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Research Article

Analytical Solution of the Fractional Linear Time-Delay Systems and their Ulam-Hyers Stability

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We introduce the delayed Mittag-Leffler type matrix functions, delayed fractional cosine, and delayed fractional sine and use the Laplace transform to obtain an analytical solution to the IVP for a Hilfer type fractional linear time-delay system $D_{0,t}^{\mu,\nu}z(t) + Az(t) + \Omega z(t-h) = f(t)$ of order $1 < \mu < 2$ and type $0 \le \nu \le 1$, with nonpermutable matrices A and Ω. Moreover, we study Ulam-Hyers stability of the Hilfer type fractional linear time-delay system. Obtained results extend those for Caputo and Riemann-Liouville type fractional linear time-delay systems with permutable matrices and new even for these fractional delay systems.

1. Introduction

Khusainov et al. [1] studied the following Cauchy problem for a second order linear differential equation with pure delay:

$$\begin{cases} x''(t) + \Omega^2 x(t-\tau) = f(t), & t \ge 0, \tau > 0, \\ x(t) = \varphi(t), x'(t) = \varphi'(t), & -\tau \le t \le 0, \end{cases} \tag{1}$$

where $f : [0,\infty) \longrightarrow \mathbb{R}^n$, Ω is a $n \times n$ nonsingular matrix, τ is the time delay and φ is an arbitrary twice continuously differentiable vector function. A solution of (1) has an explicit representation of the form ([1], Theorem 2):

$$\begin{aligned} x(t) &= (\cos_\tau \Omega t)\varphi(-\tau) + \Omega^{-1}(\sin_\tau \Omega t)\varphi'(-\tau) \\ &+ \Omega^{-1}\int_{-\tau}^{0} \sin_\tau \Omega(t-\tau-s)\varphi''(s)ds \\ &+ \Omega^{-1}\int_{0}^{t} \sin_\tau \Omega(t-\tau-s)f(s)ds, \end{aligned} \tag{2}$$

where $\cos_\tau \Omega : \mathbb{R} \longrightarrow \mathbb{R}^{n\times n}$ and $\sin_\tau \Omega : \mathbb{R} \longrightarrow \mathbb{R}^{n\times n}$ denote the delayed matrix cosine of polynomial degree $2k$ on the intervals $(k-1)\tau \le t < k\tau$ and the delayed matrix sine of polynomial degree $2k+1$ on the intervals $(k-1)\tau \le t < k\tau$, respectively.

It should be stressed out that pioneer works [1, 2] led to many new results in integer and noninteger order time-delay differential equations and discrete delayed system; see [3–18]. These models have applications in spatially extended fractional reaction-diffusion models [19], oscillating systems [20, 21], numerical solutions [22], and so on.

Introducing the fractional analogue delayed matrices cosine/sine of a polynomial degree, see Formulas (5) and (6), Liang et al. [23] gave representation of a solution to the initial value problem (3):

Theorem 1 (see [23]). *Let $h > 0$, $\varphi \in C^2([-h, 0], \mathbb{R}^n)$, Ω be a nonsingular $n \times n$ matrix. The solution $x : [-h,\infty) \longrightarrow \mathbb{R}^n$ of the initial value problem*

$$\begin{cases} {}^C D_{-h}^{\alpha} x(t) + \Omega^2 x(t-h) = 0, & t \ge 0, h > 0, \\ x(t) = \varphi(t), x'(t) = \varphi'(t), & -h \le t \le 0, \end{cases} \tag{3}$$

has the form

$$x(t) = (\cos_{h,\alpha}\Omega t^{\alpha})\varphi(-h) + \Omega^{-1}(\sin_{h,\alpha}\Omega(t-h)^{\alpha})\varphi'(0) + \Omega^{-1}\int_{-h}^{0}\cos_{h,\alpha}\Omega(t-h-s)^{\alpha}\varphi'(s)ds, \tag{4}$$

where $\cos_{h,\alpha}\Omega t^{\alpha}$ *is the fractional delayed matrix cosine of a polynomial of degree* $2k\alpha$ *on the intervals* $(k-1)h \le t < kh$, $\sin_{h,\alpha}\Omega t^{\alpha}$ *is the fractional delayed matrix sine of a polynomial of degree* $(2k+1)\alpha$ *on the intervals* $(k-1)h \le t < kh$ *defined as follows*

