le M \le \left(\frac{\pi}{T}\right)^2, \tag{33}$$

where

$$W = \left\{ (t,x,y) \in \widehat{E} \,\middle|\, y \in [-R,R] \right\}, \tag{34}$$

and there exists $L \in [M, (\pi/T)^2]$ *such that*

$$N \leq H(L), \quad \text{with} \quad H(L) = \frac{(L-M)}{\sqrt{L}} \cot\left(\frac{T\sqrt{L}}{2}\right). \quad (35)$$

Then, the boundary value problem (18) has at least one solution ψ such that

$$\eta \leq \psi(t) \leq \zeta, \quad \forall t \in [0, T]. \quad (36)$$

Proof. Suppose that $G(x) > 0$ and $\phi'(x) \geq 0$ with isolated critical points in $\widehat{I}$. This implies that there exists only one pair of values ζ, η in $\widehat{I}$ such that

$$\begin{aligned} \phi(\zeta) &= F_{\max}, \\ \phi(\eta) &= F_{\min}. \end{aligned} \quad (37)$$

Notice that $\eta \leq \zeta$ because ϕ is monotone non-decreasing function and $F_{\min} \leq F_{\max}$. Additionally η, ζ are upper and lower solutions of (18) because

$$\begin{aligned} f(t, \zeta, 0) &= \frac{K(\zeta)G(\zeta) - F(t)}{G(\zeta)} = \frac{F_{\max} - F(t)}{G(\zeta)} \geq 0, \quad \forall t \in R, \\ f(t, \eta, 0) &= \frac{K(\eta)G(\eta) - F(t)}{G(\eta)} = \frac{F_{\min} - F(t)}{G(\eta)} \leq 0, \quad \forall t \in R, \end{aligned} \quad (38)$$

which proves part (‡). In order to prove part (‡‡), first notice that

$$\left|\partial_y f(t, x, y)\right| = |c(t, x)| \leq \max_{\widehat{E}} |c(t, x)| = N, \quad \forall (t, x, y) \in \widehat{E}. \quad (39)$$

with

$$\widehat{E} = [0, T] \times [\eta, \zeta] \times R. \quad (40)$$

Then, by Theorem 1 part (†), there exists $R > 0$ such that for any solution $u(t)$ of (18) such that $\eta \leq u(t) \leq \zeta$ on $[0, T]$, we have $-R \leq \dot{u}(t) \leq R$ on $[0, T]$, and following Theorem 13 and its remark, the value R is the unique positive solution of

$$R - \ln(R + 1) = \widehat{a}(\zeta - \eta). \quad (41)$$

Notice that by the mean value theorem:

$$\sigma(t) := \frac{f(t, \zeta, 0) - f(t, \eta, 0)}{\zeta - \eta} \leq \max_W |\partial_x f(t, x, y)| \leq M \leq L, \quad \forall t \in [0, T]. \quad (42)$$

Therefore, by Theorem 1, the boundary value problem (18) has at least one solution ψ such that

$$\eta \leq \psi(t) \leq \zeta, \quad \forall t \in [0, T]. \quad (43)$$

□

2.1. Multiplicity and Stability of Periodic Solution for Duffing Equations. We end this section by showing some results about multiplicity and stability of periodic solutions for the Duffing-type equation:

$$\ddot{x} + c\dot{x} + g(t, x) = 0, \quad (44)$$

where $c > 0$ and $g : \mathbb{R} \times]l_1, l_2[\longrightarrow \mathbb{R}$, $-\infty \leq l_1 < l_2 \leq \infty$, a continuous function, T-periodic with respect to t, and having a continuous partial derivative with respect to x. The following notation will be used throughout the rest of the paper.

The positive part of a given a function f is defined as $f_+(t) := \max\{f(t), 0\}$.

(1) Given a pair of function $f, g \in L^p[0, T]$, we write $f \succ g$, if $f \geq g$ for almost every t and $f > g$ in a subset of positive measure

(2) $L^p(\mathbb{R}/T\mathbb{Z})$: T-periodic function $f \in L^p[0, T]$ with the norm

$$\|f\|_{L^p} := \|f\|_{L^p[0,T]}, \quad p \in [1, \infty]. \quad (45)$$

(3) For some $p \in [1, \infty]$ and $p_* = p/(p-1)$, $K(q)$ denotes the best Sobolev constant in the following inequality:

$$C\|u\|_{L^q}^2 \leq \|\dot{u}\|_{L^2}^2, \quad \forall u \in H_0^1([0, T]). \quad (46)$$

Let us start with some results over the linear differential operator:

$$\begin{aligned} L_a : \mathscr{W} &\longrightarrow L^1(0, T) \\ \omega &\longrightarrow L_a[\omega] = \ddot{\omega} + c\dot{\omega} + a(t)\omega, \end{aligned} \quad (47)$$

where

$$\mathscr{W} = \left\{\omega \in W^{2,1}(0, T) : \omega(0) = \omega(T), \dot{\omega}(0) = \dot{\omega}(T)\right\}, \quad (48)$$

c is a positive constant and $a \in \Gamma_{p,c}$, with

$$\Gamma_{p,c} = \left\{a \in L^p(0, T) : \left\|\left(a - c^2/4\right)_+\right\|_{L^p} < K(2p_*)\right\}, \quad (49)$$

for some $p \in [1, \infty]$.

Proposition 4. *Suppose that* $a, a_1, a_2 \in \Gamma_{p,c}$ *for some* $p \in [0, \infty[$. *Then, we have the following conclusions:*

(1) *Each possible T-periodic solution* $\omega = \omega(t)$ *of* $L_a[\omega] = 0$ *is either trivial or different from zero for each* $t \in [0, T]$

(2) *If* $a_2 \succ a_1$, *then* $L_{a_i}[\omega] = 0 (i = 1, 2)$ *cannot admit nontrivial T-periodic solutions simultaneously*

(3) $L_a[\omega] = 0$ *does not admit negative Floquet's multipliers*

(i) *If* $\bar{a} = 1/T \int_0^T a(t)dt$ *satisfies* $\bar{a} > c^2/4$, *then* $L_a[\omega] = 0$ *does not admit real Floquet's multipliers, i.e.,* $L_a[w] = 0$ *is elliptic and has no nontrivial T-periodic solutions*

The trivial solution $\omega \equiv 0$ *is locally exponentially asymptotically stable with rate of exponential decay* $c/2$.

The proof of Proposition 4 can be found in several papers, see, for example, [13–16]. It is worth pointing out that the arguments in [14–16] are not precise but easy to fix by choosing correctly the set $\Gamma_{p,c}$.

3. An Improvement for the Linear Damping Case

In this section, we consider the Duffing equation:

$$\ddot{x} + c\dot{x} + x = \frac{\beta \mathcal{V}^2(t)}{(1-x)^2}, \tag{50}$$

with $x \in]-\infty,-1[$. This equation corresponds to the Nathanson model (6) with linear damping force $F_D(x, \dot{x}) = c\dot{x}$, $c \geq 0$, and $c \in R$. The existence and stability of periodic solutions of (50) have been considered in [9] for the case $c = 0$ and also in [7] for $c > 0$. The results exposed here respect to (50) have the purpose to combine the ideas found in the mentioned papers and the results of Theorems 15 and 16 in the Appendix.

