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Research Article

Modified Oscillation Results for Advanced Difference Equations of Second-Order

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In this paper, we present a new method to establish the oscillation of advanced second-order difference equations of the form $\Delta(\eta(\ell)\Delta v(\ell)) + \rho(\ell)v(\sigma(\ell)) = 0$, using the ordinary difference equation $\Delta(\eta(\ell)\Delta v(\ell)) + q(\ell)v(\ell+1) = 0$. The obtained results are new and improve the existing criteria. We provide examples to illustrate the main results.

1. Introduction

This paper is concerned with the second-order advanced difference equation

$$\Delta(\eta(\ell)\Delta v(\ell)) + \rho(\ell)v(\sigma(\ell)) = 0, \ell \geq \ell_0, \tag{1}$$

where $\ell \in \mathbb{N}(\ell_0) = \{\ell_0, \ell_0+1, \ell_0+2, \cdots,\}$, ℓ_0 is a nonnegative integer, and

(C_1) $\{\eta(\ell)\}$, $\{\rho(\ell)\}$, and $\{q(\ell)\}$ are positive real sequences for $\ell \geq \ell_0$

(C_2) $\{\sigma(\ell)\}$ is a monotone increasing sequence of integers with $\sigma(\ell) \geq \ell + 1$ for $\ell \geq \ell_0$

(C_3) $\Omega(\ell) = \sum_{s=\ell_0}^{\ell-1} 1/\eta(s) \longrightarrow \infty$ as $\ell \longrightarrow \infty$

By a solution of (E), we mean a nontrivial sequence $\{v(\ell)\}$ that satisfies (E) for all $\ell \geq \ell_0$. A solution $\{v(\ell)\}$ of (E) is said to be *oscillatory* if it is neither eventually negative nor eventually positive. Otherwise, it is said to be *nonoscillatory*. Equation (1) is called oscillatory if all its solutions are oscillatory.

Oscillation phenomena take part in different models described by various differential equations, partial differential equations, and dynamic equations on time scales; see, for instance, the papers [1–6] for more details. In particular, we refer the reader to the papers [4, 6] for models from mathematical biology and physics where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. In recent years, there have appeared several criteria on the oscillation of (1) for the retarded case, that is, $\sigma(\ell) \leq \ell - 1$ and $\lim_{\ell \longrightarrow \infty}\sigma(\ell) = \infty$, using either comparison methods or/and Riccati transformation technique. On the majority, these studies use the comparison methods, which is considered to be the most powerful tool in the oscillation theory of difference equations (see, for example, [7–17] and the references cited therein). Another particular method appearing in several studies is also the summation averaging method (see, for example, [1, 18–21] and the references cited therein).

From the literature, it is well known that not many oscillation results are available by using comparison methods. In [22, 23], the authors obtained oscillation of the advanced difference equation (1) from that of the ordinary difference equation

$$\Delta(\eta(\ell)\Delta v(\ell)) + \rho(\ell)v(\ell+1) = 0, \tag{2}$$

without explicitly using the information about the advanced argument $\{\sigma(\ell)\}$. Very recently in [22], the authors studied the oscillation of (E) by assuming that $\eta(\ell) \geq 1$ for all $\ell \geq \ell_0$.

In this paper, we present a new method which produces the oscillation of (1) without the restriction $\eta(\ell) \geq 1$. Thus, our results generalize and complement to those reported in [14, 15, 22, 23].

2. Main Results

Without loss of generality, in studying the nonoscillatory solutions of (1), we can restrict our attention only to positive solutions.

Lemma 1. *Let $\{\upsilon(\ell)\}$ be a positive solution of (1). Then, $\eta(\ell)\Delta\upsilon(\ell) > 0$ and $\Delta(\eta(\ell)\Delta\upsilon(\ell)) < 0$, eventually.*

Proof. The proof can be found in Lemma 1, [22], and the details are omitted. □

Next we present an oscillation criterion for equation (2) which will be used to prove our main results.

Lemma 2. *Assume that*

$$\lim_{\ell \longrightarrow \infty} \sum_{s=\ell_0}^{\ell} \rho(s) \tag{3}$$

is convergent and $\{\upsilon(\ell)\}$ is a positive solution of (E_1). Then, there is an integer $\ell_1 \in \mathbb{N}(\ell_0)$ such that

$$\omega(\ell) \geq \sum_{s=\ell}^{\infty} \rho(s) + \sum_{s=\ell}^{\infty} \frac{\omega(s+1)\omega(s)}{\eta(s)}, \text{for } \ell \in \mathbb{N}(\ell_1), \tag{4}$$

where $\omega(\ell) = \eta(\ell)\Delta\upsilon(\ell)/\upsilon(\ell)$ for $\ell \geq \ell_1$.

