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Research Article

A Fractional Model for the Dynamics of Smoking Tobacco Using Caputo–Fabrizio Derivative

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In this paper, we propose a Caputo–Fabrizio fractional derivative mathematical model consisting of smoker, people exposed to secondhand smoker, people exposed to thirdhand smoker, and quitters. Secondhand smoke exposure consists of an unintentional inhalation of smoke that occurs close to people smoking and/or in indoor environments where tobacco was recently used, and thirdhand smoke consists of pollutants that remain on surfaces and in dust after tobacco has been smoked, are reemitted into the gas phase, or react with other compounds in the environment to form secondary pollutants. The solution of the proposed model, which is carried out using a fixed-point theorem and an iterative method, exists and is unique. Furthermore, the model is biologically meaningful, that is, positive and bounded. The reproduction number R_0 is determined from the model. If $R_0 < 1$, the smoking-free equilibrium point is asymptotically stable, and if $R_0 > 1$, the smoking-free equilibrium point is unstable. The results confirm that the smoking-free equilibrium point becomes increasingly stable as the fractional order is increased. Numerical simulations are performed using a three-step Adams-Moulton predictor-corrector method for a range of fractional orders to show the effects of varying the fractional order and to support the theoretical results.

1. Introduction

Smoking tobacco is epidemic and one of the biggest public health problems in the world nowadays [1] because rates of cigarette smoking are still too high [2]. It causes cancers and heart and lung diseases [3] and kills more than 8 million people a year [1]. The smoke emitted from the burning end of a cigarette where the smoke is exhaled by the smoker is also the cause of cancers and heart and lung disease and causes around 1.2 million deaths. This exposure is referred to as secondhand smoke. It is an unintentional inhalation of smoke that occurs close to people smoking and/or in indoor environments where tobacco was recently used [1].

Smoking tobacco leaves chemical residue on surfaces including floors, carpets, furniture, and clothing where smoking has occurred. The chemicals live long after the smoke itself has been cleared from the environment. People are exposed in this phenomenon called thirdhand smoke. It is increasingly recognized as a potential danger, especially to children because they ingest residues that get on their hands after crawling on floors or touching walls and furniture [4]. A study which was conducted on mice showed that thirdhand smoke exposure has several behavioral and physical health impacts, including hyperactivity and adverse effects on the liver and lungs [5]. In this perspective, thirdhand smoke is a research agenda starting from 2011 [6]. So researches are conducted on thirdhand smoke and its health consequences [7, 8]. Due to this, we motivate to study a fractional model for the dynamics of smoking consisting people exposed to secondhand and thirdhand smoke as a compartment.

Different scholars have studied the mathematical model of smoking with integer-order derivatives [9–18] and non-integer-order derivatives [19–23]. In this article, we modify and extend the work in [9] by adding one compartment (people exposed to thirdhand smoke) and considering the new model with non-integer-order derivatives, specifically the Caputo–Fabrizio derivative.

The rest of the paper are arranged as follows. In Section 2, we discuss about the Caputo–Fabrizio fractional derivative and integral. In Section 3, we present the dynamics of the Caputo–Fabrizio fractional model of smoking. In Section 4, the existence and uniqueness of the solution of the proposed model are presented. Section 5 is devoted to the invariant region, positivity, and boundedness of the proposed model. Section 6 is devoted to equilibrium points and reproduction number. In Section 7, we present the stability of equilibrium points. Numerical methods and numerical results and discussions are included in Sections 8 and 9, respectively. Lastly, conclusions are given in Section 10.

2. Preliminary

Fractional calculus is as old as classical calculus and yet a novel topic. The origins can be traced back to the end of the seventeenth century, and it has been developed up to nowadays. It deals with the study of fractional-order integrals and derivatives and generalizes the ordinary integral and differential operators [24]. Fractional differential equations have been applied to formulate problems arising in engineering, physics, economics, and chemistry. From a modeling point of view, it has been better compared with integer-order derivatives [25, 26]. Though there are different definitions of fractional integral and derivatives of a function, in this paper, we use the Caputo–Fabrizio integral and derivative of a function. For $0<\delta<1$, Caputo and Fabrizio in [27] introduced a new definition of fractional derivative with smooth kernel, that is,

$$ {}^{\mathrm{CF}}D_0^{\delta}f(t)=\frac{(2-\delta)M(\delta)}{2(1-\delta)}\int_0^t f'(\xi)\left[e^{\left(\frac{-\delta(y-\xi)}{1-\delta}\right)}\right]d\xi,\ t\geq 0, \quad (1) $$

where $M(\delta)$ is a normalization constant depending on δ. Later, Losada and Nieto in [28] investigated a new definition of fractional integral corresponding to the Caputo–Fabrizio derivative as

$$ {}^{\mathrm{CF}}I_0^{\delta}f(t)=\frac{2(1-\delta)}{(2-\delta)M(\delta)}f(t)+\frac{2\delta}{(2-\delta)M(\delta)}\int_0^t f(\xi)d\xi,\ t\geq 0. \quad (2) $$

By imposing

$$ \frac{2(1-\delta)}{(2-\delta)M(\delta)}+\frac{2\delta}{(2-\delta)M(\delta)}=1, \quad (3) $$

an explicit formula for $M(\delta)$ can be obtained, that is,

$$ M(\delta)=\frac{2}{2-\delta}. \quad (4) $$

Due to this, Losada and Nieto in [28] proposed the following definitions.

Definition 1. Let $0<\delta<1$. The fractional Caputo–Fabrizio integral of order δ of a function f is defined by

$$ {}^{\mathrm{CF}}I_0^{\delta}f(t)=(1-\delta)f(t)+\delta\int_0^t f(\xi)d\xi,\quad t\geq 0. \quad (5) $$

Definition 2. Let $0<\delta<1$. The fractional Caputo–Fabrizio derivative of order δ of a function f is given by

$$ {}^{\mathrm{CF}}D_0^{\delta}f(t)=\frac{1}{1-\delta}\int_0^t f'(\xi)\left[e^{\left(\frac{-\delta(y-\xi)}{1-\delta}\right)}\right]d\xi,\quad t\geq 0. \quad (6) $$

Remark 3. ${}^{\mathrm{CF}}I_0^{\delta}({}^{\mathrm{CF}}D_0^{\delta}(f(t)))=f(t)-f(0)$.