$$\cos_{h,\alpha}\Omega t^{\alpha} := \begin{cases} \Theta, & -\infty < t < -h, \\ I, & -h \le t < 0, \\ I - \Omega^2\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + (-1)^k\Omega^{2k}\dfrac{(t-(k-1)h)^{2k\alpha}}{\Gamma(2k\alpha+1)}, & (k-1)h \le t < kh, \end{cases} \tag{5}$$

$$\sin_{h,\alpha}\Omega t^{\alpha} := \begin{cases} \Theta, & -\infty < t < -h, \\ \Omega\dfrac{(t+h)^{\alpha}}{\Gamma(\alpha+1)}, & -h \le t < 0, \\ \Omega\dfrac{(t+h)^{\alpha}}{\Gamma(\alpha+1)} - \Omega^3\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + (-1)^k\Omega^{2k+1}\dfrac{(t-(k-1)h)^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, & (k-1)h \le t < kh, \end{cases} \tag{6}$$

and I *is the identity matrix and* Θ *is the null matrix.*

Mahmudov in [16] studied the following R-L linear fractional differential delay equation of order $1 < 2\alpha \le 2$ by introducing the concept of fractional delayed matrix cosine $\cos_{h,\alpha,\beta}\{A, \Omega; t\}$ and sine $\sin_{h,\alpha,\beta}\{A, \Omega; t\}$ ([16], Definitions 2 and 3).

$$\begin{cases} {}^{RL}D^{\alpha}_{-h+}\left({}^{RL}D^{\alpha}_{-h+}\right)x(t)A^2x(t) + \Omega^2x(t-h) = f(t), t \ge 0, h > 0, \\ x(t) = \varphi(t), \left(I^{1-\alpha}_{-h+}x\right)(-h^+) = \varphi(-h), -h \le t \le 0, \\ {}^{RL}D^{\alpha}_{-h+}x(t) = {}^{RL}D^{\alpha}_{-h+}\varphi(t), \\ \left(I^{1-\alpha}_{-h+}(D^{\alpha}_{-h+}x)\right)(-h^+) = {}^{RL}D^{\alpha}_{-h+}\varphi(-h), -h \le t \le 0, \end{cases} \tag{7}$$

where ${}^{RL}D^{\alpha}_{-h+}$ stands for the R-L fractional derivative of order $0 < \alpha \le 1$ with lower limit $-h$, $A, \Omega \in \mathbb{R}^{n\times n}$, $f \in C([0, \infty), \mathbb{R}^n)$, and $\varphi \in C^1([-h, 0], \mathbb{R}^n)$. Obviously, the derivative can be started at $-h$ instead of 0, since the function $x(t)$ governed by (7) actually originates at $-h$. However, as is known, changing the starting point of the derivative modifies the derivative and leads to a different problem. In this article, we study the case when the derivative started at 0.

Recently, Liu et al. [24, 25] have studied the analytical representations of the solutions of the inhomogeneous Caputo type fractional delay oscillating differential equations.

Motivating by the above works, we study the following Hilfer type linear fractional differential time-delay equation of order $1 < \mu < 2$ and type $0 \le \nu \le 1$:

$$\begin{cases} D^{\mu,\nu}_{0,t}z(t) + Az(t) + \Omega z(t-h) = f(t), t \in (0, T], \\ z(t) = \varphi(t), -h \le t \le 0, \\ D^{-(2-\mu)(1-\nu)+1,\nu}_{0,t}z(t)\Big|_{t=0} = a_1, \\ D^{-(2-\mu)(1-\nu),\nu}_{0,t}z(t)\Big|_{t=0} = a_0, \end{cases} \tag{8}$$

where $D^{\mu,\nu}_{0,t}$ stands for the Hilfer fractional derivative of order $1 < \mu < 2$ and type $0 \le \nu \le 1$ with lower limit 0, $A, \Omega \in \mathbb{R}^{d\times d}$, $f \in C([0, T], \mathbb{R}^d)$, and $\varphi \in C^1([-h, 0], \mathbb{R}^d)$.