Proof. From Lemma 1, there is an $\ell_1 \in \mathbb{N}(\ell_0)$ such that $\omega(\ell) > 0$ for all $\ell \geq \ell_1$. Taking into account the fact that $\eta(\ell)\Delta\upsilon(\ell)$ is positive and decreasing, we have

$$\begin{aligned}
\Delta\omega(\ell) &= \frac{\Delta(\eta(\ell)\Delta\upsilon(\ell))}{\upsilon(\ell+1)} - \frac{\eta(\ell)\Delta\upsilon(\ell)}{\upsilon(\ell)\upsilon(\ell+1)}\Delta\upsilon(\ell) \\
&= -\rho(\ell) - \frac{\omega(\ell)\eta(\ell)\Delta\upsilon(\ell)}{\eta(\ell)\upsilon(\ell+1)} \\
&\leq -\rho(\ell) - \frac{\omega(\ell)\omega(\ell+1)}{\eta(\ell)}.
\end{aligned} \tag{5}$$

□

Summing up the last inequality from ℓ to j and then taking $j \longrightarrow \infty$, we obtain

$$\omega(\ell) \geq \sum_{s=\ell}^{\infty} \rho(s) + \sum_{s=\ell}^{\infty} \frac{\omega(s)\omega(s+1)}{\eta(s)}, \tag{6}$$

for all $\ell \in \mathbb{N}(\ell_1)$.

Now let us define

$$Q(\ell, 0) = \sum_{s=\ell}^{\infty} \rho(s), j = 0, \ell \geq \ell_1, \tag{7}$$

$$Q(\ell, j) = \sum_{s=\ell}^{\infty} \frac{Q(s+1\,; j-1)Q(s\,; j-1)}{\eta(s)} + Q(\ell\,; 0), j \geq 1, \ell \geq \ell_1. \tag{8}$$

Theorem 3. *Let condition (3) hold. Then, (2) is oscillatory provided one of the following two conditions holds:*

(H_1) There exists an integer $j \in \mathbb{N}$ such that $Q(\ell\,; 0), \cdots, Q(\ell\,; j-1)$ defined by (8) satisfying

$$\sum_{s=\ell_1}^{\infty} \frac{Q(s+1, j-1)Q(s, j-1)}{\eta(s)} = \infty. \tag{9}$$

(H_2) There exists an integer $\ell \geq \ell_1$ such that

$$\lim_{j \longrightarrow \infty} \sup Q(\ell\,; j) = \infty. \tag{10}$$

Proof. Assume, for the sake of contradiction, that equation (2) is nonoscillatory. Let (H_1) hold, and let $\{\upsilon(\ell)\}$ be a positive solution of (2) for all $\ell \in \mathbb{N}(\ell_1)$. By Lemma 2, we have for $j = 0$

$$\omega(\ell) \geq \sum_{s=\ell}^{\infty} \rho(s) + \sum_{s=\ell}^{\infty} \frac{\omega(s)\omega(s+1)}{\eta(s)} \geq Q(\ell, 0), \ell \in \mathbb{N}(\ell_1). \tag{11}$$

Thus,

$$\sum_{s=\ell_1}^{\infty} \frac{Q(s+1, 0)Q(s, 0)}{\eta(s)} \leq \sum_{s=\ell_1}^{\infty} \frac{\omega(s+1)\omega(s)}{\eta(s)} \leq \omega(\ell_1) \tag{12}$$

by (11), which contradicts (9). Similarly for $j \geq 1$ from (8) and (4), we have

$$\omega(\ell) \geq Q(\ell, j) \text{ for } j = 1, 2, \cdots \tag{13}$$

and hence,

$$\sum_{s=\ell_1}^{\infty} \frac{Q(s+1, j)Q(s, j)}{\eta(s)} \leq \sum_{s=\ell_1}^{\infty} \frac{\omega(s+1)\omega(s)}{\eta(s)} \leq \omega(\ell_1) < \infty, \tag{14}$$

which is again a contradiction.