3. Dynamic System with the Caputo–Fabrizio Derivative

In this section, we describe the fractional smoking model described by the Caputo–Fabrizio derivative. The model will be introduced by adding the compartment people exposed to thirdhand smoker and by changing the integer-order derivative with the Caputo–Fabrizio derivative in [9]. In Table 1, we describe the variables and parameters to form the mathematical model that represents the dynamics of transmission of the habit of smoking.

We assume that an individual does not belong to compartment S and T at the same time. The diagram in Figure 1 describes the habit of smoking.

The new fractional smoking model using the Caputo–Fabrizio derivative can be written as

$$ {}^{\mathrm{CF}}D_0^{\delta}S(t)=\pi_1-[\delta_1 I+\alpha+\beta_1]S, \quad (7) $$

$$ {}^{\mathrm{CF}}D_0^{\delta}T(t)=\pi_2-[\delta_2 I+\alpha+\beta_2]T, \quad (8) $$

$$ {}^{\mathrm{CF}}D_0^{\delta}Q(t)=\varepsilon_1 I-[\varepsilon_2 I+\alpha+\beta_3+\gamma_3]Q, \quad (9) $$

$$ {}^{\mathrm{CF}}D_0^{\delta}I(t)=[\delta_1 S+\delta_2 T+\varepsilon_2 Q-\varepsilon_1-\alpha-\gamma_4]I, \quad (10) $$

with initial conditions

$$ \begin{aligned} S(0)&=S_0,\\ T(0)&=T_0,\\ Q(0)&=Q_0,\\ I(0)&=I_0. \end{aligned} \quad (11) $$

We note that the dimension of the left-hand side of (7)–(10) is people per unit time.

4. Existence and Uniqueness

In this section, we will show the existence and uniqueness of the system (7)–(10) with initial conditions (11). To prove it, we will use a fixed-point theory that is applied in [19, 29].

Table 1: Description of variables and parameters.

Variables/parameters	Description	Value
S	Density of people exposed to secondhand smoker	$S \geq 0$
T	Density of people exposed to thirdhand smoker	$T \geq 0$
I	Smoker	$I \geq 0$
Q	Quitter	$Q \geq 0$
π_1	Rate of healthy people to S	$\pi_1 > 0$
π_2	Rate of healthy people to T	$\pi_2 > 0$
α	Natural death rate	$0 < \alpha < 1$
β_1	The exit rate of S to healthy people	$0 \leq \beta_1 \leq 1$
β_2	The exit rate of T to healthy people	$0 \leq \beta_2 \leq 1$
β_3	The exit rate of Q to healthy people	$0 \leq \beta_3 \leq 1$
γ_3	Death rate as a result of member of Q	$0 \leq \gamma_3 \leq 1$
γ_4	Death rate as a result of I	$0 \leq \gamma_4 \leq 1$
δ_1	Infection rate from S to I	$0 < \delta_1 \leq 1$
δ_2	Infection rate from T to I	$0 < \delta_2 \leq 1$
ε_1	Exit rate from I to Q	$0 \leq \varepsilon_1 \leq 1$
ε_2	Infection rate from Q to I	$0 \leq \varepsilon_2 \leq 1$

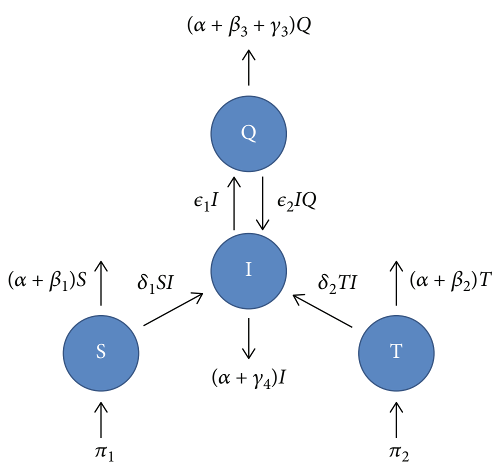

Figure 1: Smoking diagram.

Applying Remark 3, equations (7)–(10), respectively, become

$$\left.\begin{aligned} S(t) &= S(0) + {}^{CF}I_0^{\delta}\{\pi_1 - [\delta_1 I + \alpha + \beta_1]S\},\\ T(t) &= T(0) + {}^{CF}I_0^{\delta}\{\pi_2 - [\delta_2 I + \alpha + \beta_2]T\},\\ Q(t) &= Q(0) + {}^{CF}I_0^{\delta}\{\varepsilon_1 I - [\varepsilon_2 I + \alpha + \beta_3 + \gamma_3]Q\},\\ I(t) &= I(0) + {}^{CF}I_0^{\delta}\{[\delta_1 S + \delta_2 T + \varepsilon_2 Q - \varepsilon_1 - \alpha - \gamma_4)]I\}. \end{aligned}\right\} \tag{12}$$

We denote S by H^1, T by H^2, Q by H^3, and I by H^4.

Let

$$\left.\begin{aligned} K_1(t, H^1) &= \pi_1 - [\delta_1 H^4 + \alpha + \beta_1]H^1,\\ K_2(t, H^2) &= \pi_2 - [\delta_2 H^4 + \alpha + \beta_2]H^2,\\ K_3(t, H^3) &= \varepsilon_1 H^4 - [\varepsilon_2 H^4 + \alpha + \beta_3 + \gamma_3]H^3,\\ K_4(t, H^4) &= [\delta_1 H^1 + \delta_2 H^2 + \varepsilon_2 H^3 - \varepsilon_1 - (\alpha + \gamma_4)]H^4. \end{aligned}\right\} \tag{13}$$

Then, (12) becomes

$$H^i(t) = H^i(0) + (1-\delta)K_i(t, H^i) + \delta\int_0^t K_i(y, H^i)\,dy, \quad i = 1, 2, 3, 4. \tag{14}$$

We now suppose

$$\|H^i(t)\| \leq \theta_i, \tag{15}$$

for $i = 1, 2, 3, 4$. We have the following theorem.

Theorem 4. *For $i = 1, 2, 3$, and 4, K_i satisfy the Lipschitz condition in the second variable, that is,*

$$\|K^i(t, H_2^i) - K^i(t, H_I^i)\| \leq M_i\|H_2^i - H_I^i\|, \tag{16}$$

where

$$\begin{aligned} M_1 &= \alpha + \beta_1 + \eta_1 + \delta_1\theta_4, \\ M_2 &= \delta_1\theta_1 + \varepsilon_2\theta_3 + \varepsilon_1 + \alpha + \beta_4, \\ M_3 &= \alpha + \beta_3 + \eta_3 + \varepsilon_2\theta_4, \\ M_4 &= \alpha + \beta_2 + \eta_2 + \delta_2\theta_1. \end{aligned} \tag{17}$$

In addition, K_i is contraction if $0 \le M_i < 1$ for $i = 1, 2, 3, 4$.