Delayed perturbation of Mittag-Leffler matrix functions serves as a suitable tool for solving linear fractional continuous time-delay equations. First, the delayed matrix exponential function (delayed matrix Mittag-Leffler function) was defined to solve linear purely delayed (fractional) systems of order one. Then, the second order differential systems with pure delay were considered and suitable delayed sine and cosine matrix functions were introduced in [1]. Later, Liang et al. [23] introduced the fractional analogue of delayed cosine/sine matrices and obtained an explicit solution of the sequential fractional Caputo type equations with pure delay—the case $A = \Theta$ (zero matrix). Recently, Mahmudov [26] introduced the fractional analogue of delayed matrices cosine/sine in the case when A and Ω commutes to solve the sequential Riemann-Liouville type linear time-delay system. It should be noticed that the model investigated here is not sequential and differs from that of discussed in [23, 26]. For the sake of completeness, we also refer to studies of discrete/continuous variants of delayed matrices used to obtain exact solution to linear difference equations with delays and related problems [1–18, 23–26].

The main contributions of this paper is presented as below:

(i) We introduce the delayed Mittag-Leffler type matrix function $Y^h_{\mu,\gamma}(A,\Omega;t)$ by means of the determining function $Q^{A,\Omega}_{k,m}$. We give an exact analytical solution of the Hilfer type fractional problem (8) with non-permutable matrices A,Ω by using $Y^h_{\mu,\gamma}(A,\Omega;t)$ and study their Ulam-Hyers stability. Obtained results are new even for Caputo and Riemann-Liouville type fractional linear time-delay systems, since matrices A,Ω are nonpermutable

(ii) Although the problem considered by us is fractional of order $1<\mu<2$ and type $0\le\nu\le 1$, our approach is also applicable to the classical second-order equations. Thus, our results are new even for the classical second order oscillatory system

The article contains significant updates in the theory of fractional differential equations with delay and is structured in the following way. Section 2 is a preparatory section in which we introduce the main definitions and recall concepts of the fractional calculus, the Laplace transform, and the Ulam-Hyers stability. Moreover, we introduce the delayed Mittag-Leffler type matrix function of two parameters $Y^h_{\mu,\gamma}: [0,\infty)\longrightarrow\mathbb{R}^d$ and study their properties. In Section 3, we provide the analytical representation formulas of classical solutions to linear homogeneous and nonhomogeneous Hilfer type linear fractional differential time-delay equation of order $1<\mu<2$ and type $0\le\nu\le 1$ using the delayed Mittag-Leffler type matrix function $Y^h_{\mu,\gamma}(A,\Omega;t)$. Finally, we study the Ulam-Hyers stability of the problem (8).

2. Auxiliary Lemmas

We introduce a concept of delayed Mittag-Leffler type matrix function of two parameters

Definition 2. Delayed Mittag-Leffler type matrix function of two parameters $Y^h_{\mu,\gamma}: [0,\infty)\longrightarrow\mathbb{R}^d$ is defined as follows:

$$Y^h_{\mu,\gamma}(A,\Omega;t)=Y^h_{\mu,\gamma}(t):=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k Q^{A,\Omega}_{k,m}\frac{(t-mh)_+^{k\mu+\gamma-1}}{\Gamma(k\mu+\gamma)},\tag{9}$$

where $(t)_+=\max\{0,t\}$ and

$$Q_{k,m}=Q^{A,\Omega}_{k,m}=\sum_{j=m}^{k}A^{k-j}\Omega Q^{A,\Omega}_{j-1,m-1},.\tag{10}$$

$$Q^{A,\Omega}_{0,m}=Q^{A,\Omega}_{k,-1}=\Theta,\ Q^{A,\Omega}_{k,0}=A^k\,Q^{A,\Omega}_{0,0}=I,\ k=0,1,2,\cdots,m=0,1,2,\cdots\tag{11}$$

In this definition, the determining function $Q^{A,\Omega}_{k,m}$ plays the role of a kernel, see [17, 26]. It is clear that

$$\begin{aligned}&Q^{A,\Omega}_{k+1,m}=AQ^{A,\Omega}_{k,m}+\Omega Q^{A,\Omega}_{k,m-1},\\&Q^{A,\Omega}_{0,m}=Q^{A,\Omega}_{k,-1}=\Theta,\ Q^{A,\Omega}_{0,0}=I,\\&k=0,1,2,\cdots,m=0,1,2,\cdots\end{aligned}\tag{12}$$

We state the main novelties of our article as below:

(i) We introduce a novel delayed Mittag-Leffler type matrix function $Y^h_{\mu,\gamma}(A,\Omega;t)$.