Next, suppose that (H_2) hold. Clearly, in view of $\omega(\ell) \geq Q(\ell\,; j)$ for $j = 1, 2, \cdots$, we get from (4) that $\lim_{j \longrightarrow \infty} Q(\ell_1\,; j) \leq \omega(\ell_1) < \infty$, which is a contradiction. The proof of the theorem is complete. □

Theorem 4. *Assume that*

$$\Omega(\ell)\sum_{s=\ell}^{\infty}\rho(s)\geq\beta>\frac{1}{4}, \tag{15}$$

eventually. Then, (E) *is oscillatory.*

Proof. From (15) and (8) for $j=0$, we have $Q(\ell\,;0)\geq\beta/\Omega(\ell)$. From (8) for $j=1$, we get

$$\begin{aligned}Q(\ell\,;1)&=\sum_{s=\ell}^{\infty}\frac{Q(s+1,0)Q(s,0)}{\eta(s)}+Q(\ell,0)\\&\geq\beta^{2}\sum_{s=\ell}^{\infty}\frac{1}{\Omega(s+1)\Omega(s)\eta(s)}+\frac{\beta}{\Omega(\ell)}\\&=\beta^{2}\sum_{s=\ell}^{\infty}\Delta\left(\frac{-1}{\Omega(s)}\right)+\frac{\beta}{\Omega(\ell)}\\&=\frac{\beta+\beta^{2}}{\Omega(\ell)}.\end{aligned}\tag{16}$$

Hence,

$$Q(\ell\,;1)\geq\frac{\beta_{1}}{\Omega(\ell)}, \tag{17}$$

where $\beta_1=\beta+\beta^2$.
In general,

$$Q(\ell\,;j)=\sum_{s=\ell}^{\infty}\frac{Q(s+1,j-1)Q(s,j-1)}{\eta(s)}+Q(\ell\,;0)\geq\frac{\beta_{j}}{\Omega(\ell)}, \tag{18}$$

where $\beta_j=\beta+\beta_{j-1}^2$ and $j=1,2,\cdots$. Clearly, $\beta=\beta_0<\beta_1<\beta_2<\cdots$. If β_j converges to some positive number α, then

$$\alpha=\beta_{0}+\alpha^{2}. \tag{19}$$

But there is no real positive solution for such an equation $\beta_0>1/4$. Thus, $\lim_{j\longrightarrow\infty}\beta_j=\infty$. Then, we have $\lim_{j\longrightarrow\infty}Q(\ell\,;j)=\infty$. Thus, from Theorem 3, it follows that (2) is oscillatory. Now using Theorem 3.5 of [23], we see that (1) is oscillatory. The proof of the theorem is complete. □

Next assume that the opposite condition of (15), namely,

$$\Omega(\ell)\sum_{s=\ell}^{\infty}\rho(s)\geq\beta\,\text{but}\,\beta\leq\frac{1}{4}, \tag{20}$$

holds.

Theorem 5. *Let* $\{v(\ell)\}$ *be a positive solution of (1) and*

$$\Omega(\ell)\sum_{s=\ell}^{\infty}\rho(s)\geq\beta>0, \tag{21}$$

eventually. Then, there is an integer L such that for $\ell\geq L$,

$$\left\{\frac{v(\ell)}{\Omega^{\beta}(\ell)}\right\}, \tag{22}$$

is monotonically nondecreasing.

Proof. The proof is similar to that of Theorem 3 of [22], and hence, the details are omitted. □

Next we state a comparison result, containing an advanced argument.

Theorem 6. *Let (21) hold. If the difference equation*

$$\Delta(\eta(\ell)\Delta v(\ell))+\left(\frac{\Omega(\sigma(\ell))}{\Omega(\ell+1)}\right)^{\beta}\rho(\ell)v(\ell+1)=0 \tag{23}$$

is oscillatory, then (1) is oscillatory.

Proof. The proof is similar to that of Theorem 4 in [22] and hence omitted. □

The above theorem ensures that any oscillation criterion established for (23) leads to an oscillation criterion for (1).

Theorem 7. *Let (21) hold. Assume that there is a constant* β_1 *such that*

$$\Omega(\ell)\sum_{s=\ell}^{\infty}\left(\frac{\Omega(\sigma(s))}{\Omega(s+1)}\right)^{\beta}\rho(s)\geq\beta_{1}>\frac{1}{4}, \tag{24}$$

eventually. Then, (1) is oscillatory.

Proof. The condition (24) guarantees that (23) oscillates, which in turn implies that (1) is oscillatory. This completes the proof. □

Next, we provide an example, illustrating this result.

Example 1. Consider the second-order advanced Euler type difference equation

$$\Delta\left(\frac{1}{(\ell+1)}\Delta v(\ell)\right)+\frac{b}{\ell(\ell+1)(\ell+2)}v(\lambda\ell)=0,\ \ell\geq1, \tag{25}$$

with $b>0$ and $\lambda\geq2$ is an integer.