Proof. We consider only K_1. The other can be done analogously. Let H_1^1 and H_2^1 be two functions. Then,

$$\begin{aligned} \left\|K^1\left(t, H_2^1\right) - K^1\left(t, H_1^1\right)\right\| &= \left\|\left[\delta_1 H^4 + \alpha + \beta_1\right] H_2^1 - \left[\delta_1 H^4 + \alpha + \beta_1\right] H_1^1\right\| \\ &\le \left(\delta_1\left\|H^4\right\| + \alpha + \beta_1 + \gamma_1\right)\left\|H_2^1 - H_1^1\right\|, \end{aligned} \tag{18}$$

by triangle inequality. □

We now define the following recurrence formula:

$$H_n^i(t) = (1-\delta)K_i\left(t, H_{n-1}^i\right) + \delta\int_0^t K_i\left(y, H_{n-1}^i\right)dy, \tag{19}$$

where $H_0^i(t) = H^i(0)$, $i = 1, 2, 3, 4$.

Let $\phi_n^i(t) = H_n^i(t) - H_{n-1}^i(t)$. Then, $H_n^i(t) = \sum_{j=0}^n \phi_j^i(t)$, where $\phi_0^i(t) = H^i(0)$.

Lemma 5. *Let M_i be as defined in (17). We have*

$$\left\|\phi_n^i(t)\right\| \le \left\|H^i(0)(1-\delta+\delta t)^n M_i^n\right\|. \tag{20}$$

Proof. Since K_i satisfies the Lipschitz condition with Lipschitz condition M_i, we have

$$\begin{aligned} \left\|H_n^i - H_{n-1}^i\right\| &= \left\|(1-\delta)K_i\left(t, H_{n-1}^i\right) + \delta\int_0^t K_i\left(y, H_{n-1}^i\right)dy - \left((1-\delta)K_i\left(t, H_{n-2}^i\right) + \delta\int_0^t K_i\left(y, H_{n-2}^i\right)dy\right)\right\| \\ &\le (1+\delta+\delta t)M_i\left\|H_{n-1}^i - H_{n-1}^i\right\|. \end{aligned} \tag{21}$$

It follows that

$$\|\phi_n(t)\| \le (1+\delta+\delta t)M_i\|\phi_{n-1}(t)\|. \tag{22}$$

Applying (22) recursively, we get

$$\|\phi_n(t)\| \le \left\|H^i(0)\right\|(1+\delta+\delta t)^n M_i^n. \tag{23}$$

We can easily show that

$$H^i(t) - H^i(0) = H_n^i(t) + B_n^i(t), \tag{24}$$

where

$$\begin{aligned} B_n^i(t) = (1-\delta)&\left(K_i\left(t, H^i\right) - K_i\left(t, H_{n-1}^i\right)\right) \\ &+ \delta\int_0^t \left(K_i\left(y, H^i\right) - K_i\left(y, H_{n-1}^i\right)\right)dy. \end{aligned} \tag{25}$$

□

For the purpose of the next theorem, we state and prove the following lemma.

Lemma 6. *Let $B_i^n(t)$, M_i, and θ_i be defined as in (25), (17), and (15), respectively. Then,*

$$\left\|B_n^i(t)\right\| \le (1-\delta+\delta t)^{n+1}M_i^{n+1}\theta_i. \tag{26}$$

Proof.

$$\begin{aligned} \left\|B_n^i(t)\right\| &= \left\|(1-\delta)\left(K_i\left(t, H^i\right) - K_i\left(t, H_{n-1}^i\right)\right) + \delta\int_0^t \left(K_i\left(y, H^i\right) - K_i\left(y, H_{n-1}^i\right)\right)dy\right\| \\ &\le (1-\delta)M_i\left\|H^i - H_{n-1}^i\right\| + \delta M_i\int_0^t \left\|H^i - H_{n-1}^i\right\|dy \\ &\le 1-\delta)M_i\left\|H^i - H_{n-1}^i\right\| + \delta M_i t\left\|H^i - H_{n-1}^i\right\|. \end{aligned} \tag{27}$$

Applying (27) recursively, we get

$$\left\|B_n^i(t)\right\| \le (1-\delta+\delta t)^{n+1}M_i^{n+1}\theta_i. \tag{28}$$

□

Theorem 7. *If $(1-\delta+\delta t)M_i < 1$, then $\lim_{n\longrightarrow\infty}\left\|B_n^i(t)\right\| = 0$. Consequently, the solution of the system (7)–(10) exists.*

Proof. Using Lemma 6, $\lim_{n\longrightarrow\infty}\left\|B_n^i(t)\right\| = 0$ when $(1-\delta+\delta t)M_i < 1$. It follows that $\lim_{n\longrightarrow\infty}B_n = 0$. Hence, using (24), $\lim_{n\longrightarrow\infty}H_n^i(t) = H^i(t) - H^i(0)$. □

Theorem 8. *If $(1-\delta+\delta t)M_i \le 1$, then the solution of the system (7)–(10) is unique.*

Proof. Let $H^i(t)$ and $R^i(t)$ be solutions of the system (7)–(10). We notice that

$$\begin{aligned} \left\|H^i(t) - R^i(t)\right\| &= \left\|(1-\delta)\left(K_i\left(t, H^i\right) - K_i\left(t, R^i\right)\right) + \delta\int_0^t \left(K_i\left(y, H^i\right) - K_i\left(y, R^i\right)\right)dy\right\| \\ &\le (1-\delta)M_1\left\|H^i(t) - R^i(t)\right\| + \delta M_i\int_0^t \left\|H^i(t) - R^i(t)\right\|dy \\ &\le ((1-\delta)+\delta t)M_i\left\|H^i(t) - R^i(t)\right\|. \end{aligned} \tag{29}$$

It follows that $\|H^i(t) - R^i(t)\|(1 - M_i(1 - \delta + \delta t)) \leq 0$. So if $(1 - \delta + \delta t)M_i \leq 1$, then $\|H^i(t) - R^i(t)\| = 0$. Hence, $H^i(t) = R^i(t)$. □

5. Invariant Region, Positivity, and Boundedness

The dynamics of the Caputo–Fabrizio fractional model (7)–(10) is explored in a feasible region $\Omega \subset \mathbb{R}^4_+$ such that

$$\Omega = \left\{ x(t) \in \mathbb{R}^4_+ : N(t) \leq \frac{1}{1 + \nu(1 - \delta)} ((1 - \delta)(\pi_1 + \pi_2) + N(0)) \right\}, \tag{30}$$

where $x(t) = (S(t), T(t), Q(t), I(t))$ and $N(t) = S(t) + T(t) + Q(t) + I(t)$.