(ii) If $\Omega=\Theta$, then

$$\begin{aligned}Y^h_{2,1}\left(A^2,\Theta;t\right)&=\sum_{k=0}^{\infty}(-1)^kA^{2k}\frac{t^{2k}}{(2k)!}=\cos(At),\\AY^h_{2,2}\left(A^2,\Theta;t\right)&=A\sum_{k=0}^{\infty}(-1)^kA^{2k}\frac{t^{2k+1}}{(2k+1)!}=\sin(At),\\Y^h_{\mu,1}\left(A^2,\Theta;t\right)&=\sum_{k=0}^{\infty}(-1)^kA^{2k}\frac{t^{k\mu}}{\Gamma(k\mu+1)},\\Y^h_{\mu,2}\left(A^2,\Theta;t\right)&=\sum_{k=0}^{\infty}(-1)^kA^{2k}\frac{t^{k\mu+1}}{\Gamma(k\mu+2)},\ 1<\mu<2.\end{aligned}\tag{13}$$

$Y^h_{\mu,1}(A^2,\Theta;t)$ and $Y^h_{\mu,2}(A^2,\Theta;t)$ can be called fractional cosine and sine for $1<\mu<2$. Similar cosine/sine matrix functions were defined in [16, 23] to solve $1<2\mu<2$ order sequential fractional differential equations.

(i) If $A=\Theta$, then we have

$$Q^{A,\Omega}_{m,m}=\Omega^m,\ Y^h_{\mu,\gamma}(A,\Omega;t)=\sum_{m=0}^{\infty}(-1)^m\Omega^m\frac{(t-mh)_+^{m\mu+\gamma-1}}{\Gamma(m\mu+\gamma)}.\tag{14}$$

Moreover,

$$\begin{aligned}Y^h_{\mu,1}\left(\Theta,\Omega^2;t\right)&=\sum_{m=0}^{\infty}(-1)^m\Omega^{2m}\frac{(t-mh)_+^{m\mu}}{\Gamma(m\mu+1)}=\cos^h_\mu(\Omega;t),\\\Omega Y^h_{\mu,2}\left(\Theta,\Omega^2;t\right)&=\Omega\sum_{m=0}^{\infty}(-1)^m\Omega^{2m}\frac{(t-mh)_+^{m\mu}}{\Gamma(m\mu+2)}=\Omega\sin^h_\mu(\Omega;t).\end{aligned}\tag{15}$$

Similar delayed cosine/sine matrix functions were defined in [4, 16] to solve $1<2\mu<2$ order sequential fractional linear differential equations with pure delay.

Before introducing properties of $Y^h_{\mu,\gamma}(A,\Omega;t)$, we recall the definition of the Hilfer fractional derivative and Ulam-Hyers stability:

Definition 3. Let $m \in \mathbb{N}, m-1 < \mu < m$, $0 \le \nu \le 1$, $a \in \mathbb{R}$, and $f \in C^m[a,b]$. Then the Hilfer fractional derivative of f of order μ and type ν is given by

$$D_{a,t}^{\mu,\nu} f(t) \coloneqq I_{a,t}^{\nu(m-\mu)} \frac{d^m}{dt^m} I_{a,t}^{(1-\nu)(m-\mu)} f(t), \tag{16}$$

where

$$I_{a,t}^{\gamma} f(t) \coloneqq \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds \tag{17}$$

is the R-L fractional integral of f of order $\gamma > 0$.

The main tool we use in this paper is the Laplace transform $F(s) \coloneqq L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt, Res > a$, which is defined for an exponentially bounded function f. Here are some of properties of the Laplace transform.

Lemma 4. *The following equalities hold true for sufficiently large Re(s) and appropriate functions f, g:*

(i) $L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}, a, b \in \mathbb{R}$

(ii) $L^{-1}\{e^{-sh} s^{-1}\} = 1, t \ge h \ge 0$

(iii) $L^{-1}\{F(s)G(s)\} = (f * g)(t)$

(iv) $L\{D_{0,t}^{\mu,\nu} f(t)\} = s^{\mu} L\{f(t)\} - \sum_{k=0}^{m-1} s^{m(1-\nu)+\mu\nu-k-1} I_{0,t}^{(1-\nu)(m-\mu)-k} f(0)$

