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Research Article

Common Fixed Point Theorems for Generalized Contractive Pair of Mappings in a Metric Space and Their Application to Fractional Calculus

Priya Goel,[1] **Manoj Kumar**,[2] **Dimple Singh**,[1] **and Kamal Kumar**[1]

[1]*Department of Mathematics, Amity School of Applied Sciences, Amity University Haryana, India*
[2]*Department of Mathematics, Baba Mastnath University, Rohtak, India*

Correspondence should be addressed to Dimple Singh; dsingh@ggn.amity.edu and Kamal Kumar; kkumar10@ggn.amity.edu

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In this manuscript, we have established relation-theoretic version of some common fixed point results in metric space for generalized $\beta-\phi-Z$-contractive pair of mappings furnished with an arbitrary binary relation $\mathfrak{R}$. Recently, the concept of binary relation is well known leading trend in fixed point theory. Our results extend and unify several fixed point theorems present in the literature. An illustrative example is given to support our main theorem. Finally, we exploit our main result for proving existence and uniqueness results to established the solution of a fractional differential equation of Caputo type.

1. Introduction

Fixed point theory is a very extensive and active area of research which promises the existence and uniqueness of solution to many mathematical problems in various field of sciences. It begins with fixed point and proceeds through the Banach contraction principle [1]. Banach contraction principle is the most conspicuous tool in the field of fixed point theory which states that every contraction defined on a complete metric space possesses a unique fixed point that is a self- mapping U defined on a complete metric space (S, σ) has a unique fixed point if there exists $h \in (0, 1)$ such that

$$\sigma(Ur, Us) < h\sigma(r, s), \forall r, s \in S. \quad (1)$$

Furthermore, $\lim_{\varrho} U_{\varrho} r = t$ for all r in S, where $U_{\varrho}(\varrho \geq 1)$ is the ϱth iteration of U. It ensures us about the existence and uniqueness of solutions to substantial problems in various directions of mathematics. Banach contraction principle is employed in various field of mathematics as well as other domain of sciences and proved many new fixed point results and their uniqueness related to contraction type of mappings. In 1967, the first coincidence fixed point result is proved by Machuca [2], which was later improved by Goebel [3]. Further, in 1976, Jungck [4] has established the first common fixed point theorem. The principle of Banach contraction is generalized by many authors in numerous ways. Ran and Reurings [5] and Nieto and Rodriguez-Lopez [6] extended the Banach contraction principle in a new way by showing that if the metric space is endowed with an ordered binary relation, then it is sufficient to assume that the contraction condition holds only for those comparable elements. In 2006, T. Bhaskar and Lakshmikantham [7] established a fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order. In 2007, Ben-El-Mechaiekh [8] generalized Ran–Reurings' fixed point theorem using the concept of transitive binary relation in a complete metric space. Alam and Imdad [9] have presented a new variant of Banach contraction principle in a complete metric space under an universal binary relation and also utilized their result for transitive, strict order, preorder binary relation, etc. in 2015.

In 2012, Wardowski [10] introduced the concept of F- contraction and stated a fixed point theorem for F-contraction in complete metric spaces. In 2014, Wardowski and Van Dung [11] introduced the notion of F-weak contraction and stated a fixed point theorem for F-weak contraction in metric space. Jleli and Samet [12] gave a generalization of Banach contraction principle in complete generalized metric space in 2014. Later, in 2015, Aggarwal et al. [13] discussed the existence and uniqueness of the common fixed point of expansive mappings in the complete G-metric space. In 2018, Imdad et al. [14] demonstrated the idea of weak θ-contraction on metric space by generalizing the idea of θ-contraction introduced by JLeli and Samet and proved some relation-theoretical fixed point results for weak θ- contraction in a generalized metric space without completeness property with application in fractional calculus. In 2019, Alfaqih et al. [15] gave the notion of $(F, \mathscr{R}_g)$- contraction and proved some common and coincident fixed point results in metric space endowed with binary relations.

In 2015, Khojasteh et al. [16] initiated the idea of Z -contraction by introducing a new function, known as simulation function. They generalized the Banach contraction principle by originating the idea of Z-contraction and proved some fixed point results in metric spaces. Moreover, many other researchers proved many coincidence and common fixed point results in various metric spaces using Z -contraction. Shahzad and Karapinar [17] demonstrated some coincidence fixed point results in metric spaces using the concept of Z-contractions. In b-metric spaces, existence and the uniqueness of few operators are introduced by Rashid et al. [18]. They established a new type of contractive condition by combining the idea of simulation functions with admissible functions. Argoubi et al. [19] slightly modified the definition of simulation function by withdrawing a condition that is $\xi(0,0)=0$ and proved some fixed point results for a pair of nonlinear operators satisfying nonlinear contraction results in partial ordered metric spaces based on Z-contraction.

In 2012, Samet et al. [20] initiated the idea of α-admissible functions and proposed a new category of contractive type mapping, known as $\alpha-\phi$-contractive type mapping. With the help of α- admissibility, they established some fixed point results for $\alpha-\phi$- contractive type mappings. Motivated Samet et al. [20], Durmaz et al. [21] attained the existence and uniqueness of the solution of a fourth order two-point boundary value problem. Further, Wardowski and Van Dung [11] gave a generalized form of $\alpha-\phi$-contractive type mappings and obtained various fixed point results. Recently, Sarwar et al. [22] have proposed some fixed point results in metric interval and normed interval spaces using the concept of simulation functions and α-admissibility in 2021. Also, in 2021, Kumar and Sharma [23] illustrated the idea of generalized $\alpha-\phi-Z$-contractive type mapping and proved some fixed point results in metric spaces. Here, the results are proved in terms of α-admissibility condition and a condition of existence a subsequence for some convergent sequence and for the validity of their results; they have also demonstrated some fixed point results in metric spaces endowed with a partial order.

Fixed point theory also plays a vital role in establishing the existence and the uniqueness of the solution of a fractional differential equation. In 1996, Delbosco and Rodino [24] established the existence and the uniqueness of the solution of a nonlinear fractional differential equation. In 2009, Belmekki et al. [25], attained the existence of periodic solution of a nonlinear fractional differential equation. In [26], Zada et al. proved a coupled fixed point theorem and attained the solution of fractional variable order hybrid differential equations in 2021. In addition, Abdo et al. [27], in 2021, developed the theory of nonlinear system of pantograph-type fractional differential equations and proved fixed point result with the use of Banach contraction principle and Krasnoselskii's fixed point theorem. In 2021, Aydi et al. established some fixed point results in extended b-metric space and existence and the uniqueness of a system of nonlinear fractional differential equation is obtained. For instance, the Caputo nonlinear fractional differential equation proposed by Kilbas, is given as

$$ {}^{C}D_{p}^{\gamma} r(\mathbf{t}) = \begin{cases} h(\mathbf{t}, r(\mathbf{t}))\,; \mathbf{t} \in (0,1), 1<\gamma\leq 2, \\ r(0)=0, r(1)=\displaystyle\int_0^w r(p)dp, 0<w<1, \end{cases} \tag{2} $$

where ${}^{C}D_{p}^{\gamma}$ represents the Caputo fractional derivative of order γ, and $h:[0,1]\longrightarrow R$ is a continuous function. This nonlinear fractional differential equation can also be written as

$$ \begin{aligned} r(\mathbf{t}) = {} & \frac{1}{\Gamma(\gamma)}\int_0^{\mathbf{t}} (\mathbf{t}-p)^{\gamma-1} h(p, r(p))dp \\ & - \frac{2\mathbf{t}}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1} h(p, r(p))dp \\ & + \frac{2\mathbf{t}}{(2-w^2)\Gamma(\gamma)}\int_0^w \left[\int_0^p (p-z)^{\gamma-1} h(z, r(z))dz\right] dp. \end{aligned} \tag{3} $$

In this paper, we have extended and generalized the results proved by Kumar and Sharma in [23] in 2021. Our current study provides a new approach for proving fixed point results in terms of binary relation by relaxing the condition of existing a subsequence for some convergent sequence and modifying the condition of β-admissibility. Our results show some fixed point results for generalized $\beta-\phi-$Z- contractive pair of mappings in a new manner using the conditions in terms of $\mathfrak{R}$-continuity and relation $\mathfrak{R}$ in terms of closeness of functions. We also explore an example (Example 21) which shows that our results are better in comparison to the existing result [23]. In addition, we have attained the existence and the uniqueness of the solution of fractional differential equations in the context of our main result, and hence our study is also proved to be useful in finding the existence and the uniqueness of solutions of integral equations, boundary value problems, and fractional differential equations.

