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Research Article

Stability Analysis of the Crank-Nicolson Finite Element Method for the Navier-Stokes Equations Driven by Slip Boundary Conditions

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This paper is devoted to the study of numerical approximation for a class of two-dimensional Navier-Stokes equations with slip boundary conditions of friction type. The objective is to establish the well-posedness and stability of the numerical scheme in L^2-norm and H^1-norm for all positive time using the Crank-Nicholson scheme in time and the finite element approximation in space. The resulting variational structure dealing with is in the form of inequality, and obtaining H^1-estimate is more involved because of the presence of the nondifferentiable term appearing at the boundary where slip occurs. We prove that the numerical scheme is stable in L^2 and H^1-norms with the aid of different versions of discrete Grownwall lemmas, under a CFL-type condition. Finally, some numerical simulations are presented to illustrate our theoretical analysis.

1. Introduction

We consider the Navier-Stokes equations of viscous incompressible fluids:

$$u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \text{ in } Q = \Omega \times \mathbb{R}^+, \tag{1}$$

$$\text{div } u = 0 \text{ in } Q, \tag{2}$$

with the impermeability boundary condition

$$u_n = u \cdot n = 0 \text{ on } S \times \mathbb{R}^+, \tag{3}$$

and the slip boundary condition

$$\left.\begin{array}{l} \left|(\sigma n)_\tau\right| \le g, \\ \left|(\sigma n)_\tau\right| < g \Rightarrow u_\tau = 0, \\ \left|(\sigma n)_\tau\right| = g \Rightarrow u_\tau \ne 0, -(\sigma n)_\tau = g \dfrac{u_\tau}{|u_\tau|}, \end{array}\right\} \text{on } S \times (0,\infty). \tag{4}$$

On the remaining part of the boundary, Γ, we assume Dirichlet boundary condition, i.e.,

$$u = 0 \text{ on } \Gamma \times \mathbb{R}^+. \tag{5}$$

Finally, the initial condition is given by

$$u(x,0) = u_0(x) \text{ on } \bar{\Omega}. \tag{6}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain, with boundary $\partial\Omega$. It is assumed that $\partial\Omega$ is made of two components S, and Γ with $\partial\bar{\Omega} = S \bar{\cup} \Gamma$, and $S \cap \Gamma = \varnothing$. ν is a positive quantity representing the viscosity coefficient, $u_0 : \Omega \longrightarrow \mathbb{R}^2$ is the initial velocity, and $g : S \times (0,\infty) \longrightarrow (0,\infty)$ is the barrier or threshold function. The velocity of the fluid is u and p stands for the pressure, while f is the external force. Furthermore, n is the outward unit normal to the boundary $\partial\Omega$ of Ω, $u_\tau = u - u_n n$ is the tangential component of the velocity u, and $(\sigma n)_\tau = \sigma n - (n \cdot \sigma n)n$ is the tangential traction. Of course, $\sigma = -pI + 2\nu\varepsilon(u)$ is the Cauchy stress tensor, where I is the identity matrix, and $\varepsilon(u) = 1/2(\nabla u + (\nabla u)^T)$.

It can easily be shown that (4) is equivalent to

$$-(\sigma n)_\tau \in g\partial|u_\tau| \text{ on } S\times(0,\infty), \tag{7}$$

where the symbol $\partial|.|$ is the subdifferential of the real value function $|.|$, with $|u|^2 = u\cdot u$. We recall that if X is the Hilbert space with $x_0 \in X$, then,

$$y \in \partial\Psi(x_0) \Leftrightarrow \Psi(x) - \Psi(x_0) \ge y\cdot(x - x_0), \quad \text{for all } x \in X. \tag{8}$$

It should be mentioned that different boundary conditions describe different physical phenomena. The slip boundary condition of friction type can be justified by the fact that frictional effects of the fluid at the pores of the solid can be very important. Many studies have focused on the properties of the solution of the resulting boundary value problem, for example, existence, uniqueness, regularity, and continuous dependence on data, for Stokes, Navier-Stokes, and Brinkman–Forchheimer equations under such boundary condition. Details can be found in [1–4] among others. In [5], a generalization of the boundary condition (4) is formulated and analyzed for the steady Stokes flow, while the case of Navier-Stokes equations has been examined in [6]. There are numerous works devoted to the development of efficient schemes for the nonstationary Navier-Stokes problem dealing with Dirichlet or periodic boundary conditions; some works can be found in [7–11]. It should also be mentioned that there are other works dealing with Navier-Stokes equations with time fractional derivatives (see for instance [12] and references therein). For the time fractional operators, details can be found in [13, 14].

