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Research Article

Benefits of Fluctuating Exchange Rates on the Investor's Wealth

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We consider a problem of maximizing the utility of an agent who invests in a stock and a money market account incorporating proportional transaction costs ($\lambda > 0$) and foreign exchange rate fluctuations. Assuming a HARA utility function $U(c) = c^p/p$ for all $c \geq 0$, $p < 1$, $p \neq 0$, we suggest an approach of determining the value function. Contrary to fears associated with exchange rate fluctuations, our results show that these fluctuations can bring about tangible benefits in one's wealth. We quantify the level of these benefits. We also present an example which illustrates an investment strategy of our agent.

1. Introduction

The study by [1] on portfolio optimization ignored the effects of transaction costs and concluded that the optimal policy is achieved by keeping a constant proportion of wealth in stock throughout the investment period. As a sequel to this study, several studies [2, 3] have shown how Merton's result can be altered when the investment is subjected to penalties under different strategies. Some studies [2] on investment subjected to transaction costs under different utility functions have shown how conclusions vary depending on whether the transaction costs are fixed or proportional or whether the utility function is of the exponential or logarithmic type (see, for example, [4] and the references therein). While most studies consider the problem of maximizing wealth when the only trading penalty results from transaction costs [5, 6], other studies (see, for example [7]) consider an investor holding bond and stock securities who consumes from an investment subjected to transaction costs and who receives a stochastic income that cannot be replicated by trading the available securities.

Some of these studies [4] have used a martingale and convex duality approach which allows very weak assumptions to be made about the dynamics of the underlying asset prices. Other studies, such as [2, 3], have used the dynamic programming principle approach which imposes a Markovian structure on the underlying asset prices. The latter approach is preferred by many researchers because it converts the problem to one of solving a partial differential equation which may be solvable either analytically or numerically.

Magill and Constantinides [2] considered the Merton problem in the presence of transaction costs and concluded through a heuristic argument that when the transaction cost is proportional to the amount transacted, the domain splits into three regions comprising the upper selling region, the lower buying region, and the middle no-trade region which is wedge-shaped. However, the study neither suggested how to compute the location of the wedge boundaries nor specified what the controlled processes should do when it reaches them. At the time this study appeared, the theory of local time and reflecting diffusion had not been developed. Hence, the authors could not make recommendations on how to compute the location of the wedge boundaries.

Davis and Norman [8] provided formulation and analysis which included an algorithm and numerical computations of the optimal policy. Their work motivated [9] to use the viscosity solution techniques instead of the classic stochastic control approach. [10] analyzed and derived the optimal transaction policy in an explicit form when risky assets are correlated and are subjected to fixed transaction costs in an infinite time horizon. However, [10] concluded that when asset returns are uncorrelated, the optimal investment policy is to keep the level of investment in each risky asset between two constant levels and, upon reaching either of these thresholds, to either buy or sell in order to remain between the thresholds. This analysis reviewed transaction cost as an important factor affecting trading volume which could significantly diminish the importance of the stock return predictability.

Bichuch [5] and Janecek and Shreve [6] considered two scenarios where an agent invested in a stock and a money market account in order to maximize wealth under two exit times, namely, either infinite time horizon or fixed maturity time T, in the presence of transaction costs ($\lambda > 0$). Their main goal was to provide a heuristic and rigorous derivation of the value function in powers of λ and to find for what power of λ the transaction costs are proportional to the amount invested. [11] heuristically studied the effect on one's investment which is subjected to either fixed or proportional transaction costs and recommended caution regarding the expansion of the value function as the two approaches (fixed or proportional transaction costs) yielded results that were at variance. In the presence of the proportional transaction costs, the investor's optimal investment dictates when to buy and sell. That is, the investor must buy some stocks as soon as the target amount falls below the lower bound or sell stocks when the risky asset rises beyond the upper bound.

Doctor et al. [12] analyzed the optimal portfolio selection problem of maximizing the utility of an agent who invests in a stock and a money market account in the presence of proportional transaction cost $\lambda > 0$ and foreign exchange rate. The stock price followed a (generalised) geometric Itô-Lévy process. Even though the stock price may have jumps, the research paper showed that if the jumps are small in absolute value, then the total payoff increases exponentially.

This study is motivated by the worsening investment climate in most African countries caused, among other things, by poor monetary policies and failure of fiscal policies to control budget deficits [13]. This has resulted in unstable local currencies and high inflation which in turn have greatly eroded the value of domestic debt instruments. Investors in African bonds have been subjected to negative real interest rates and perceived unfair government regulation on stock market returns [14].

Specifically, this study considers the effect of foreign exchange fluctuations on the investor's wealth. In a nutshell, the investor's interest is to invest her wealth in a foreign market to offset the loss due to the declining value of the local currency. The focus of our study concerns a risk averse trader who invests in the money market account (local government bond) and the stock when the two securities are held in two different currencies.

This study is applicable to many situations in developing countries, but we shall give examples of two countries in Southern Africa, namely, Botswana which is seen as a success story and Zimbabwe which has faced economic hardships [13].

Specifically, we have set up a hypothesised investment model as in [5] where now the securities are affected by foreign exchange fluctuations and ask how future payoffs are altered by these fluctuations. We present the strategy which illustrates how cumulative buying of a stock $(-1/\lambda < \xi \leq \xi_1(t))$ and cumulative selling of stocks $(\xi_2(t) \leq \xi < 1/\lambda)$ can result in a gain for the investor. We ask if our model can explain extreme economic scenarios in Africa.

The paper is organized in the following manner. In Section 2, we describe the problem and general market model together with its characterizations. Section 3 solves the investor's optimal investment problem in the absence of the transaction costs providing a benchmark for the subsequent analysis. In Section 4, we transform the problem into one with two variables and illustrate the benefits of fluctuating foreign exchange rates. In Section 5, we give conclusion. Lastly, we provide the references.