In this paper, we begin with introduction in Section 1. In Section 2, we have stated some fundamental definitions related to our work. In Section 3, we have demonstrated a relation-theoretical common fixed point result and its uniqueness for generalized $\beta-\phi-Z$-contractive pair of mappings with the help of binary relation. In Section 4, we have proved some more fixed point results with the help of our main result that is proved in Section 3. In the support of our main theorem, we have also provided an example in Section 5, for which our main result is applicable in finding the existence and the uniqueness of the solution for a given mathematical problem. In Section 6, we have applied our main result in finding the existence and the uniqueness of the solutions of fractional differential equation in Caputo sense. We have also given an example with graphical representation for the reliability of our attained result in fractional calculus.

2. Preliminaries

In this part, we have stated a few definitions associated with our work.

Definition 1 (see [28]). A subset of $S \times S$ is termed as binary relation $\mathfrak{R}$ where S is any nonempty set.

Definition 2 (see [29]). A sequence $(s_\varrho) \in S$ is called $\mathfrak{R}$-preserving if $(s_\varrho, s_{\varrho+1}) \in \mathfrak{R}$ for all $\varrho \in N_0$, where $\mathfrak{R}$ is a binary relation defined on a nonempty set S.

Definition 3 (see [29]). Let (S, σ) be any metric space and a binary relation $\mathfrak{R}$ defined on set S. Thus, if each $\mathfrak{R}$-preserving cauchy sequence in S converges to a point in S, then (S, σ) is called $\mathfrak{R}$-complete.

Definition 4 (see [29]). Let us consider a metric space (S, σ) and a binary relation $\mathfrak{R}$ defined on set S. Then, a self-mapping $U : S \longrightarrow S$ is known as $\mathfrak{R}$-continuous at s^* whenever $s_\varrho \longrightarrow s^*$ for any $\mathfrak{R}$-preserving sequence (s_ϱ), then $U(s_\varrho) \longrightarrow U(s^*)$ and if U is continuous at every point of S, then it is called $\mathfrak{R}$-continuous.

Definition 5 (see [16]). A simulation function is a mapping $\xi : [0,\infty) \times [0,\infty) \longrightarrow \mathbf{R}$ requiring the following assumptions:

(1) $\xi(0, 0) = 0$;

(2) $\xi(r, s) < s - r$, for each $r, s > 0$;

(3) If we have two sequences (U_ϱ) and $(V_\varrho) \in (0,\infty)$ in such a manner that $\lim_{\varrho\longrightarrow\infty}(U_\varrho) = \lim_{\varrho\longrightarrow\infty}(V_\varrho) = l \in (0,\infty)$, then

$$\lim_{\varrho\longrightarrow\infty}\left(U_\varrho, V_\varrho\right) < 0. \tag{4}$$

Note: Z denotes the classes of all the simulation functions $\xi : [0,\infty) \times [0,\infty) \longrightarrow \mathbf{R}$.

Definition 6 (see [20]). If we consider a self-mapping $U : S \longrightarrow S$ and a function $\beta : S \times S \longrightarrow [0,+\infty)$, then U is called β-admissible if

$$\beta(r, s) \geq 1 \Rightarrow \beta(Ur, Us) \geq 1, \forall r, s \in S. \tag{5}$$

Definition 7 (see [30]). Let $U, V : S \times S \longrightarrow [0,\infty)$ be two self-mappings, and $\beta : S \times S \longrightarrow [0,+\infty)$ is a function. Then, U is called β-admissible with respect to V if

$$\beta(Vr, Vs) \geq 1 \Rightarrow \beta(Ur, Us) \geq 1, \forall r, s \in S. \tag{6}$$

Definition 8 (see [20]). Φ, is the collection of functions $\phi : [1,\infty) \longrightarrow [1,\infty)$ where ϕ is required to hold:

(1) $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing

(2) $\sum_{\varrho=1}^{+\infty}\phi^\varrho(s) < \infty$ for all $s > 0$, where $\phi^\varrho(s)$ is the ϱth iterate of ϕ.

Definition 10 (see [20]). Consider a metric space (S, σ) and a self-mapping $U : S \longrightarrow S$. Then, the self-mapping U is known as $\beta-\phi$-contractive mapping if there exist two functions $\phi \in \Phi$ and $\beta : S \times S \longrightarrow [0,+\infty)$ in such a manner that

$$\beta(r, s)\sigma(Ur, Us) \leq \phi(\sigma(r, s)), \forall r, s \in S. \tag{7}$$

Definition 11 (see [30]). Consider a metric space (S, σ) and two self-mappings $U, V : S \longrightarrow S$. Then, the pair (U, V) is said to be generalized $\beta-\phi$-contractive pair of mappings if there exist two functions $\phi \in \Phi$ and $\beta : S \times S \longrightarrow [0,+\infty)$ such that

$$\beta(Vr, Vs)\sigma(Ur, Us) \leq \phi\{M(Vr, Vs)\}] \geq 0, \forall r, s \in S, \tag{8}$$

where

$$M(Vr, Vs) = \max\left\{\sigma(Vr, Vs), \frac{\sigma(Vr, Ur) + \sigma(Vs, Us)}{2}, \cdot \frac{\sigma(Vr, Us) + \sigma(Vs, Ur)}{2}\right\}. \tag{9}$$

Definition 12 (see [23]). Consider a metric space (S, σ) and two self-mappings $U, V : S \longrightarrow S$. Then, the pair (U, V) is said to be generalized $\beta-\phi-Z$-contractive pair of mappings with respect to ξ if

$$\xi[\beta(Vr, Vs)\sigma(Ur, Uv), \phi\{M(Vr, Vs)\}] \geq 0, \forall r, s \in S, \tag{10}$$

where $\beta : S\times S \longrightarrow [0,\infty)$ and $\phi \in \Phi$ and

$$M(Vr, Vs) = \max\left\{\sigma(Vr, Vs), \frac{\sigma(Vr, Ur)+\sigma(Vs, Us)}{2}, \frac{\sigma(Vr, Us)+\sigma(Vs, Ur)}{2}\right\}. \quad (11)$$

3. Main Results

In this part, we have proved few common fixed point results and their uniqueness for generalized $\beta-\phi-$Z-contractive pair of mappings by using the concept of binary relation and the simulation function.

Theorem 13. *If we assume that (S,σ) be any $\mathfrak{R}$-complete metric space equipped with a binary relation $\mathfrak{R}$ defined on S and two self-mappings $U, V : S \longrightarrow S$ are such that $U(S) \subseteq V(S)$, and the pair (U, V) is a generalized $\beta-\phi-$Z-contractive pair of mappings. Then, U and V have a unique common fixed point if we have the following assumptions:*

(1) $S(U,\mathfrak{R})$ is nonempty

(2) $\mathfrak{R}$ is U-closed and V-closed

(3) $\beta(r,s)\geq 1, \forall r, s\in S$;

(4) U and V are $\mathfrak{R}$-continuous

Proof. In condition (1) we have given that $S(U,\mathfrak{R})$ is nonempty i.e., $S(U,\mathfrak{R})\neq\phi$. So, let $r_0\in S(U,\mathfrak{R})$ i.e., $(r_0, Ur_0)\in\mathfrak{R}$ and construct a Picard sequence $(r_\varrho), \varrho=0,1,2,\cdots\cdots$ such that $r_{\varrho+1}=Ur_\varrho$; $\forall\varrho\in N$, and also it is given in condition (3) that $\mathfrak{R}$ is U-closed, so we get

$$(r_0, ur_0), (Ur_0, U^2r_0), (U^2r_0, U^3r_0), \cdots, (U^\varrho r_0, U^{\varrho+1}r_0) = (r_\varrho, r_{\varrho+1})\in\mathfrak{R};\forall\varrho\in N. \quad (12)$$

Thus, $\{r_\varrho\}$ is a $\mathfrak{R}$-preserving sequence. Now, from the given condition (i), we have $U(S)\subseteq V(S)$, therefore, we can choose a point $r_1\in S$ such that $Ur_0=Vr_1$. Hence, by continuing the same process and selecting the points $r_2, r_3, \cdots\cdots, r_\varrho, \cdots\cdots\in S$, we get

$$Ur_\varrho = Vr_{\varrho+1}, \text{for all } \varrho=0,1,2,\cdots\cdots. \quad (13)$$

□

Here, we have two cases.