The subject of the present work is to establish the well-posedness and stability of the numerical scheme on L^2-norm and H^1-norm for all positive times of the two-dimensional problems (1)-(4) using the Crank-Nicholson scheme in time and the finite element approximation in space. The resulting variational structure dealing with is in the form of inequality, and obtaining H^1-estimate is more involved because of the presence of the nondifferentiable term appearing at the boundary where slip occurs.

2. Preliminaries and Variational Formulation

In this section, we introduce notations and some results that will be used throughout our work. We also formulate various weak formulations and discuss (recall) some existence results. For the mathematical setting of the problem, we need to introduce the following spaces:

$$\begin{aligned} V &= \{v \in H^1(\Omega)^2, v|_\Gamma = 0, v\cdot n|_S = 0\},\\ V_\sigma &= \{v \in V, \text{div } v = 0\},\\ H &= \{v \in L^2(\Omega)^2, \text{div } v = 0, v|_\Gamma = 0, v\cdot n|_S = 0\},\\ M &= L_0^2(\Omega). \end{aligned} \tag{9}$$

V' is the dual space of V, and the duality pairing between V' and V is denoted by $\langle\cdot\rangle$. Throughout the paper, we assume that Ω is bounded, convex planar domain with polygonal boundary. As usual, $\phi(t)$ stands for the function $x\in\Omega \mapsto \phi(x,t)$. Next, we introduce the Stokes operator A by following the approach adopted in [15, 16]. We denote by $\mathscr{P} : L^2(\Omega)^2 \longrightarrow H$ the Helmholtz projection operator, which is bounded projection associated to the Helmholtz decomposition of $L^2(\Omega)^2$. We define the Stokes operator as follows $A : V \longrightarrow V'$ such that $A = -\mathscr{P}\Delta$, with domain given as follows, $D(A) = \{v \in V, \text{such that } Av \in H\}$. Now, assuming that Γ is C^2 and S is C^3, then $D(A) \subset H^2(\Omega)^2$ since $\|w\|_2 \le C\|Aw\|$, and one has

$$\lambda_1 \int_\Omega |v|^2 dx \le \int_\Omega |\nabla v|^2 dx, \quad \text{for all } v \in V, \tag{10}$$

$$\lambda_1 \int_\Omega |\nabla v|^2 dx \le \int_\Omega |Av|^2 dx, \quad \text{for all } v \in D(A), \tag{11}$$

where λ_1 is the first eigenvalue of the Stokes operator A. It should be noticed that thanks to (10), $\|\nabla v\|$ is a norm on V equivalent to the usual H^1-norm.

The Stokes operator $A : D(A) \longrightarrow H$ is self-adjoint, positive with a compact inverse A^{-1} which is self-adjoint as a mapping from H to H.

We recall some classical bilinear and trilinear forms (see [17, 18])

$$\begin{aligned} &b : V\times M \longrightarrow \mathbb{R} \text{ with } b(u,p) = (\text{div } u, p),\\ &a : V\times V \longrightarrow \mathbb{R} \text{ with } a(u,v) = \nu(\varepsilon(u),\varepsilon(v)) = 2\nu((u,v)),\\ &d : V\times V\times V \longrightarrow \mathbb{R} \text{ with } d(u,v,w) = ((u\cdot\nabla)v, w). \end{aligned} \tag{12}$$

We denote by B a bilinear operator from $V\times V$ into V' such that

$$\langle B(u,v), w\rangle = d(u,v,w), \quad \text{for all } u,v,w \in V. \tag{13}$$

The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup conditions; i.e., there exists a positive constant β such that

$$\beta\|p\| \le \sup_{u\in V} \frac{b(u,p)}{\|u\|_1}, \quad \text{for all } p \in L_0^2(\Omega). \tag{14}$$

As a readily obtainable consequence of Korn's inequality (11), there exists a positive constant α such that

$$a(v,v) \ge \alpha\|v\|_1^2, \quad \text{for all } v \in V. \tag{15}$$

The trilinear form $d(\cdot,\cdot,\cdot)$ is continuous on $H^1(\Omega)^3$ and enjoys the following properties:

$$|d(u,v,w)| \le c_d\|u\|^{1/2}\|Au\|^{1/2}\|v\|_1\|w\|, \quad \text{for all } u \in D(A), v \in V, w \in H, \tag{16}$$

$$|d(u,v,w)| \le c_d \|u\|^{1/2}\|u\|_1^{1/2}\|v\|_1\|w\|^{1/2}\|w\|_1^{1/2}, \quad \text{for all } u,v,w \in V, \tag{17}$$

$$d(u,v,v) = 0, \quad \text{for all } u,v \in V_\sigma, \tag{18}$$

$$d(u,v,w) = -d(u,w,v), \quad \text{for all } u,v,w \in V_\sigma. \tag{19}$$

We will make reference to the following inequalities:

$$2(u-v,u) = \|u\|^2 - \|v\|^2 + \|u-v\|^2, \quad \text{for all } u,v \in L^2(\Omega), \tag{20}$$