2. The Model

We consider a market consisting of two investment opportunities, a money market account and a stock. Suppose that the riskless account is in the local currency (a requirement in most third world countries), and the stock price is quoted in a foreign currency. The two assets are assumed to grow at interest rates r_1 and r_2, respectively. Due to uncertainty about the future exchange rates, the money market account is risky in relation to the foreign currency. The investor's wealth is determined by converting to a common currency, say the foreign currency. Converting to foreign currency is the preferred option for most African countries whose currencies continuously lose value against Western currencies. Let D_t be the rate of exchange at time $t \in [0, T]$ with dynamics described by a diffusion process:

$$dD(t) = D(t)[\mu_1 dt + \sigma_1 dB_1(t)], \tag{1}$$

where $B_1(t)$ is the Brownian motion. Let the bond share price $R^b(t)$ at time $t \in [0, T]$ reported in units of local currency be given by

$$dR^b(t) = r_1 R^b(t)dt, R^b(0) = 1. \tag{2}$$

Thus, the share price of the money market at time t in foreign currency is $R^b(t)D(t) = R(t)$, where

$$dR(t) = R(t)[(r_1 + \mu_1)dt + \sigma_1 dB_1(t)], R(0) = D(0) = r_0. \tag{3}$$

We have opted to convert the returns from the bond, if any, to foreign currency, as mentioned before, as this gives the real value of the investor's wealth. We note that in the bond markets, neither the stock nor the exchange rate is tradable. However, $\{R_t\}_{t\geq 0}$ can be viewed as tradable. The share price of stock, S_t, which is in dollars at any time $t \geq 0$, is assumed to follow a geometric Brownian motion:

$$dS(t) = \mu_2 S(t)dt + \sigma_2 S(t)dB_2(t), S(0) = s_0, \tag{4}$$

where $\mu_j > r_j > 0$ and $\sigma_j > 0$ for $j = 1, 2$ are (constants) known as mean rates of return and volatilities of the exchange rate and stock, respectively. $\{B_i(t)\}_{t\ge 0}$ is the standard Brownian motion on a filtered probability space $(\Omega, F, \{F_t\}_{t\ge 0}, \mathbb{P})$ with $B_i(0)=0$, $i=1,2$, almost surely. The correlation between $\{B_1(t)\}_{t\ge 0}$ and $\{B_2(t)\}_{t\ge 0}$ is given by

$$B_1(t)=\beta B_2(t)+\sqrt{1-\beta^2}B(t), -1\le\beta\le 1, \quad (5)$$

where β is the correlation coefficient and $B(t)$ is a Brownian motion independent of $B_2(t)$. Note that if $\beta=1$, then the bond is perfectly correlated to the stock, and if $\beta=-1$, it is perfectly anticorrelated to the stock price. Define the dynamics of the bond and stock holdings as

$$\begin{aligned} dX_t &= (r_1+\mu_1)X_t dt+\sigma_1 X_t dB_1(t)-(1+\lambda)dL_t \\ &\quad +(1-\lambda)dM_t, X_0=x, \\ dY_t &= Y_t[\mu_2 dt+\sigma_2 dB_2(t)]+dL_t-dM_t, Y_0=y, \end{aligned} \quad (6)$$

where $\lambda\in[0,1)$ accounts for transaction costs paid from the money market, L_t represents the cumulative dollar value of stock purchased up to time t, and M_t is the cumulative dollar value of stock sold. We have adopted the dynamic programming approach since it is not the wealth bounds that are of interest but the effect of currency fluctuations and how it alters the investment. Our approach is different from [5] who used a power series approach.

The agent must choose a policy consisting of two adapted processes L_t and M_t that are nondecreasing and right-continuous with left limits and $L_{0-}=M_{0-}=0$. Note that purchase of dL_t units of stock requires a payment of $(1+\lambda)dL_t$ from the money market account while sale of dM_t units of stock realizes $(1-\lambda)dM_t$ in cash. The investor's net wealth in monetary terms at time $t\in[0,T]$ is

$$W_t=X_t+Y_t-\lambda|Y_t|. \quad (7)$$

Define the solvency region

$$Q_\lambda := \{(x,y)\,;\, x+(1+\lambda)y\ge 0, x+(1-\lambda)y\ge 0\} \quad (8)$$

as the set of positions from which the agent can move to gain in wealth. The policy $(L_s, M_s)_{s\in[t,T]}$ is admissible for the initial position (t,x,y) if (X_s, Y_s) starting from $(X_{t\_}, Y_{t\_})=(x,y)$ remains in $\bar{Q}_\lambda \forall s\in[t,T]$. Since the agent may choose to rebalance her position, we have set the initial time to be $t\_$. We let $A(t,x,y)$ be the set of all such admissible policies.

Let $U(c)$ be the agent's utility function given by $U(c)=c^p/p$ for all $c\ge 0$, $p<1$, $p\ne 0$. Define the value function as the supremum of the utility of the final cash position, after the agent liquidates her stock holdings as

$$V(t,x,y)=\sup_{(L,M)\in A(t,x,y)} \mathbb{E}^{x,y}[U(X_T+Y_T-\lambda|Y_T|)|F_t], \quad (9)$$

for $(t,x,y)\in[0,T]\times\bar{Q}_\lambda$. The auxiliary or discounted value function is given by

$$v_\rho(t,x,y)=\sup_{(L,M)\in A(t,x,y)} \mathbb{E}^{x,y}\left[e^{-\rho(T-t)}U(X_T+Y_T-\lambda|Y_T|)\Big|F_t\right], \quad (10)$$

for $(t,x,y)\in[0,T]\times\bar{Q}_\lambda$ and $\rho\ge 0$. Here, ρ is the discounting factor and $\mathbb{E}^{x,y}$ denotes the conditional expectation at time t given the initial endowments as $X_t=x$ and $Y_t=y$. We express (9) as

$$V(t,x,y)=e^{\rho(T-t)}v_\rho(t,x,y), \rho\ge 0, (t,x,y)\in[0,T]\times\bar{Q}_\lambda. \quad (11)$$

As will be proved in the next section, the problem involving fluctuating exchange rate but no transaction costs ($\lambda=0$) has the explicit solution

$$v_\rho(t,x,y)=\frac{1}{p}e^{pA(T-t)}(x+y)^p, \rho\ge 0, (t,x,y)\in[0,T]\times\bar{S}_\lambda, \quad (12)$$

where

$$\begin{aligned} A &= -\frac{\rho}{p}+\theta(\mu_2-\alpha)+\alpha+\frac{p-1}{2}\left(\sigma_1^2\theta(1-)^2+2\beta\sigma_1\sigma_2\theta(1-\theta)+\sigma_2^2\theta^2\right), \\ \theta &= \frac{\sigma_1^2-\beta\sigma_1\sigma_2}{\sigma_1^2-2\beta\sigma_1\sigma_2+\sigma_2^2}-\frac{\mu_2-\alpha}{(p-1)\left(\sigma_1^2-2\beta\sigma_1\sigma_2+\sigma_2^2\right)}, \\ \alpha &= r_1+\mu_1. \end{aligned} \quad (13)$$

θ is the Merton proportion when the investment is affected by fluctuations in the exchange rate, that is, fraction of the wealth invested in stock.

