

Research Article

Quasisimple Wave Solutions of Euler's System of Equations for Ideal Gas

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Quasisimple wave solutions of Euler's system of equations for ideal gas are investigated under the assumption of spherical and cylindrical symmetries. These solutions are proved to be stabilized into sound wave solutions and cavitation. It is proved that if initial conditions from outside the invariant region approach to transitional solution, then reciprocal of the self-similar parameter goes to infinity. However, when initial conditions stabilize into sound waves or cavitation, then reciprocal of self-similar parameter approaches finite value. Further, it is proved that initial conditions can be parametrized so that some of the initial conditions stabilize into sound wave solutions. The rest of the initial conditions are proved to be stabilized into cavitation. This extends the work of G. I. Taylor to the case of cavitation. It is proved that quasisimple wave solutions exist for the balance laws comprised of Euler's system of equations in the case of cylindrically and spherically symmetric cases. The description applies to the motion of cylindrical and spherical piston in real life. In particular, self-similar description of appearance of vacuum in the motion of cylindrical and spherical piston is given.

1. Introduction

Euler's system of equations for compressible fluids is a system of quasilinear partial differential equations which has occupied center stage in both the spheres of pure and applied mathematics. Problem of solving Euler's equations in several dimensions is one of the most difficult problem in analysis of partial differential equations owing to the enigmatic role played by the convective term in the system. Central questions of existence and uniqueness of solutions for a general initial value problem and mixed initial boundary value problem remain largely open with limited success in some particular cases. So far, the analysis of this system under variety of initial and boundary conditions remained ad hoc and guided by physics of the problem owing to the difficulties in identifying correct function spaces for the existence of solutions. This difficulty is essentially due to the fact that the form of the functions solutions may take are largely unknown, and in several cases, in fact, solutions even admit discontinuities. Appearance of discontinuities adds even more difficulties in the analysis and makes the questions even more challenging.

Among many, one of the approaches that has gained much attention in recent times is use of artificial neural networks [1–4]. Nonlinearities in the problem make solving and analysis initial boundary value problems or boundary value problems quite difficult. Hence, researchers in the field try to analysis by means of finding numerical solution to the problem. One of the method used by Shafiq et al. [3] to carry out numerical process is the use of Galerkin weighted residual method. In this method, the unknown solution is anticipated in terms linear combinations of certain appropriately chosen trial functions. Upon substituting, these linear combinations into the problem to be solved leads to residuals. Coefficients of anticipated solution then adjusted in such a way that the residual will be minimum. Such a linear combination is achieved by means of artificial neural networks. However, such linear combination is indeed a solution requires validation and actually mathematical proof too. If the problem is framed in the language of Sobolev spaces then already available knowledge of wavelet bases of Sobolev spaces makes writing explicit mathematical proof of ANN-based argument easier. It should be born in mind that

analyzing wavelet bases of Sobolev spaces is just one aspect of entire argument and one can think of other spaces like Orlicz spaces or weighted Sobolev spaces depending upon the problem.

In the present article, we have tried to analyze self-similar solutions of Euler's system of equations with cylindrically symmetric and spherically symmetric geometries. Assumption of self-similarity of the solution simplify the situation to certain extent and has become popular among researchers in the subject. One of the most striking uses of self-similar solutions in the case of system of hyperbolic conservation laws is the solution of Riemann problem by Lax and its subsequent use in proving Glimm's existence theorem for the solution of system of hyperbolic conservation laws in one space dimension. While this development was taking place, simple wave solutions of the system of conservation laws occupied important place in the analysis of these equations especially in the case of Euler's system of equations both in one space and several space dimensions. One of the reason for the popularity of simple wave solutions is that they model several physical phenomena taking place in compressible fluids. One important step in this kind of analysis is the observation (justified by symmetry considerations) that self-similarity parameter can be taken as x/t^β where β may take values other than 1. Sachdev et al. [5] have discussed multitude of possibilities self-similar solutions offer for system of Euler equations with spherical and cylindrical symmetry. Under the assumption of self-symmetry, the original system of partial differential equations become the system of ordinary differential equations. Phase plane analysis of system of ordinary differential equations obtained for axially symmetric flows in two space dimensions for the ideal gas is carried out by Zheng [6] and serves important source of material in this field. Apart from their importance in the mathematical analysis of flows, self-similar solutions have also served to demonstrate physical phenomena governed by solutions of system of Euler's equations especially in the case of problems related to imploding and exploding shock waves. Monograph written by Sachdev [7, 8] in this regard is a good reference for the solutions of Euler's system of equations with spherical and cylindrical symmetry depending on self-similar variable x/t^β . Euler's system of equations in two space dimensions is studied in great details in recent times by Zheng [9, 10] and several authors [11, 12]. Desale and Potadar [13] have analyzed self-similar motion of uniformly advancing piston. The same authors Desale and Potadar [14] have carried out analysis of Euler's system of equations in two space dimensions in the case of nonideal gases (real gases).