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{q\varepsilon^{q/p}}b^q, \quad \text{for all } a,b,\epsilon > 0 \text{ and } \frac{1}{p}+\frac{1}{q} = 1. \tag{21}$$

We assume that $f \in L^\infty(0,\infty;L^2(\Omega)^2)$, and we set $\|f\|_\infty := \|f\|_{L^\infty(0,\infty;L^2(\Omega)^2)}$. We also assume that $u_0 \in L^2(\Omega)^2$. With the above notations, we introduce the following variational formulation for (1)-(6): Find $(u(t),p(t)) \in V \times M$ such that

$$u(0) = u_0 \text{ in } \Omega, \tag{22}$$

and for a.e. t, with $t \ge 0$

$$\begin{cases} \text{for all } (v,q) \in V \times M, \\ \left\langle u'(t), v-u(t)\right\rangle + a(u(t),v-u(t)) + d(u(t),u(t),v-u(t)) - b(v-u(t),p(t)) + J(v) - J(u(t)) \ge (f(t),v-u(t)), \\ b(u(t),q) = 0, \end{cases} \tag{23}$$

where $J(v) = (g,|v_\tau|)_S$.

Note that since the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (14), the variational inequality problem (23) is equivalent to the following:

Find $u(t) \in V_\sigma$ such that

$$u(0) = u_0 \text{ in } \Omega, \tag{24}$$

and for a.e. t, with $t \ge 0$

$$\begin{cases} \text{for all } v \in V_\sigma, \\ \left\langle u'(t), v-u(t)\right\rangle + a(u(t),v-u(t)) + d(u(t),u(t),v-u(t)) + J(v) - J(u(t)) \ge (f(t),v-u(t)). \end{cases} \tag{25}$$

The problem of existence and uniqueness of (24)-(25) can be stated as follows and has been proved in Kashiwabara [3].

Theorem 1. *Assume*

$$\begin{aligned} & f \in H^1(\mathbb{R}^+, L^2(\Omega)^2), \\ & g \in H^1(\mathbb{R}^+, L^2(S)^2) \text{ with } g(0) \in H^1(S), \\ & u_0 \in H^2(\Omega)^2 \cap V_\sigma, \text{and slip boundary condition (1.4) is satisfied at } t=0, \text{ i.e.,} \\ & |\sigma_\tau(u_0)| \le g(0) \text{ and } \sigma_\tau(u_0)u_{0\tau} + g|u_{0\tau}| = 0 \text{ a.e. on } S. \end{aligned} \tag{26}$$

Then, there exists a unique solution u of problem (24)-(25) such that

$$u \in L^\infty(\mathbb{R}^+, V_\sigma), \text{ and } u' \in L^\infty(\mathbb{R}^+, L^2(\Omega)^2) \cap L^2(\mathbb{R}^+, V_\sigma). \tag{27}$$

Let $\{t_n / t_n = nk, n \in \mathbb{N}\}$ be a uniform partition of $\mathbb{R}^+$ with a given time step k. We consider a time discretization of (24)-(25) using the Crank-Nicolson scheme. Find $u^n \in V$ such that

$$u^0 = u_0 \text{ in } \Omega, \tag{28}$$

and for all $n \ge 1$

$$\begin{cases} \text{for all } v \in V_\sigma, \\ \left(\dfrac{u^n - u^{n-1}}{k}, v-u^n\right) + \dfrac{1}{2}a(u^n+u^{n-1}, v-u^n) + \dfrac{1}{4}d(u^n+u^{n-1}, u^n+u^{n-1}, v-u^n) + J(v) - J(u^n) \ge (f^n, v-u^n), \end{cases} \tag{29}$$

where $f^n = (1/k)\int_{t_{n-1}}^{t_n} f(t)dt$. We want to show that the solution u^n of (28)-(29) is uniformly bounded for all $n \geq 0$, in both the L^2- and H_0^1-norms. In what follows, we discretize in space and derived such a result assuming some kind of stability condition.

