Research Article

Travelling Wave Analysis of a Diffusive COVID-19 Model

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Received 19 July 2022; Accepted 16 September 2022; Published 7 October 2022

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In this paper, a mathematical model based on a system of nonlinear parabolic partial differential equations is developed to investigate the effect of human mobility on the dynamics of coronavirus 2019 (COVID-19) disease. Positivity and boundedness of the model solutions are shown. The existence of the disease-free, the endemic equilibria, and the travelling wave solutions of the model are shown. From the numerical analysis, it is shown that human mobility plays a crucial role in the disease transmission. Therefore, interventions that affect diffusion (human mobility), such as lock-down, travel restrictions, and cessation of movement, may play a significant role in controlling and preventing the spread of COVID-19.

1. Introduction

COVID-19 disease, caused by SARS-COV-2 virus emerged from the city of Wuhan, Hubei Province, China in late December 2019 and quickly spread all over the world [1]. This global spread was greatly facilitated by movement of infectious individuals from Wuhan city (the epicenter) to other parts of the world where there were no COVID-19 infections. The disease was declared a global health pandemic by the World Health Organization (WHO) on 11th of March, 2020. The virus is primarily transmitted to humans through inhaling respiratory droplets produced by sneezing, coughing, and conversing with an infectious person (direct transmission) or indirectly through contact with contaminated surfaces (indirect transmission) [2]. COVID-19 incubation period, which is the time between virus exposure and symptom onset, is on average 5–6 days, but can be as long as 14 days [3]. Fever, a dry cough, and fatigue are the most common symptoms of COVID-19. Body aches and pains, a sore throat, diarrhoea, headache, loss of taste and smell, breathing difficulties, a rash on the skin, and discoloration of the fingers and toes are some of the other symptoms [1, 2].

There were no antiviral prophylaxis or therapeutics available during the early phases of the COVID-19 outbreak, so all affected countries relied on nonpharmaceutical interventions such as isolation and quarantine of infected individuals, contact tracing, and population lockdown [4]. These interventions are aimed at containing and mitigating the epidemic, that was a threat to many health-care systems. These preventive measures appeared to be effective in lowering the number of fatalities and hospitalizations [4, 5]. COVID-19 vaccine candidates have been developed and are already in use [6]. Vaccines such as BNT162b2 (Pfizer), mRNA1273 (Moderna), ChAd0x1 nCoV-19 (Astra Zeneca), Sputnik, Sinopharm, and Johnson & Johnson vaccines are currently being administered worldwide.

The dynamics of COVID-19 pandemic have been complex in both time and space. These dynamics are caused by a number of factors, including the spatial distribution of the population’s social and economic levels and the patterns of human mobility within a given country [7]. Human mobility may be due to travel, migration between localities, countries, or towns. When infected people travel to infection-free areas, there is a good risk that they will pass their infections on to locals. The transfer of infections from a region of high concentration to a region of low concentration causes a wave of transfer of infection.

A variety of mathematical models have been used widely in various biological applications such as ecology, population dynamics, tumor growth (cancer), immunology, and epidemiology [8, 9]. A number of epidemics have been
modeled using reaction-diffusion equations with the focus to demonstrate the existence of travelling wave solutions in work such as in references [10, 11]. Reaction-diffusion epidemic models are generally standard epidemic models that assume there is movement and interaction of individuals in a given population in space.

Human mobility restrictions such as lock-down, travel restrictions, and cessation of movement are regarded as an effective strategy for controlling disease spread. However, it is still unclear whether mobility restriction is a proportional response to control the ongoing COVID-19 pandemic [12]. In this paper, the existence of travelling wave solutions as well as the effect of human mobility (diffusion) on the transmission dynamics of COVID-19 disease are investigated.