In the case of cylindrical symmetry and spherical symmetry, Euler's system becomes a system of balance laws. Self-similar solutions to a system of conservation laws are known as simple waves, and self-similar solutions to balance laws fall in the category of quasisimple waves. For introductory description of quasisimple waves, we refer to Courant and Friedrichs [15]. Sound wave solutions correspond to nonzero density whereas vacuum corresponds to zero density and hence zero sound speed. Thus, zero density and zero

sound speed contrast the case of infinite shock speed and also known as degenerate sound speed. Under the assumption of self-similarity, these balance laws become system of ordinary differential equations which can be thought of as a dynamical system or a vector field. In this paper, we have found that complete solutions stabilizing into sound wave solutions and complete solutions stabilizing into vacuum (degenerate sound speed) are characterized by the equilibrium (stationary) points to which these respective complete solutions go into when self-similar parameter is made tend to infinity. Thus, nonzero sound speed corresponds to a stationary point of underlying dynamical system and zero sound speed corresponds to other stationary point of the same dynamical system. Other peculiarities of derived dynamical system are discussed at length in the article. In Section 1, we have given introduction and the Euler's system is reduced to the system of ordinary differential equations under the assumption of self-similarity. Stationary points are also determined in Section 1, and in Section 2, linearization at stationary points is carried out. In Section 3, phase plane analysis is carried out; in the following sections, phase plane analysis is interpreted to understand the mechanics of quasisimple waves which are introduced subsequently. Potential of velocity is discussed in Section 5, and in Section 6, parametrization of initial conditions is given; for different ranges of initial conditions, solutions lead to different stationary points. In Section 7, formal existence of quasisimple waves is proved. Chapter is concluded in Section 8.

2. Basic Equations

Euler's system of equations in three space dimensions under the assumption of cylindrical symmetry ($m = 1$) and spherical symmetry ($m = 2$) are given as follows. These equations form a system of balance laws for which we seek self-similar solutions.

$$\begin{aligned} \frac{\partial}{\partial t}(r^m \rho) + \frac{\partial}{\partial r}(r^m \rho u) &= 0, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial r}(p + \rho u^2) + m \frac{\rho u^2}{r} &= 0, \\ \frac{\partial}{\partial t}\left(r^m \rho \left(e + \frac{u^2}{2}\right)\right) + \frac{\partial}{\partial r}\left(r^m \rho u \left(e + \frac{p}{\rho} + \frac{u^2}{2}\right)\right) &= 0. \end{aligned} \quad (1)$$

For ideal gas, we have the following standard formulae:

$$e = \frac{p}{(\gamma - 1)\rho}, \quad \rho = \frac{\gamma p}{c^2}. \quad (2)$$

We now introduce self-similar variable $\xi = r/t$ in terms of which the above equations become the following system of ordinary differential equations.

$$\begin{aligned} (u - \xi)u_\xi + \frac{c^2}{\gamma p} p_\xi &= 0, \\ (u - \xi)p_\xi - \frac{2p}{c}(u - \xi)c_\xi + pu_\xi &= -\frac{mup}{\xi}, \\ (u - \xi)p_\xi + \gamma pu_\xi &= -\frac{m\gamma pu}{\xi}. \end{aligned} \tag{3}$$

The same system is written in matrix form as below so that upon applying Cramer's rule, we get equations for u_ξ, c_ξ, p_ξ .

$$\begin{pmatrix} u - \xi & \frac{c^2}{\gamma p} & 0 \\ p & u - \xi & -\frac{2p}{c}(u - \xi) \\ \gamma p & u - \xi & 0 \end{pmatrix} \begin{pmatrix} u_\xi \\ p_\xi \\ c_\xi \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{mup}{\xi} \\ -\frac{m\gamma pu}{\xi} \end{pmatrix}. \tag{4}$$