3. No Transaction Costs Case

In the case $\lambda=0$, the investment problem is to choose an admissible proportion θ that maximizes the auxiliary value function

$$V(t,w)=\mathbb{E}\left[e^{-\rho(T-t)}U(W_T)\Big|F_t\right], \quad (14)$$

given that the total wealth W_t evolves according to the equation below:

$$dW_t=W_t[(\theta(\mu_2-\alpha)+\alpha)dt+\sigma_1(1-\theta)dB_1(t)+\sigma_2\theta dB_2(t)], W_0=w. \quad (15)$$

We note that the squared variation of the wealth process when the Brownian motions correlate is

$$(dW_t)^2=W_t^2\left(\sigma_1^2(1-\theta)^2+2\beta\sigma_1\sigma_2\theta(1-\theta)+\sigma_2^2\theta^2\right)dt, \quad (16)$$

employing the following rules: $(dB_2(t))^2=(dB_1(t))^2=dt$, $(dt)^2=dt\cdot dB_1(t)=dt\cdot dB_2(t)=0$, and $dB_1(t)\cdot dB_2(t)=\beta d$

t. As an example, we have maximized (14) over the proportion θ for the wealth process (15) for a specific utility function $U(c) = c^p/p$ for all $c \geq 0$, $p < 1$, $p \neq 0$. The following lemma takes into account the influence of exchange rate fluctuations on the wealth process.

Lemma 1 (modified Merton results). *Suppose $\lambda = 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\mu_1 > 0$, and $0 \neq p < 1$. Then, a risk averse investor's value function is given by*

$$v_\rho(t, w) = \frac{1}{p} e^{pA(T-t)} w^p, \tag{17}$$

where

$$A = -\frac{\rho}{p} + \theta(\mu_2 - \alpha) + \alpha + \frac{p-1}{2}\left(\sigma_1^2(1-\theta)^2 + 2\beta\sigma_1\sigma_2\theta(1-\theta) + \sigma_2^2\theta^2\right), \tag{18}$$

and the exchange rate affecting optimal proportion invested in the stock is given by

$$\theta = \frac{\sigma_1^2 - \beta\sigma_1\sigma_2}{\sigma_1^2 - 2\beta\sigma_1\sigma_2 + \sigma_2^2} - \frac{\mu_2 - r_1 - \mu_1}{(p-1)\left(\sigma_1^2 - 2\beta\sigma_1\sigma_2 + \sigma_2^2\right)}. \tag{19}$$

Proof of Lemma 1 (the optimal strategy). Applying the Itô lemma on the value function V given in (14), we have

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial w} dW_t + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(dW_t)^2. \tag{20}$$

We obtain the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\dot{V} + W_t(\theta(\mu_2 - \alpha) + \alpha) V_w + \frac{1}{2}\left(\sigma_1^2(1-\theta)^2\right. \\ \left. + 2\beta\sigma_1\sigma_2\theta(1-\theta) + \sigma_2^2\theta^2\right) W_t^2 V_{ww} = 0,\end{aligned} \tag{21}$$

where the rule $\mathbb{E}(dB_1(t)) = \mathbb{E}(dB_2(t)) = 0$ has been used. The optimal strategy is given by differentiating (21) with respect to θ. That is,

$$\begin{aligned}W_t(\mu_2 - \alpha)V_w + \left(-\sigma_1^2 + \theta\left(\sigma_1^2 - 2\beta\sigma_1\sigma_2 + \sigma_2^2\right) + 2\beta\sigma_1\sigma_2\right) \\ \cdot W_t^2 V_{ww} = 0.\end{aligned} \tag{22}$$

Solving for θ yields

$$\theta = \frac{\left(\sigma_1^2 - \beta\sigma_1\sigma_2\right) W_t V_{ww} - (\mu_2 - \alpha) V_w}{\left(\sigma_1^2 - 2\beta\sigma_1\sigma_2 + \sigma_2^2\right) W_t V_{ww}}. \tag{23}$$

The risk averse investor's value function is given as

$$V(t, w) = \frac{1}{p} e^{-\rho(T-t)} h(t) w^p, p < 1, p \neq 0. \tag{24}$$

Substituting (24) into (21), we obtain

$$h(t) = e^{p(A_0 + \rho/p)(T-t)} \tag{25}$$

and the optimal portfolio as in (19). Finally, the value function is

$$\begin{aligned}V(t, w) &= \frac{1}{p} e^{\rho(T-t)} e^{p(A_0 - \rho/p)(T-t)} w^p \\ &= e^{\rho(T-t)}\left[\frac{1}{p} e^{p(A_0 - \rho/p)(T-t)} w^p\right] \\ &= e^{\rho(T-t)}\left[\frac{1}{p} e^{pA(T-t)} w^p\right] = e^{\rho(T-t)} v_\rho,\end{aligned} \tag{26}$$

where

$$A_0 = \theta(\mu_2 - \alpha) + \alpha + \frac{p-1}{2}\left(\sigma_1^2(1-\theta)^2 + 2\beta\sigma_1\sigma_2\theta(1-\theta) + \sigma_2^2\theta^2\right), \tag{27}$$

and A is as in (18). □

Remark 2. The risk averse investor of Lemma 1 intends to realize a value $v_\rho(t, w)$ in (19). She invests a proportion θ in (19) in a stock. By discounting her stock investment at the rate $e^{\rho(T-t)}$, she hopes to realize a value v_ρ in (19) at the terminal time T.

Remark 3.