2. Model Formulation

In this study the transmission dynamics of COVID-19 is assumed to be determined by the spatial location, , as well as time, . Despite the fact that the underlying conditions mentioned in the previous section predisposes someone to a higher risk of contracting COVID-19 disease, because the focus is on human mobility, the study assumes the differential susceptibility. Individuals in this category are referred to as susceptible individuals, which are denoted by the symbol . When susceptible individuals are exposed to the virus, they undergo an incubation period, which is the time between virus exposure and onset of symptoms. This is on average 5–6 days, but can be as long as 14 days [3]. The incubating individuals here are categorised as the exposed class, denoted by . Following the end of the exposure period, the incubating individuals transit into the infection class denoted by . The infectious potential of COVID-19 is significantly greater just before or within the first five days of symptom onset [13]. Therefore, the infected class in this study will include both individuals with and without symptoms. Depending on the severity of symptoms and the intervention strategies in place, the infected individuals may die or recover from the disease. Thus, the class denotes the number of individual who have recovered from the infection.

It is assumed that susceptible humans are recruited at a constant rate . The susceptible humans become infected through a force of infection \((\beta SI)/(1 + \eta I))\), where \(\beta\) is the transmission coefficient, and \(\eta\) is the half-maximal human saturation constant of the infected individuals. It is considered to be of Michaelis–Menten form to account for saturation of the human infection [14]. The saturated incidence given by \((\beta SI)/(1 + \eta I))\) is reasonable due to the fact that as the infected individuals increase, they reach a saturation point. The number of infected individuals decreases due to psychological effects, behavioral changes, or preventive measures taken by the affected individuals [15]. The rate of progression from the exposed to infected classes is assumed to be . The rates of natural and disease-induced mortality are \(\mu\) and \(\delta\), respectively. The recovery rate from the infection is taken as \(\lambda\). The diffusivity of susceptible, exposed, infected, and recovered individuals are \(d_1\), \(d_2\), \(d_3\), and \(d_4\), respectively.

The corresponding proposed mathematical equations arising from the preceding description can be described by the following system of partial differential equations:

\[
\frac{\partial S}{\partial t} - d_1 \nabla^2 S = \Lambda - \frac{\beta SI}{1 + \eta I} - \mu S, \tag{1}
\]

\[
\frac{\partial E}{\partial t} - d_2 \nabla^2 E = \frac{\beta SI}{1 + \eta I} - (\mu + \epsilon)E, \tag{2}
\]

\[
\frac{\partial I}{\partial t} - d_3 \nabla^2 I = \epsilon E - (\mu + \lambda + \delta)I, \tag{3}
\]

\[
\frac{\partial R}{\partial t} - d_4 \nabla^2 R = \lambda I - \mu R, \tag{5}
\]

where \(\{S(x, t), E(x, t), I(x, t), R(x, t)\} \in \Psi \times T \subset \mathbb{R}^4 \times \mathbb{R}\) and \(\Psi\) are bounded domains in \(\mathbb{R}^4\). Human mobility is considered in one-dimensional space i.e., \(\nabla^2 = \partial_x^2\). The result can be extended in higher-dimensional space, such as 2-dimensional space.

3. Well-Posedness of the Model

To ensure that the solutions of the model reflect the biological reality, the fundamental question of positivity and boundedness of the solutions is addressed in this section. The model is analyzed with zero flux boundary conditions (no movement across the boundary of \(\partial \Psi\)).

\[
\frac{\partial S}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n}, \tag{6}
\]

where \((\partial/(\partial n))\), denotes the outward normal derivative on \(\partial \Psi\). Let the initial conditions be given by

\[
S(x, 0) = S(0, 0) \geq 0, E(x, 0) = E(0, 0) \geq 0, I(x, t) = I(x, 0) \geq 0, R(x, t) = R(x, 0) \geq 0. \tag{7}
\]

For \(x \in (-\infty, +\infty), \Psi\) is a bounded domain in \(\mathbb{R}^4\) with smooth boundary \(\partial \Psi\), and \(t \geq 0\).

Proposition 1. Suppose that the initial condition (7) holds, then the solutions of model (5) are nonnegative in \([0, +\infty)\) for all \(t \geq 0\).