Case I: If $Ur_{\varrho+1}=Ur_\varrho$ for some ϱ, then from (13) equation, we have $Ur_{\varrho+1}=Vr_{\varrho+1}$ for some ϱ. Thus, U and V have a common fixed point $r=r_{\varrho+1}$.

Case II: If $Ur_{\varrho+1}\neq Ur_\varrho$ and it is given that pair (U, V) is a generalized $\beta-\phi-$Z- contractive pair of mappings, then

$$\xi(\beta(Vr, Vs)\sigma(Ur, Us), \phi(M(Vr, Vs)))\geq 0, \quad (14)$$

$$\phi(M(Vr, Vs)) - \beta(Vr, Vs)\sigma(Ur, Uv)\geq 0, \quad (15)$$

$$\beta(Vr, Vs)\sigma(Ur, Us)\leq\phi(M(Vr, Vs)). \quad (16)$$

For $r=r_\varrho$ and $s=r_{\varrho+1}$, the given inequality becomes

$$\beta(Vr_\varrho, Vr_{\varrho+1})\sigma(Ur_\varrho, Ur_{\varrho+1})\leq\phi(M(Vr_\varrho, Vr_{\varrho+1})) \quad (17)$$

Now since $U, V : S\longrightarrow S$ are two self-mappings, so

$$U(r), V(r)\in S;\forall r\in S \quad (18)$$

and from given condition (3), we have

$$\beta(r,s)\geq 1, \forall r, s\in S. \quad (19)$$

Hence, by comparing Equations (18) and (19), we get

$$\beta(Vr, Vs)\geq 1, \forall r, s\in S. \quad (20)$$

Using Condition (20) in Equation (14), we attain

$$\sigma(Ur_\varrho, Ur_{\varrho+1})\leq\phi(M(Vr_\varrho, Vr_{\varrho+1})); \quad (21)$$

where

$$\begin{aligned} M(Vr_\varrho, Vr_{\varrho+1}) &= \max\left\{\sigma(Vr_\varrho, Vr_{\varrho+1}), \frac{\sigma(\mathrm{Vr}_\varrho, \mathrm{Ur}_\varrho)+\sigma(\mathrm{Vr}_{\varrho+1}, \mathrm{Ur}_{\varrho+1})}{2} + \frac{\sigma(Vr_\varrho, Ur_{\varrho+1})+\sigma(Vr_{\varrho+1}, Ur_\varrho)}{2}\right\}, \\ &= \max\left\{\sigma(Ur_{\varrho-1}, Ur_\varrho), \frac{\sigma(Ur_{\varrho-1}, Ur_\varrho)+\sigma(Ur_\varrho, Ur_{\varrho+1})}{2} + \frac{\sigma(Vr_\varrho, Ur_{\varrho+1})+\sigma(Vr_{\varrho+1}, Ur_\varrho)}{2}\right\}, \\ &\leq \max\left\{\sigma(Ur_{\varrho-1}, Ur_\varrho), \sigma(Ur_\varrho, Ur_{\varrho+1})\right\}. \end{aligned} \quad (22)$$

Now if $\max\{\sigma(Ur_{\varrho-1}, Ur_\varrho), \sigma(Ur_\varrho, Ur_{\varrho+1})\} = \sigma(Ur_\varrho, Ur_{\varrho+1})$; then we get,

$$\sigma(Ur_\varrho, Ur_{\varrho+1})\leq\phi(\sigma(Ur_\varrho, Ur_{\varrho+1})) < \sigma(Ur_\varrho, Ur_{\varrho+1});\forall\varrho\geq 1, \quad (23)$$

which is a contradiction.

$$\text{Hence, } \max\left\{\sigma(Ur_{\varrho-1}, Ur_\varrho), \sigma(Ur_\varrho, Ur_{\varrho+1})\right\} = \sigma(Ur_{\varrho-1}, Ur_\varrho). \quad (24)$$

Thus, we have

$$\sigma(Ur_\varrho, Ur_{\varrho+1})\leq\phi(\sigma(Ur_{\varrho-1}, Ur_\varrho));\forall\varrho\geq 1. \quad (25)$$

By repeating the same process for $r = r_{\varrho-1}, s = r_{\varrho}$, and so on, we get

$$\begin{gathered}\sigma\left(Ur_{\varrho-1}, Ur_{\varrho}\right) \leq \phi\left(\sigma\left(Ur_{\varrho-2}, Ur_{\varrho-1}\right)\right),\\ \sigma\left(Ur_{\varrho-2}, Ur_{\varrho-1}\right) \leq \phi\left(\sigma\left(Ur_{\varrho-3}, Ur_{\varrho-2}\right)\right),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.\\ \sigma(Ur_2, Ur_1) \leq \phi(\sigma(Ur_1, Ur_0)).\end{gathered} \tag{26}$$

Thus, by substituting all these inequalities in Equation (25), we get

$$\sigma\left(Ur_{\varrho}, Ur_{\varrho+1}\right) \leq \phi^{\varrho}(\sigma(Ur_0, Ur_1));\forall\varrho \geq 1. \tag{27}$$

Using triangular inequality and above inequality, we get:

$$\sigma\left(Ur_{\varrho}, Ur_{\varrho+k}\right) \leq \sum_{p=\varrho}^{\infty} \phi^{p}(\sigma(Ur_1, Ur_0));\forall k \geq 1. \tag{28}$$

By letting, $p \longrightarrow \infty$, we get (Ur_{ϱ}) which is a cauchy sequence in (S, σ), and since $Ur_{\varrho} = r_{\varrho+1}$, so $(r_{\varrho+1})$ i.e., (r_{ϱ}) is a cauchy sequence, and sequence(r_{ϱ}) is also the $\mathfrak{R}$-preserving. So, (r_{m}) is the $\mathfrak{R}$-preserving cauchy sequence in (S, σ), and (S, σ) is the $\mathfrak{R}$-complete. So, (r_{ϱ}) is the convergent sequence in S, that is, $\lim_{\varrho\longrightarrow\infty} r_{\varrho} \longrightarrow p$, and $p \in S$. Since U is the $\mathfrak{R}$-continuous from the given condition (5), we have

$$p = \lim_{\varrho\longrightarrow\infty} r_{\varrho} = \lim_{\varrho\longrightarrow\infty} r_{\varrho+1} = \lim_{\varrho\longrightarrow\infty} Ur_0 = U \lim_{\varrho\longrightarrow\infty} r_{\varrho} = Up. \tag{29}$$

Therefore, p is a fixed point of U. Now, since $Ur_{\varrho} = V r_{\varrho+1}$; $\forall m = 0, 1, 2, \cdots\cdots$ and (Ur_{ϱ}) is a cauchy sequence in S. Hence, $(Vr_{\varrho+1})$ i.e. (Vr_{ϱ}) is also a cauchy sequence in S, and V is the $\mathfrak{R}$-continuous from the given condition (4), so

$$\begin{aligned} p &= lim_{\varrho\longrightarrow\infty} r_{\varrho} = \lim_{\varrho\longrightarrow\infty} r_{\varrho+1}\\ &= \lim_{\varrho\longrightarrow\infty} Vr_{\varrho+1} = V \quad \lim_{\varrho\longrightarrow\infty} r_{\varrho+1} = Vp.\end{aligned} \tag{30}$$

Thus, p is also a fixed point of V. Hence, p is the common fixed point of U and V.