Lemma 9. *The region $\Omega \subset \mathbb{R}^4_+$ is positively invariant with nonnegative initial conditions for model (7)–(10) in $\mathbb{R}^4_+$.*

Proof. After adding the components of human population in model (7)–(10), we get

$$ {}^{CF}D_0^\delta N(t) = {}^{CF}D_0^\delta S(t) + {}^{CF}D_0^\delta T(t) + {}^{CF}D_0^\delta Q(t) + {}^{CF}D_0^\delta I(t). \tag{31}$$

Then, we have

$$ {}^{CF}D_0^\delta N(t) + \nu N(t) \leq \pi_1 + \pi_2. \tag{32}$$

By applying Laplace transform and then its inverse, we obtain

$$N(t) \leq \frac{1}{1 + \nu(1 - \delta)} ((1 - \delta)(\pi_1 + \pi_2) + N(0)). \tag{33}$$

Thus, the solution of the model (7)–(10) with the nonnegative conditions in Ω remains in Ω. So, the region Ω is positively invariant and attracts all the solutions in $\mathbb{R}^4_+$. Now, for the positivity of the system solution, let

$$\begin{aligned} \mathbb{R}^4_+ &= \{x(t) \in | x(t) \geq 0\}, \\ x(t) &= (S(t), T(t), Q(t), I(t))^t. \end{aligned} \tag{34}$$

□

Corollary 10 (see [30]). *Suppose $g(t) \in C[a, b]$ and ${}^{CF}D_0^\delta g(t) \in C[a, b]$, where $\delta \in [0, 1]$. Then,*

(1) if ${}^{CF}D_0^\delta g(t) \geq 0$, $\forall y \in [a, b]$, then $g(t)$ is nondecreasing

(2) if ${}^{CF}D_0^\delta g(t) \leq 0$, $\forall y \in [a, b]$, then $g(t)$ is nonincreasing

Theorem 11. *If the initial population sizes of the model are positive, then the solution is positive and bounded at any time.*

Proof. We observe that

$$\begin{aligned} {}^{CF}D_0^\delta S(t)\Big|_{s=0} &= \pi_1 > 0, \\ {}^{CF}D_0^\delta T(t)\Big|_{T=0} &= \pi_2 > 0, \\ {}^{CF}D_0^\delta Q(t)\Big|_{Q=0} &= \varepsilon_1 I \geq 0, \\ {}^{CF}D_0^\delta I(t)\Big|_{I=0} &= 0. \end{aligned} \tag{35}$$

By Corollary 10, we have

$$S(t) > 0, T(t) > 0, Q(t) > 0, I(t) > 0, \text{ for } t > 0. \tag{36}$$

Since $N(t) \leq (1/(1 + \nu(1 - \delta)))((1 - \delta)(\pi_1 + \pi_2) + N(0))$, the solution is bounded. □

6. Equilibrium Points and Reproduction Number

Let us denote $A_1 = \alpha + \beta_1$, $A_2 = \alpha + \beta_2$, $A_3 = \alpha + \beta_3 + \gamma_3$, $A_4 = \alpha + \gamma_4$, and $B = \varepsilon_1 + A_4$. The smoking-free equilibrium point is $M_0 = (S^0, T^0, Q^0, I^0)$, where

$$\begin{aligned} S^0 &= \frac{\pi_1}{\alpha + \beta_1}, \\ T^0 &= \frac{\pi_2}{\alpha + \beta_2}, \\ Q^0 &= 0, \\ I^0 &= 0. \end{aligned} \tag{37}$$

We need to distinguish new infections from all other changes in population to compute reproduction number [31]. Let F be the rate of appearance of new infections in each compartment and V be the rate of transfer of individuals into and out of each compartment by all other means. It is assumed that each function is continuously differentiable at least twice in each variable. The smoking transmission model (7)–(10) can be written as

$$ {}^{CF}D_0^\delta x(t) = F - V, \tag{38}$$

where $x(t) = (S(t)\ T(t)\ Q(t)\ T(t))^t$,

$$\begin{aligned} F &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_1 IS + \delta_2 IT + \varepsilon_2 IQ \end{pmatrix}, \\ V &= \begin{pmatrix} (\delta_1 I + A_1)S - \pi_1 \\ (\delta_2 I + A_2)T - \pi_2 \\ (\varepsilon_2 I + A_3)Q - \varepsilon_1 I \\ BI \end{pmatrix}. \end{aligned} \tag{39}$$

The Jacobian matrices of F and V at the smoking-free equilibrium point are, respectively,

$$DF = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta_1\pi_1}{A_1} + \frac{\delta_2\pi_2}{A_2} \end{pmatrix},$$
$$DV = \begin{pmatrix} A_1 & 0 & 0 & \frac{\delta_1\pi_1}{A_1} \\ 0 & A_2 & 0 & \frac{\delta_2\pi_2}{A_2} \\ 0 & 0 & A_3 & -\varepsilon_1 \\ 0 & 0 & 0 & B \end{pmatrix}. \quad (40)$$

We see that

$$(DF)(DV)^{-1} = \frac{1}{A_1A_2A_3B} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1A_2A_3\left(\frac{\delta_1\pi_1}{A_1} + \frac{\delta_2\pi_2}{A_2}\right) \end{pmatrix}. \quad (41)$$

The reproduction number of a fractional dynamical system can be found using the next-generation matrix method [29, 32]. So, the reproduction number of the model (7)–(10) can be computed using the next-generation matrix method and is given by the spectral radius of $(DF)(DV)^{-1}$ [33], that is,

$$R_0 = \rho\left((DF)(DV)^{-1}\right) = \frac{1}{B}\left(\frac{\delta_1\pi_1}{A_1} + \frac{\delta_2\pi_2}{A_2}\right). \quad (42)$$

If $I \neq 0$, then

$$\delta_1 S + \delta_2 T + \varepsilon_2 Q = \varepsilon_1 + \alpha + \gamma_4. \quad (43)$$

Substituting

$$S = \frac{\pi_1}{\delta_1 I + A_1}, \quad T = \frac{\pi_2}{\delta_2 I + A_2}, \quad Q = \frac{\varepsilon_1 I}{\varepsilon_2 I + A_3}, \quad (44)$$

in (43), we get

$$\frac{\delta_1\pi_1}{\delta_1 I + A_1} + \frac{\delta_2\pi_2}{\delta_2 I + A_2} + \frac{\varepsilon_1\varepsilon_2 I}{\varepsilon_2 I + A_3} = B. \quad (45)$$

Let $G(x) = (\delta_1\pi_1/\delta_1 x + A_1) + (\delta_2\pi_2/\delta_2 x + A_2) + (\varepsilon_1\varepsilon_2 x/\varepsilon_2 x + A_3) - B$. Then, the following holds true.