Proof. Model (5) can be written as an abstract Banach space \(X = \mathcal{C}(\Psi) \times \mathcal{C}(\Psi)\) in the form

\[
u' = Au(t) + F(u(t)), \quad t > 0, u(0) = u^0 \in X, \tag{8}
\]

where \(u = (S, E, I, R)^T\), \(u(0) = (S(x, 0), E(x, 0), I(x, 0) R(x, 0))^T\), \(Au(t) = (d_1S, d_2E, d_3I, d_4R)^T\), and
The solutions of model (5) are bounded in the region $\Psi \times T$ for all $t \geq 0$.

Proof. To check for the boundedness, adding all the equations in model (5) and setting $D = \max \{d_1, d_2, d_3, d_4\}$ to obtain

$$
\frac{dN(x,t)}{dt} \leq A - \mu N + D V^2 N. \tag{11}
$$

The inequality (11) has a unique solution of the form

$$
\frac{dN(x,t)}{dt} \leq A - \mu N + D V^2 N \leq A - \mu N + \int_{-\infty}^{\infty} \frac{\mu N}{\sqrt{4Dt}} e^{-x^2/4Dt} dx, \tag{12}
$$

where the fundamental solution of inequality (12) is given by

$$
K(t,x) = \begin{cases} 
\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}, & x \in \mathbb{R}, \quad t > 0, \\
0, & x \in \mathbb{R}, \quad t < 0.
\end{cases} \tag{13}
$$

For one-dimensional reaction-diffusion equation, equation (13) satisfies the following

$$
K(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx = 1. \tag{14}
$$

Hence

$$
\frac{dN}{dt} \leq A - \mu N. \tag{15}
$$

Since $N(t,x) > 0$, solving this inequality (15) and taking the limit as $t \to \infty$. Hence by variation of constants formula, yields

$$
\limsup_{t \to \infty} N(t,x) \leq \frac{A}{\mu}. \tag{16}
$$

Hence $N(t,x)$ is bounded. □

From Proposition 1 and Proposition 2, it is clear that solutions of model (5) are positive and bounded for $t \geq 0$. Therefore, model (5) is mathematically and epidemiologically meaningful and it is now sufficient to consider its solutions in $\Psi \times T$.

4. Existence of Equilibrium Points

The disease-free equilibrium point is the state at which there is no infection spreading in the population.

Proposition 3. Model (5) has a disease-free equilibrium denoted by $E_0 = \{(\Lambda/\mu), 0, 0, 0\}$ where $S_0 = \{(\Lambda/\mu), \text{ and } E = I = R = 0\}$.

Proof. Setting $d_i = 0$ for $i = 1, 2, 3, 4$ in model (5), then from first equation of model (5) becomes

$$
S^* = \frac{\Lambda \beta (1 + \eta I^*)}{\beta I^* + \mu (1 + \eta I^*)}. \tag{17}
$$

Substituting equation (17) into second equation of model (5) yields

$$
E^* = \frac{\beta \Lambda I^*}{(\mu + \epsilon) (\beta I^* + \mu (1 + \eta I^*)}. \tag{18}
$$

Now substituting equation (18) into the third equation of model (5) yields

$$
I^* = \frac{\mu (R_0 - 1)}{(\beta + \mu \eta)}. \tag{19}
$$

It is clear from equation (19) that $I^* > 0$ when $R_0 > 1$. This implies that there exists an endemic equilibrium for model (5) whenever $R_0 > 1$. □
5. Existence of the Travelling Wave Solution

Traveling wave solutions have a specific shape and incorporate time and spatial variables through a moving coordinate. A solution of this type describes the spatial transition from one steady state to another. The traveling wave fronts that connect disease-free and endemic states are determined in this section. Model (5) travelling wave solutions are shown using the approach used in references [10, 11].

Define a new variable of the form \( z = k(x - vt) \), \( v > 0 \), where \( v \) is the propagating wave speed of model (5) in one-dimensional space and \( k \) is the wave number. Let \( S(x, t) = S(z), E(x, t) = E(z), I(x, t) = I(z), \) and \( R(x, t) = R(z) \). Then model (5) can be transformed into the following set:

\[
-vkS' - d_k^2 S'' = \Lambda - \frac{\beta SI}{1 + \eta I} - \mu S - vkS' - d_k^2 E''
\]

\[
= \frac{\beta SI}{1 + \eta I} - (\mu + \epsilon)E - vkI' - d_k^2 I''
\]