(v) $L^{-1}\{1\} = \delta(t)$,*where* $\delta(t)$ *is Dirac delta distribution*

(vi) $L^{-1}\{e^{-nsh} s^{-n}\} = (t-nh)_+^{n-1}/(n-1)!, h > 0, n \in \mathbb{N}$

(vii) $L^{-1}\{e^{-sh} F(s)\} = f(t-h), h \ge 0$

(viii) $L^{-1}\{e^{-sh} s^{\alpha\gamma-\beta} (s^{\alpha} I - A)^{-\gamma}\} = (t-h)^{\beta-1} E_{\alpha,\beta}^{\gamma}(A (t-h)^{\alpha})$, $t \ge h$, *where* $E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} (z^{\alpha k}/\Gamma(\alpha k + \beta))((\gamma)_k/k!)$ *is the three parameter Mittag-Leffler function,* $\alpha, \beta, \gamma > 0$, $t \in \mathbb{R}$ *and* $(\gamma)_k \coloneqq \gamma(\gamma+1)\cdots(\gamma+k-1)$

Definition 5. System (8) is Ulam-Hyers stable on $[0, T]$ if there exists $C > 0$ such that for any $\varepsilon > 0$ and for any function $z^*(t)$ satisfying inequality

$$\left\| D_{0,t}^{\mu,\nu} z^*(t) + A z^*(t) + \Omega z^*(t-h) - f(t) \right\| \le \varepsilon, \tag{18}$$

and the initial conditions in (8), there is a solution $z(t)$of (8) such that

$$\|z^*(t) - z(t)\| \le C\varepsilon, \tag{19}$$

for every $t \in [0, T]$.

We reduce the notations of $Y_{\mu,\gamma}^h(A, \Omega; t)$, $Q_{k,m}^{A,\Omega}$ to a mere $Y_{\mu,\gamma}^h(t)$, and $Q_{k,m}$ in the sequel.

Theorem 6. *The following formulae hold:*

The function $Y_{\mu,\gamma}^h(\cdot)$ *is continuous on* $(0, +\infty)$,

$$\begin{aligned} &\frac{d}{dt} Y_{\mu,\gamma+1}^h(t) = Y_{\mu,\gamma}^h(t), , \\ &\frac{d}{dt} Y_{\mu,\gamma+2}^h(t) = Y_{\mu,\gamma+1}^h(t) \text{ for all } t \in . \\ &D_{0,t}^{\mu,\nu} Y_{\mu,\gamma}^h = -A Y_{\mu,\gamma}^h(t) - \Omega Y_{\mu,\gamma}^h(t-h). \end{aligned} \tag{20}$$

Proof. The proofs of the properties 1 and 2 are obvious. Proof of the property 3 is based on the following formula

$$D_{0,t}^{\mu,\nu} t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} t^{\alpha-\mu}, t > 0, n-1 < \mu \le n, 0 \le \nu \le 1, \alpha > -1. \tag{21}$$

The main tool we use in this paper is the Laplace transform $F(s) \coloneqq L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt, Res > a$, which is defined for an exponentially bounded function f. □

Lemma 7. *We have*

$$\begin{aligned} &L^{-1}\left\{ \left(e^{-hs} (s^{\mu} I + A)^{-1} \Omega \right)^m s^{\mu-\gamma} (s^{\mu} I + A)^{-1} \right\} \\ &\quad = \sum_{k=0}^{\infty} (-1)^{k-m} Q_{k,m} \frac{(t-mh)_+^{k\mu+\gamma-1}}{\Gamma(k\mu+\gamma)}, \end{aligned} \tag{22}$$

where $Q_{k,m}$ *is defined in (10).*

Proof. For $n = 0$ by Lemma 4(viii) we have

$$\begin{aligned} &L^{-1}\left\{ s^{\mu-\gamma} (s^{\mu} I + A)^{-1} \right\} = t^{\gamma-1} E_{\mu,\gamma}(-A t^{\mu}), \\ &L^{-1}\left\{ e^{-sh} (s^{\mu} I + A)^{-\gamma} \right\} = (t-h)_+^{\mu-1} E_{\mu,\mu}(-A(t-h)^{\mu}), t \ge h. \end{aligned} \tag{23}$$