In this system of equations, we denote for brevity as follows:

$$AU = b. \tag{5}$$

This is a system of linear equations. Determinant of the underlying matrix A is nonzero for $u \neq \xi$. We already have seen that $u = \xi$ is a solution only in the case $m = 0$, the degenerate case which is not of concern at this moment.

$$\det(A) = -\frac{2p}{c}(u - \xi)(c^2 - (u - \xi)^2). \tag{6}$$

Thus, $u \neq \xi$ means nontrivial unique solution exist which we have written below:

$$\begin{aligned} u_\xi &= -\frac{muc^2}{\xi(c^2 - (u - \xi)^2)}, \\ c_\xi &= \frac{mc(u - \xi)u(\gamma - 1)}{2\xi(c^2 - (u - \xi)^2)}, \\ p_\xi &= -\frac{m\gamma pu(u - \xi)}{\xi(c^2 - (u - \xi)^2)}. \end{aligned} \tag{7}$$

We now introduce following variables on the lines of Courant and Friedrichs [15], Sachdev [7], and Zheng [6]. The central point of our discussion is to extend discussion carried out by Taylor [16] for sound waves to include the case of cavitation.

$$\begin{aligned} s &= \frac{1}{\xi}, I = us, K = cs, J = ps^2, \\ s \frac{dI}{ds} &= \frac{(1 - I)^2 - K^2 - mK^2}{(1 - I)^2 - K^2} I, \\ s \frac{dK}{ds} &= \frac{2((1 - I)^2 - K^2) - m(1 - I)I(\gamma - 1)}{2((1 - I)^2 - K^2)} K, \\ s \frac{dJ}{ds} &= \frac{2((1 - I)^2 - K^2) + m\gamma(1 - I)I}{(1 - I)^2 - K^2} J. \end{aligned} \tag{8}$$

Introduce new variable τ so that

$$\begin{aligned} \frac{ds}{d\tau} &= s((1 - I)^2 - K^2), \\ A &= 2((1 - I)^2 - K^2) - m(\gamma - 1)(1 - I)I, \\ B &= (1 - I)^2 - K^2 - mK^2, \\ C &= 2((1 - I)^2 - K^2) + m\gamma(1 - I)I. \end{aligned} \tag{9}$$

Then, we have

$$\begin{aligned} \frac{dI}{d\tau} &= BI, \\ \frac{dK}{d\tau} &= AK, \\ \frac{dJ}{d\tau} &= CJ. \end{aligned} \tag{10}$$

Equilibrium points of this system are $(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, -1, 0), Q$ and Q' where Q, Q' are defined as follows:

$$\begin{aligned} Q &= \left(\frac{2m}{1 + \gamma - m + m\gamma}, \frac{(\gamma - 1)\sqrt{1 + m}}{1 + \gamma - m + m\gamma}, 0 \right), \\ Q' &= \left(\frac{2m}{1 + \gamma - m + m\gamma}, \frac{-(\gamma - 1)\sqrt{1 + m}}{1 + \gamma - m + m\gamma}, 0 \right). \end{aligned} \tag{11}$$

System (10) of ordinary differential equations defines a dynamical system symmetric with respect K axis. Equilibrium point $(1, 0, 0)$ is repeated twice. Note that for $\gamma = 1$, equilibrium points Q and Q' degenerate to $(1, 0, 0)$. It is well known in the literature that $\gamma = 1$ corresponds to isothermal case, and it is no surprise that (24) too give rise to self-similar solutions of system of Euler's equations with spherical and cylindrical symmetry obey condition of being isothermal. Figure 1 illustrates the vector field determined by (11) in the first quadrant of the $I - K$ plane. The point Q represents transitional solution. The region represents invariant region for the vector field (10). The points $(0, 1)$ and $(1, 0)$ and Q are equilibrium points which are separated by separatrices. In the following sections, we have carried out phase plane analysis of the vector field (10).