\[
= \epsilon E - (\mu + \lambda + \delta)I - vkR' - d_k^2 R''
\]

\[
= \lambda I - \mu R,
\]

where \( ' \) denotes the derivative with respect to the new variable \( z \). System (20) can be expressed as follows:

\[
d_k^2 I'' + vkS' + \Lambda - \frac{\beta SI}{1 + \eta I} - \mu S
\]

\[
= 0d_k^2 E'' + vkE' + \frac{\beta SI}{1 + \eta I} - (\mu + \epsilon)E
\]

\[
= 0d_k^2 I'' + vkI' + \epsilon E - (\mu + \lambda + \delta)I
\]

\[
= 0d_k^2 R'' + vkR' + \lambda I - \mu R = 0,
\]

which can further be expressed as

\[
S'' + H_1 S' + F_1(S, E, I, R)
\]

\[
= 0E'' + H_2 E' + F_2(S, E, I, R)
\]

\[
= 0I'' + H_3 I' + F_3(S, E, I, R)
\]

\[
= 0R'' + H_4 R' + F_4(S, E, I, R) = 0,
\]

where \( F_1 = 1d_k^2 \left( \Lambda - \frac{\beta SI}{1 + \eta I} - \mu S \right) \),

\[
F_2 = 1d_k^2 \left( \frac{\beta SI}{1 + \eta I} - (\mu + \epsilon)E \right),
\]

\[
F_3 = 1d_k^2 (\mu E - (\mu + \lambda + \delta)I),
\]

\[
F_4 = 1d_k^2 (\lambda I - \mu R),
\]

\[
H_1 = \frac{v}{d_k}, \quad i = 1, 2, 3, 4.
\]

Setting the variables \( x_1 = S', x_2 = E', x_3 = I', \) and \( x_4 = R' \), then model (22) is transformed into a system of first order differential equations.

\[
Y = [x_1, S, x_2, E, x_3, I, x_4, R] \in \mathbb{R}^8,
\]

\[
\frac{dY}{dz} = f(Y) =
\]

\[
\begin{bmatrix}
-H_1 x_1 - F_1 \\
x_1 \\
-H_2 x_2 - F_2 \\
x_2 \\
-H_3 x_3 - F_3 \\
x_3 \\
-H_4 x_4 - F_4 \\
x_4 
\end{bmatrix}
\]

with boundary conditions

\[
\lim_{z \to \infty} \begin{bmatrix}
x_1, S, x_2, E, x_3, I, x_4, R
\end{bmatrix} = E^0,
\]

\[
\lim_{z \to -\infty} \begin{bmatrix}
x_1, S, x_2, E, x_3, I, x_4, R
\end{bmatrix} = E_\ast,
\]

where \( E^0 \) is the disease-free equilibrium point and \( E_\ast \) is the endemic equilibrium point. Then a travelling wave solution is a trajectory that joins \( E^0 \) and \( E_\ast \).

Computing the Jacobian matrix of equation (25) yields

\[
J =
\begin{bmatrix}
-H_1 - \frac{1}{d_k^2} \left( \frac{\beta I}{(1 + \eta I)} - \mu \right) & 0 & 0 & 0 & -\frac{1}{d_k^2} \frac{\beta SI}{(1 + \eta I)^2} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{d_k^2} \frac{\beta I}{(1 + \eta I)} & -H_2 (\mu + \epsilon) & 0 & -\frac{\beta SI}{d_k^2 (1 + \eta I)^2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \epsilon & -H_3 (\mu + \lambda + \delta) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & -H_4 \frac{\mu}{d_k^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix}
\]
Evaluating matrix (28) at the Disease-free equilibrium

\[ E^0 = \left( \frac{\Lambda}{\mu}, 0, 0, 0, 0 \right) \]
yields

\[
\begin{pmatrix}
-\frac{v}{d_1 k} & \frac{\mu}{d_1 k^2} & 0 & 0 & 0 & \frac{\beta \Lambda}{\mu d_1 k^2} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{v}{d_2 k} & \frac{(\mu + \varepsilon)}{d_2 k^2} & 0 & -\frac{\beta \Lambda}{\mu d_2 k^2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\varepsilon}{d_3 k} & -\frac{v}{d_3 k} & \frac{(\mu + \lambda + \delta)}{d_3 k^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\lambda}{d_4 k^2} & -\frac{v}{d_4 k} & \frac{\mu}{d_4 k^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\[ J_{E^0} = \ (28) \]