3. Numerical Scheme

For the spatial discretization, we introduce the general framework as in, e.g., [18, 19]. We consider a family of finite element spaces $V_{\sigma h} \subset L^2(\Omega)^2$, each of which is endowed with two scalar products, $(\cdot,\cdot)_h$ and $((\cdot,\cdot))_h$, with the corresponding norms, $\|\cdot\|_h$ and $\|\cdot\|_{1,h}$ which mimic the L^2- and H_0^1-norms. These norms are related as follows:

$$\|u_h\|_h \leq K_1 \|u_h\|_{1,h}, \quad \text{for all } u_h \in V_{\sigma h}, \tag{30}$$

$$\|u_h\|_{1,h} \leq S(h) \|u_h\|_h, \quad \text{for all } u_h \in V_{\sigma h}, \tag{31}$$

where K_1 is independent of h and $S(h)$ is such that

$$S(h) \longrightarrow \infty \text{as } h \longrightarrow 0. \tag{32}$$

We assume that the operator A satisfies the same properties on $V_{\sigma h}$ as on V. We also assume that a trilinear continuous form $d(\cdot,\cdot,\cdot)$ enjoys the same properties on $V_{\sigma h} \times V_{\sigma h} \times V_{\sigma h}$ with the constant c_d independent of h.

We introduce the so-called restriction operators $r_h : V_\sigma \longrightarrow V_{\sigma h}$ and assume that, if $u_0 \in V_\sigma \cap C^1(\bar{\Omega})^2$, then,

$$\|r_h u_0\|_h \leq K_2 \|u_0\|_{C^1(\bar{\Omega})^2}, \tag{33}$$

with the constant K_2 being independent of h (see, e.g., [18]).

As for the temporal discretization, we consider the following scheme, a discrete version of (24)-(25): Find $u_h^n \in V_{\sigma h}$ such that

$$u_h^0 = r_h u_0, \tag{34}$$

and for all $n \geq 1$

$$\begin{cases} \text{for all } v_h \in V_{\sigma h}, \\ \left(\dfrac{u_h^n - u_h^{n-1}}{k}, v_h - u_h^n\right)_h + \dfrac{1}{2} a\left(u_h^n + u_h^{n-1}, v_h - u_h^n\right) + \dfrac{1}{4} d\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, v_h - u_h^n\right) + J(v_h) - J(u_h^n) \geq (f_h^n, v_h - u_h^n)_h. \end{cases} \tag{35}$$

Remark 2. For existence and uniqueness of the solution of (34)-(35), one observes that the variational inequality (34)-(35) can be seen as a case of following modified variational formulation associated to the stationary Navier-Stokes equations with slip boundary condition type.

$$\begin{cases} \text{Find } v_h \in V_{\sigma h}, \\ \mathbb{T}(v_h, w_h - v_h) + \mathbb{D}(v_h, v_h, w_h - v_h) + j(w_h) - j(v_h) \geq (F, w_h - v_h), \quad \text{for all } w_h \in V_{\sigma h}, \end{cases} \tag{36}$$

where $\mathbb{T}(u, v) = (u, v) + (k/2)a(u, v)$, $\mathbb{D}(u, v, w) = (k/4)d(u, v, w)$, $j(v) = kJ(v)$ and $F = kf_h^n$. Following [6, 20, 21], (36) admits a unique solution $v_h \in V_{\sigma h}$.

4. The $(V_h, \|\cdot\|_h)$-Stability

We start this section by performing the stability analysis of the scheme (34)-(35) in $(V_{\sigma h}, \|\cdot\|_h)$ and show that the solution is uniformly bounded, provided that a stability CFL-type condition is satisfied.

Lemma 3. *Let $M \geq K_1^2 \sqrt{2} \|f\|_\infty / \nu$ be arbitrarily fixed and assume that $\|u_0\| \leq M$ and*

$$kS^2(h) \leq \frac{4\nu}{15\left(c_d^2 M^2 + \nu^2\right)}. \tag{37}$$

Then, for any integer $n \geq 1$, we have

$$\|u_h^n\|_h^2 \leq (1 + Ck)^{-n} \|u_0\|^2 + C\|f\|_\infty^2 [1 - (1 + Ck)^{-n}], \tag{38}$$

$$\|u_h^n\|_h \le M, \tag{39}$$

$$J(u_h^n) \le \frac{K_1^2}{2\nu}\|f\|_\infty^2, \tag{40}$$

$$k\sum_{j=i}^{n}\left\|u_h^j\right\|_{1,h}^2 \le C\left(M^2 + (n-i+1)k\|f\|_\infty^2\right), \quad \text{for all } i = 1, \cdots, n, \tag{41}$$

$$\sum_{i=1}^{n}\left\|u_h^i - u_h^{i-1}\right\|_h^2 \le C\left(M^2 + nk\|f\|_\infty^2\right). \tag{42}$$