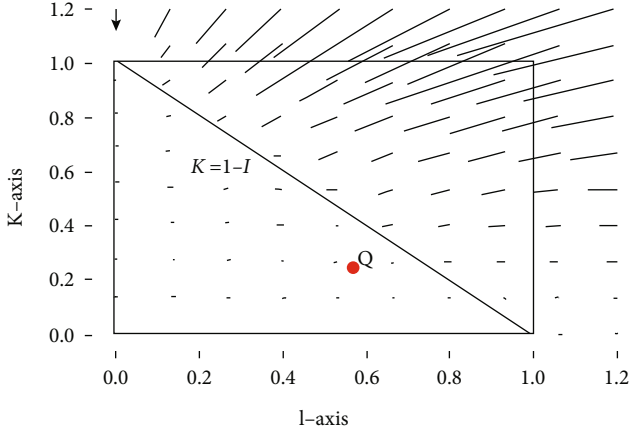


FIGURE 1: Vector field (10): lengths of arrows indicate the rate at which flow is proceeding toward $K = 1 - I$.

3. Linearization of (11)

In this section, we have provided linearization of (10) at different critical points. The words critical points, equilibrium points, and stationary points are synonymous to each other and used interchangeably.

3.1. *Linearization at the Equilibrium Point (0, 1, 0).* (10) linearizes to the following at (0, 1, 0):

$$\begin{pmatrix} \frac{dI}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{dKJ}{d\tau} \end{pmatrix} = \begin{pmatrix} -m & 0 & 0 \\ -\gamma - 3 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I \\ K - 1 \\ J \end{pmatrix} + \text{second-order terms.} \quad (12)$$

Eigenvalues of linear part of the preceding system of equations are $-m$, -4 and 0 . For $m = 1$ and $m = 2$, we have two-dimensional stable manifold corresponding to both the negative eigenvalues and a slow manifold corresponding to zero eigenvalue.

3.2. *Linearization at the Equilibrium Point (0, 0, 0).* (10) linearizes to the following at (0, 0, 0):

$$\begin{pmatrix} \frac{dI}{d\tau} \\ \frac{dK}{d\tau} \\ \frac{dJ}{d\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} I \\ K \\ J \end{pmatrix} + \text{second-order terms.} \quad (13)$$

3.3. *Linearization at the Equilibrium Point Q.* (10) linearises to the following at Q:

$$\begin{pmatrix} \frac{d(I-a)}{d\tau} \\ \frac{d(K-b)}{d\tau} \\ \frac{dKJ}{d\tau} \end{pmatrix} = \begin{pmatrix} L & M & 0 \\ N & P & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I-a \\ K-b \\ J \end{pmatrix} + \text{second-order terms,}$$

$$a = \frac{2m}{1 + \gamma - m + m\gamma},$$

$$b = \frac{(\gamma - 1)\sqrt{1+m}}{1 + \gamma - m + m\gamma},$$

$$L = \frac{-4(\gamma - 1)m(m+1)}{(\gamma + 1)^2 m^2 + \gamma^2 + 2(\gamma^2 - 1)m - 2\gamma + 1},$$

$$M = \frac{-4m(\gamma - 1)(m + \sqrt{m+1})}{(\gamma + 1)^2 m^2 + \gamma^2 + 2(\gamma^2 - 1)m - 2\gamma + 1},$$

$$N = -\sqrt{m+1} \frac{\gamma^3 + \gamma^2 + (\gamma^3 - \gamma^2 - \gamma + 1)m - 5\gamma + 3}{(\gamma + 1)^2 m^2 + \gamma^2 + 2(\gamma^2 - 1)m - 2\gamma + 1},$$

$$P = -4 \frac{\gamma^2 + (\gamma - 1)^2 m - 2\gamma + 1}{(\gamma + 1)^2 m^2 + \gamma^2 + 2(\gamma^2 - 1)m - 2\gamma + 1}. \quad (14)$$

3.4. *Linearization at the Equilibrium Point (1, 0, 0).* Linearization of (10) at (1, 0, 0) has zero linear part. In this section, we have presented basic calculations concerning linearization of the vector field (10) around its critical points. In the following section, we will carry out phase plane analysis of vector field given by (10) with the help of the above calculations.

4. Phase Plane Analysis of Equation (10)

Phase plane analysis of (10) poses no difficulty except that its linearization at (1, 0) has zero linear part. Simple analysis shows that (1, 0) is an attracting equilibrium point, and there is a separatrix that connects (1, 0) to equilibrium point Q. There is also separatrix that connects Q to attracting equilibrium point (0, 1). Equilibrium point (0, 0) is a source as all the eigenvalues of linearized part of (10) at (0, 0) are all positive. Since (0, 0) is a repelling equilibrium point and (0, 1) is an attracting equilibrium point, we find that there is a separatrix that connects (0, 0) to (0, 1). Thus, we have an invariant region Ω that is bounded by separatrices which connect one equilibrium point to the other. However, there is no separatrix that connects equilibrium point (0, 0) to Q and we see that there are loops at (1, 0) and it will be argued below that some of the integral curves emanating in the neighborhood of (1, 0) take s to become infinitely large to reach (1, 0) back again. The invariant region is given in Figure 2.