Let $Q_{k,0} = A^k$. For $n = 1$, we use the convolution property (Lemma 4(iii)) of the Laplace transform to get

$$\begin{aligned} &L^{-1}\left\{ e^{-hs} (s^{\mu} I + A)^{-1} \Omega s^{\mu-\gamma} (s^{\mu} I + A)^{-1} \right\}, \\ &= L^{-1}\left\{ e^{-hs} (s^{\mu} I + A)^{-1} \Omega \right\} * L^{-1}\left\{ s^{\mu-\gamma} (s^{\mu} I + A)^{-1} \right\}, \\ &= \int_0^t (s-h)_+^{\mu-1} E_{\mu,\mu}(-A(s-h)^{\mu}) \Omega (t-s)^{\gamma-1} E_{\mu,\gamma}(-A(t-s)^{\mu}) ds, \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k A^k \Omega (-1)^j A^j}{\Gamma(\mu k + \mu) \Gamma(\mu j + \gamma)} \int_h^t (s-h)^{\mu k + \mu - 1} (t-s)^{\mu j + \gamma - 1} ds, \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k A^k \Omega (-1)^j A^j \frac{(t-h)_+^{\mu k + \mu j + \mu + \gamma - 1}}{\Gamma(\mu k + \mu j + \mu + \gamma)}, \\ &= \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} A^{k-j} \Omega A^j \frac{(t-h)_+^{\mu k + \mu + \gamma - 1}}{\Gamma(\mu k + \mu + \gamma)}, \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{j=0}^{k-1} A^{k-1-j} \Omega A^j \frac{(t-h)_+^{\mu k + \gamma - 1}}{\Gamma(\mu k + \gamma)}. \end{aligned} \tag{24}$$

Now, to use the mathematical induction, suppose that it holds for $n = m$. Then convolution property yields

$$\begin{aligned}
&L^{-1}\left\{\left(e^{-hs}(s^{\mu}I+A)^{-1}\Omega\right)^{m+1}s^{\mu-\gamma}(s^{\mu}I+A)^{-1}\right\},\\
&=L^{-1}\left\{e^{-hs}(s^{\mu}I+A)^{-1}\Omega\right\}*L^{-1}\left\{\left(e^{-hs}s^{-\beta}(s^{\mu}I+A)^{-1}\Omega\right)^{m}s^{\mu-\gamma}(s^{\mu}I+A)^{-1}\right\},\\
&=\int_{h}^{t}(s-h)_{+}^{\mu-1}E_{\mu,\mu}(-A(s-h)^{\mu})\Omega\sum_{j=0}^{\infty}(-1)^{j}Q_{j+m,m}\frac{(t-s-mh)_{+}^{\mu j+\mu m+\gamma-1}}{\Gamma(\mu j+\mu m+\mu)}ds,\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k}A^{k}\Omega(-1)^{j}Q_{j+m,m}\int_{h}^{t}\frac{(t-s-h)_{+}^{k\mu+\mu-1}}{\Gamma(k\mu+\mu)}\frac{(s-mh)_{+}^{\mu j+\mu m+\gamma-1}}{\Gamma(\mu j+\mu m+\gamma)}ds,\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k}A^{k}\Omega(-1)^{j}Q_{j+m,m}\int_{mh}^{t-h}\frac{(t-s-h)_{+}^{k\mu+\mu-1}}{\Gamma(k\mu+\mu)}\frac{(s-mh)_{+}^{\mu j+\mu m+\gamma-1}}{\Gamma(\mu j+\mu m+\gamma)}ds,\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k}A^{k}\Omega(-1)^{j}Q_{j+m,m}\frac{(t-(m+1)h)_{+}^{k\mu+j\mu+(m+1)\mu+\gamma-1}}{\Gamma(k\mu+j\mu+(m+1)\mu+\gamma)},\\
&=\sum_{k=m+1}^{\infty}(-1)^{k-m-1}\sum_{j=0}^{k-m-1}A^{k-j}\Omega Q_{j+m,m}\frac{(t-(m+1)h)_{+}^{k\mu+j\mu+(m+1)\mu+\gamma-1}}{\Gamma(k\mu+j\mu+(m+1)\mu+\gamma)},
\end{aligned}\tag{25}$$

what was to be proved. □

Lemma 8. *We have*

$$\begin{aligned}
Y_{\mu,\gamma}^{h}(t)&:=L^{-1}\left\{s^{\mu-\gamma}\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}\right\},\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{k}Q_{k,m}\frac{(t-mh)_{+}^{k\mu+\gamma-1}}{\Gamma(k\mu+\gamma)}.
\end{aligned}\tag{26}$$