Uniqueness of common fixed point. Let $q \in S$ be any other common fixed point point of U and V other than p. Then, by definition of common fixed point, we have

$$Uq = Vq = q \text{ and also } Up = Vp = p \tag{31}$$

So, we have to prove $p = q$ i.e., $\sigma(p, q) = 0$. By choosing $r = p$ and $s = r_{\mathrm{m}}$, we attain

$$\begin{aligned}\sigma\left(p, r_{\varrho+1}\right) &= \sigma\left(Up, Ur_{\varrho}\right) \leq \phi\left(M\left(\nu p, Vr_{\varrho}\right)\right), \text{where}\\ M\left(Vp, Vr_{\varrho}\right) &= \max\left\{\sigma\left(Vp, Vr_{\varrho}\right), \frac{\sigma(Vp, Up) + \sigma\left(Vr_{\varrho}, Ur_{\varrho}\right)}{2}, \frac{\sigma\left(Vp, Ur_{\varrho}\right) + \sigma\left(Vr_{\varrho}, Up\right)}{2}\right\},\\ &= \max\left\{\sigma\left(p, r_{\varrho}\right), \frac{\sigma\left(r_{\varrho}, r_{\varrho+1}\right)}{2}\right\}.\end{aligned} \tag{32}$$

Thus, if $\max\{\sigma(p, r_{\varrho}), 1/2\sigma(r_{\varrho}, r_{\varrho+1})\} = 1/2\sigma(r_{\varrho}, r_{\varrho+1})$, then

$$\sigma\left(p, r_{\varrho+1}\right) \leq \phi\left(\frac{1}{2}\sigma\left(r_{\varrho}, r_{\varrho+1}\right)\right) < \sigma\left(r_{\varrho}, r_{\varrho+1}\right). \tag{33}$$

By letting $\varrho \longrightarrow \infty$, we get $0 < 0$, which is a contradiction. Hence,

$$\begin{gathered}\max\left\{\sigma\left(p, r_{\varrho}\right), \frac{1}{2}\sigma\left(r_{\varrho}, r_{\varrho+1}\right)\right\} = \sigma\left(p, r_{\varrho}\right).\\ \text{Hence, } \sigma\left(p, r_{\varrho+1}\right) \leq \phi\left(\sigma\left(p, r_{\varrho}\right)\right).\end{gathered} \tag{34}$$

By continuing the same process, we get

$$\begin{aligned}&\sigma\left(p, r_{\varrho+1}\right) \leq \phi^{\varrho}\{\sigma(p, r_0)\};\forall\varrho \in \mathrm{N}\\ &\Rightarrow \sigma\left(p, r_{\varrho+1}\right) \longrightarrow 0 \text{ as } \varrho \longrightarrow \infty.\end{aligned} \tag{35}$$

Similarly, we can prove that $\sigma(q, r_{\varrho+1}) \longrightarrow 0$ as $\varrho \longrightarrow \infty$.

$\sigma(p, q) \leq \sigma(p, r_{\varrho+1}) + \sigma(q, r_{\varrho+1})$ [By triangular inequality].

$\sigma(p, q) \longrightarrow 0$ as $n \longrightarrow \infty$ i.e., $\sigma(p, q) = 0$ i.e. $p = q$.

Thus, U and V have a unique common fixed point.

4. Consequences

Here, we have stated some other results with the help of Theorem (13).

Theorem 14. *Consider a $\mathfrak{R}$-complete metric space (S, σ) where $\mathfrak{R}$ is a binary relation defined on set S and $U, V : S \longrightarrow S$ are two self-mappings with $\phi \in \Phi$ in such a manner that $\sigma(Ur, Us) \leq \lambda(\phi(M(Vr, Vs)))$. Then, U and V have a* *unique common fixed point if we have the following assumptions:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $U(S) \subseteq V(S)$

(3) $\mathfrak{R}$ is U-closed and V-closed

(4) U and V are $\mathfrak{R}$-continuous

Proof. The result will hold directly from the given Theorem (13) by taking $\xi(r, s) = \lambda s - r$ for all $r, t > 0$ and $\lambda \in (0, 1)$ and $\beta(r, s) = 1$ for all $r, s \in S$. □

Theorem 15. *Let (S, σ) be any $\mathfrak{R}$-complete metric space where $\mathfrak{R}$ is a binary relation defined on set S, and $U : S \longrightarrow S$ is a self- mapping with $\phi \in \Phi$ such that $\sigma(Ur, Us) \leq \lambda(\phi(M(r, s)))$. Then, U has a unique fixed point if the following propositions hold:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $\mathfrak{R}$ is U-closed

(3) U is $\mathfrak{R}$-continuous

Proof. The result will hold directly from the given Theorem (13) by taking $\xi(r, s) = \lambda s - t$ for all $r, s > 0$ and $\lambda \in (0, 1)$, $\beta(r, s) = 1$ for all $r, s \in S$ and $Tr = r, \forall r \in S$. □

Theorem 16. *Suppose that (S, σ) be an $\mathfrak{R}$-complete metric space where $\mathfrak{R}$ is a binary relation defined on set S, and U, $V : S \longrightarrow S$ are two mappings satisfying $\sigma(Ur, Us) \leq \phi(\sigma(Vr, Vs))$ with $\phi \in \Phi$. Now if we have the following conditions:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $U(S) \subseteq V(S)$

(3) $\mathfrak{R}$ is U-closed and V-closed

(4) U and V are $\mathfrak{R}$-continuous

then, U and V have a unique common fixed point.

Proof. The result can be hold directly from the given Theorem (13) by taking $\beta(r, s) = 1$ for each $r, s \in S$ and $M(Vr, Vs) = \sigma(Vr, Vs)$. □

Theorem 17. *Assume that (S, σ) be any $\mathfrak{R}$-complete metric space equipped with a binary relation $\mathfrak{R}$ defined on set S, and $U : S \longrightarrow S$ is a self-mapping satisfying the condition $\sigma(Ur, Us) \leq \phi(\sigma(r, s))$ where $\phi \in \Phi$. Thus, if we have the following assumptions:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $\mathfrak{R}$ is U-closed

(3) U is $\mathfrak{R}$-continuous

then, U has a unique fixed point.

Proof. The result can be attained directly from the given Theorem (13) for $\beta(r, s) = 1$ for each $r, s \in \lambda$, $M(Vr, Vs) = \sigma(Vr, Vs)$ and the self-map V as $Vr = r$. □

Theorem 18. *Consider a $\mathfrak{R}$-complete metric space (S, σ) equipped with a binary relation $\mathfrak{R}$ defined on set S, and $U : S \longrightarrow S$ be a self-mappings in such a manner that $\sigma(Ur, Us) \leq \sigma(r, s)$, Then, U has a unique fixed point if we have the following propositions:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $\mathfrak{R}$ is U-closed

(3) U is $\mathfrak{R}$-continuous

Proof. The result will hold directly from the given Theorem (13) by taking $\beta(r, s) = 1, M(Vr, Vs) = \sigma(Vr, Vs), \phi = 1$, and $Vr = r \forall r, s \in U$. □

Theorem 19. *Let us suppose that (S, σ) be any $\mathfrak{R}$-complete metric space where $\mathfrak{R}$ is a binary relation defined on set S, and $U : S \longrightarrow S$ is a self-mapping with $\phi \in \Phi$ such that $\sigma(Ur, Us) \leq \lambda[\sigma(r, Ur) + \sigma(s, Us)]$. Then, U has a unique fixed point if we have the following conditions:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $\mathfrak{R}$ is U-closed

(3) U is $\mathfrak{R}$-continuous

Proof. The result can be attained directly from the given Theorem (13) by taking $\xi(s, r) = 2\lambda r - s, \lambda \in (0, 1/2), \beta(r, s) = 1, \phi = 1$ and $Vr = r$ for every $r, s \in U$. □

Theorem 20. *Let us suppose that (S, σ) be any $\mathfrak{R}$-complete metric space equipped with a binary relation $\mathfrak{R}$ defined on set S, and $U : S \longrightarrow S$ is a self-mapping such that $\sigma(Ur, Us) \leq \lambda[\sigma(r, Us) + \sigma(s, Ur)]$. Then, U has a unique fixed point if we have the following conditions:*

(1) $S(U, \mathfrak{R})$ is nonempty

(2) $\mathfrak{R}$ is U-closed

(3) U is $\mathfrak{R}$-continuous

Proof. The result will hold directly from the given Theorem (13) by taking $\xi(r, s) = 2\lambda s - r$ for all $r, s > 0$ and $\lambda \in (0, 1/2)$, $\beta(r, s) = 1, \phi = 1$ and $Vr = r$ for all $r, s \in S$. □

5. Example

Here, in this section, we have demonstrated an example in support of Theorem (13) which shows that our main Theorem (13) is applicable to determine the existence of a unique common fixed point for the given problem, but the existing result in [31] is not applicable.