Now $\Omega(\ell) \simeq \ell^2/2$. With $\beta = \mathrm{b}/4$ and by Theorem 7, equation (25) is oscillatory provided that

$$b\lambda^{\mathrm{b}/4} > 1. \tag{26}$$

For example, if $b = 1/2$, then it is required that $\lambda \geq 18$.

Note that the result in [22] cannot be applicable to (25) since $\eta(\ell) = 1/(\ell+1) < 1$.

If condition (24) fails to hold ($\beta_1 \leq 1/4$), then we can derive a new oscillation criterion using the constant β_1.

Theorem 8. *Let (21) hold. Assume that $\{v(\ell)\}$ is a positive solution of (1) and*

$$\Omega(\ell) \sum_{s=\ell}^{\infty} \left(\frac{\Omega(\sigma(s))}{\Omega(s+1)} \right)^{\beta} \rho(s) \geq \beta_1 > 0, \tag{27}$$

eventually. Then,

$$\left\{ \frac{v(\ell)}{\Omega^{\beta_1}(\ell)} \right\} \tag{28}$$

is monotonically nondecreasing.

Proof. Use [22], Theorem 2.6, to complete the proof. □

Theorem 9. *Let (21) and (24) hold. If the difference equation*

$$\Delta(\eta(\ell)\Delta v(\ell)) + \left(\frac{\Omega(\sigma(\ell))}{\Omega(\ell+1)} \right)^{\beta_1} \rho(\ell) v(\ell+1) = 0 \tag{29}$$

is oscillatory, then so is (1).

Theorem 10. *Let (21) and (24) hold. If there exists a constant β_2 such that*

$$\Omega(\ell) \sum_{s=\ell}^{\infty} \left(\frac{\Omega(\sigma(s))}{\Omega(s+1)} \right)^{\beta_1} \rho(s) \geq \beta_2 > \frac{1}{4} \tag{30}$$

eventually, then (1) is oscillatory.

The proofs of Theorems 9 and 10 follow from Theorems 6 and 7, and hence, the details are omitted.

Example 2. Consider the difference equation (25). For this equation $\beta_1 = b\lambda^{2b}$. By Theorem 10, we see that (25) is oscillatory provided that

$$b\lambda^{2\beta_1} > \frac{1}{4}. \tag{31}$$

Since $\beta_1 > b$, Theorem 10 improves Theorem 7.

For convenience, let us use the additional condition that there is a positive constant δ such that

$$\frac{\Omega(\sigma(\ell))}{\Omega(\ell+1)} \geq \delta > 1, \tag{32}$$

eventually. In view of (21), conditions (24) and (30) can be written in simpler form as

$$\beta_1 = \delta^{\beta}\beta > \frac{1}{4}, \tag{33}$$

$$\beta_2 = \delta^{\beta_1}\beta > \frac{1}{4}, \tag{34}$$

respectively. Repeating the above process, we have the increasing sequence $\{\beta_j\}_{j=0}^{\infty}$ defined as $\beta_0 = \beta$

$$\beta_{j+1} = \delta^{\beta_j}\beta. \tag{35}$$

Now as in [22], Theorem 2.9, one can generalize the oscillation criteria obtained in Theorems 7 and 10.

Theorem 11. *Let (21) and (32) hold. If there exists a positive integer m such that $\beta_j \leq 1/4$ for $j = 0, 1, \cdots, m-1$ and $\beta_m > 1/4$, then (1) is oscillatory.*

Example 3. Consider the second-order advanced difference equation

$$\Delta\left(\frac{1}{\ell+1} \Delta v(\ell) \right) + \frac{0.5}{\ell(\ell+1)(\ell+2)} v(5\ell) = 0, \ell \geq 1. \tag{36}$$

For this equation $\beta_0 = 1/8$ and $\delta = 25$. Through direct calculations, we get

$$\begin{aligned} \beta_1 &= 0.18692, \\ \beta_2 &= 0.22815. \end{aligned} \tag{37}$$

Thus, Theorems 7 and 10 fail for equation (36). But

$$\beta_3 = 0.26053 > 0.25, \tag{38}$$

and hence, Theorem 11 guarantees the oscillation of (36).

3. Conclusion

In this paper, we have derived a new comparison method to obtain the oscillation of second-order advanced difference equation which removed the restriction imposed on the coefficient $\{\eta(\ell)\}$ such that $\eta(\ell) \geq 1$ as in [22]. Thus, the oscillation criteria obtained in this paper improved and complemented to the existing results. It is an interesting problem to extend the results of this paper to equation (1) when it is in noncanonical form.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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