5. Infiniteness of s to Reach Q

Here, we argue that s needs to become infinitely large along an integral curve that emanates from any point in the $I - K$ phase plane. One of the reason for this is that certain of the eigenvalues of linear part of the vector field (10) at Q are

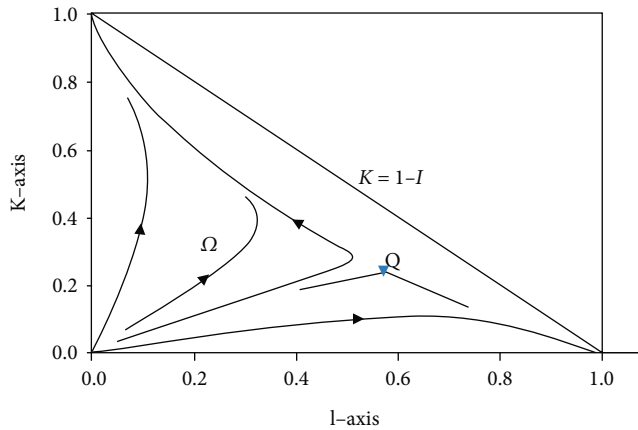


FIGURE 2: Integral curves of the vector field (10) in the invariant region Ω .

positive and Q is a repelling equilibrium point in those directions. The same thing can be argued in the following way as well. Relationship between s and τ is given by the following equation:

$$\frac{ds}{d\tau} = s((1 - I)^2 - K^2), \tag{15}$$

which implies that

$$\ln \frac{s}{s_0} = \int_{\tau_0}^{\tau} ((1 - I)^2 - K^2) dt. \tag{16}$$

We see that in the small neighborhood of Q , the expression $(1 - I)^2 - K^2$ is nonzero that remains bounded below by a positive constant, and hence, the integral (16) implies that $s \rightarrow \infty$ as $\tau \rightarrow \infty$. These results are summarized in the form of the following theorem.

Theorem 1. *It takes $s \rightarrow \infty$ in order to reach Q from the initial conditions from within the invariant region as well from the initial conditions which lie outside the invariant region.*

Remark 2. There is unique point in the phase plane in the region Ω for which integral curve emanating from it reaches to Q , and it takes s to become infinitely large to reach there. In the same way, there is a unique point in the complement of the region Ω for which integral curve emanating from it takes s to become infinitely large to reach Q .

6. Potential for the Velocity of a Sound Wave

Radially expanding surfaces produce outward going waves. These waves are governed by wave equation with cylindrical symmetry ($m = 1$) and spherically symmetry ($m = 2$) and given by

$$\frac{\partial^2 \Phi}{\partial t^2} = a^2 \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{m}{r} \frac{\partial \Phi}{\partial r} \right). \tag{17}$$

General expression for potential of outward going waves is given as follows:

$$\Phi = \frac{1}{r^{m/2}} f(r - at), \tag{18}$$

where a is the speed of sound and r is radial coordinate. Huygens principle of wave propagation is different for waves in even number of spatial dimensions and odd number of spatial dimensions. Because of this principle, the above expression is exact form $m = 2$ while it represents fairly accurate form of the potential for $m = 1$ for large values of radius (that is for $r \rightarrow \infty$). Detailed justification of how progressive wave solutions are to be interpreted for $m = 1$ is given by Whitham [17]. It should be noted that for small values of r , in the case of cylindrical symmetry, amplitude of outward moving waves have nonzero azimuthal component whose contribution diminishes when r becomes arbitrarily large. Diminishing of this azimuthal component as r becomes infinite makes use of progressive solutions even in the case of $m = 1$ justified. We have the following equations:

$$u = \left(-\frac{m}{2} \right) r^{(-m/2)-1} f(r - at) - r^{-m/2} f'(r - at),$$

$$p - p_0 = -\rho a r^{-m/2} f'(r - at). \tag{19}$$

If R is the radial coordinate of the front, which, by expansion, produces waves, R is function of t and then the condition required to be satisfied at the front is

$$\frac{dR}{dt} = -\left(\frac{m}{2} \right) R^{-(m+2)/2} f(R - at) - r^{-m/2} f'(R - at). \tag{20}$$