(a) Note that with $\sigma_1 = \mu_1 = 0$, expression (19) reduces to the original Merton proportion

(b) If the exchange rate is purely deterministic, that is, $\sigma_1 = 0$, $\mu_1 > 0$, then

$$\theta = \frac{\mu_2 - (r_1 + \mu_1)}{(1-p)\sigma_2^2}. \tag{28}$$

We note the following:

(i) If $\mu_2 > r_1 + \mu_1$, then the investor should invest a fraction θ of her wealth in stock and remaining $1 - \theta$ in bond

(ii) If $\mu_2 = r_1 + \mu_1$, then $\theta = 0$ and the investor should sell stocks and invest in the bond account

(iii) If $\mu_2 < r_1 + \mu_1$, then $\theta < 0$ and the investment strategy requires borrowing funds which in our case we do not allow

(c) For $\sigma_1 > 0$ and $\mu_1 > 0$, we note the following observations:

(i) If $(1-p)(\sigma_1^2 - \beta\sigma_1\sigma_2) \leq \mu_1$, the investment is not benefitting from fluctuations in the exchange rate and the investor is advised to invest *smaller* fraction of the wealth in stock

(ii) If $(1-p)(\sigma_1^2 - \beta\sigma_1\sigma_2) > \mu_1$, then the effect of fluctuations in the foreign exchange is beneficial and the investor is advised to invest a *larger* portion of the wealth in stock

(d) As $\sigma_1 \longrightarrow \infty$, $\mu_1 \longrightarrow \infty$, then the portfolio fraction $\theta \longrightarrow 1$.

4. The Case with Transaction Costs

In contrast to the previous section, the introduction of transaction costs does not allow for a single HJB equation but rather a pair of HJB equations, each of which applies in a different region. The approach adopted is to find a transformation which helps us to use the simple ideas of the previous section.

Theorem 4 (see [5]). *The value function $v(t,x,y)$ defined by equation (10) is a viscosity solution of the following HJB equation*

$$\min\left\{-v_t - K(v), -(1-\lambda)v_x + v_y, (1+\lambda)v_x - v_y\right\} = 0, \quad (29)$$

on $[0,T]\times Q_\lambda$ where K is the second-order differential operator given by

$$K(v) = \frac{1}{2}\left[\sigma_2^2 y^2 v_{yy}(t,x,y) + \sigma_1^2 x^2 v_{xx}(t,x,y)\right] + \alpha x v_x(t,x,y) + \mu_2 y v_y(t,x,y) + \beta\sigma_1\sigma_2 xy v_{xy}(t,x,y) - \rho v(t,x,y), \quad (30)$$

with the terminal condition

$$v(T,x,y) = U(x+y-\lambda|y|), (x,y)\in\bar{Q}_\lambda, \quad (31)$$

and with the property $v(t,\gamma x,\gamma y) = \gamma^p v(t,x,y)$ for $\gamma > 0$.

The admissible policy satisfies $A(t,\gamma x,\gamma y) = \{(\gamma L,\gamma M) : (L,M)\in A(t,x,y)\}$.

Remark 5. We note that the discriminant $(\beta^2-1)\sigma_1^2\sigma_2^2x^2y^2$ allows us to classify the second-order differential operator K with respect to the correlation parameter as follows:

(1) If $(\beta^2-1)\sigma_1^2\sigma_2^2x^2y^2 > 0$ which implies that $\beta^2-1>0$ or $\beta\in(-\infty,-1)\cup(1,\infty)$, $\forall(x,y)\in Q_\lambda$, then the operator is hyperbolic. This case cannot apply for a β interpreted as a correlation, and so we do not consider it further here

(2) If $(\beta^2-1)\sigma_1^2\sigma_2^2x^2y^2 < 0$ which implies that $\beta^2-1<0$ or $\beta\in(-1,1)$, $\forall(x,y)\in Q_\lambda$, then we have an elliptic operator. This case applies when the random drivers of the stock price and the foreign exchange rate are not perfectly correlated

(3) If $(\beta^2-1)\sigma_1^2\sigma_2^2x^2y^2 = 0$ which implies that $\beta^2-1=0$ or $\beta\in\{-1,1\}$, $\forall(x,y)\in Q_\lambda$, then the operator is parabolic. This applies when the stock and the foreign exchange drivers are more in lock step because they are perfectly correlated or perfectly uncorrelated.

As the correlation parameter satisfies $|\beta|\le 1$, we can focus our attention on the parabolic and elliptic cases.

Defining

$$\xi = \frac{y}{x+y}, \qquad 1-\xi = \frac{x}{x+y}, \quad (32)$$

we can transform the value function $v(t,x,y)$ to one with two variables, that is,

$$(x+y)^{-p}v(t,x,y) = v(t,\xi), (t,\xi)\in[0,T]\times\bar{Q}_v, \bar{Q}_v = \left[\frac{-1}{\lambda},\frac{1}{\lambda}\right]. \quad (33)$$

The problem in Theorem 4 can now be converted to the following one.

Theorem 6. *The value function $v(t,\xi)$ is the viscosity solution of the HJB equation:*

$$L(v) = \min\left\{-v_t - P(v), \lambda p v(t,\xi) + (1-\lambda\xi)v_\xi(t,\xi), \lambda p v(t,\xi) - (1+\lambda\xi)v_\xi(t,\xi)\right\} = 0, \quad (34)$$

on $[0,T]\times\bar{Q}_v$ with

$$P(v) = p\left\{A + \frac{(p-1)}{2}\left(\sigma_1^2+\sigma_2^2-2\beta\sigma_1\sigma_2\right)(\xi-\theta)^2\right\}v(t,\xi) + \xi(1-\xi)\left[(p-1)\left(\sigma_1^2+\sigma_2^2-2\beta\sigma_1\sigma_2\right)(\xi-\theta) - \beta\sigma_1\sigma_2\xi\right] \cdot v_\xi(t,\xi) - \frac{\xi^2(1-\xi)^2}{2}\left(\sigma_2^2+\sigma_1^2-2\beta\sigma_1\sigma_2\right)v_{\xi\xi}(t,\xi), \quad (35)$$

and the terminal condition

$$v(T,\xi) = U(1-\lambda|\xi|), \xi\in\bar{Q}_v. \quad (36)$$

Lemma 7. *Given the transformed version of the value function (12) without the influence of transaction costs (as in (33)), then the following holds:*

$$\beta = \frac{\sigma_2^2+\sigma_1^2}{2\sigma_2\sigma_1} \quad \text{or } \xi = \theta, \quad (37)$$

where $\sigma_2,\sigma_1>0$ and θ is the portfolio choice.