Theorem 5. *Assume the following conditions:*

(I) $0 < \mathcal{V}^2_{\min} < \mathcal{V}^2_{\max} \leq 4/27\beta$

(II) $1 - 2\beta\mathcal{V}^2(t) \in \Gamma_{p,c}$ *and* $c^2/4 + 2\beta\mathcal{V}^2(t) < 1$ *for all* $t \in [0, T]$ *and some* $p \in [1,\infty]$

Then, Equation (50) has exactly two positive T-periodic solutions ψ_1 *and* ψ_2 *such that*

$$\eta_1 \leq \psi_1(t) \leq \zeta_1 \leq \frac{1}{3} \leq \zeta_2 \leq \psi_2(t) \leq \eta_2, \quad \forall t \in [0, T], \tag{51}$$

with η_i, ζ_i, $i = 1, 2$ *the corresponding solutions of*

$$x(1-x)^2 = \beta\mathcal{V}^2_{\min} \quad \text{and} \quad x(1-x)^2 = \beta\mathcal{V}^2_{\max}, \tag{52}$$

in $]0, 1[$, *respectively. Moreover,* ψ_1 *is asymptotically stable, and* ψ_2 *is unstable.*

Additionally, if

$$\frac{c^2}{4} < \frac{1-3\zeta_1}{1-\zeta_1}, \tag{53}$$

then ψ_1 *is locally exponentially asymptotically stable with rate of exponential decay* $c/2$.

Proof. We divide the proof in 4 steps following the ideas in [9].

Step 1: Constant lower and upper solutions

Equation (50) is Equation (18) with

$$c(t, x) = c, \quad K(x) = x, \quad G(x) = (1-x)^2, \quad F(t) = \beta\mathcal{V}^2(t). \tag{54}$$

In such a case, $\phi(x) = K(x)G(x)$ is given by $\phi(x) = x(1-x)^2$. Direct computations prove that $\phi(x)$ is monotone increasing in $\widehat{I} =]0, 1/3[$ and monotone decreasing in $I =]1/3, 1[$ with isolated critical points at $x = 1/3$ and $x = 1$. Therefore, by the assumption (I) and Theorems 2 and 3, it follows directly that the solutions of

$$\begin{aligned} x(1-x)^2 &= F_{\min}, \\ x(1-x)^2 &= F_{\max}, \end{aligned} \tag{55}$$

provide constant upper solutions η_i, $i = 1, 2$ with $0 < \eta_1 < 1/3 < \eta_2 < 1$ and constant lower solutions ζ_i, $i = 1, 2$ with $0 < \zeta_1 < 1/3 < \zeta_2 < 1$, respectively. Moreover,

$$\eta_1 < \zeta_1 \leq \frac{1}{3} \leq \zeta_2 < \eta_2 < 1. \tag{56}$$

Step 2: Existence of periodic solutions

Applying Theorem 2, there exists at least one T-periodic solution ψ_2 of (50) such that

$$\zeta_2 \leq \psi_2(t) \leq \eta_2, \quad \forall t \in [0, T]. \tag{57}$$

In order to apply Theorem 15 (part A) on the set

$$\tilde{E} = \{(t, x) \in \mathbb{R} \times]0, 1/3] | \eta_1 \leq x \leq \zeta_1\}, \tag{58}$$

it is necessary to study the condition

$$\partial_x g(t, x) \leq a(t), \quad \forall (t, x) \in \tilde{E}, \tag{59}$$

for some function a such that $a \succ c^2/4$ and $a \in \Gamma_{p,c}$, some $p \in [1,\infty]$, where $g(t, x) = x - ((\beta\mathcal{V}^2(t))/((1-x)^2))$. Then, a direct computation shows that

$$\partial_x g(t, x) = 1 - \frac{2\beta\mathcal{V}^2(t)}{(1-x)^3} < 1 - 2\beta\mathcal{V}^2(t), \tag{60}$$

for all $(t, x) \in \mathbb{R} \times [0, 1[$. Let $a(t) := 1 - 2\beta\mathcal{V}^2(t)$, $t \in \mathbb{R}$. The previous inequality along with the assumptions *(I)* and

(II) imply the inequality (59) with $a > c^2/4$, and $a \in \Gamma_{p,c}$ for some $p \in [1,\infty]$. This proves the existence of at least one T-periodic solution ψ_1 of (50) which is asymptotically stable and such that

$$\eta_1 \le \psi_1(t) \le \zeta_1, \quad \forall t \in [0, T], \tag{61}$$

if the number of T-periodic solutions of (50) between η_1 and ζ_1 is finite.

Step 3: Multiplicity of periodic solutions

Assume that φ_1 and φ_2 are two different nontrivial T-periodic solutions of (50). Define $v(t) = \varphi_2(t) - \varphi_1(t)$, $t \in \mathbb{R}$. Then, v is a nontrivial T-periodic solution of the equation:

$$\ddot{v} + c\dot{v} + \tilde{a}(t)v = 0, \tag{62}$$

with $\tilde{a}(t) = \int_0^1 \partial_x g(t, \varphi_1(t) + m(\varphi_2(t) - \varphi_1(t)))dm$. By the inequality (60), the condition (II) and part 1 of Proposition 4, we conclude that $v(t) > 0$ or $v(t) < 0$ for all $t \in \mathbb{R}$. Therefore,

$$\varphi_2(t) < \varphi_1(t) \quad \text{or} \quad \varphi_1(t) < \varphi_2(t), \tag{63}$$

for all $t \in R$. Now, assume that there exists a third nontrivial T-periodic solution φ_3 of (50). The preceding arguments allow us to assume that

$$\varphi_1(t) < \varphi_2(t) < \varphi_3(t) \quad \forall t \in [0, T]. \tag{64}$$

In consequence, the nontrivial T-periodic functions $v_i(t) = \varphi_{i+1}(t) - \varphi_i(t)$, $i = 1, 2$ satisfy the equations:

$$\ddot{v}(t) + c\dot{v} + \tilde{a}_i(t)v = 0, \quad i = 1, 2, \tag{65}$$

where $\tilde{a}_i(t) = \int_0^1 \partial_x g(t, \varphi_i(t) + m(\varphi_{i+1}(t) - \varphi_i(t)))dm$. Since

$$\partial_x^2 g(t, x) = -\frac{6\beta\mathscr{V}^2(t)}{(1-x)^4}, \quad \forall (t, x) \in R \times]-\infty, 1[, \tag{66}$$

it follows that $a_2(t) < a_1(t)$ for all $t \in \mathbb{R}$. Therefore, by part 2 of Proposition 4, we reach a contradiction. This proves that there is at most two positive T-periodic solutions of (50).