(1) G is continuous on $[0.\infty)$

(2) $\lim_{x\longrightarrow 0^+} G(x) > 0$ if $R_0 > 1$

(3) $\lim_{x\longrightarrow \infty} G(x) < 0$

Consequently, if

$$\frac{\varepsilon_1\varepsilon_2 A_3}{(\varepsilon_2 x + A_3)^2} < \frac{\delta_1^2\pi_1}{(\delta_1 x + A_1)^2} + \frac{\delta_2^2\pi_2}{(\delta_2 x + A_2)^2}, \quad (46)$$

then $G(x)$ has a unique zero in $(0, \infty)$. Let I^* be the unique solution of (45). The endemic equilibrium point is $M^* = (S^*, T^*, Q^*, I^*)$, where

$$S^* = \frac{\pi_1}{\delta_1 I^* + A_1}, \quad T^* = \frac{\pi_2}{\delta_2 I^* + A_2}, \quad Q^* = \frac{\varepsilon_1 I^*}{\varepsilon_2 I^* + A_3}. \quad (47)$$

7. Stability

Consider the following fractional-order linear system described by the Caputo–Fabrizio derivative:

$$^{CF}D_0^{\delta} x(t) = Ax(t), \quad (48)$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^n \times \mathbb{R}^n$, and $0 < \delta < 1$.

Definition 12 (see [34]). The characteristic equation of system (48) is

$$|s[I - (1-\delta)A] - \delta A| = 0. \quad (49)$$

Theorem 13 (see [34]). *If $(I - (1-\delta)A)$ is invertible, then system (48) is asymptotically stable if and only if the real parts of the roots to the characteristic equation of system (48) are negative.*

We next state and prove the asymptotic stability of smoking-free equilibrium point of the dynamic system (7)–(10).

Theorem 14. *The smoking-free equilibrium point M_0 of the system (7)–(10) with $R_0 < 1$ is asymptotically stable if and only if real parts of the roots of the characteristic equation are negative.*

Proof. The characteristic equation of the linearized system (7)–(10) at smoking-free equilibrium point M_0 is

$$|s[I - (1-\delta)J(M_0)] - \delta J(M_0)| = 0, \quad (50)$$

where $J(M_0)$ is the Jacobian matrix at M_0, that is,

$$J(M_0)=\begin{pmatrix} -A_1 & 0 & 0 & -\frac{\delta_1\pi_1}{A_1} \\ 0 & -A_2 & 0 & -\frac{\delta_2\pi_2}{A_2} \\ 0 & 0 & -A_3 & \varepsilon_1 \\ 0 & 0 & 0 & B(R_0-1) \end{pmatrix}. \quad (51)$$

The roots of the characteristic equation (50) are

$$\begin{aligned} s_1 &= -\frac{\delta A_1}{1+(1-\delta)A_1} < 0, \\ s_2 &= -\frac{\delta A_2}{1+(1-\delta)A_2} < 0, \\ s_3 &= -\frac{\delta A_3}{1+(1-\delta)A_3} < 0, \\ s_4 &= \frac{\delta B(R_0-1)}{1-B(1-\delta)(R_0-1)} < 0. \end{aligned} \quad (52)$$

□

8. Numerical Methods

In this section, we will use the three step Adams-Moulton predictor-corrector methods to determine the unknowns since it is superior to the three-step Adams-Bashforth predictor-corrector method that was applied in [29]. The truncation error of Adams-Bashforth methods is of $O(h^3)$, and we will show in what follows the truncation error of Adams-Moulton methods is of $O(h^4)$. Let $t_j=t_0+jh$, where $h=(t_{\text{end}}-t_0)/n$ and $j=0,1,2,\cdots,n$, be the discretization of the interval $[t_0=0, t_{\text{end}}]$. We will define the recursive formula as follows.

$$\begin{aligned} H^i(t_{j+1})-H^i(t_j) &= (1-\delta)\left[K_i(t_j,H^i)-K_i(t_{j-1},H^i)\right] \\ &\quad + \delta\int_{t_j}^{t_{j+1}} K_i(y,H^i)dy, i=1,2,3,4. \end{aligned} \quad (53)$$

We approximate $K_i(t,H^i)$ by a Lagrange polynomial of degree 3

$$P_{i,3}(\tau)=\sum_{k=j-2}^{j+1} L_j K_i(t_j,H^i), \quad (54)$$

where

$$L_{i(\tau)}=\prod_{k=j-2,k\neq i}^{j+1}\frac{\tau-t_k}{t_i-t_k}, \quad i=j-2,j-1,j,j+1. \quad (55)$$

Let $u=(\tau-t_j)/h$. Then,

$$\begin{aligned} \int_{t_j}^{t_{j+1}} K_i(y,H^i)dy &= \int_{t_j}^{t_{j+1}}\sum_{k=j-2}^{j+1} L_j K_i(t_j,H^i) \\ &= \frac{h}{24}\left(K_{i,j-2}-5K_{i,j-1}+19K_{i,j}+9K_{i,j+1}\right). \end{aligned} \quad (56)$$

Substituting (56) into (53), we get

$$\begin{aligned} H^i(t_{j+1}) &= H^i(t_j)+(1-\delta)\left[K_i(t_j,H^i)-K_i(t_{j-1},H^i)\right]+\delta\frac{h}{24} \\ &\quad \cdot\left(K_{i,j-2}-5K_{i,j-1}+19K_{i,j}+9K_{i,j+1}\right), i=1,2,3,4. \end{aligned} \quad (57)$$

Equation (57) is a type of three-step Adams-Moulton method. It is an implicit method because its right-hand side contains $H^i(t_{j+1})$. So the left-hand side of (57) can be calculated using Adams-Bashforth methods

$$\begin{aligned} H^i(t_{j+1}) &= H^i(t_j)+(1-\delta)\left[K_i(t_j,H^i)-K_i(t_{j-1},H^i)\right]+\delta\frac{h}{12} \\ &\quad \cdot\left(5K_{i,j-2}-16K_{i,j-1}+23K_{i,j}\right), i=1,2,3,4. \end{aligned} \quad (58)$$