Proof. It is easy to see that

$$\begin{aligned}
&L^{-1}\left\{s^{\mu-\gamma}\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}\right\},\\
&=L^{-1}\left\{s^{\mu-\gamma}\left((s^{\mu}I+A)I+(s^{\mu}I+A)(s^{\mu}I+A)^{-1}\Omega e^{-hs}\right)^{-1}\right\},\\
&=L^{-1}\left\{\left(I+(s^{\mu}I+A)^{-1}\Omega e^{-hs}\right)^{-1}s^{\mu-\gamma}(s^{\mu}I+A)^{-1}\right\},\\
&=L^{-1}\left\{\sum_{m=0}^{\infty}e^{-mhs}(-1)^{n}\left((s^{\mu}I+A)^{-1}\Omega\right)^{m}s^{\mu-\gamma}(s^{\mu}I+A)^{-1}\right\},\\
&=\sum_{m=0}^{\infty}L^{-1}\left\{e^{-mhs}(-1)^{m}\left((s^{\mu}I+A)^{-1}\Omega\right)^{m}s^{\mu-\gamma}(s^{\mu}I+A)^{-1}\right\}.
\end{aligned}\tag{27}$$

Hence, by Lemma 7 we have

$$\begin{aligned}
&L^{-1}\left\{s^{\mu-\gamma}\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}\right\},\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{k}Q_{k,m}\frac{(t-mh)_{+}^{k\mu+\gamma-1}}{\Gamma(k\mu+\gamma)}.
\end{aligned}\tag{28}$$

□

3. Exact Analytical Solution and Ulam-Hyers Stability

We obtain the exact analytical solution of the Hilfer type fractional second order problem (8) using delayed Mittag-Leffler type matrix function $Y_{\mu,\gamma}^{h}(A,\Omega;t)$ and study their Ulam-Hyers stability.

Theorem 9. *The analytical solution of the initial value problem (8) has the form*

$$\begin{aligned}
z(t)&=Y_{\mu,(\mu-2)(1-\nu)+1}^{h}(t)\left(I_{0,t}^{(1-\nu)(2-\mu)}\varphi\right)(0)\\
&\quad+Y_{\mu,(\mu-2)(1-\nu)+2}^{h}(t)\left(I_{0,t}^{(1-\nu)(2-\mu)-1}\varphi\right)(0)\\
&\quad-\int_{-h}^{0}Y_{\mu,\mu}^{h}(t-s-h)\Omega\varphi(s)ds+\int_{0}^{t}Y_{\mu,\mu}^{h}(t-s)f(s)ds.
\end{aligned}\tag{29}$$

Proof. Assume that the function f and the solution of (8) is exponentially bounded. By applying the Laplace transform to the both sides of (8), we obtain the following relation

$$L\left\{D_{0,t}^{\mu,\nu}z(t)\right\}+AL\{z(t)\}+\Omega L\{z(t-h)\}=L\{f(t)\}.\tag{30}$$

It follows that

$$\begin{aligned}
\left(s^{\mu}I+A+\Omega e^{-hs}\right)Z(s)&=s^{2(1-\nu)+\mu\nu-1}\left(I_{0,t}^{(1-\nu)(2-\mu)}\varphi\right)(0)\\
&\quad+s^{2(1-\nu)+\mu\nu-2}\left(I_{0,t}^{(1-\nu)(2-\mu)-1}\varphi\right)(0)-\Omega\int_{0}^{\infty}e^{-st}z(t-h)dt+F(s),
\end{aligned}\tag{31}$$

where $Z(s)=L\{z(t)\}$, $F(s)=L\{f(t)\}$. For sufficiently large s, such that

$$\left\|A+\Omega e^{-hs}\right\|<s^{\mu},\tag{32}$$

the matrix $s^{\mu}I+A+\Omega e^{-hs}$ is invertible, and it holds that

$$\begin{aligned}
Z(s)&=s^{2(1-\nu)+\mu\nu-1}\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}\left(I_{0,t}^{(1-\nu)(m-\mu)}\varphi\right)(0)\\
&\quad+s^{2(1-\nu)+\mu\nu-2}\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}\left(I_{0,t}^{(1-\nu)(m-\mu)-1}\varphi\right)(0)\\
&\quad-\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}\Omega\Psi(s)+\left(s^{\mu}I+A+\Omega e^{-hs}\right)^{-1}F(s).
\end{aligned}\tag{33}$$