Remark 8.

(1) (35) reduces to [5] if $\sigma_1 = \mu_1 = 0$; that is, the effect of the exchange rate is removed

(2) However, we want to illustrate the benefits of fluctuating exchange rates under a special scenario when $P_0(\nu(t,\xi)) = 0$ and $\nu_t(t,\xi) + P_1(\nu(t,\xi)) = 0$. We consider the following:

$$\nu_t(t,\xi) + P(\nu) = \nu_t(t,\xi) + P_0(\nu) + P_1(\nu), \tag{38}$$

where

$$P_0(\nu(t,\xi)) = \frac{p(p-1)}{2} B\left(\xi^2 \nu_{\xi\xi}(t,\xi) + \xi\nu_\xi(t,\xi) + \left(\xi^2 - \theta^2\right)\nu(t,\xi)\right),$$

$$\begin{aligned} P_1(\nu(t,\xi)) &= p[A + (p-1)\pi(\xi-\theta)B]\nu(t,\xi) \\ &\quad + \xi\left[(p-1)B\left(\xi(1-\xi+\theta) - \theta - \frac{p}{2}\right) - \xi(1-\xi)\beta\sigma_1\sigma_2\right] \\ &\quad \cdot \nu_\xi(t,\xi) + \frac{\xi^2}{2}B\left(\left(2p^2-2p-1\right)(2-\xi)\xi - \left(p^2-p-1\right)\right)\nu_{\xi\xi}(t,\xi), \end{aligned} \tag{39}$$

with $B = \sigma_1^2 + \sigma_2^2 - 2\beta\sigma_1\sigma_2$. Specifically, the idea is to find the solution $\nu(t,\xi)$, given that $(t,\xi) \in [0,T] \times [-1/\lambda, 1/\lambda]$.

We shall show that the problem stated in Remark (8) highlights the benefits of a fluctuating foreign exchange. For $(t,\xi) \in [0,T] \times Q_\nu$, we also define the following first-order differential operators:

$$\begin{aligned} (S\nu)(t,\xi) &= \lambda p\nu(t,\xi) + (1-\lambda\xi)\nu_\xi(t,\xi), \\ (B\nu)(t,\xi) &= \lambda p\nu(t,\xi) - (1+\lambda\xi)\nu_\xi(t,\xi). \end{aligned} \tag{40}$$

Let the functions $\xi_1(t)$ and $\xi_2(t)$ be such that $0 \le \xi_1(t) \le \xi_2(t) < \infty$. Then, the no-trade region is the domain

$$\{(t,\xi) \mid t \in [0,T], \xi \in Q_\nu, \xi_1(t) < \xi < \xi_2(t)\}. \tag{41}$$

Within this region, $-\nu_t - P(\nu) = 0$. This has the following interpretation: If $\xi < \xi_1(t)$, the investor should buy the stock in order to move to the boundary $\xi_1(t)$ or inside the no-trade region. If $\xi_2(t) < \xi$, the investor should sell stock to move to the boundary $\xi_2(t)$ or inside the no-trade region.

We see from Theorem 6 that for all $t \in [0,T]$, we have to solve

$$\lambda p\nu(t,\xi) - (1+\lambda\xi)\nu_\xi(t,\xi) = 0, -\frac{1}{\lambda} < \xi \le \xi_1(t), \tag{42}$$

$$\left\{\begin{array}{c} \xi^2\nu_{\xi\xi}(t,\xi) + \xi\nu_\xi(t,\xi) + \left(\xi^2-\theta^2\right)\nu(t,\xi) = 0 \\ \nu_t(t,\xi) + P_1(\nu(t,\xi)) = 0 \end{array}\right\}, \xi_1(t) \le \xi \le \xi_2(t), \tag{43}$$

$$\lambda p\nu(t,\xi) + (1-\lambda\xi)\nu_\xi(t,\xi) = 0, \xi_2(t) \le \xi < \frac{1}{\lambda}. \tag{44}$$

Equations (42) and (44) have the solutions

$$\nu(t,\xi) = \nu(t,\xi_1)\left(\frac{1+\lambda\xi}{1+\lambda\xi_1(t)}\right)^p, \xi \in \left(-\frac{1}{\lambda}, \xi_1(t)\right], \tag{45}$$

$$\nu(t,\xi) = \nu(t,\xi_2)\left(\frac{1-\lambda\xi}{1-\lambda\xi_2(t)}\right)^p, \xi \in \left[\xi_2(t), \frac{1}{\lambda}\right), \tag{46}$$

respectively.

We can determine the lower and upper boundaries for our problem from (45) and (46) through a limit process as shown below. However, the choice of the boundaries depends on the amount of risk the investor is willing to take. Essentially, one can set the trading margins within the lower and upper boundaries. The extreme lower and upper boundaries can be approximated as follows:

$$F_{\text{Buy}}(\xi) = \lim_{p \longrightarrow 0}\left(\frac{1+\lambda\xi}{1+\lambda\xi_1}\right)^p = 1, \tag{47}$$

which is a horizontal line, and

$$F_{\text{Sell}}(\xi) = \lim_{p \longrightarrow 1}\left(\frac{1-\lambda\xi}{1-\lambda\xi_2}\right)^p = \frac{1-\lambda\xi}{1-\lambda\xi_2} \tag{48}$$

which is a linear function with gradient $-\lambda/1-\lambda\xi_2$. The intersection point of the two linear functions is $(\xi_2, 1)$, implying that the upper extreme boundary is a decreasing linear function. This, in turn, means that the returns for the investor decrease as $\xi(t)$ increases. We propose in this study to use the intersection point $(\xi_2, 1)$ as the point to exit the market. The investor is advised to change the strategy before this point is reached.