We now write the methods as follows:

$$\begin{aligned} H^i(t_{j+1}) &= H^i(t_j)+(1-\delta)\left[K_i(t_j,H^i)-K_i(t_{j-1},H^i)\right]+\delta\frac{h}{24} \\ &\quad \cdot\left(K_{i,j-2}-5K_{i,j-1}+19K_{i,j}+9\widehat{K}_{i,j+1}\right), i=1,2,3,4, \end{aligned} \quad (59)$$

where $\widehat{K}_{i,j+1}=K_i(t_{j+1},\widehat{H}^i(t_{j+1}))$ and

$$\begin{aligned} \widehat{H}^i(t_{j+1}) &= H^i(t_j)+(1-\delta)\left[K_i(t_j,H^i)-K_i(t_{j-1},H^i)\right] \\ &\quad +\delta\frac{h}{12}\left(5K_{i,j-2}-16K_{i,j-1}+23K_{i,j}\right), i=1,2,3,4. \end{aligned} \quad (60)$$

In (60), $H^i(t_1)$ and $H^i(t_2)$ can be computed using the method

$$H^i(t_{j+1})=H^i(t_j)+(1-\delta+\delta h)K_i(t_j,H^i). \quad (61)$$

The truncation error for the three-step Adams-Moulton methods can be estimated by using the error estimate for the Lagrange interpolating polynomial, namely,

$$K_i(t,H^i)=P_{i,3}(t)+E_{i,3}(t), \quad (62)$$

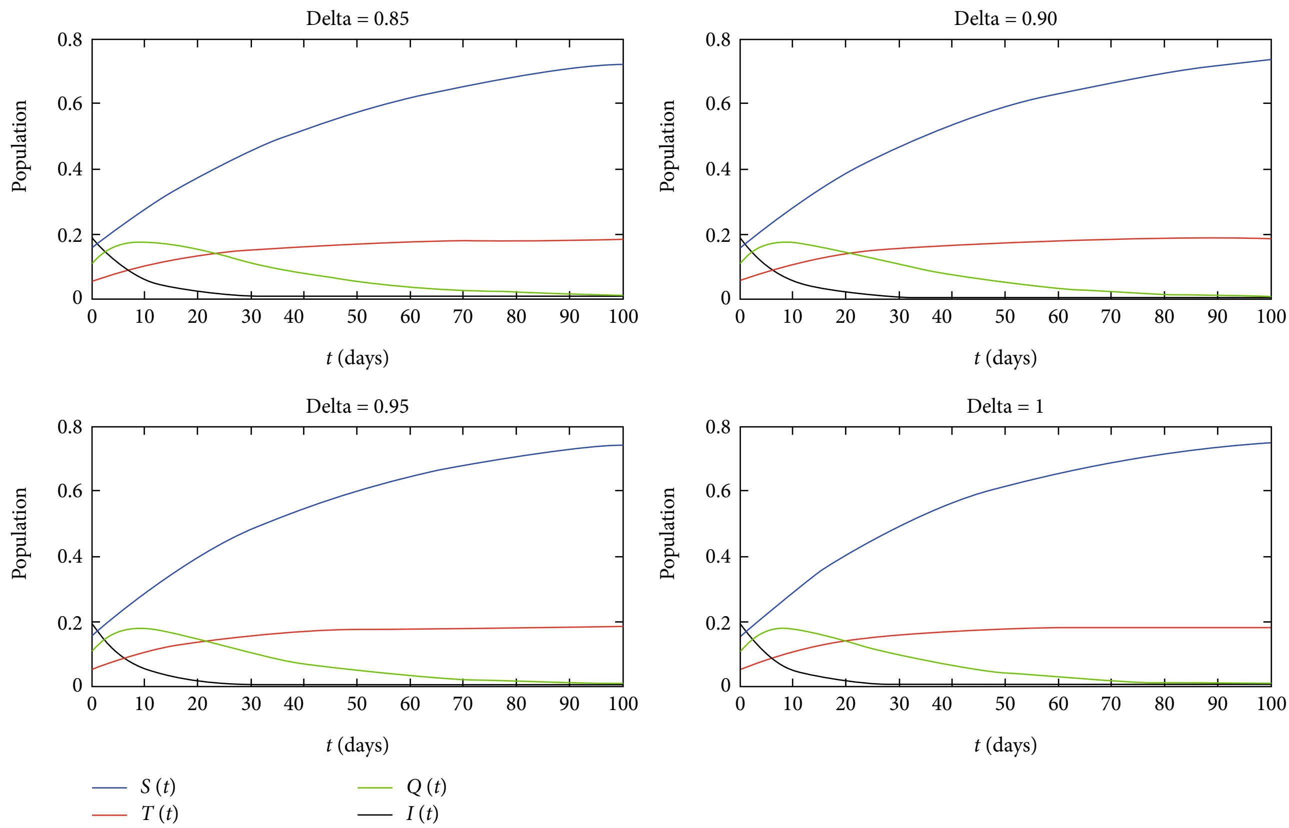

Figure 2: Time series plot of all variables for distinct values of δ.