To sum up, under the assumptions (I) and (II), Equation (50) has exactly two positive T-periodic solutions which are precisely the functions ψ_1 and ψ_2 satisfying

$$\eta_1 \le \psi_1(t) \le \zeta_1 \le \frac{1}{3} \le \zeta_2 \le \psi_2(t) \le \eta_2, \quad \forall t \in [0, T]. \tag{67}$$

Moreover, by Step 1, ψ_1 is asymptotically stable, and ψ_2 is unstable.

Step 4: Exponential stability

Finally, we want to apply Theorem 16. We need to find a lower bound of $\partial_x g(t, x)$ for all $(t, x) \in \tilde{E}$. Then, direct computations show that

$$\partial_x g(t, x) > 1 - \frac{2\beta\mathscr{V}^2(t)}{(1-\zeta_1)^3} > 1 - \frac{2\beta\mathscr{V}^2_{\max}}{(1-\zeta_1)^3} = \frac{1-3\zeta_1}{1-\zeta_1} > 0, \quad \forall (t, x) \in \tilde{E}. \tag{68}$$

Then, by (60) and the previous inequality, we have

$$0 < \frac{1-3\zeta_1}{1-\zeta_1} < \partial_x g(t, x) < 1 - 2\beta\mathscr{V}^2(t), \tag{69}$$

for all $(t, x) \in \tilde{E}$. Define $l(t) \coloneqq 1 - ((2\beta\mathscr{V}^2(t))/((1-\zeta_1)^3))$; therefore, $l \in C(\mathbb{R}/T\mathbb{Z})$. From (53), we can deduce that $\bar{l} > c^2/4$. Then, by Theorem 16, the T-periodic function ψ_1 is exponentially asymptotically stable with rate of exponential decay $c/2$. This completes the proof. □

Remark 6. Condition (53) can be replaced by

$$\bar{\mathscr{V}}^2 < \frac{(1-\zeta_2)^3}{2\beta}\left(1 - \frac{c^2}{4}\right), \tag{70}$$

where $\bar{\mathscr{V}}^2 = 1/T\int_0^T \mathscr{V}^2(t)dt$.

Additionally, respect to the results over the Nathanson model with constant damping given in [7, 9], the criteria that we illustrated over the function $\partial_x g(t, x) - c^2/4$ in Theorem 5 have the advantage that considers the L^p norms ($p \in [1,\infty]$), and not over the supremum of its range. In consequence, Theorem 5 leads to a refinement of the results founded in [7, 9].

Example 1. The values required to determine the existence of ψ_1 are $F_{\max}$, $F_{\min,}$ and T. In order to test different combinations of parameters, let

$$\frac{2}{27} = \frac{F_{\max} + F_{\min}}{2} \qquad A = \frac{F_{\max} - F_{\min}}{2}. \tag{71}$$

If $A \le 2/27$, then $F_{\max} \le 4/27$. Figure 3 shows the combination of parameters that allowed to prove the existence and exponentially asymptotical stability, only existence, or did not allow to prove the existence of ψ_1 with Theorem 5.

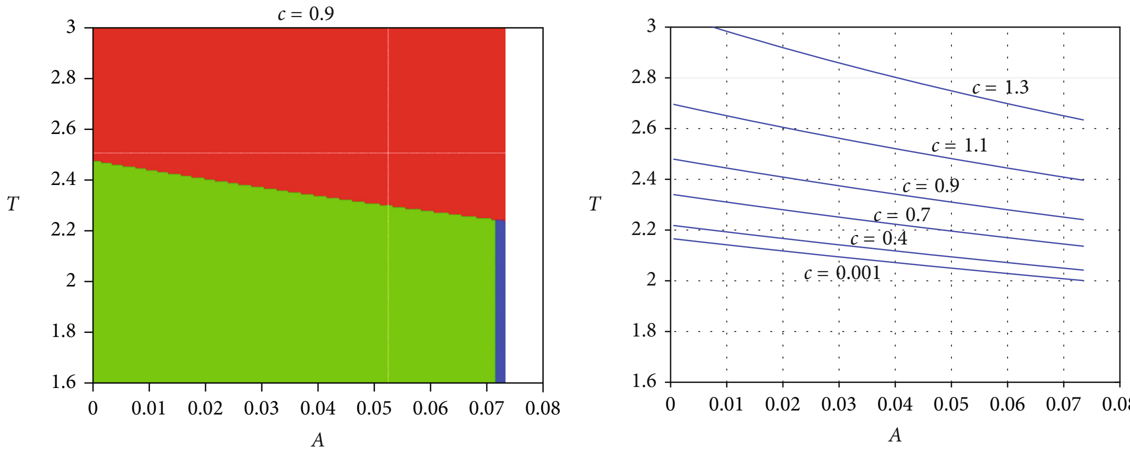
To test the exponentially asymptotical stability property of one of the combination of parameters, let $c = 0.9$, $A = 0.07$, $T = 2$, and $\mathscr{V}(t) = 0.1579 \cos(2\pi t/T) + 0.2217$. Following the results depicted in Figure 3, with that combination, it is possible to prove the existence of ψ_1, and it is exponentially asymptotically stable.

Let the error between any other solution φ of (50) and the periodic solution ψ_1 be defined as

$$\varepsilon(t) = |\varphi(t) - \psi_1(t)| + |\dot{\varphi}(t) - \dot{\psi}_1(t)|. \tag{72}$$

Following the results of Theorem 5, ψ_1 is locally exponentially asymptotically stable with rate of exponential decay $c/2$. Then, there exists an adequate positive value d such that

$$\hat{\varepsilon}(n) \coloneqq \ln(\varepsilon(nT)) \le \ln d + \frac{cT}{2}n \eqqcolon \hat{v}(n). \tag{73}$$



(a) (b)

FIGURE 3: (a) With $c = 0.9$. The values of A and T that satisfied the conditions of Theorem 7 for existence and exponentially asymptotical stability are plotted in the green area. In blue, only where the conditions of Theorem 7 for the existence are satisfied and in red where they are not. (b) Boundary between satisfying and not satisfying the conditions for existence of Theorem 7 for different values of c.

For this example, Figure 4 depicts $\widehat{v}(n)$ in blue for 10 different solutions of (50) and $\widehat{v}$ in red with $\ln d = -2$.

4. On the Squeeze Film Damping Case

In this section, we present some analytic and numerical validation on the existence and linear stability of periodic solutions for the Nathanson model under the squeeze film damping effect. Therefore, the boundary value problem

$$\ddot{x} + \frac{\kappa}{(1-x)^3}\dot{x} + x = \frac{\beta\mathscr{V}^2(t)}{(1-x)^2}, \tag{74}$$

with $\kappa = \gamma/d^3 > 0$ and $x \in]-\infty,1[$.