Proof. We first establish the relation (52) below and next use it to handle the proof by induction. First, let $v_h = 0$ and $v_h = 2u_h^n$ in (35); one has

$$\begin{aligned}&\frac{1}{k}\left(u_h^n - u_h^{n-1}, u_h^n\right)_h + \frac{1}{2}a\left(u_h^n + u_h^{n-1}, u_h^n\right)\\ &\quad + \frac{1}{4}d\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right)\\ &\quad + J(u_h^n) = (f_h^n, u_h^n)_h,\end{aligned} \tag{43}$$

that is

$$\begin{aligned}&2\left(u_h^n - u_h^{n-1}, u_h^n\right)_h + ka\left(u_h^n + u_h^{n-1}, u_h^n\right)\\ &\quad + \frac{k}{2}d\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right)\\ &\quad + 2kJ(u_h^n) = 2k(f_h^n, u_h^n)_h.\end{aligned} \tag{44}$$

Using relation (20), we have

$$2\left(u_h^n - u_h^{n-1}, u_h^n\right)_h = \|u_h^n\|_h^2 - \left\|u_h^{n-1}\right\|_h^2 + \left\|u_h^n - u_h^{n-1}\right\|_h^2. \tag{45}$$

Using Cauchy-Schwarz inequality, (31) and (21), we write

$$\begin{aligned}ka\left(u_h^n + u_h^{n-1}, u_h^n\right) &= 2\nu k\|u_h^n\|_{1,h}^2 + ka\left(u_h^{n-1} - u_h^n, u_h^n\right)\\ &\ge 2\nu k\|u_h^n\|_{1,h}^2 - \nu k\|u_h^n\|_{1,h}\left\|u_h^n - u_h^{n-1}\right\|_{1,h}\\ &\ge 2\nu k\|u_h^n\|_{1,h}^2 - \nu kS(h)\|u_h^n\|_{1,h}\left\|u_h^n - u_h^{n-1}\right\|_h\\ &\ge 2\nu k\|u_h^n\|_{1,h}^2 - \frac{1}{6}\left\|u_h^n - u_h^{n-1}\right\|_h^2\\ &\quad - \frac{3}{2}\nu^2k^2S^2(h)\|u_h^n\|_{1,h}^2,\end{aligned} \tag{46}$$

and the right hand side of (44) is bounded as follows:

$$2k(f_h^n, u_h^n)_h \le 2K_1k\|f_h^n\|_h\|u_h^n\|_{1,h} \le \nu k\|u_h^n\|_{1,h}^2 + \frac{K_1^2}{\nu}k\|f\|_\infty^2. \tag{47}$$

To bound the nonlinear term $d(\cdot,\cdot,\cdot)$ in (44), we write it as

$$\begin{aligned}\frac{k}{2}d\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right) &= kd\left(u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right)\\ &\quad + \frac{k}{2}d\left(u_h^n - u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right).\end{aligned} \tag{48}$$

Using (18) and (19), we have

$$\begin{aligned}d\left(u_h^{n-1}, u_h^n, u_h^n - u_h^{n-1}\right) &= d\left(u_h^{n-1}, u_h^n, u_h^n\right) - d\left(u_h^{n-1}, u_h^n, u_h^{n-1}\right)\\ &= d\left(u_h^{n-1}, u_h^n, u_h^n\right) + d\left(u_h^{n-1}, u_h^{n-1}, u_h^n\right)\\ &= d\left(u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right).\end{aligned} \tag{49}$$

Hence, using (16),(17), and (21) and recalling (31), we obtain the following bounds:

$$\begin{aligned}kd\left(u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right) &= kd\left(u_h^{n-1}, u_h^n, u_h^n - u_h^{n-1}\right)\\ &\le c_dkS(h)\left\|u_h^{n-1}\right\|_h\|u_h^n\|_{1,h}\left\|u_h^n - u_h^{n-1}\right\|_h\\ &\le \frac{1}{6}\left\|u_h^n - u_h^{n-1}\right\|_h^2\\ &\quad + \frac{3}{2}c_d^2k^2S(h)^2\left\|u_h^{n-1}\right\|_h^2\|u_h^n\|_{1,h}^2,\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{k}{2}d\left(u_h^n - u_h^{n-1}, u_h^n + u_h^{n-1}, u_h^n\right) &= -\frac{k}{2}d\left(u_h^n - u_h^{n-1}, u_h^n, u_h^{n-1}\right)\\ &\le c_dkS(h)\left\|u_h^n - u_h^{n-1}\right\|_h\|u_h^n\|_{1,h}\left\|u_h^{n-1}\right\|_h\\ &\le \frac{1}{6}\left\|u_h^n - u_h^{n-1}\right\|_h^2\\ &\quad + \frac{3}{8}c_d^2k^2S(h)^2\left\|u_h^{n-1}\right\|_h^2\|u_h^n\|_{1,h}^2.\end{aligned} \tag{51}$$