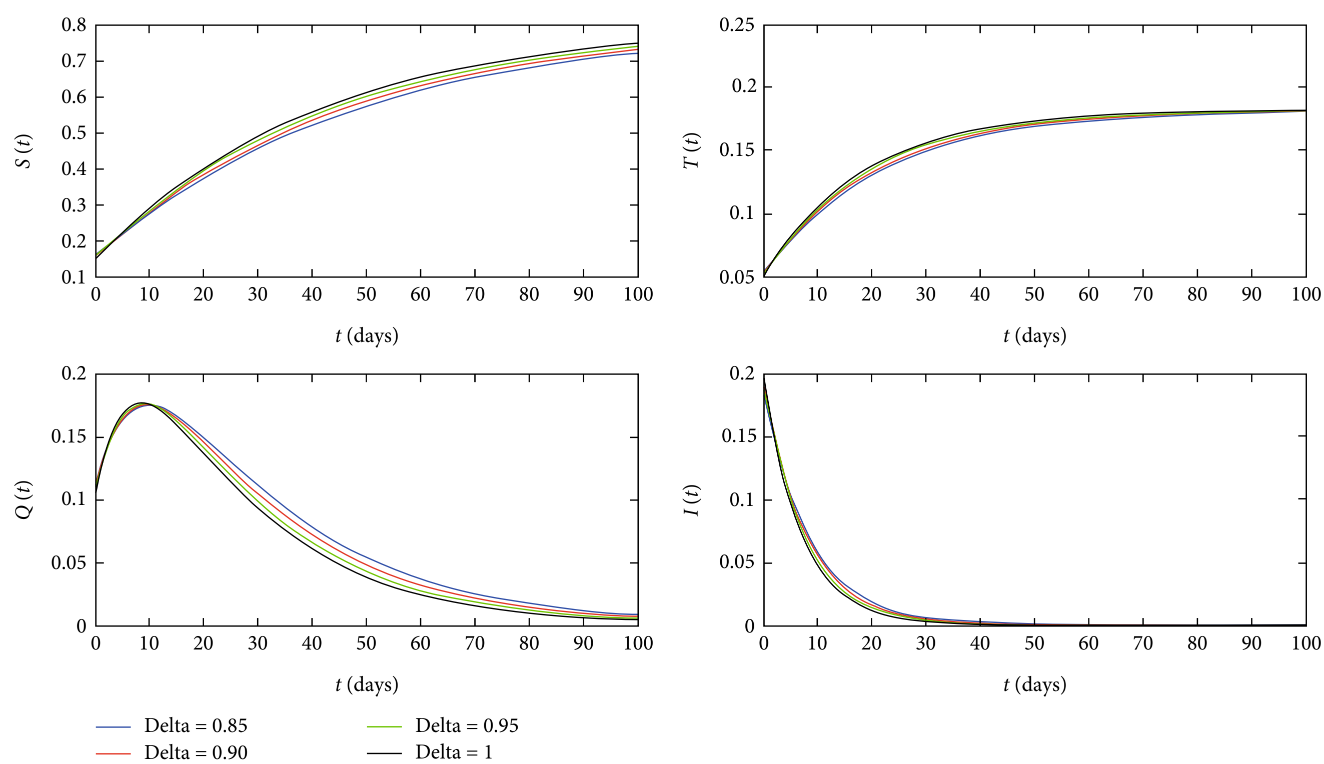

Figure 3: Time series plot of each of $S(t)$, $T(t)$, $Q(t)$, and $I(t)$ for distinct values of δ.

where

$$E_{i,3}(t) = \frac{f(4)\left(\xi_j, H^i\right)}{24}\left(t - t_{j-2}\right)\left(t - t_{j-1}\right)\left(t - t_j\right) \cdot \left(t - t_{j+1}\right), \quad \xi_j \in \left(t_{j-2}, t_{j+1}\right). \tag{63}$$

We notice that

$$\int_{t_j}^{t_{j+1}} E_{i,3}(t)dt = \int_{t_j}^{t_{j+1}} \frac{f^{(4)}\left(\xi_j, H^i\right)}{24}\left(t - t_{j-2}\right)\left(t - t_{j-1}\right)\left(t - t_j\right) \cdot \left(t - t_{j+1}\right)dt = h^5 \frac{f^{(4)}\left(\xi_j, H^i\right)}{24}\int_0^1 (u+2) \cdot (u+1)u(u-1)du \text{ by taking } u = \left(t - t_{j-2}\right)/h = -h^5 \frac{19}{30}\frac{f^{(4)}\left(\xi_j, H^i\right)}{24}. \tag{64}$$

We denote the right-hand side of (59) by $\tilde{H}^i$. The total truncation error of the use of formula (59) is

$$\frac{H^i\left(t_{j+1}\right) - \tilde{H}^i}{h} = -\delta \frac{19h^4}{720} f^{(4)}\left(\xi_j, H^i\right). \tag{65}$$

9. Numerical Results and Discussions

For the purpose of numerical simulations, we utilize the values of the initial conditions $S(0) = 0.15, T(0) = 0.05, Q(0) = 0.1, I(0) = 0.2$ and parameters $\pi_1 = 0.02, \pi_2 = 0.01, \varepsilon_1 = 0.15, \varepsilon_2 = 0.068, \delta_1 = 0.02, \delta_2 = 0.01, \alpha = 0.005, \beta_1 = 0.02, \beta_2 = 0.05, \beta_3 = 0.04, \gamma_3 = 0.0015, \gamma_4 = 0.003$. Thus, we have

$$M_0 = \left(S^0, T^0, Q^0, I^0\right) = (0.7692, 0.1815, 0, 0), \quad R_0 = 0.1124 < 1. \tag{66}$$

The solutions of model (7)–(10) are computed using the corrector three-step Adams-Moulton methods (59) and predictor three-step Adams-Bashforth methods (60). We use MATLAB software to plot Figures 2 and 3. As we notice in Figure 2, the number of quitters $Q(t)$ and smokers $I(t)$ goes to zero when $t \longrightarrow \infty$. Hence, the smoking-free equilibrium point is asymptotically stable for $\delta = 0.85, 0.90, 0.95, 1$, and S and T eventually disappear from the system.

Remark 15. $\delta = 1$ represents the standard derivative.

In Figure 3, $S(t), T(t), Q(t)$, and $I(t)$ are plotted for different values of δ. We can see that as δ increases, $S(t)$ and $T(t)$ increase and converge to smoking-free equilibrium point $S^0 = 0.7692$ and $T^0 = 0.1815$, respectively, whereas quitters $Q(t)$ and smokers $I(t)$ decrease and converge to smoking-free equilibrium point $Q^0 = 0$ and $I^0 = 0$, respectively.

10. Conclusions

In this work, we investigated a Caputo–Fabrizio fractional derivative smoking model containing smoker, people exposed to secondhand smoker, people exposed to thirdhand smoker, and quitters. The existence and uniqueness of solution of the proposed model have been shown with the help of the fixed-point theorem. Moreover, the positivity and boundedness of the model were discussed. We also calculated the reproduction number of the model and showed that the smoking-free equilibrium point is asymptotically stable when $R_0 < 1$. Furthermore, the three-step Adams-Moulton method was applied to compute the numerical solutions. The solution depends on the fractional order δ and as $\delta \longrightarrow 1^-$ the solutions close to the smoking-free equilibrium point. So we recommend the Caputo–Fabrizio derivative to study a model that represents a real-world problem.

Data Availability

Data sharing is not applicable for this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally in preparing and writing this manuscript. They read and approved the final manuscript.

References

[1] Global Burden of Disease, *Global Burden of Disease*, Institute for Health Metrics and Evaluation, Washington, DC, 2019.

[2] I. T. Agaku, B. A. King, C. G. Husten et al., "Tobacco product use among adults—United States, 2012-2013," *MMWR. Morbidity and Mortality Weekly Report*, vol. 63, no. 25, pp. 542–547, 2014.