Our approach to the existence of solutions of (74) is again throughout the Theorems 2 and 3. Therefore, from the notation of those theorems,

$$\begin{aligned} c(t,x) &= \frac{\kappa}{(1-x)^3}, \\ G(x) &= (1-x)^2, \\ K(x) &= x, \\ F(t) &= \beta\mathscr{V}^2(t), \end{aligned} \tag{75}$$

with $G(x) > 0$ for all x in $]-\infty, 1[$. As before, the function ϕ is given by

$$\phi(x) = x(1-x)^2, \quad x \in R. \tag{76}$$

Recall that ϕ is monotone non-decreasing in $]-\infty,1/3]$ and monotone non-increasing in the interval $[1/3, 1[$ with a unique local maximum at $x = 1/3$.

Theorem 7. *Assume that $0 < \mathscr{V}^2_{\min} \le \mathscr{V}^2_{\max} \le 4/27\beta$. Then, there exists ζ_i, η_i with $i = 1, 2$ such that*

$$0 < \eta_1 < \zeta_1 \le 1/3, \quad \text{and} \quad 1/3 \le \zeta_2 < \eta_2 < 1, \tag{77}$$

where ζ_i and η_i satisfy

$$\phi(\eta_i) = \beta\mathscr{V}^2_{\min}, \quad \phi(\zeta_i) = \beta\mathscr{V}^2_{\max}, \quad i = 1, 2. \tag{78}$$

Furthermore,

(1) The problem (74) admits a T-periodic solution ψ_2 such that

$$\zeta_2 \le \psi_2(t) \le \eta_2 \quad \forall t \in [0, T], \tag{79}$$

(2) Let

$$N = \frac{\kappa}{(1-\zeta_1)^3}, \quad \widehat{a} = \max\{N, \zeta_1 - \eta_1\}, \quad M = \frac{3\kappa R}{(1-\zeta_1)^4} + \frac{1-3\eta_1}{1-\eta_1}, \tag{80}$$

with $R > 0$ the unique solution of

$$R - \ln(R+1) = \widehat{a}(\zeta_1 - \eta_1). \tag{81}$$

Assume that the following conditions hold:

$$M \le (\pi/T)^2 \quad \text{and} \quad N \le H(L_*) = \frac{L_* - M}{\sqrt{L_*}} \cot\left(\frac{T\sqrt{L_*}}{2}\right), \tag{82}$$

with $L_* \in [M, (\pi/T)^2]$ the unique value that satisfies $H'(L) = 0$ which is equivalent to

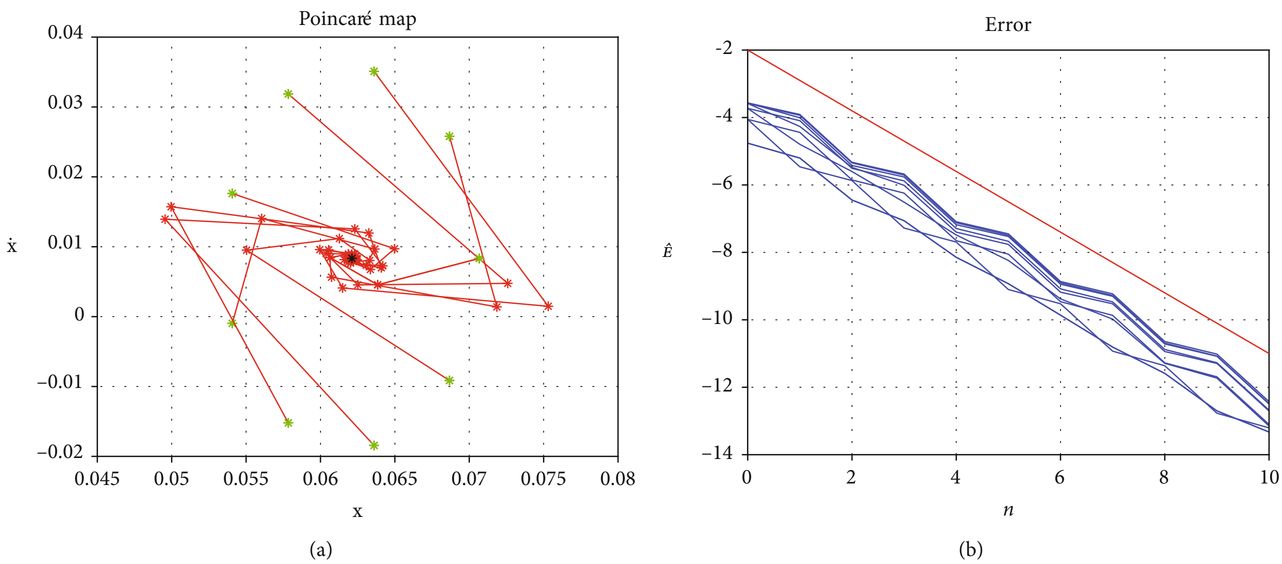


FIGURE 4: (a) Poincaré map of nine different solutions of (50). The periodic solution is painted in black, while the others are in red with their initial point in green. (b) Plot in blue of $\widehat{\varepsilon}$ for nine different solutions of the differential equation (50). Plot in red of (73) which shows that it is an upper bound of $\widehat{\varepsilon}$.

$$\sin\left(T\sqrt{L_*}\right) = T\sqrt{L_*}\left(\frac{L_* - M}{L_* + M}\right). \tag{83}$$

Then, there exists a periodic solution ψ_1 of (74) such that

$$\eta_1 \le \psi_1(t) \le \zeta_1, \quad \forall t \in [0, T]. \tag{84}$$

Proof. Under the existence of ζ_i, η_i follows directly from the monotonicity properties of the function ϕ in each of the considered intervals.

For part *1* by Theorem 2 applied over $I = [1/3, 1[$, the problem (74) admits constant lower and upper solutions that correspond to ζ_2 and η_2, respectively. Additionally, there exists a solution ψ_2 of (6) such that

$$\zeta_2 \le \psi_2(t) \le \eta_2, \quad \forall t \in [0, T]. \tag{85}$$

For part 2 by Theorem 3 (part (‡)) applied over $\widehat{I} = [0, 1/3]$, the problem (74) admits constant lower and upper solutions that correspond to ζ_1 and η_1, respectively. Furthermore, since $\phi(\eta_1) = \beta\mathcal{V}^2_{\min}$, we have

$$\begin{aligned} x - \frac{\beta\mathcal{V}^2(t)}{(1-x)^2} &\le \zeta_1 - \frac{\beta\mathcal{V}^2_{\min}}{(1-\eta_1)^2} \\ &= \zeta_1 - \eta_1, \quad \forall (t, x) \in [0, T] \times [\eta_1, \zeta_1]. \end{aligned} \tag{86}$$

Define,

$$\begin{aligned} N &:= \max_{R\times[\eta_1,\zeta_1]} \left|\frac{\kappa}{(1-x)^3}\right| = \frac{\kappa}{(1-\zeta_1)^3}, \\ \widehat{a} &:= \max\{N, \zeta_1 - \eta_1\}. \end{aligned} \tag{87}$$

From here, we are able to find a unique positive constant R that satisfies

$$R - \ln(R+1) = \widehat{a}(\zeta_1 - \eta_1). \tag{88}$$

Now consider the set

$$W = \{(t, x, y) : t \in [0, T], x \in [\eta_1, \zeta_1], y \in [-R, R]\}. \tag{89}$$