Gathering (44)-(51), we obtain

$$\begin{aligned}&\|u_h^n\|_h^2 - \left\|u_h^{n-1}\right\|_h^2 + \frac{1}{2}\left\|u_h^n - u_h^{n-1}\right\|_h^2\\ &\quad + \nu k\left\{1 - \frac{15}{8\nu}kS(h)^2\left(c_d^2\left\|u_h^{n-1}\right\|_h^2 + \nu^2\right)\right\}\|u_h^n\|_{1,h}^2\\ &\quad + 2kJ(u_h^n) \le \frac{K_1^2}{\nu}k\|f\|_\infty^2.\end{aligned} \tag{52}$$

Note that according to CFL-condition (37), if

$$\|u_h^n\|_h \le M, \tag{53}$$

then

$$0 \le \left\{1 - \frac{15}{8\nu}kS(h)^2\left(c_d^2\left\|u_h^{n-1}\right\|_h^2 + \nu^2\right)\right\} \le \frac{1}{2}. \tag{54}$$

We now use the induction. It is clear that (38) and (39) hold for $n = 0$. Then assuming that (38) holds for $n = 0, \cdots, m-1$, for $m \ge 2$, we see under the assumption of Lemma 3 that (39) holds for $n = 0, \cdots, m-1$. Then (52), together with (39) and (37), implies

$$\|u_h^n\|_h^2 - \|u_h^{n-1}\|_h^2 + \frac{1}{2}\|u_h^n - u_h^{n-1}\|_h^2 + \frac{\nu}{2}k\|u_h^n\|_{1,h}^2 + 2kJ(u_h^n) \le \frac{K_1^2}{\nu}k\|f\|_\infty^2, \quad \text{for all } n = 1, \cdots, m. \tag{55}$$

If we drop the last term on the left hand side and rewrite the remaining equation with n replaced by j and take the sum with $j = i, \cdots, n$, for some $1 \le i \le n$, we obtain

$$\|u_h^n\|_h^2 + \frac{1}{2}\sum_{j=i}^{n}\left\|u_h^j - u_h^{j-1}\right\|_h^2 + \frac{\nu}{2}k\sum_{j=i}^{n}\left\|u_h^j\right\|_{1,h}^2 \le M^2 + \frac{K_1^2}{\nu}(n-i+1)k\|f\|_\infty^2, \tag{56}$$

and hence, (41) and (42) hold for all $n = 1, \cdots, m-1$.

Now using (30), relation (55) implies

$$\|u_h^n\|_h^2 \le \frac{1}{(1 + (\nu/2K_1^2)k)}\|u_h^{n-1}\|_h^2 + \frac{K_1^2}{(1 + (\nu/2K_1^2)k)\nu}k\|f\|_\infty^2, \quad \text{for all } n = 1, \cdots, m. \tag{57}$$

Using recursively (57), we obtain

$$\begin{aligned}\|u_h^m\|_h^2 &\le \frac{1}{(1 + (\nu/2K_1^2)k)^m}\|u_h^0\|_h^2 + \frac{K_1^2}{\nu}k\|f\|_\infty^2\sum_{i=1}^{m}\frac{1}{(1 + (\nu/2K_1^2)k)^i}, \\ &\le \left(1 + \frac{\nu}{2K_1^2}k\right)^{-m}\|u^0\|^2 + C\|f\|_\infty^2\left[1 - \left(1 + \frac{\nu}{2K_1^2}k\right)^{-m}\right].\end{aligned} \tag{58}$$

□

Thus, (38) holds for $n = m$.

5. The $(V_h, \|\cdot\|_{1,h})$-Stability

For proving the uniform bound of u_h^n in $(V_h, \|\cdot\|_{1,h})$ for all $n \ge 1$, we first show that it is bounded on any finite interval of time. Then we extend the result to the infinite time using the discrete uniform Gronwall lemma.

Lemma 4. *Let $M \ge K_1^2\sqrt{2}(\|f\|_\infty + C_1\|Ag\|_S)/\nu$ be arbitrarily fixed and assume that $\|u_0\| \le M$, and assume also that the CFL-condition (37) is satisfied. Assume that k also satisfies*

$$k \le \frac{4K_1^2}{\nu} := \kappa_1. \tag{59}$$

Assume also that for some n the following is true:

$$K_3M^2k\left[L_1\|u_h^{n-1}\|_{1,h}^2 + \frac{2\kappa_1}{\nu}\left(\|f\|_\infty^2 + C_1\|Ag\|_S^2\right)\right] \le \frac{1}{6}, \tag{60}$$

where $L_1 = 2 + 3(c_d^2M^2/\nu^2)$, $C_1 = C^2/\lambda_1$ is given by (65) and K_3 is given by (64). Then,

$$\|u_h^n\|_{1,h}^2 \le \|u_h^{n-1}\|_{1,h}^2\left[1 + K_4M^2k\left(\|u_h^{n-1}\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right)\right] + K_5k\left(\|f\|_\infty^2 + \|Ag\|_S^2\right), \tag{61}$$

where K_4 and K_5 are positive constants independent of h and n.