The no-trade region equation (43) is of Bessel type, and we can therefore solve it explicitly. Setting $\nu(t,\xi) = K_\theta(\xi)l(t)$ such that $K_\theta(\xi), l(t) \ne 0$, then equation (43) becomes

$$\xi^2 K''_\theta(\xi) + \xi K'_\theta(\xi) + \left(\xi^2 - \theta^2\right)K_\theta(\xi) = 0, \xi \ge 0, 0 \le \theta \le 1. \tag{49}$$

The general solution of (49) of order θ is

$$K_\theta(\xi) = \begin{cases} C_1 J_\theta(\xi) + C_2 Y_\theta(\xi), & \theta \in \mathbb{Z}, \\ C_3 J_\theta(\xi) + C_4 J_{-\theta}(\xi), & \theta \notin \mathbb{Z}, \end{cases} \tag{50}$$

where

$$\begin{aligned} J_\theta(\xi) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1+\theta+k)}\left(\frac{\xi}{2}\right)^{2k+\theta}, \\ J_{-\theta}(\xi) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1-\theta+k)}\left(\frac{\xi}{2}\right)^{2k-\theta}, \\ Y_\theta(\xi) &= \frac{J_\theta(\xi)\cos(\theta\pi) - J_{-\theta}(\xi)}{\sin(\theta\pi)}, \end{aligned} \tag{51}$$

and $\mathbb{Z}$ is an integer, that is, in our case for $\theta \in \mathbb{Z}$ implies $\theta \in \{0, 1\}$ and for $\theta \notin \mathbb{Z}$ implies $0 < \theta < 1$. Here, $\Gamma(\cdot)$ is a gamma function and arbitrary real numbers $C_i, i = 1, \cdots, 4$, do not depend on the argument θ. From equation (50) and the derivatives thereof, the equation $v_t(t,\xi) + P_1(v(t,\xi)) = 0$ gives

$$l(t) = l(T) \exp\{L_\theta(\xi)(T-t)\}, t \in [0, T], \tag{52}$$

where

$$L_\theta(\xi) = \frac{h_2(\xi)}{K_\theta(\xi)} K''_\theta(\xi) + \frac{h_1(\xi)}{K_\theta(\xi)} K'_\theta(\xi) + h_0(\xi), \tag{53}$$

with

$$h_0(\xi) = p\left[A + (p-1)\theta(\xi-\theta)\left(\sigma_1^2 + \sigma_2^2 - 2\beta\sigma_1\sigma_2\right)\right], \tag{54}$$

$$\begin{aligned} h_1(\xi) = \xi\Big[&(p-1)\left(\sigma_1^2+\sigma_2^2-2\beta\sigma_1\sigma_2\right)\left(\xi(1-\xi+\theta)-\theta-\frac{p}{2}\right) \\ &-\xi(1-\xi)\beta\sigma_1\sigma_2\Big], \end{aligned} \tag{55}$$

$$\begin{aligned} h_2(\xi) = &\frac{\xi^2}{2}\left(\sigma_1^2+\sigma_2^2-2\beta\sigma_1\sigma_2\right) \\ &\cdot\left(\left(2p^2-2p-1\right)(2-\xi)\xi-\left(p^2-p-1\right)\right), \end{aligned} \tag{56}$$

$$K_\theta(\xi) = \begin{cases} C_1 J_\theta(\xi) + C_2 Y_\theta(\xi), & C_1, C_2 \in \mathbb{R}, \\ C_3 J_\theta(\xi) + C_4 J_{-\theta}(\xi), & C_3, C_4 \in \mathbb{R}. \end{cases} \tag{57}$$

Therefore, assuming $l(T) = 1$, then

$$\begin{aligned} v(t,\xi) &= K_\theta(\xi)\cdot\exp\{L_\theta(\xi)(T-t)\} \\ &= \exp\{pA(T-t)\} \\ &\quad\cdot\exp\left\{\left[\frac{h_2(\xi)}{K_\theta(\xi)}K''_\theta(\xi) + \frac{h_1(\xi)}{K_\theta(\xi)}K'_\theta(\xi) + h_0^{å}(\xi)\right](T-t)\right\} \\ &\quad\times K_\theta(\xi), \end{aligned} \tag{58}$$

where

$$h_0^{å}(\xi) = p(p-1)\theta(\xi-\theta)\left(\sigma_1^2+\sigma_2^2-2\beta\sigma_1\sigma_2\right). \tag{59}$$

4.1. Case 1 ($\theta \in \{0, 1\}$). These are cases whereby the trader invests in one asset only; that is, for $\theta = 0$, the trader invests her wealth in money market account only while for $\theta = 1$, the investor invests in stock only. The two cases are not of interest in our analysis.

4.2. Case 2 ($0 < \theta < 1$). We state the following lemma.

Lemma 9. *Suppose $\sigma_i > 0, i \in \{1, 2\}$, $0 \neq p < 1$ and $\beta \in [-1, 1]$, then the value function in the no-trade region $\xi \in [\xi_1(t), \xi_2(t)]$ is given by*

$$\begin{aligned} v(t,\xi) &= \exp\{pA(T-t)\} \\ &\quad\cdot\exp\left\{\left[\frac{h_2(\xi)}{K_\theta(\xi)}K''_\theta(\xi) + \frac{h_1(\xi)}{K_\theta(\xi)}K'_\theta(\xi) + h_0^{å}(\xi)\right](T-t)\right\} \\ &\quad\times K_\theta(\xi), \end{aligned} \tag{60}$$

where $h_2(\xi)$, $h_1(\xi)$, $h_0^{å}(\xi)$, $K_\theta(\xi)$ are as in (56), (55), (59), and (57), respectively. Moreover, the optimal proportion is as in (17).