In order to fulfill the all the conditions in (‡‡) in Theorem 3, for the value $M > 0$, it is necessary to show that

$$\begin{aligned} &\max_W \left| \frac{\partial}{\partial x}\left(\frac{\kappa}{(1-x)^3}\dot{x} + x - \frac{\beta\mathcal{V}^2(t)}{(1-x)^2}\right)\right| \\ &\le \frac{3\kappa}{(1-\zeta_1)^4} R + \frac{1-3\eta_1}{1-\eta_1} = M. \end{aligned} \tag{90}$$

Indeed, notice that

$$\begin{aligned} &\max_W \left| \frac{\partial}{\partial x}\left(\frac{\kappa}{(1-x)^3}\dot{x} + x - \frac{\beta\mathcal{V}^2(t)}{(1-x)^2}\right)\right| \\ &= \max_W \left| \frac{3\kappa}{(1-x)^4}\dot{x} + 1 - \frac{2\beta\mathcal{V}^2(t)}{(1-x)^3}\right| \\ &\le \max_W \left| \frac{3\kappa}{(1-x)^4}\right| R + \max_W \left| 1 - \frac{2\beta\mathcal{V}^2(t)}{(1-x)^3}\right|. \end{aligned} \tag{91}$$

The function $3\kappa/(1-x)^4$ is monotone increasing in the domain $]-\infty, 1[$; then

$$\frac{3\kappa}{(1-x)^4} \le \frac{3\kappa}{(1-\zeta_1)^4}, \forall x \in [\eta_1, \zeta_1]. \tag{92}$$

Additionally,

$$1-\frac{2F_{\max}}{(1-\zeta_1)^3}\le 1-\frac{2\beta\mathcal{V}^2(t)}{(1-x)^3}\le 1-\frac{2F_{\min}}{(1-\eta_1)^3}. \quad (93)$$

Since $\phi(\zeta_1)=F_{\max}$ and $\phi(\eta_1)=F_{\min}$, then

$$\begin{aligned}0 &\le \frac{1-3\zeta_1}{(1-\zeta_1)}=1-\frac{2\zeta_1}{(1-\zeta_1)}\le 1-\frac{2\beta\mathcal{V}^2(t)}{(1-x)^3}\\ &\le 1-\frac{\eta_1}{(1-\eta_1)}=\frac{1-3\eta_1}{(1-\eta_1)},\end{aligned} \quad (94)$$

because $\zeta_1\in\,]0,1/3]$. Hence,

$$\max_W\left|\frac{\partial}{\partial x}\left(\frac{\kappa}{(1-x)^3}\dot{x}+x-\frac{\beta\mathcal{V}^2(t)}{(1-x)^2}\right)\right|\le M, \quad (95)$$

Then, by Theorem 3, there exists a solution ψ_1 of (74) such that

$$\eta_1\le\psi_1(t)\le\zeta_1,\quad \forall t\in[0,T]. \quad (96)$$

□

Notice that it is fairly simply to prove the existence of ψ_2. However, to prove the existence of ψ_1 requires intermediate computations of values that depend not only on parameters such as κ, β and T but also in other intermediate values. Therefore, here we present the steps that allow us to prove the existence of ψ_1 for a set of parameters $(\kappa, T, F(t,T))$.

(1) Find $F_{\min}$ and $F_{\max}$.

(2) If $0<F_{\min}\le F_{\max}\le 4/27$, find ζ_1, η_1 in $]0,1/3]$ such that

$$\begin{aligned}\phi(\zeta_1)&=F_{\max},\\ \phi(\eta_1)&=F_{\min}.\end{aligned} \quad (97)$$

(3) Compute N and $\hat{a}$.

(4) Find the positive value $R>0$ such that

$$R-\ln(R+1)=\hat{a}(\zeta_1-\eta_1). \quad (98)$$

(5) Compute the value M given in Theorem 7

(6) If $M\le(\pi/T)^2$, find L_* such that

$$\sin\left(T\sqrt{L_*}\right)=T\sqrt{L_*}\left(\frac{L_*-M}{L_*+M}\right). \quad (99)$$

(7) Finally, if $N\le H(L_*)$ we can conclude the existence of ψ_1.

If any of the conditions given in the Steps (3), (12) and (16) are not satisfied, then we cannot use Theorem 7 to conclude the existence of ψ_1. Example 2 is based on these steps for different parameters $(\kappa, T, F(t,T))$.

4.1. Linear Stability. As a final contribution of this work, we provide some results about the linear stability of any periodic solution ψ_1 of (74) located in $[\eta_1,\zeta_1]$. Our approach is based on the analysis of the linear equation corresponding to the given periodic solution. Direct computations shows that the associated Hill's equation is given by

$$\begin{aligned}&\ddot{w}+\frac{\kappa}{(1-\psi_1(t))^3}\dot{w}+q(t)w=0,\ \text{with } q(t)\\ &=\frac{3\kappa}{(1-\psi_1(t))^4}\dot{\psi}_1(t)+1-\frac{2\beta\mathcal{V}^2(t)}{(1-\psi_1(t))^3}.\end{aligned} \quad (100)$$

Proposition 8. *Under the assumption of Theorem 7, then any possible periodic solution ψ_1 of (74) located $[\eta_1,\zeta_1]$ is locally asymptotically stable if*

$$\frac{1-3\zeta_1}{1-\zeta_1}>\frac{\kappa}{2(1-\zeta_1)^4}\left(3R+\frac{\kappa}{2(1-\zeta_1)^2}\right), \quad (101)$$

$$\left(\frac{\pi}{T}\right)^2\ge\frac{1-3\eta_1}{1-\eta_1}+\frac{\kappa}{2(1-\zeta_1)^4}\left(3R-\frac{\kappa(1-\zeta_1)^4}{2(1-\eta_1)^6}\right). \quad (102)$$

Proof. Under the change of variables,

$$w(t)=u(t)v(t),\quad v(t)=e^{-P(t)},\quad P(t)=\frac{1}{2}\int\frac{\kappa}{(1-\psi_1(t))^3}dt, \quad (103)$$

Equation (100) can be written as

$$\ddot{u}+Q(t)u=0, \quad (104)$$

with $Q(t)$ given by

$$Q(t)=\frac{3\kappa}{2(1-\psi_1(t))^4}\dot{\psi}_1(t)-\frac{\kappa^2}{4(1-\psi_1(t))^6}+1-\frac{2\beta\mathcal{V}^2(t)}{(1-\psi_1(t))^3}. \quad (105)$$

Notice that if any solution $u(t)$ of (104) is bounded, then any solution $w(t)$ of (100) converges to zero. Indeed, notice that

$$\eta_1\le\psi_1(t)\le\zeta_1\quad\text{and}\quad|\dot{\psi}_1(t)|<R,\quad\forall t\in R, \quad (106)$$

with $0<\eta_1<\zeta_1\le 1/3$. Direct computations show that

$$\frac{\kappa t}{2(1-\eta_1)^3}\le P(t)\le\frac{\kappa t}{2(1-\zeta_1)^3},\quad\forall t\in R, \quad (107)$$