Proof. Let $v_h = u_h^n - A(u_h^n + u_h^{n-1})$ in (35); we obtain

$$\begin{aligned}&\frac{1}{k}\left(u_h^n - u_h^{n-1}, A(u_h^n + u_h^{n-1})\right)_h + \frac{\nu}{2}\|A(u_h^n + u_h^{n-1})\|_h^2 \\ &+ \frac{1}{4}d\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, A(u_h^n + u_h^{n-1})\right) \le J\left(u_h^n - A(u_h^n + u_h^{n-1})\right) \\ &- J(u_h^n) + \left(f_h^n, A(u_h^n + u_h^{n-1})\right)_h,\end{aligned} \tag{62}$$

that is

$$\begin{aligned}&\|u_h^n\|_{1,h}^2 - \|u_h^{n-1}\|_{1,h}^2 + \frac{\nu}{2}k\|A(u_h^n + u_h^{n-1})\|_h^2 \\ &+ \frac{1}{4}kd\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, A(u_h^n + u_h^{n-1})\right) \le kJ\left(A(u_h^n + u_h^{n-1})\right) \\ &+ k\left(f_h^n, A(u_h^n + u_h^{n-1})\right)_h.\end{aligned} \tag{63}$$

Using relations (16) and (21) and the uniform bound (39), we majorize the trilinear form as

$$\begin{aligned}&\frac{1}{4}kd\left(u_h^n + u_h^{n-1}, u_h^n + u_h^{n-1}, A(u_h^n + u_h^{n-1})\right) \\ &\le \frac{1}{4}kc_d\|u_h^n + u_h^{n-1}\|_h^{3/2}\|u_h^n + u_h^{n-1}\|_{1,h}\|A(u_h^n + u_h^{n-1})\|_h^{3/2} \\ &\le \frac{1}{4}kc_d\sqrt{M}\Big\{\|u_h^n\|_{1,h}\|A(u_h^n + u_h^{n-1})\|_h^{3/2} \\ &\quad + \|u_h^{n-1}\|_{1,h}\|A(u_h^n + u_h^{n-1})\|_h^{3/2}\Big\} \\ &\le \frac{\nu}{8}k\|A(u_h^n + u_h^{n-1})\|_h^2 + K_3M^2k\|u_h^{n-1}\|_{1,h}^4 \\ &\quad + K_3M^2k\|u_h^n\|_{1,h}^4,\end{aligned} \tag{64}$$

where $K_3 = 27c_d^4/16\nu^3$.

$$\begin{aligned} J\left(A\left(u_h^n+u_h^{n-1}\right)\right) &= \left(Ag, A^{-1}\left|A\left(u_{\tau h}^n+u_{\tau h}^{n-1}\right)\right|\right)_S \\ &\leq \|Ag\|_S \left\|A^{-1}\left|A\left(u_{\tau h}^n+u_{\tau h}^{n-1}\right)\right|\right\|_S \\ &\leq \|Ag\|_S \left\|u_{\tau h}^n+u_{\tau h}^{n-1}\right\|_S \\ &\leq C\|Ag\|_S \left\|\nabla\left(u_h^n+u_h^{n-1}\right)\right\| \\ &\leq \frac{C}{\sqrt{\lambda_1}}\|Ag\|_S \left\|A\left(u_h^n+u_h^{n-1}\right)\right\| \\ &\leq \frac{\nu}{8}\left\|A\left(u_h^n+u_h^{n-1}\right)\right\|_h^2 + C_1\frac{2}{\nu}\|Ag\|_S^2, \end{aligned} \tag{65}$$

where $C_1 = C^2/\lambda_1$.