Requirement of self-similarity amounts to dR/dt is constant. Because of self-similarity, cylindrical wave fronts ($m = 1$) and spherical wave fronts ($m = 2$) are expanding at uniform speed. At $t = 0$, we can assume that $R = 0$ and the radius at time t can be expressed in the form $R = \alpha at$, where α is a nondimensional constant, like, reciprocal of Mach number. Put $w = R - at$ and we use $R = \alpha at$.

$$\alpha \alpha = -\left(\frac{m}{2} \right) R^{((-m/2)-1)} f(w) - r^{-m/2} f'(w), \tag{21}$$

which results into the following:

$$\left(\frac{w}{\alpha - 1} \right)^{m/2} f'(w) + \frac{m}{2} \left(\frac{w}{\alpha - 1} \right)^{(m+2)/2} f(w) + \alpha \alpha = 0. \tag{22}$$

Note that both u and $p - p_0$ are constant when r/at is constant at the surface of the expanding front $u = r/t$. Because of this in our notation, $I = 1$ corresponds to boundary condition at the piston. In the sound wave solution, α specifies the speed of outward expansion of the front (cylindrical or spherical) as a fraction of the speed of sound. In the complete solution, K represents the ratio of local speed of sound to the speed of sound. Correspondence between the complete solution and the sound wave solution or cavitation is attained when the

arbitrary constant α is defined by the relation $K_0 = \alpha^{-1}$, where K_0 is the value of K at $I = 1$.

7. Parametrization of Initial Conditions

Correspondence between complete solution and the sound wave solution is accomplished through α . This correspondence is elaborately explained in Taylor [16] for nondegenerate sound speeds, i.e., $K \neq 0$. We extend this work to include the case of degenerate sound speed $K = 0$. Case of degenerate sound speed is also known as cavitation as this case corresponds $\rho = 0$. Phase plane analysis of 23 makes us to state the following theorem.

Theorem 3. *Initial conditions form outside of the invariant region can be parametrized by α so that complete solutions corresponding to some of the initial conditions stabilize into sound waves corresponding to equilibrium point $I = 0, K = 1$ and others stabilize into cavitation which corresponds to the equilibrium point $I = 1, K = 0$.*

In this way, we establish that correspondence between complete solution and sound wave solution obtained through α also takes into account the case of degenerate sound speed or cavitation with the difference that nondegenerate sound wave solutions are the one which stabilize to the equilibrium point $I = 0, K = 1$ whereas degenerate sound waves or cavitation is achieved by the solutions which stabilize into equilibrium point $I = 1, K = 0$. Figure 3 shows integral curves of the vector field (10) for different values of α^{-1} . It is thus diagrammatically displayed the way in which α parametrizes integral curves of (10) and hence the self-similar solutions of Euler’s system of equations in cylindrically symmetric and spherically symmetric geometries. We can see homoclinic trajectories at equilibrium point $(1, 0)$ in the $I - K$ plane.

8. Existence of Quasisimple Waves

Under the assumption of cylindrical and spherical symmetry, Euler’s system of equations ceases from being system of conservation laws as nonzero source terms appear on right hand side of the equations and it becomes system of balance laws. Solutions in this case depending on self-similar variable r/t are now known as quasisimple waves. Simple waves are the solutions which depend on one parameter in state space (hodograph plane in two dimensions). In general, solutions can be sought which depend on two parameters, and in such a case, they are known as double waves. In this fashion, in general, one may seek triple waves and so on. In general, solutions of this sort admitted by system of balance laws are known as quasisimple waves. A beautiful introduction to quasisimple waves is given by Courant and Friedrichs [15]; Sheshadri and Sachdev [8] have also dealt with these kind of waves. In Section 3, we have characterized initial states which correspond to sound wave solution and we have characterized stated which eventually stabilize into cavitation. In this article, we have proved that appropriate states stabilize either into sound wave or