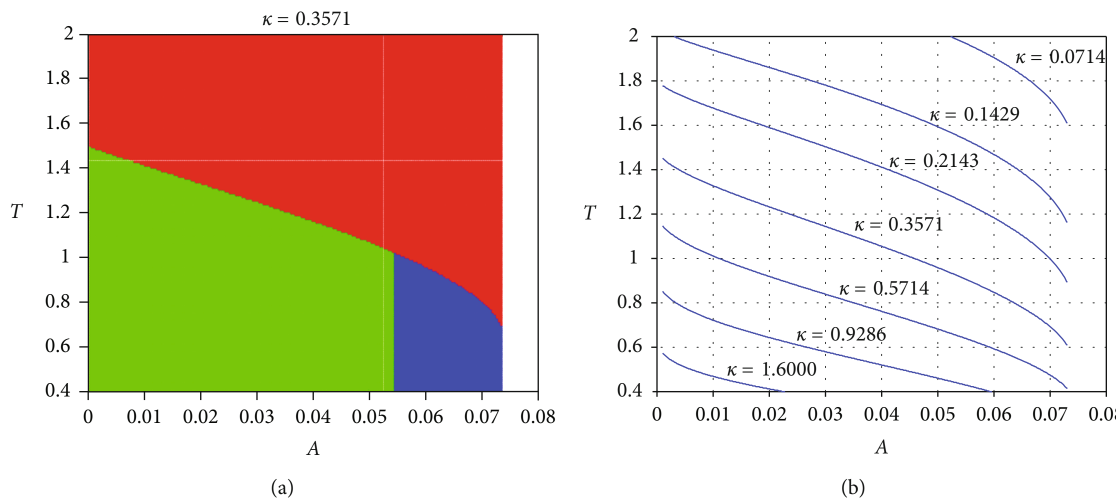


FIGURE 5: (a) Areas in the plane A and T where the conditions for the existence are satisfied (blue) and where they are not (red) with $\kappa = 0.3674$. (b) Boundary between satisfying and not satisfying the conditions of Theorem 7 for different values of κ.

then, $v(t) \longrightarrow 0$ if $t \longrightarrow \infty$. Moreover,

$$\begin{aligned} -\frac{3\kappa R}{2(1-\zeta_1)^4} &\le \frac{3\kappa\dot{\psi}_1(t)}{2(1-\psi_1(t))^4} \le \frac{3\kappa R}{2(1-\zeta_1)^4}, \\ -\frac{\kappa^2}{4(1-\zeta_1)^6} &\le -\frac{\kappa^2}{4(1-\psi_1(t))^6} \le -\frac{\kappa^2}{4(1-\eta_1)^6}, \end{aligned} \tag{108}$$

and also,

$$0 < \frac{1-3\zeta_1}{1-\zeta_1} \le 1 - \frac{2\beta\mathcal{V}^2(t)}{(1-\psi_1(t))^3} \le \frac{1-3\eta_1}{1-\eta_1}, \tag{109}$$

for all $t \in R$. From here, we deduce the following:

$$\begin{aligned} \frac{1-3\zeta_1}{1-\zeta_1} - \frac{\kappa}{2(1-\zeta_1)^4}\left(3R + \frac{\kappa}{2(1-\zeta_1)^3}\right) &\le Q(t), \quad \forall t \in R, \\ Q(t) \le \frac{1-3\eta_1}{1-\eta_1} + \frac{\kappa}{2(1-\zeta_1)^4}\left(3R - \frac{\kappa(1-\zeta_1)^4}{2(1-\eta_1)^6}\right), &\quad \forall t \in R. \end{aligned} \tag{110}$$

From the assumptions (101), (102) follows directly

$$0 < Q(t) \le \left(\frac{\pi}{T}\right)^2, \quad \forall t \in R. \tag{111}$$

□

From the previous computations and Theorem 18, we deduce that ψ_1 is locally asymptotically stable.

Remark 9. To arrive to the conclusion of Proposition 8, we used the results of Theorem 18 that resemble the Lyapunov-Zukovskii's criteria for stability. We could also arrive to the linear stability of the periodic solution ψ_1 of (74) by asking the following conditions:

$$\begin{aligned} \frac{1-3\zeta_1}{1-\zeta_1} &> \frac{\gamma}{2(1-\zeta_1)^4}\left(3R + \frac{\gamma}{2(1-\zeta_1)^3}\right), \\ \left(\frac{2}{T}\right)^2 &\ge \frac{1-3\eta_1}{1-\eta_1} - \frac{\gamma^2}{4(1-\eta_1)^6}. \end{aligned} \tag{112}$$

We arrive to this conclusion by Theorem 18 with the conditions that resemble the Lyapunov-Borg's criteria for stability.

Example 2. Following the same definition (71) from Example 1 for parameter A, in Figure 5, we display the combination of parameters that allowed or did not allow to prove the existence of ψ_1 for Equation (74) by means of Theorem 7. If the existence is guaranteed, we attempt to prove with Proposition 8 that ψ_1 is asymptotically stable. The numerical results of Figure 5 indicate that if we take low values of κ, it seems to be more possible to prove the existence of ψ_1 with Theorem 7.

5. Conclusions and Outlook

In this work, we have rigorously shown the existence of at least two positive periodic solutions for the Nathanson model under squeeze damping forces, as a direct consequence of a periodic voltage load with a maximum value under $V_0 = \sqrt{4/27\beta}$ known as *pull-in voltage*, which is precisely the critical voltage associated with the pull-in phenomenon. The location and the L^∞-norm of these solutions are provided, and we have also been able to give an algorithm to show the numerical conditions of Theorem 7. We also note that these analytical and numerical computations could be reproduced for other types of MEMS devices and new algorithms could be developed to show the existence and linear stability. Future work could explore the upper boundaries of the number of positive periodic solutions, and more interestingly, the possibility of periodic oscillations with a negative or nonconstant sign, for example, for the comb-drive model under squeeze damping force. We also revisited the Nathanson model under linear damping. Although this problem is considered in [7–9], Theorem 5 improves the existence and stability results as a result of appropriate conditions over the voltage load and the viscous damping coefficient c, providing new and significant knowledge of the dynamics of this model.

Appendix

A.1. The Upper and Lower Solution Method

Consider the boundary value problem

$$\ddot{x} + f(t, x, \dot{x}) = 0, \quad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T) = 0. \tag{A.1}$$

where $D \subset R \times]l_1, l_2[\times R$ is an open connected set with $-\infty \leq l_1 < l_2 \leq \infty$ and $f : D \longrightarrow R$, is a continuous function.We have that there exists a constant $R > 0$ such that $\|\dot{u}\|_\infty < R$.