[3] National Center for Chronic Disease Prevention and Health Promotion (US) Office on Smoking and Health, *The Health Consequences of Smoking—50 Years of Progress: A Report of the Surgeon General*, Centers for Disease Control and Prevention (US), Atlanta (GA), 2014, http://www.ncbi.nlm.nih.gov/books/NBK179276/.

[4] National Institution on Drug Abuse, "Tobacco, Nicotine, and E-Cigarettes Research Report," 2022.

[5] M. Martins-Green, N. Adhami, M. Frankos et al., "Cigarette smoke toxins deposited on surfaces: implications for human health," *PLoS One*, vol. 9, no. 1, article e86391, 2014.

[6] P. Jacob, N. L. Benowitz, H. Destaillats et al., "Thirdhand smoke: new evidence, challenges, and future directions," *Chemical research in toxicology*, vol. 30, no. 1, pp. 270–294, 2017.

[7] J. E. Drehmer, B. H. Walters, E. Nabi-Burza, and J. P. Winickoff, "Guidance for the clinical management of thirdhand smoke exposure in the child health care setting," *Journal of Clinical Outcomes Management*, vol. 24, no. 12, pp. 551–559, 2017.

[8] G. Ferrante, M. Simoni, F. Cibella et al., "Third-hand smoke exposure and health hazards in children," *Monaldi archives for chest disease*, vol. 79, no. 1, pp. 38–43, 2013.

[9] B. Fekede and B. Mebrate, "Sensitivity and mathematical model analysis on secondhand smoking tobacco," *Journal of the Egyptian Mathematical Society*, vol. 28, no. 1, p. 50, 2020.

[10] S. Habte, F. Gidaf, H. Siraw et al., "Sensitivity and mathematical model analysis on smoking with health education effect," *Communications in Mathematical Biology and Neuroscience*, vol. 2022, p. 68, 2022.

[11] C. C. Garsow, G. J. Salivia, and A. R. Herrera, *Mathematical Models for the Dynamics of Tobacoo Use, Recovery and Relapse*, Technical Report Series BU-1505-M, Cornell University, UK, 2000.

[12] O. Sharomi and A. B. Gumel, "Curtailing smoking dynamics: a mathematical modeling approach," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 475–499, 2008.

[13] A. Zeb, A. Bano, E. Alzahrani, and G. Zaman, "Dynamical analysis of cigarette smoking model with a saturated incidence rate," *AIP Advances*, vol. 8, no. 4, article 045317, 2018.

[14] A. Zeb, G. Zaman, and S. Momani, "Square-root dynamics of a giving up smoking model," *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 5326–5334, 2013.

[15] A. Alzahrani and A. Zeb, "Detectable sensation of a stochastic smoking model," *Open Mathematics*, vol. 18, no. 1, pp. 1045–1055, 2020.

[16] A. Zeb and A. Alzahrani, "Non-standard finite difference scheme and analysis of smoking model with reversion class," *Results in Physics*, vol. 21, p. 103785, 2021.

[17] E. Alzahrani and A. Zeb, "Stability analysis and prevention strategies of tobacco smoking model," *Alzahrani and Zeb Boundary Value Problems*, vol. 2020, no. 1, p. 3, 2020.

[18] A. Zeb, S. Kumar, A. Tesfay, and A. Kumar, "A stability analysis on a smoking model with stochastic perturbation," *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 32, no. 3, pp. 915–930, 2022.

[19] J. Singh, D. Kumar, M. A. Qurashi, and D. Baleanu, "A new fractional model for giving up smoking dynamics," *Advances in Difference Equations*, vol. 2017, p. 88, 2017.

[20] V. S. Erturk, G. Zaman, and S. Momani, "A numeric-analytic method for approximating a giving up smoking model containing fractional derivatives," *Computers & Mathematcs with Applications*, vol. 64, no. 10, pp. 3065–3074, 2012.

[21] H. Alrabaiah, A. Zeb, E. Alzahrani, and K. Shahe, "Dynamical analysis of fractional-order tobacco smoking model containing snuffing class," *Alexandria Engineering Journal*, vol. 60, pp. 3669–3678, 2012.

[22] M. Kocabiyik, M. Y. Ongun, Department of Mathematics, Burdur Mehmet Akif Ersoy University, Burdur, Turkey, and Department of Mathematics, Süleyman Demirel University, Isparta, Turkey, "Construction a distributed order smoking model and its nonstandard finite difference discretization," *AIMS Mathematics*, vol. 7, no. 3, pp. 4636–4654, 2022.

[23] A. Zeb, S. Kumar, and T. Saeed, "A robust computational dynamics of fractional order smoking model with relapse habit," *Fractals*, vol. 30, no. 1, article 2240034, 2022.

[24] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg Dordrecht London New York, 2010.

[25] I. Podlubny, *Fractional Differential Equations*, vol. 198, Academics Press, San Diego, CA, USA, 1999.

[26] A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Nederland, 2006.

[27] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation & Applications*, vol. 1, no. 2, pp. 73–85, 2015.

[28] J. Losada and J. J. Nieto, "Properties of a new fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 87–92, 2015.

[29] E. J. Moore, S. Sirisubtawee, and S. Koonprasert, "A Caputo–Fabrizio fractional differential equation model for HIV/AIDS with treatment compartment," *Advances in Difference EquationsAdv. Difference Equ.*, vol. 2019, no. 1, p. 200, 2019.

[30] H. Zine, E. M. Lotfi, D. F. M. Torres, and N. Yousfi, "Taylor's formula for generalized weighted fractional derivatives with nonsingular kernels," *Axioms*, vol. 11, no. 5, p. 231, 2022.

[31] P. Van den Driessche, "Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission," *Mathematical Biosciences*, vol. 180, no. 1–2, pp. 29–48, 2002.

[32] P. Ding and Z. Wang, "Dynamic analysis of a delayed fractional infectious disease model with saturated incidence," *Fractal and Fractional*, vol. 6, no. 3, p. 138, 2022.

[33] O. Diekmann, J. A. P. Heesterbeek, and J. A. J. Metz, "On the definition and the computation of the basic reproduction ratio R0 in models for infectious diseases in heterogeneous populations," *Journal of mathematical biology*, vol. 28, no. 4, pp. 365–382, 1990.

[34] H. Li, J. Cheng, H.-B. Li, and S.-M. Zhong, "Stability analysis of a fractional-order linear system described by the Caputo–Fabrizio derivative," *Mathematics*, vol. 7, no. 2, p. 200, 2019.