Example 21. Consider the space $S = [0,\infty)$ with usual metric σ and a binary relation $\mathfrak{R}$ defined on S. Then, a pair of mappings $U, V : S \longrightarrow S$ defined as

$$Ur = \begin{cases} 0, & \text{if } r \in [0,1], \\ 2, & \text{if } r \in (1,\infty). \end{cases}$$
$$Vr = \begin{cases} 0, & \text{if } r \in [0,1], \\ 1, & \text{if } r \in (1,2], \\ 2, & \text{if } r \in (2,3], \\ 4, & \text{if } r \in (3,\infty), \end{cases} \quad (36)$$

have a unique common fixed point with $\beta(s, r) = 1$.

Proof. For $r = 1.5$ and $s = 2$, we have $Ur = 2$, $Vr = 1$, $Us = 2$, and $Vs = 1$. Hence,

$$\beta(Vr, Vs) = \beta(1,1) = 1,$$
$$\beta(Ur, Us) = \beta(2,2) = \frac{1}{6}. \quad (37)$$

Thus, $\beta(Vr, Vs) \geq 1 \nRightarrow \beta(Ur, Us) \geq 1$ and hence, U is not β- admissible with respect to V. Hence, solution of this mathematical problem can not be find out with the help of existing result Theorem (2.2) in [32].

(1) Clearly, $S(U, \mathfrak{R})$ is nonempty

(2) Since $U(S) = \{0, 2\}$, $V(S) = \{0, 1, 2, 4\}$ and hence clearly, we have $U(S) \subseteq V(S)$.

(3) Since for all $r \in \mathfrak{R}$, we have $(r, Ur), (r, Vr) \in \mathfrak{R}$. Thus, $\mathfrak{R}$ is $U-$*closed* and V-closed

(4) Also, U and V are $\mathfrak{R}$-continuous, as for any sequence (r_n) with $r_n \longrightarrow r$, we have $Ur_n \longrightarrow Ur$ and $Vr_n \longrightarrow Vr$.

(5) $M(Vr, Vs) = 4$ and max $(\sigma(Ur, Us)) = 2$.

$\beta(Vr, Vs)\sigma(Ur, Us) \leq 2 < 4 = M(Vr, Vs)$.
$\beta(Vr, Vs)\sigma(Ur, Us) < M(Vr, Vs)$.
Now by choosing an monotonically increasing function $\phi(t) = e^t$, we obtain
$\beta(Vr, Vs)\sigma(Ur, Us) < \phi(M(Vr, Vs))$.
Thus, $\xi(\beta(Vr, Vs)\sigma(Ur, Us), \phi(M(Vr, Vs))) \geq 0$.
Hence, pair (U, V) is generalized $\beta - \phi -$ Z-contractive pair of mappings. Hence, all the conditions are satisfied of the above Theorem (13) and thus U and V have a unique common fixed point. Here, in our example, U and V have a unique common fixed point 0. □

Graphical representation of solution: the graphical solution of Example 21 is given in Figure 1, which clearly shows that U and V have a unique common fixed point 0.

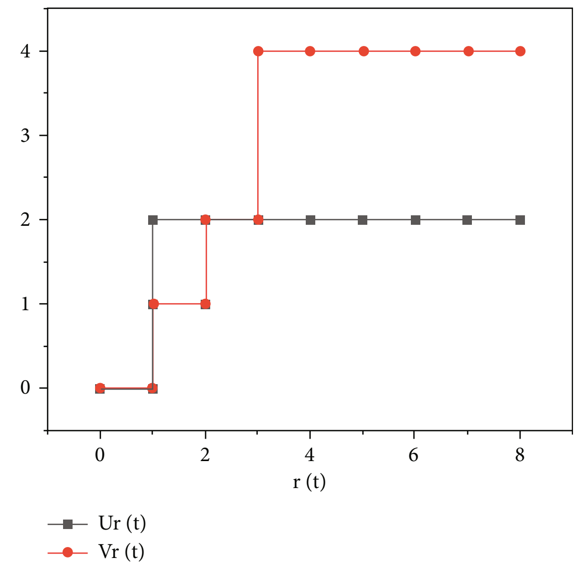

Figure 1: Graphical solution of the Example 21.

6. Application

6.1. Existence and Uniqueness of a Common Solution of Nonlinear Fractional Differential Equation of Caputo Type. The aim of this part is to introduce an application of Theorem (13) to obtain a common solution of nonlinear fractional differential equation of Caputo type for a pair of generalized $\beta - \phi - z$-contractive pair of mapping in metric space. The Caputo nonlinear fractional differential equation proposed by Kilbas, is given as follows:

$$ {}^{C}D_{p}^{\alpha} r(\mathfrak{t}) = \begin{cases} h(\mathfrak{t}, r(\mathfrak{t})), \mathfrak{t} \in (0,1), 1 < \gamma \leq 2, \\ r(0) = 0, r(1) = \displaystyle\int_0^w r(p)dp, 0 < w < 1, \end{cases} \quad (38)$$

where ${}^{C}D_{p}^{\gamma}$ represents the Caputo fractional derivative of order γ and $h : [0,1] \longrightarrow \mathbf{R}$ is a continuous function. This nonlinear fractional differential equation can also be represented in the form as follows:

$$r(\mathfrak{t}) = \frac{1}{\Gamma(\gamma)} \int_0^{\mathfrak{t}} (\mathfrak{t} - p)^{\gamma-1} h(p, r(p))dp - \frac{2\mathfrak{t}}{(2 - w^2)\Gamma(\gamma)} \int_0^1 (1-p)^{\gamma-1} h(p, r(p))dp + \frac{2\mathfrak{t}}{(2 - w^2)\Gamma(\gamma)} \int_0^w \left[\int_0^p (p-z)^{\gamma-1} h(z, r(z))dz\right] dp. \quad (39)$$

A function $r(\mathfrak{t})$ is a solution of above defined nonlinear fractional differential equation whenever it is the solution of the fractional integral Equation (39) and vice-versa.

Theorem 22. *Consider the space of all continuous functions $S = C[0,1]$ constructed on closed interval $[0,1]$, equipped with a binary relation $\mathfrak{R} = (C[0,1] \times C[0,1])$. Let $C([0,1], S)$ be* *the Banach space of all continuous functions from* $[0,1]$ *into* S *with norm* $\|r\| = sup_{t\in[0,1]}|r(t)|\,;\forall r(t)\in C[0,1]$. *This space defines the metric as follows:*

$$\sigma(r,s) = sup_{t\in[0,1]}|r(t)-s(t)|;\forall r,s\in S. \quad (40)$$

Now if we construct two self-maps $U, V : S \longrightarrow S$ *as*

$$Ur(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p,r(p))dp - \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}h(p,r(p))dp + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\left[\int_0^p (p-z)^{\gamma-1}h(z,r(z))dz\right]dp, \quad (41)$$

and

$$Vr(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p,r(p))dp, \quad (42)$$

satisfying that $U(S)\subseteq V(S)$. *Then,* U *and* V *have a unique common fixed point if we have the following assumptions:*

(1) $\beta(r,s)=1, \forall r,s\in S$,

(2) *there exist a continuous function* $h:[0,1]\times\mathbf{R}\longrightarrow\mathbf{R}$ *satisfying that* $|h(t,r(t))-h(t,s(t))| = k|r(t)-s(t)|$, *where* k *is some constant*

(3) $t^\gamma(2-w^2)(\gamma+1)+2t(1+\gamma+\gamma^2)/(\gamma+1)(2-w^2)\le 1$.