Definition 10 Lower and upper solution. A function $\zeta \in C^2(]0, T[) \cap C^1([0, T])$ is called lower solution of (A.1) relative to the domain D if

$$\ddot{\zeta}(t) + f\left(t, \zeta(t), \dot{\zeta}(t)\right) \geq 0 \quad \text{for all} \quad t \in]0, T[. \tag{A.2}$$

$\zeta(t) \in]l_1, l_2[$ for all $t \in]0, T[$, and $\zeta(0) = \zeta(T)$, $\dot{\zeta}(0) \geq \dot{\zeta}(T)$.

A function $\eta \in C^2(]0, T[) \cap C^1([0, T])$ is called upper solutions of (A.1) relative to the domain D if all the previous conditions hold with the reverse inequalities.

The lower and upper solutions are *well-ordered* if

$$\zeta(t) \leq \eta(t), \quad \forall t \in [0, T]. \tag{A.3}$$

Meanwhile, the lower and upper solutions are in the *reversed order* if

$$\eta(t) \leq \zeta(t), \quad \forall t \in [0, T]. \tag{A.4}$$

Given $\zeta, \eta \in C([0, T])$ such that $\zeta \leq \eta$, define the set

$$E_{\zeta,\eta} := \left\{(t, x, y) \in [0, T] \times \mathbb{R}^2 \middle| \zeta(t) \leq x \leq \eta(t)\right\}. \tag{A.5}$$

Definition 11 Nagumo condition. Let $f : E_{\zeta,\eta} \longrightarrow \mathbb{R}$ continuous. The function f satisfies the Nagumo condition on $E_{\zeta,\eta}$ if there exists a positive continuous function $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that

$$\int_0^\infty \frac{s}{\rho(s)} ds = \infty \quad \text{and} \quad |f(t, x, y)| \leq \rho(|y|), \tag{A.6}$$

for all $(t, x, y) \in E_{\zeta,\eta}$.

Theorem 12. *Let ζ and η be lower and upper solution of (10) such that $\zeta(t) \leq \eta(t)$ for all t in $[0, T]$. If f satisfies the Nagumo condition in $E_{\zeta,\eta}$, then the problem (5) has at least one solution $u \in C^2([0, T])$ such that*

$$\zeta(t) \leq u(t) \leq \eta(t), \quad \forall t \in [0, T]. \tag{A.7}$$

Theorem 13. *Let $f : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ a continuous function with a continuous partial derivative on the third variable such that $|\partial_y f(t, x, y)| \leq N$ in D where $N \geq 0$ and D is a connected domain. Consider*

$$E := [0, T] \times [\alpha, \beta] \times \mathbb{R} \subset D, \tag{A.8}$$

where $T, \alpha, \beta \in R$. Then, for any solution $u(t) \in [\alpha, \beta]$ for all t in $[0, T]$ of any of the following problems:

$$\ddot{x} \geq f(t, x, \dot{x}), \quad x(0) = x(T), \tag{A.9}$$

$$\ddot{x} \leq f(t, x, \dot{x}), \quad x(0) = x(T), \tag{A.10}$$

$$\ddot{x} = f(t, x, \dot{x}), \quad x(0) = x(T). \tag{A.11}$$

For sake of completeness, we will make the proof for the boundary value problem (A.9) following the ideas of the proof of Proposition I-4.5 in [12]. The procedures for (A.10) and (A.11) are similar.

Proof. Since $|\partial_y f(t, x, y)| \leq N$, then by the mean value theorem, we have that

$$-\rho(|\vartheta|) \leq f(t, x, y) \leq \rho(|\vartheta|), \tag{A.12}$$

with $\rho(\vartheta) = N\vartheta + \max |f(t, x, 0)|$. Notice that

$$\int_0^\infty \frac{s}{\rho(s)} ds = \infty. \tag{A.13}$$

Define $R > 0$ such that

$$\int_0^R \frac{s}{\rho(s)} ds > \beta - \alpha. \tag{A.14}$$

Let u_1 be a solution of (A.9) such that $u_1(t) \in [\alpha, \beta]$ for all t in $[0, T]$. Suppose that there exists $t_1 \in [0, T]$ such that $\dot{u}_1(t_1) = R$. Let $t_0 \in [0, T]$ be the closest zero to t_1; then, $\dot{u}_1(t) > 0$ for all t in $]\min\{t_0, t_1\}, \max\{t_0, t_1\}[$. Notice that

$$\int_0^R \frac{s}{\rho(s)} ds = \int_{\dot{u}_1(t_0)}^{\dot{u}_1(t_1)} \frac{s}{\rho(s)} ds = \int_{t_1}^{t_0} \frac{\dot{u}_1 \ddot{u}_1}{-\rho(|\dot{u}_1|)} dt, \tag{A.15}$$

since

$$-\rho(|\dot{u}_1|) \leq f(t, u_1, \dot{u}_1) \leq \ddot{u}_1, \tag{A.16}$$

then, $1 \geq \ddot{u}_1/(-\rho(|\dot{u}_1|))$ and

$$\int_{t_1}^{t_0} \frac{\dot{u}_1 \ddot{u}_1}{-\rho(|\dot{u}_1|)} dt \leq \int_{t_1}^{t_0} \dot{u}_1 dt = u_1(t_0) - u_1(t_1) \leq \beta - \alpha < \int_0^R \frac{s}{\rho(s)} ds, \quad (A.17)$$

hence,

$$\int_0^R \frac{s}{\rho(s)} ds < \int_0^R \frac{s}{\rho(s)} ds, \quad (A.18)$$

which is a contradiction. □

Suppose that there exists $t_1 \in [0, T]$ such that $\dot{u}_1(t_1) = -R$. Let $t_0 \in [0, T]$ be the closest zero to t_1; then, $\dot{u}_1(t) < 0$ for all t in $]\min\{t_0, t_1\}, \max\{t_0, t_1\}[$. Notice that

$$\int_0^R \frac{s}{\rho(s)} ds = \int_{-\dot{u}_1(t_0)}^{-\dot{u}_1(t_1)} \frac{s}{\rho(s)} ds = \int_{t_1}^{t_0} \frac{(-\dot{u}_1)(-\ddot{u}_1)}{-\rho(|\dot{u}_1|)} dt, \quad (A.19)$$

since

$$-\rho(|\dot{u}_1|) \leq f(t, u_1, \dot{u}_1) \leq \ddot{u}_1, \quad (A.20)$$

then $1 \geq \ddot{u}_1/(-\rho(|\dot{u}_1|))$ and

$$\int_{t_1}^{t_0} \frac{\dot{u}_1 \ddot{u}_1}{-\rho(|\dot{u}_1|)} dt \leq \int_{t_1}^{t_0} \dot{u}_1 dt = u_1(t_0) - u_1(t_1) \leq \beta - \alpha < \int_0^R \frac{s}{\rho(s)} ds, \quad (A.21)$$

hence

$$\int_0^R \frac{s}{\rho(s)} ds < \int_0^R \frac{s}{\rho(s)} ds, \quad (A.22)$$

which is a contradiction.