Lemma 10. *From Lemma (9), equations (45) and (46) become*

$$\begin{aligned} v(t,\xi) &= v(t,\xi_1)\left(\frac{1+\lambda\xi}{1+\lambda\xi_1(t)}\right)^p \\ &= K_\theta(\xi_1)\exp\{L_\theta(\xi_1)(T-t)\} \\ &\quad\cdot\left(\frac{1+\lambda\xi}{1+\lambda\xi_1(t)}\right)^p, \xi \in \left(-\frac{1}{\lambda}, \xi_1(t)\right], \end{aligned} \tag{61}$$

$$\begin{aligned} v(t,\xi) &= v(t,\xi_2)\left(\frac{1-\lambda\xi}{1-\lambda\xi_2(t)}\right)^p \\ &= K_\theta(\xi_2)\exp\{L_\theta(\xi_2)(T-t)\} \\ &\quad\cdot\left(\frac{1-\lambda\xi}{1-\lambda\xi_2(t)}\right)^p, \xi \in \left[\xi_2(t), \frac{1}{\lambda}\right), \end{aligned} \tag{62}$$

where

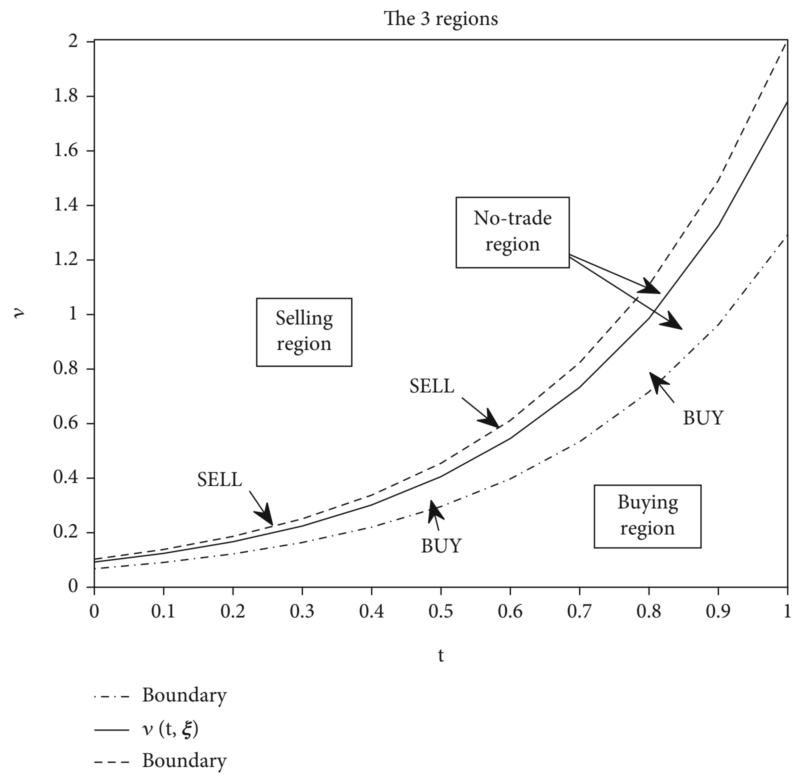
$$L_\theta(\xi_1) = \frac{h_2(\xi_1)}{K_\theta(\xi_1)}K''_\theta(\xi_1) + \frac{h_1(\xi_1)}{K_\theta(\xi_1)}K'_\theta(\xi_1) + h_0(\xi_1), \tag{63}$$

$$L_\theta(\xi_2) = \frac{h_2(\xi_2)}{K_\theta(\xi_2)}K''_\theta(\xi_2) + \frac{h_1(\xi_2)}{K_\theta(\xi_2)}K'_\theta(\xi_2) + h_0(\xi_2), \tag{64}$$

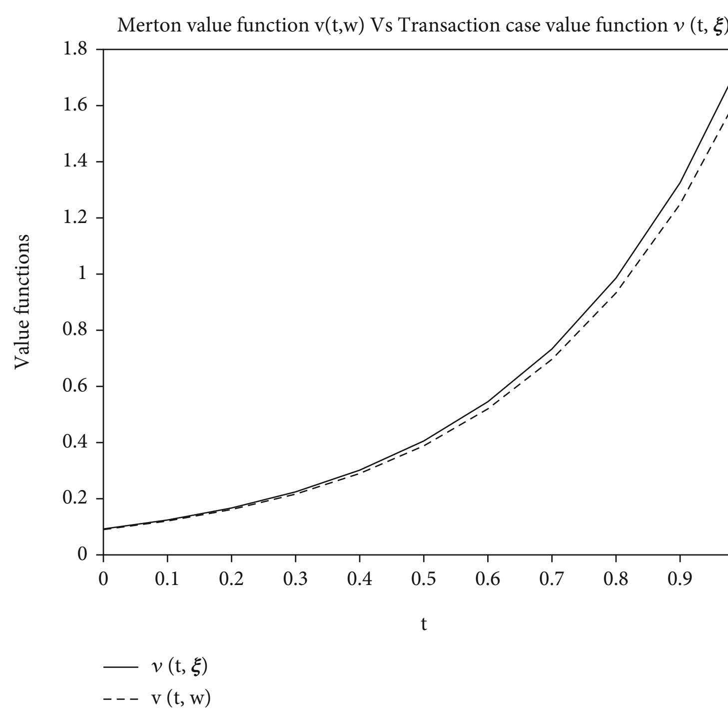
respectively.

Figure 1(a) illustrates the evolution of the investor's cumulative wealth and the strategy of selling stocks in the upper region and buying stocks in the lower region so that the cumulative wealth process is reflected into the no-trade region from below and above. The trading boundaries are marked for $p = 0.5$. We note that the investor who is less averse to risk can choose higher values of p. Despite intervention from the investor, the wealth process shows rising returns. However, the investor must trade with caution as the returns will decrease in time as suggested by the limiting process $F_{\text{Sell}}(\xi)$ and $F_{\text{Buy}}(\xi)$ in equations (48) and (47).

Figure 1(b) compares the Merton value function with the reflected wealth process in the no-trade region. Clearly, both wealth processes are rising in the no-trade region, but the reflected process outperforms the Merton process as time increases.



(a)



(b)

Figure 1: Plots of modified Merton value function and the value function in the three regions with $\lambda = 0.5$. $\mu_1 = 0.02$, $\mu_2 = 0.2$, $\sigma_1 = 0.25$, $\sigma_2 = 0.5$, $r_1 = 0.09$, $\rho = 3$, $p = 0.5$, and $\beta = 0.2$. Then, the portfolio becomes $\theta = 0.8286$.