Proof. Obviously, $C([0,1],S)$ is a complete metric space.

(1) since $U : S\longrightarrow S$ is a self map, and $\mathfrak{R} = \{S\times S\}$, then for any $r_0\in S$, we have $Ur_0\in S$. Thus, $(r_0, Ur_0)\in\mathfrak{R}$, and hence $S(U,\mathfrak{R})$ is nonempty

(2) for any $r_0\in S, Ur_0, U2r_0, \cdots.., U^Q r_0, U^{Q+1}r_0, \cdots\in S$ as $U:S\longrightarrow S$ is a self-map. Therefore, $(r_0, Ur_0), (Ur_0, U^2r_0), \cdots\cdots\cdots, (U^Q r_0, U^{Q+1}r_0)\in\mathfrak{R}$ and hence $\mathfrak{R}$ is U-closed. In the same pattern, we can easily prove that $\mathfrak{R}$ is V-closed.Thus, $\mathfrak{R}$ is U-closed as well as V-closed

(3) from given condition (1), we have $\beta(r,s)=1$, i.e. $\beta(r,s)\ge 1$, for each $r,s\in S$.

(4) it is only required to prove that pair (U,V) is generalized $\beta-\phi-z$-contractive pair of mappings

$$\sigma(Vr(t),Vs(t)) = sup_{t\in[0,1]}|Vr(t)-Vs(t)|,$$

$$= sup_{t\in[0,1]}\left|\frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p,r(p))dp - \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p,s(p))dp\right|,$$

$$= \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}sup_{t\in[0,1]}|h(p,r(p))dp - h(p,s(p))dp|,$$

$$= \frac{k}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}sup_{t\in[0,1]}|r(p)dp - s(p)dp|,$$

$$= \frac{k}{\Gamma(\gamma)}\|r-s\|_\infty\int_0^t (t-p)^{\gamma-1}dp = \frac{k\|r-s\|_\infty p^\gamma}{\gamma[\Gamma(\gamma)]^2}.$$

$$\begin{aligned}\sigma(Ur(t),Us(t)) &= \sup_{t\in[0,1]}|Ur(t)-Us(t)|,\\ &= \sup_{t\in[0,1]}\left|\frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p,r(p))dp\right.\\ &\quad - \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}h(p,r(p))dp\\ &\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\left[\int_0^p (p-z)^{\gamma-1}h(z,r(z))dz\right]dp\\ &\quad - \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p,s(p))dp\\ &\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}h(p,s(p))dp\\ &\quad \left.- \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\left[\int_0^p (p-z)^{\gamma-1}h(z,s(z))dz\right]dp\right|.\end{aligned}$$

$$\begin{aligned}\sigma(Ur(t),Us(t)) &= \sup_{t\in[0,1]}\left|\frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}[h(p,r(p))-h(p,s(p))]dp\right.\\ &\quad - \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}[h(p,r(p))-h(p,s(p))]dp\\ &\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\left[\int_0^p (p-z)^{\gamma-1}[h(z,r(z))\right.\\ &\quad - h(z,s(z))]dz]dp|,\\ &\le \sup_{t\in[0,1]}\frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}|h(p,r(p))-h(p,s(p))|dp\\ &\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}|h(p,r(p))\\ &\quad - h(p,s(p))|dp + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\\ &\quad \cdot\left[\int_0^p (p-z)^{\gamma-1}|h(z,r(z))-h(z,s(z))|dz\right]dp,\\ &\le \frac{k}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}\sup_{t\in[0,1]}|r(p)-s(p)|dp\\ &\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}\sup_{t\in[0,1]}|r(p)-s(p)|dp\\ &\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\left[\int_0^p (p-z)^{\gamma-1}\sup_{t\in[0,1]}|r(z)-s(z)|dz\right]dp.\end{aligned}$$

$$\begin{aligned}\sigma(Ur(t),Us(t)) &\le \frac{k\|r-s\|_\infty}{\Gamma(\gamma)}\left[\int_0^t (t-p)^{\gamma-1}dp - \frac{2t}{(2-w^2)}\int_0^1 (1-p)^{\gamma-1}dp\right.\\ &\quad \left.+ \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w\left[\int_0^p (p-z)^{\gamma-1}dz\right]dp\right],\\ &= \frac{k\|r-s\|_\infty}{\gamma\Gamma(\gamma)^2}\left[\frac{t^\gamma(2-w^2)(\gamma+1)+2t(1+\gamma+\gamma^2)}{(\gamma+1)(2-w^2)}\right].\end{aligned} \quad (43)$$

By using given condition (c), we get

$$\begin{aligned}\sigma(Ur(\mathrm{t}), Us(\mathrm{t})) &\le \frac{k\|r-s\|_{\infty}}{\gamma\Gamma(\gamma)^2},\\ &= \sigma(Vr(\mathrm{t}), Vs(\mathrm{t})),\\ &\le M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t}))).\end{aligned} \tag{44}$$

Now if we assume that $\phi(\mathrm{t}) = e^{\mathrm{t}}$, then by assuming $M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t}))) = \mathrm{t}$, we get

$$\begin{aligned} &e^{M(\sigma(Vr(\mathrm{t}),Vs(\mathrm{t})))} > M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t}))),\\ &\text{i.e.,}\phi(M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t})))) > M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t}))).\end{aligned} \tag{45}$$

Then, by comparing Equations (45) and (6.6), we get

$$\sigma(Ur(t), Us(t)) \le \phi(M(\sigma(Vr(t), Vs(t)))),$$

$$\begin{aligned}&\beta(Vr(\mathrm{t}), Vs(\mathrm{t}))\sigma(Ur(\mathrm{t}), Us(\mathrm{t}))\\ &\quad\le \phi(M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t})))), [\beta(Vr(\mathrm{t}), Vs(\mathrm{t})) = 1],\end{aligned}$$

$$\xi(\beta(Vr(\mathrm{t}), Vs(\mathrm{t}))\sigma(Ur(\mathrm{t}), Us(\mathrm{t})), \phi(M(\sigma(Vr(\mathrm{t}), Vs(\mathrm{t}))))) \ge 0. \tag{46}$$

This shows that pair (U, V) is generalized $\beta-\phi-z$ -contractive pair of mapping. Therefore, all the hpothesis of Theorem (13) are satisfied, and hence, U and V have a unique fixed point. □

Example 23. Consider the space of all continuous functions $S = C[0, 1]$, defined on closed interval $[0, 1]$ endowed with a metric σ defined as $\sigma(r, s) = sup_{\mathrm{t}\in[0,1]}|r(\mathrm{t}) - s(\mathrm{t})|$, and a binary relation $\mathfrak{R} = \{C[0, 1] \times C[0, 1]\}$. Then, the two self-maps $U, V : S \longrightarrow S$ are constructed as

$$\begin{aligned} Vr(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1}h(p, r(p))dp,\\ \text{and } \mathrm{Ur}(\mathrm{t}) &= \frac{1}{\Gamma(\gamma)}\int_0^{\mathrm{t}} (\mathrm{t}-p)^{\gamma-1}h(p, r(p))dp\\ &\quad - \frac{2\mathrm{t}}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1}h(p, r(p))dp\\ &\quad + \frac{2\mathrm{t}}{(2-w^2)\Gamma(\gamma)}\int_0^w \left[\int_0^p (p-z)^{\gamma-1}h(z, r(z))dz\right]dp,\end{aligned} \tag{47}$$

with $\nu = 1/2, \gamma = 3/2$ and $\mathrm{t} \in [0, 1]$, have a unique common fixed point for $r(\mathrm{t}) = 2\mathrm{t} + 1/\mathrm{t}$, $s(\mathrm{t}) = 4 + 6\mathrm{t} + 8\mathrm{t}^2 + 1/\mathrm{t}$ and $h(\mathrm{t}, z(\mathrm{t})) = 1/2[r(\mathrm{t}) - 1/\mathrm{t}]$.