Remark 14. Notice that an alternative definition for the function ρ is

$$\rho(\vartheta) = \hat{a}(\vartheta + 1), \quad (A.23)$$

with $\hat{a} = \max\{N, \max|f(t, x, 0)|\}$. This allows us to compute R as the positive real value that satisfies the inequality.

$$\int_0^R \frac{s}{\rho(s)} ds = \frac{1}{\hat{\alpha}}(R - \ln(R+1)) \geq \beta - \alpha. \quad (A.24)$$

In particular, we can select R such that $R - \ln(R+1) = \hat{a}(\beta - \alpha)$.

A.2. Multiplicity and Stability of Periodic Solution for Duffing Equations

We finish this section showing some results that provide a connection between lower and upper solution method and the multiplicity and stability of periodic solutions of the Duffing equation:

$$\ddot{x} + c\dot{x} + g(t, x) = 0, \quad (A.25)$$

where $c > 0$ and $g : \mathbb{R} \times]l_1, l_2[\longrightarrow \mathbb{R}$, $-\infty \leq l_1 < l_2 \leq \infty$, a continuous function, T-periodic with respect to t and having a continuous partial derivative with respect to x. Consider the linear differential operator:

$$L_a : \mathscr{W} \longrightarrow L^1(0, T) \\ \omega \longrightarrow L_a[\omega] = \ddot{\omega} + c\dot{\omega} + a(t)\omega, \quad (A.26)$$

where

$$\mathscr{W} = \{\omega \in W^{2,1}(0, T) : \omega(0) = \omega(T), \quad \dot{\omega}(0) = \dot{\omega}(T)\}, \quad (A.27)$$

c is a positive constant and $a \in \Gamma_{p,c}$, with

$$\Gamma_{p,c} = \left\{a \in L^p(0, T) : \left\|(a - c^2/4)_+\right\|_{L^p} < K(2p_*)\right\}, \quad (A.28)$$

for some $p \in [1, \infty]$ and $p_* = p/(p-1)$. Here, $K(q)$ is the best Sobolev constant in the following inequality:

$$C\|u\|_{L^q}^2 \leq \|\dot{u}\|_{L^2}^2, \quad \forall u \in H_0^1([0, T]). \quad (A.29)$$

Explicitly (see [17]),

$$K(q) = \begin{cases} \dfrac{2\pi}{qT^{1+2/q}} \left(\dfrac{2}{2+q}\right)^{1-2/q} \left(\dfrac{\Gamma(1/q)}{\Gamma(1/2 + (1/q))}\right)^2, & 1 \leq q < \infty, \\ 4/T, & q = \infty. \end{cases} \quad (A.30)$$

(A) If $\zeta > \eta$ and for some $1 \leq p \leq \infty$, there exists $a \in \Gamma_{p,c}$ with $a \succ c^2/4$ and verifying

$$g_x(t, x) \leq a(t) \quad \text{a.e.} \quad \forall x \in [\eta(t), \zeta(t)], \quad (A.32)$$

Theorem 15. *Let $g : R \times]l_1, l_2[\longrightarrow R$, $-\infty \leq l_1 < l_2 \leq \infty$ a continuous function, T-periodic with respect to t and having a continuous partial derivative with respect to x. Assume that ζ and η are a couple of lower and upper solutions, respectively, of the boundary value problem:*

$$\ddot{x} + c\dot{x} + g(t, x) = 0, \quad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \quad (A.31)$$

with $c > 0$ and $\zeta(t), \eta(t) \in]l_1, l_2[$ for all $t \in R$.

Then, (A.31) has at least an asymptotically stable solution (which is T-periodic) ψ, such that

$$\eta(t) < \psi(t) < \zeta(t), \quad \forall t \in R, \quad (A.33)$$

provided that the number of T-periodic solutions between η and ζ is finite.

(B) If $\zeta < \eta$, then (A.31) has at least an unstable solution (which is T-periodic) φ such that

$$\zeta(t) < \varphi(t) < \eta(t), \quad \forall t \in R, \tag{A.34}$$

provided that the number of T-periodic solutions between ζ and η is finite.

Proof. For part (A), the proof can be found in [15, 18], and for part (B) the proof can be found in [19]. □

Another and more accurate results about the stability of periodic solutions of Duffing equations like (A.31) are found in [14, 16] and indicate the following.

Theorem 16. *Assume that $g_x(t,x)$ exists and satisfies*

$$l(t) \le g_x(t,x) \le u(t), \quad \forall (t,x) \in [0,T] \times R, \tag{A.35}$$

where l and u are T-periodic functions such that $\bar{l} > c^2/4$ and $u \in \Gamma_{p,c}$ for some $p \in [1,\infty]$. Then (A.31) has a unique and locally exponentially asymptotical stable T-periodic solution $x_0(t)$. And the rate of exponential decay of $x_0(t)$ is $c/2$.

Remark 17. Following the lines of the proof of Theorem 16 given in [14, 16] it is easy to deduce that the exponential decay of $x_0(t)$ only requires that

$$l(t) \le g_x(t, x_0(t)) \le u(t), \quad \forall t \in \mathbb{R}, \tag{A.36}$$

This follows if we are able to located $x_0(t)$ for all $t \in R$ and compute a lower bound of $g_x(t, x_0(t))$ for all $t \in \mathbb{R}$.

Theorem 18 Stability test for Hill's equations. *Let Q be a T-periodic function such that*

$$Q \not\equiv 0, \quad Q \in L^1(0,T), \quad \int_0^T Q(t)dt > 0. \tag{A.37}$$

Assume that $Q \in L^p(0,T)$ for some $p \in [1,\infty]$, the Hill's equation:

$$\ddot{u} + Q(t)u = 0, \tag{A.38}$$

is stable (elliptic) when

$$\|Q_+\|_{L^\alpha} < K(2\alpha_*) \quad if \quad 1 < \alpha \le \infty, \tag{A.39}$$

or

$$\|Q_+\|_{L^\alpha} \le K(\infty) = \frac{4}{T} \quad if \quad \alpha = 1, \tag{A.40}$$

Furthermore, the upper bounds $K(2\alpha_*)$ for $\|Q_+\|_{L^\alpha}$ are the best possible.

If $\alpha = 1$, $Q(t) \ge 0$ (i.e., $Q(t) > 0$ for all $t \in R$ on a subset of positive measure), Theorem 18 establishes that

$$\|Q_+\|_{L^1} = \int_0^T Q(t)dt \le \frac{4}{T}, \tag{A.41}$$

then, (A.38) is elliptic, which corresponds to the Lyapunov-Borg's stability criteria (see [20]). If $\alpha = \infty$, Theorem 18 established that if $Q \in L^1(\mathbb{R} \setminus T\mathbb{Z})$ (i.e., Q is T-periodic and $Q \in L^1(0,T)$) such that

$$Q \not\equiv 0, \quad 0 < \int_0^T Q(t)dt, \quad \|Q_+\|_{L^\infty} < K(2) = \left(\frac{\pi}{T}\right)^2, \tag{A.42}$$

then (A.38) is elliptic, which corresponds to the Lyapunov-Zukovskii's stability criteria. The proof of Theorem 18 can be found in [21, 22].

Data Availability

No data is used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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