Remark 11. Moreover, comparing equation (26) in Lemma (1) and equation (60) in Lemma (9), we observe that

$$w^p = pK_\theta(\xi), \tag{65}$$

$$\rho = \frac{h_2(\xi)}{J_\theta(\xi)} J''_\theta(\xi) + \frac{h_1(\xi)}{J_\theta(\xi)} J'_\theta(\xi) + h_0^{å}(\xi) > 0. \tag{66}$$

4.3. An Example: Use of the Boundaries $\xi_i(t)$'s. At this juncture, we consider the nature of the boundaries $\xi_i(t), i \in \{1, 2\}, t \in [0, T]$. This is crucial since it serves as an indicator to the investor when to sell and when to buy. That is, if the investor's position is in the buying region (lower than $\xi_1(t)$), then she has to buy stock to move to the boundary $\xi = \xi_1(t)$ or inside the no-intervention region, where equation (61) holds. These boundaries $\xi_1(t)$ and $\xi_2(t)$ are assumed to be perfectly absorbing with no trading at the boundaries. We have not definitively determined the shapes of $\xi_1(t)$ and $\xi_2(t)$, but we can see that they are curves of parabolic shapes from Figure 1.

5. Conclusion

Despite the integration of the transaction costs and foreign exchange fluctuations in the investment model, our results show the investor's benefits resulting from the investment strategy. The strategy of a reflecting and an absorbing boundary illustrated with an example ensures that selling and buying of stocks result in a net gain for the investor. The theory of reflected processes has been applied before, but we have a unique scenario where the no-intervention region is placed between two reflecting boundaries.

This study has suggested a way of fixing the extreme boundaries within which investors can set their own trading boundaries to suit their averse to risk. The set back is that while data on buying and selling of stocks in emerging African markets is available, the corresponding bond values are not available, making it impossible to calibrate our model to actual market data.

We have opted to demonstrate the results of our model using cumulative volumes of traded assets following the work [15] which has shown the advantage of working with cumulative volume of traded assets. This approach yields conclusions that are more reliable and provide better forecasts.

We can see (Figure 1(b)) that the reflected cumulative wealth process outperforms the Merton process in the no-intervention region despite losses due to transaction costs. The fluctuations in the exchange rate for both the bond and the stock provide a safe guard for the declining local currency. Moreover, our strategy has the advantage in so far as the investor has an opportunity to raise capital from the selling of stocks before the expiry date $t = T$ and investing it elsewhere to increase the security of her overall investment.

Appendix

We assume the solution of equation (49) is of the form:

$$K_\theta(\xi) = \sum_{m=0}^{\infty} a_m \xi^{m+r}, \tag{A.1}$$

where $a_m \neq 0$, which leads to

$$\begin{aligned} &a_0\left(r^2 - \theta^2\right)\xi^r + a_1\left[(r+1)^2 - \theta^2\right]\xi^{r+1} \\ &\quad + \sum_{m=2}^{\infty}\left(a_m\left[(r+m)^2 - \theta^2\right] + a_{m-2}\right)\xi^{r+m} = 0. \end{aligned} \tag{A.2}$$

Equating coefficients of the series to zero gives

$$a_0\left(r^2 - \theta^2\right) = 0 (m = 0), \tag{A.3}$$

$$a_1\left((r+1)^2 - \theta^2\right) = 0 (m = 1), \tag{A.4}$$

$$a_m\left((r+m)^2 - \theta^2\right) + a_{m-2} = 0 (m \geq 2). \tag{A.5}$$

From (A.3), since $a_0 \neq 0$, we obtain indicial equation

$$(r - \theta)(r + \theta) = 0, \tag{A.6}$$

with indicial roots $r = \theta$ and $r = -\theta$. Setting $r = \theta$ in (A.5) gives the recurrence relation

$$a_m = \frac{-1}{m(m + 2\theta)} a_{m-2}, m \geq 2. \tag{A.7}$$

Equation (A.4) implies that odd-indexed terms are zeros, i.e., $a_1 = 0$, since $\theta \in [0, 1]$ and so $a_3 = a_5 = \cdots = 0$. For the even-indexed terms, we let $m = 2k$ and rewrite the general recurrence relation as

$$a_{2k} = \frac{-1}{2^2 k(k + \theta)} a_{2(k-1)}, k \geq 1. \tag{A.8}$$

Substituting the coefficient into (A.1), we obtain one solution of Bessel's equation:

$$K_\theta(\xi) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(1+\theta)(2+\theta)(k+\theta)} \xi^{2k+\theta}, \tag{A.9}$$

where $a_0 \neq 0$. Without loss of generality, we choose

$$a_0 = \frac{1}{2^\theta \Gamma(\theta + 1)}, \tag{A.10}$$

where $\Gamma(\cdot)$ is a gamma function. Using the basic property of the gamma function, $\Gamma(\xi + 1) = \xi\Gamma(\xi)$, we can simplify the terms in the series and obtain one solution denoted J_θ by

$$J_\theta(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1 + \theta + k)} \left(\frac{\xi}{2}\right)^{2k+\theta}. \tag{A.11}$$

Similarly, the second solution is found to be

$$J_{-\theta}(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1-\theta+k)} \left(\frac{\xi}{2}\right)^{2k-\theta}. \quad (A.12)$$

Then, the general solution is given by

$$Y_\theta(\xi) = \frac{J_\theta(\xi)\cos(\theta\pi) - J_{-\theta}(\xi)}{\sin(\theta\pi)}, \quad (A.13)$$

where θ is not an integer. Lastly, we have the general solution to (43) as

$$v(t,\xi) = K_\theta(\xi) l(t), \quad (A.14)$$

where $l(t) \neq 0$ and

$$K_\theta(\xi) = \begin{cases} C_1 J_\theta(\xi) + C_2 Y_\theta(\xi), & C_1, C_2 \in \mathbb{R}, \\ C_3 J_\theta(\xi) + C_4 J_{-\theta}(\xi), & C_3, C_4 \in \mathbb{R}. \end{cases} \quad (A.15)$$

The derivatives are as follows:

$$\begin{aligned} v_t(t,\xi) &= \dot{l}(t) K_\theta(\xi), \\ v_\xi(t,\xi) &= l(t) K'_\theta(\xi), \\ v_{\xi\xi}(t,\xi) &= l(t) K''_\theta(\xi). \end{aligned} \quad (A.16)$$

Data Availability

We have attached the MATLAB code for simulations of the results. The initial conditions are arbitrarily chosen.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

Data. In order to illustrate the effects, MATLAB program was used. The initial conditions are arbitrary selected. *(Supplementary Materials)*

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