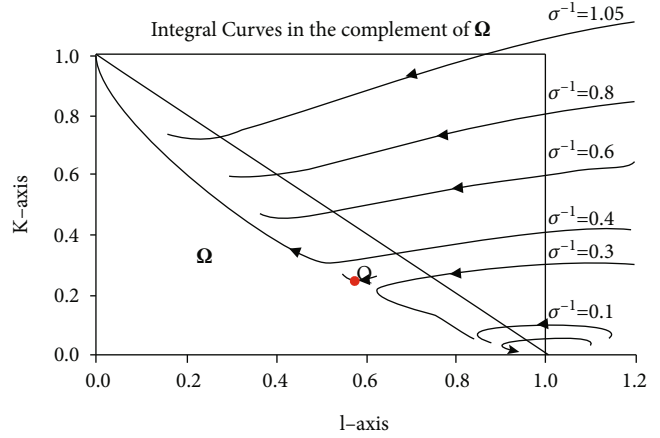


FIGURE 3: Integral curves of (10) in the complement of Ω impinging into different equilibrium points

cavitation In Section 4, we have proved that not only the solution depends on parameter but also the parameter can be taken as J , the variable which corresponds to the pressure. This section establishes that the waves are quasisimple in the sense meant by Courant and Friedrichs [15].

Theorem 4. *Quasisimple wave solutions exist for the balance laws comprised of Euler’s system of equations in the case of cylindrically symmetric case and spherically symmetric case.*

Proof. J has dimensions of density, and hence in the following analysis, we normalize J by dividing it by ρ_0 and again denote it by J . Here, ρ_0 is ambient density that is atmospheric density before the flow has taken place. I and K satisfy the following pair of ordinary differential equations:

$$\frac{dI}{d\eta} = ((1 - I)^2 - K^2 - mK^2)I, \tag{23}$$

$$\frac{dK}{d\eta} = (2((1 - I)^2 - K^2) - m(1 - I)I(\gamma - 1))K, \tag{24}$$

where η is defined by the following:

$$\frac{d\eta}{dJ} = \frac{2((1 - I)^2 - K^2) + m\gamma(1 - I)I}{J}. \tag{25}$$

□

Here, (24) is an autonomous system of ordinary differential equations and we can always solve it to get I and K as functions of η . On the other hand, (25) gives η as a function of J . Thus, J is a function of η . In the sufficiently small neighborhood of J , we see that right hand side of (25) is zero only at $J = 0$ and at $I = 0, K = 1$ and $I = 1, K = 0$. For ϵ sufficiently small, we see that right hand side of (25) remains nonzero in the open interval $(0, \epsilon)$, and by applying inverse function theorem to $\eta(J)$, we get that $\eta(J)$ is invertible and we get J as a function of η . Thus, we have expressed I, K , and J as

functions of η . This can be also looked at in the following alternative way. Also, (23) determines I and K as functions of η . Thus, (25) determines η as a function of J . Through this dependence of η on J , we have I and K as functions of J . Thus, solutions are parametrized by J and this is what we mean by quasisimple waves. Since J contains thermodynamic information of flow, it is an important observation that quasisimple waves in this case are parametrized by thermodynamic quantity.

Remark 5. As self-similar flow proceeds KE and PE are not equal. As it takes finite time to stabilize the flow into sound waves and since there is no dissipation, KE becomes equal to PE after a finite time. We want to give an estimate on s in terms of α .

9. Conclusion

Presence of vacuum in some of the solutions of piston problem is known to people for quite a some time now. Appearance of vacuum in the solution of the piston problem, and in general, in the solutions of Euler's equations, is not yet appraised with the kinetic theory of gases as it becomes difficult to address applicability of kinetic theory if density becomes zero. This is in stark contrast with the shock wave solution where density apparently becomes infinite. Vacuum can never be accompanied by a shock just ahead or behind it, and only smooth solutions (quasisimple wave solutions for example) can accommodate region fully void of any matter. In this paper, we have tried to highlight dynamical difference between traditional sound wave solutions and solutions which end up with vacuum or cavitation. Solutions stabilizing into sound wave and solutions stabilizing into cavitation correspond to different equilibrium points, and these equilibrium points show quite a different behavior from each other. This difference in behavior is justified by the presence of another equilibrium point Q and because its presence appearance of limit cycles is completely excluded. Analysis of equilibrium points becomes interesting because of the appearance of zero eigenvalue at one equilibrium point and zero linear part of the (10) at another equilibrium point makes application of Conley index theory impossible. However, authors believe that Conley index theory can be modified to take into account nonhyperbolic equilibrium points if the underlying system does not admit limit cycles.

Data Availability

All references on which the article is based are duly cited.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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