By Lemma 8

$$\begin{aligned}
z(t)&=Y_{\mu,(\mu-2)(1-\nu)+1}^{h}(t)\left(I_{0,t}^{(1-\nu)(m-\mu)}\varphi\right)(0)\\
&\quad+Y_{\mu,(\mu-2)(1-\nu)+2}^{h}(t)\left(I_{0,t}^{(1-\nu)(m-\mu)-1}\varphi\right)(0)\\
&\quad-\int_{-h}^{0}Y_{\mu,\mu}^{h}(t-s-h)\Omega\varphi(s)ds+\int_{0}^{t}Y_{\mu,\mu}^{h}(t-s)f(s)ds,
\end{aligned}\tag{34}$$

since

$$\begin{aligned} & L^{-1}\left\{\left(s^{\mu}I + A + \Omega e^{-hs}\right)^{-1}\Omega\Psi(s)\right\}, \\ & = L^{-1}\left\{\left(s^{\mu}I + A + \Omega e^{-hs}\right)^{-1}\right\} * L^{-1}\{\Omega\Psi(s)\}, \\ & = \int_0^t Y^h_{\mu,\mu}(t-s)\Omega\psi(s-h)ds, \\ & = \int_0^h Y^h_{\mu,\mu}(t-s)\Omega\varphi(s-h)ds, \\ & = \int_{-h}^0 Y^h_{\mu,\mu}(t-s-h)\Omega\varphi(s)ds. \end{aligned} \tag{35}$$

Now the assumption on the exponential boundedness can be omitted. We can easily check that (34) is a solution of (8). □

Theorem 10. *Let $1 < \mu < 2$, $0 \le \nu \le 1$, $f \in C([0,\infty), \mathbb{R}^d)$. System (8) is stable in Ulam-Hyers sense on $[0, T]$.*

Proof. Let $z^*(t)$ satisfy the inequality (18) and the initial conditions in (8). Set

$$X(t) = D^{\mu,\nu}_{0,t}z^*(t) + Az^*(t) + \Omega z^*(t-h) - f(t), t \in [0, T]. \tag{36}$$

It follows from definition 5 that $\|X(t)\| < \varepsilon$. By Theorem 9 we have

$$\begin{aligned} z^*(t) = & Y^h_{\mu,(\mu-2)(1-\nu)+1}(t)\left(I^{(1-\nu)(m-\mu)}_{0,t}\varphi\right)(0) \\ & + Y^h_{\mu,(\mu-2)(1-\nu)+2}(t)\left(I^{(1-\nu)(m-\mu)-1}_{0,t}\varphi\right)(0) \\ & - \int_{-h}^0 Y^h_{\mu,\mu}(t-s-h)\Omega\varphi(s)ds + \int_0^t Y^h_{\mu,\mu}(t-s)(f(s)-X(s))ds. \end{aligned} \tag{37}$$

Thus, we can estimate the difference $z^*(t) - z(t)$ as follows:

$$\begin{aligned} & \|z^*(t) - z(t)\|, \\ & = \left\|\int_0^t Y^h_{\mu,\mu}(t-s)X(s)ds\right\| \le \varepsilon\int_0^T \left\|Y^h_{\mu,\mu}(T-s)\right\|ds, \\ & = C\varepsilon. \end{aligned} \tag{38}$$

Then, the problem (8) is Ulam-Hyers stable on $[0, T]$. □

4. Conclusion

The article solves a problem of finding exact analytical solution of continuous linear time-delay systems using the delayed Mittag-Leffler type matrix functions of two variables. In articles [1, 6] delayed exponential is suggested to obtain an exact solution of delayed first order continuous equations. Similar results for sequential Caputo type and Riemann-Liouville type fractional linear time-delay systems of order $1 < 2\alpha < 2$ were obtained in [23, 26]. These results are obtained either for systems with pure delay or under the condition of commutativity of A and Ω. In this article, we drop the commutativity condition. The result has been obtained by defining the new delayed Mittag-Leffler matrix function and employing the Laplace transform. The work contained in this article will be useful for future research on fractional time-delay systems.

One possible direction in which to extend the results of this paper is toward fractional impulsive systems [27] and conformable fractional differential systems of order $1 < \alpha < 2$. Another challenge is to study the qualitative properties of the problem (8).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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