Proof. Since $r(\mathrm{t}) = 2\mathrm{t} + 1/\mathrm{t}$, $s(\mathrm{t}) = 4 + 6\mathrm{t} + 8\mathrm{t}^2 + 1/\mathrm{t}$ and $h(\mathrm{t}, z(\mathrm{t})) = 1/2[z(\mathrm{t}) - 1/\mathrm{t}]$. Then, $h(\mathrm{t}, r(\mathrm{t})) = \mathrm{t}$, $h(\mathrm{t}, s(\mathrm{t})) = 2 + 3\mathrm{t} + 4\mathrm{t}^2$.

$$\begin{aligned}&|r(\mathrm{t}) - s(\mathrm{t})| = 4(1 + \mathrm{t} + 2\mathrm{t}^2),\\ &|h(\mathrm{t}, r(\mathrm{t})) - h(\mathrm{t}, s(\mathrm{t}))| = 2(1 + \mathrm{t} + 2\mathrm{t}^2).\\ &\text{Hence, } |h(\mathrm{t}, r(\mathrm{t})) - h(\mathrm{t}, s(\mathrm{t}))| \le |r(\mathrm{t}) - s(\mathrm{t})|\end{aligned}$$

$$\begin{aligned} Vr(\mathrm{t}) &= \frac{1}{\Gamma(\gamma)}\int_0^{\mathrm{t}} (\mathrm{t}-p)^{\gamma-1}h(p, r(p))dp\\ &= \frac{1}{\Gamma(3/2)}\int_0^{\mathrm{t}} (\mathrm{t}-p)^{1/2}pdp,\\ &= \frac{1}{\Gamma(3/2)}\mathrm{t}^{1/2}\int_0^p \left(1-\frac{p}{\mathrm{t}}\right)^{1/2}pdp,\end{aligned}$$

$$Vs(\mathrm{t}) = \frac{1}{\Gamma(\gamma)}\int_0^{\mathrm{t}} (\mathrm{t}-p)^{\gamma-1}h(p, s(p))dp,$$

$$\begin{aligned} &= \frac{1}{\Gamma(3/2)}\int_0^p (\mathrm{t}-p)^{1/2}\left(2 + 3p + 4p^2\right)dp,\\ &= \frac{1}{\Gamma(3/2)}\mathrm{t}^{1/2}\left[2\int_0^p \left(1-\frac{p}{t}\right)^{1/2}dp + 3\int_0^p \left(1-\frac{p}{\mathrm{t}}\right)^{1/2}pdp\right.\\ &\quad \left.+ 4\int_0^p \left(1-\frac{p}{t}\right)^{1/2}p^2dp\right].\end{aligned} \tag{48}$$

By putting $\mathrm{p}/\mathrm{t} = l$, we get $p = \mathrm{t}l$, $dp = \mathrm{t}dl$, at $p = 0, l = 0$ and at $p = \mathrm{t}, l = 1$, we get

$$\begin{aligned} Vr(t) &= \frac{1}{\Gamma(3/2)}t^{5/2}\\ &\quad\int_0^1 (1-l)^{1/2}ldl; = \frac{2t^{5/2}\Gamma(3/2)\Gamma(2)}{\sqrt{\pi}\Gamma(7/2)} = \frac{8t^{5/2}}{15\sqrt{\pi}},\end{aligned}$$

$$\begin{aligned} Vs(\mathrm{t}) &= \frac{1}{\Gamma(3/2)}\mathrm{t}^{1/2}\left[2\int_0^1 (1-l)^{1/2}tdl + 3\int_0^1 (1-l)^{1/2}\mathrm{t}^2ldl\right.\\ &\quad\left.+ 4\int_0^1 (1-l)^{1/2}\mathrm{t}^3l^2dl\right],\end{aligned}$$

$$\begin{aligned} Vs(\mathrm{t}) &= \frac{4\mathrm{t}^{3/2}}{\sqrt{\pi}}\left[\frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)}\right] + \frac{6\mathrm{t}^{5/2}}{\sqrt{\pi}}\left[\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)}\right]\\ &\quad + \frac{8\mathrm{t}^{7/2}}{\sqrt{\pi}}\left[\frac{\Gamma(3/2)\Gamma(3)}{\Gamma(9/2)}\right]; = \frac{8\mathrm{t}^{3/2}}{\sqrt{\pi}}\left[\frac{1}{3} + \frac{\mathrm{t}}{5} + \frac{16\mathrm{t}^2}{105}\right].\end{aligned}$$

$$
\begin{aligned}
Ur(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1} h(p,r(p))\,dp \\
&\quad - \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1} h(p,r(p))\,dp \\
&\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w \left[\int_0^p (p-z)^{\gamma-1} h(z,r(z))\,dz\right] dp, \\
&= \frac{1}{\Gamma(3/2)}\int_0^t (t-p)^{1/2} p\,dp - \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^1 (1-p)^{1/2} p\,dp \\
&\quad + \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^w \left[\int_0^p (p-z)^{1/2} z\,dz\right] dp, \\
&= \frac{1}{\Gamma(3/2)}\left[t^{1/2}\int_0^t \left(1-\frac{p}{t}\right)^{1/2} p\,dp\right. \\
&\quad - \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^1 (1-p)^{1/2} p\,dp + \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^w \\
&\quad \cdot \left.\left[p^{1/2}\int_0^p \left(1-\frac{z}{p}\right)^{1/2} z\,dz\right] dp\right].
\end{aligned}
$$

$$
\begin{aligned}
Us(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-p)^{\gamma-1} h(p,s(p))\,dp \\
&\quad - \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^1 (1-p)^{\gamma-1} h(p,s(p))\,dp \\
&\quad + \frac{2t}{(2-w^2)\Gamma(\gamma)}\int_0^w \left[\int_0^p (p-z)^{\gamma-1} h(z,s(z))\,dz\right] dp, \\
&= \frac{1}{\Gamma(3/2)}\int_0^t (t-p)^{1/2}\left(2+3p+4p^2\right) dp \\
&\quad - \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^1 (1-p)^{1/2}\left(2+3p+4p^2\right) ds \\
&\quad + \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^w \left[\int_0^p (p-z)^{1/2}\left(2+3p+4p^2\right) dz\right] dp, \\
&= \frac{1}{\Gamma(3/2)}\left[t^{1}2\int_0^t \left(1-\frac{p}{t}\right)^{1/2}\left(2+3p+4p^2\right) dp\right. \\
&\quad - \frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^1 (1-p)^{1/2}\left(2+3p+4p^2\right) ds \\
&\quad + \left.\frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^w \left[p^{1/2}\int_0^p \left(1-\frac{z}{p}\right)^{1/2}\left(2+3z+4z^2\right) dz\right] ds\right].
\end{aligned}
\tag{49}
$$

By substituting p/t = ρ, we get $p = t\varrho$, $dp = t\,d\varrho$, and at $p = 0$, $\varrho = 0$, at $p = t$, $\varrho = 1$ and z/p = l, we get $z = pl$, $dz = p\,dl$, and at $z = 0$, $l = 0$, at $z = p$, $l = 1$, we get

$$
\begin{aligned}
Ur(t) &= \frac{2}{\sqrt{\pi}}\left[t^{5/2}\int_0^1 (1-\varrho)^{1/2}\varrho\,d\varrho - \frac{2t}{(2-w^2)}\int_0^1 (1-p)^{1/2} p\,dp\right. \\
&\quad + \left.\frac{2t}{(2-w^2)}\int_0^w \left[p^{5/2}\int_0^p (1-l)^{1/2} l\,dl\right] dp\right], \\
&= \frac{2}{\sqrt{\pi}}\left[t^{5/2}\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)} - \frac{2t}{(2-w^2)}\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)}\right. \\
&\quad + \left.\frac{2t}{(2-w^2)\Gamma(3/2)}\int_0^w p^{5/2}\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)}\,dp\right], \\
&= \frac{8t}{15\sqrt{\pi}}\left[t^{3/2} - \frac{2}{2-w^2} + \frac{4w^{7/2}}{7(2-w^2)}\right].
\end{aligned}
$$