The eigenvalues of the Jacobian matrix (28) evaluated at the disease-free equilibrium, \( \xi_1, \ldots, \xi_8 \), are

\[
\xi_{1,2} = -\frac{v \pm \sqrt{v^2 + 4\mu d_1}}{2d_1 k}, \quad \xi_{3,4} = -\frac{v \pm \sqrt{v^2 + 4(\mu + \varepsilon)d_2}}{2d_2 k}, \\
\xi_{5,6} = -\frac{v \pm \sqrt{v^2 + 4(\mu + \lambda + \delta)d_3}}{2d_3 k}, \quad \xi_{7,8} = -\frac{v \pm \sqrt{v^2 + 4\mu d_4}}{2d_4 k}.
\]

(29)

Clearly, \( \xi_1, \ldots, \xi_8 \) are real-valued eigenvalues, since all the parameters of the model are positive. Therefore, a traveling wave profile, that travels at a speed \( v > 0 \), exists that connects the disease-free and endemic equilibrium states. This means that if infectious individuals diffuse into a susceptible population, a transition zone of infection will form.

6. Numerical Simulations and Discussion

Simulation analysis of model (5) is presented to graphically illustrate the behaviour of the solutions of the model with varying rates of diffusivity. For purposes of simulation the parameter values are stated in Table 1 and unless otherwise stated the initial populations are taken to be \( S(0, x) = 3000, E(0, x) = 2000, I(0, x) = 1500, \) and \( R(0, x) = 1000 \).

Figures 1(a) and 1(b) show the simulations of the susceptible humans for varying values of diffusivity rate \( d_1 \) at 0.1 and 0.9, respectively. The number of the susceptible humans reduces faster when diffusivity rate is high and also reduces with an increase in space. When \( d_1 = 0.9 \), it would
take less than 10 days for a pool of susceptible humans to be exposed to the virus. When diffusivity rate is low, \( d_1 = 0.1 \text{ km day}^{-1} \), it would take about 20 days for a pool of susceptible humans to become exposed once they come into contact with the virus.

Figure 2(a) and 2(b) show the spatiotemporal simulations of the infected humans for varying values of diffusivity rate \( d_3 \). They show that as the rate of diffusion (human mobility), \( d_3 \) increases the number of infected individuals increases. This means that, human mobility has an effect in the transmission dynamics of COVID-19. Therefore, use of control interventions such as lock-down, travel restrictions, and cessation of movement that reduce human mobility may play a significant role in reducing COVID-19 spread. Infectious individuals will spread the infection to noninfected areas until an equilibrium is reached. The numerical results show that the disease persists in the population due to the movement of infectious individuals. The formation of a wave profile indicates that the introduction of an infective individual into a susceptible population results in infection propagation.

7. Conclusion

The effect of human mobility on COVID-19 dynamics is investigated in this paper using a nonlinear parabolic partial differential equation. Well-posedness of the model and existence of steady states are shown. Existence of traveling wave solutions of the model are shown. The model analysis revealed that when infected individuals are introduced into a fully susceptible population, there is a moving transition “a propagating wave” of infections that connects the disease-free equilibrium to the endemic equilibrium. These waves propagate at speed \( v > 0 \), which connects the two equilibria points. From the numerical analysis, it is shown
that human mobility is critical for disease transmission. As a result, control interventions such as lock-down, travel restrictions, and cessation of movement that reduce human mobility may play a significant role in reducing COVID-19 spread. These interventions are aimed at creating barriers that will slow the spread of the virus in the affected areas.

Data Availability
All the data is within the text. However, the Matlab code used in simulation can be available on request.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
The authors would like to thank Masinde Muliro University of Science and Technology, Postgraduate Merit Scholarship Award for supporting this work.

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