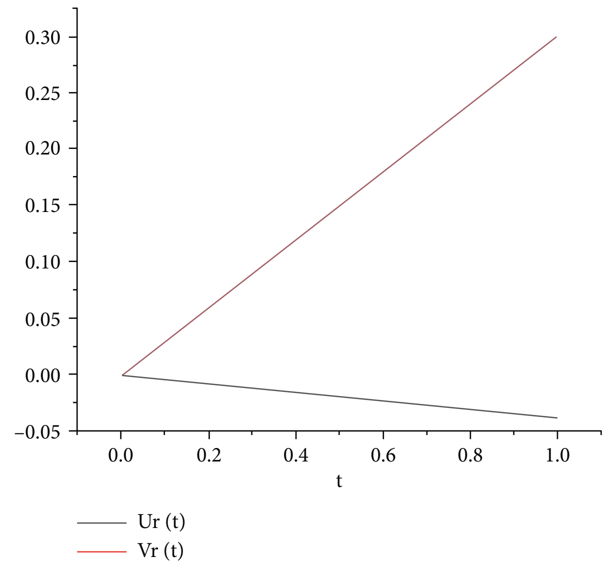

Figure 2: Graphical solution of the Example 23 for the function $r(t) = 2t + 1/t$.

$$
\begin{aligned}
Us(t) &= \frac{1}{\Gamma(3/2)}\left[t^{1/2}\int_0^t \left(1-\frac{p}{t}\right)^{1/2}\left(2+3p+4p^2\right) dp\right. \\
&\quad - \frac{2t}{(2-w^2)}\int_0^1 (1-p)^{1/2}\left(2+3p+4p^2\right) dp \\
&\quad + \left.\frac{2t}{(2-w^2)}\int_0^w \left[p^{1/2}\int_0^p \left(1-\frac{z}{p}\right)^{1/2}\left(2+3z+4z^2\right) dz\right] dp\right], \\
&= \frac{2}{\sqrt{\pi}}\left[t^{1/2}\int_0^1 (1-\varrho)^{1/2}\left(2+3t\varrho+4(t\varrho)^2\right) t\,d\varrho\right. \\
&\quad - \frac{2t}{(2-w^2)}\int_0^1 (1-p)^{1/2}\left(2+3p+4p^2\right) dp \\
&\quad + \left.\frac{2t}{(2-w^2)}\int_0^w \left[p^{1/2}\int_0^1 (1-l)^{1/2}\left(2+3lp+4(lp)^2\right) p\,dl\right] dp\right].
\end{aligned}
$$

$$
\begin{aligned}
Us(t) &= \frac{2}{\sqrt{\pi}}\left[2t^{3/2}\frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} + 3t^{5/2}\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)}\right. \\
&\quad + 4t^{7/2}\frac{\Gamma(3/2)\Gamma(3)}{\Gamma(9/2)} - \frac{2t}{(2-w^2)}\left[\frac{2\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)}\right. \\
&\quad + \left.\frac{3\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)} + \frac{4\Gamma(3/2)\Gamma(3)}{\Gamma(9/2)}\right] \\
&\quad + \frac{2t}{(2-w^2)}\int_0^w \left[p^{3/2}\frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} + 3p^{5/2}\frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)}\right. \\
&\quad + \left.\left.4p^{7/2}\frac{\Gamma(3/2)\Gamma(3)}{\Gamma(9/2)}\right]\right], = \frac{2}{\sqrt{\pi}}\left[\left[\frac{4t^{3/2}}{3} + \frac{4t^{5/2}}{5} + \frac{64t^{7/2}}{105}\right]\right. \\
&\quad - \frac{2t}{(2-w^2)}\left[\frac{4}{3} + \frac{4}{5} + \frac{64}{105}\right] + \frac{2t}{(2-w^2)}\int_0^w \left[\frac{4p^{3/2}}{3}\right. \\
&\quad + \left.\left.\frac{4p^{5/2}}{5} + \frac{64p^{7/2}}{105}\right]\right] = \frac{8t}{\sqrt{\pi}}\left[t^{1/2}\left[\frac{1}{3} + \frac{1}{5} + \frac{16t^2}{105}\right]\right. \\
&\quad - \left.\frac{48t}{35(2-w^2)} + \frac{48t}{35(2-w^2)}4w^{5/2}\left[\frac{1}{15} + \frac{w}{35} + \frac{16w^2}{105}\right]\right].
\end{aligned}
\tag{50}
$$

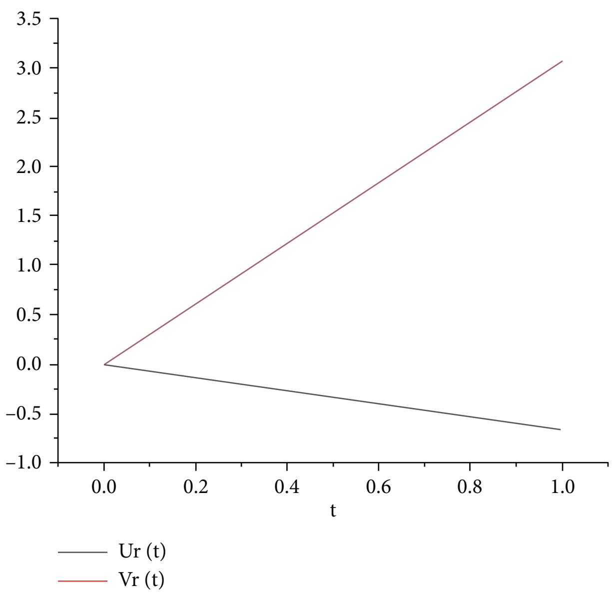

FIGURE 3: Graphical solution of the Example 23 for the function $s(\mathfrak{t}) = 4 + 6\mathfrak{t} + 8\mathfrak{t}^2 + 1/\mathrm{t}$.

Thus, all the hypotheses of Theorem (5.1) are fulfilled, and hence, U and V have a unique common fixed point. Here, clearly, U and V have a unique fixed point $\mathfrak{t} = 0$.

Graphical representation of solution: Figure 2 represents the solution of Example 23 in graphical form for the function $r(\mathfrak{t}) = 2\mathfrak{t} + 1/\mathrm{t}$, which shows that 0 is the unique common fixed point of U and V. □

Figure 3 represents the solution of Example 23 in graphical form for the function $s(\mathfrak{t}) = 4 + 6\mathfrak{t} + 8\mathfrak{t}^2 + 1/\mathrm{t}$, which shows that 0 is the unique common fixed point of U and V, and in this graph, s is represented by r.

7. Conclusion

The work presented here was carried out in the context of investigating some fixed point results for Z-contractive pair of mappings in a metric space under some binary relation assumed conditions. The notion of a binary relation is found to be more flexible in nature in the context of fixed point results. Due to the less restrictive nature of binary relations, the fixed point results attained in our paper have a much wider scope of applications. In the context of our main relation-theoretic common fixed point result, we also present an application in fractional calculus. In the future, we can try to prove the same results and many other results for some particular binary relations like transitive relations, symmetrical relations, reflexive relations, etc. The reader can also apply the concept of binary relation to other generalized metric spaces such as b-metric space, nontriangular metric space, cone metric space, and so on.

Data Availability

The research data used to support the findings of this study are currently under embargo, while the research findings are commercialized. Requests for data, 6 months after the publication of this article, will be considered by the corresponding authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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