Using Cauchy-Schwarz inequality and relation (21), we have that

$$\begin{aligned} k\left(f_h^n, A\left(u_h^n+u_h^{n-1}\right)\right)_h &\leq k\|f\|_\infty \left\|A\left(u_h^n+u_h^{n-1}\right)\right\|_h \\ &\leq \frac{\nu}{8}k\left\|A\left(u_h^n+u_h^{n-1}\right)\right\|_h^2 + \frac{2}{\nu}k\|f\|_\infty^2. \end{aligned} \tag{66}$$

Gathering relations (63)-(66), we find

$$\begin{aligned} &\|u_h^n\|_{1,h}^2 - \left\|u_h^{n-1}\right\|_{1,h}^2 + \frac{\nu}{4}k\left\|A\left(u_h^n+u_h^{n-1}\right)\right\|_h^2 \\ &\quad\leq K_3M^2k\left\|u_h^{n-1}\right\|_{1,h}^4 + K_3M^2k\|u_h^n\|_{1,h}^4 \\ &\qquad + \frac{2}{\nu}k\|f\|_\infty^2 + C_1\frac{2}{\nu}k\|Ag\|_S^2, \end{aligned} \tag{67}$$

from which we obtain

$$\begin{aligned} &K_3M^2k\|u_h^n\|_{1,h}^4 - \|u_h^n\|_{1,h}^2 + K_3M^2k\left\|u_h^{n-1}\right\|_{1,h}^4 + \left\|u_h^{n-1}\right\|_{1,h}^2 \\ &\quad + \frac{2}{\nu}k\|f\|_\infty^2 + C_1\frac{2}{\nu}k\|Ag\|_S^2 \geq 0, \quad \text{for all } n \geq 1. \end{aligned} \tag{68}$$

From (68), we have either

$$\|u_h^n\|_{1,h}^2 \leq \frac{1-\sqrt{\Delta_h^{n-1}}}{2K_3M^2k} \tag{69}$$

or

$$\|u_h^n\|_{1,h}^2 \geq \frac{1+\sqrt{\Delta_h^{n-1}}}{2K_3M^2k}, \tag{70}$$

where

$$\begin{aligned} \Delta_h^{n-1} &= 1 - 4K_3M^2k\left(K_3M^2k\left\|u_h^{n-1}\right\|_{1,h}^4 + \left\|u_h^{n-1}\right\|_{1,h}^2 + \frac{2}{\nu}k\|f\|_\infty^2 + C_1\frac{2}{\nu}k\|Ag\|_S^2\right) \\ &\geq \frac{1}{3} \text{ by (59) and (60).} \end{aligned} \tag{71}$$

Let us show that with our assumption, (70) is impossible. Taking $\boldsymbol{v}_h = u_h^{n-1}$ in (35), we find

$$\begin{aligned} &\left\|u_h^n - u_h^{n-1}\right\|_h^2 + \frac{\nu}{2}k\|u_h^n\|_{1,h}^2 - \frac{\nu}{2}k\left\|u_h^{n-1}\right\|_{1,h}^2 \\ &\quad + \frac{1}{4}kd\left(u_h^n+u_h^{n-1}, u_h^n+u_h^{n-1}, u_h^n-u_h^{n-1}\right) \\ &\quad + k\left(J(u_h^n) - J\left(u_h^{n-1}\right)\right) \leq k\left(f_h^n, u_h^n-u_h^{n-1}\right)_h. \end{aligned} \tag{72}$$

Using (30) and (21), we bound the right hand side of (72) by

$$\begin{aligned} &K_1k\|f\|\|u_h^n\|_{1,h} + K_1k\|f\|\left\|u_h^{n-1}\right\|_{1,h} \\ &\quad\leq \frac{\nu}{12}k\|u_h^n\|_{1,h}^2 + \frac{\nu}{2}k\left\|u_h^{n-1}\right\|_{1,h}^2 + \frac{7K_1^2}{2\nu}k\|f\|_\infty^2. \end{aligned} \tag{73}$$

Since $d(\cdot,\cdot,\cdot)$ is a trilinear form, we can rewrite the nonlinear term in (72) as

$$\begin{aligned} &\frac{1}{4}kd\left(u_h^n+u_h^{n-1}, u_h^n+u_h^{n-1}, u_h^n-u_h^{n-1}\right) \\ &\quad = \frac{1}{2}kd\left(u_h^n, u_h^{n-1}, u_h^n\right) - \frac{1}{2}kd\left(u_h^{n-1}, u_h^n, u_h^{n-1}\right), \end{aligned} \tag{74}$$

and using property (17), we obtain the following bounds:

$$\begin{aligned} \frac{1}{2}kd\left(u_h^n, u_h^{n-1}, u_h^n\right) &\leq \frac{1}{2}c_dk\|u_h^n\|_h\|u_h^n\|_{1,h}\left\|u_h^{n-1}\right\|_{1,h} \\ &\leq \frac{\nu}{12}k\|u_h^n\|_{1,h}^2 + \frac{3}{4\nu}c_d^2k\|u_h^n\|_h^2\left\|u_h^{n-1}\right\|_{1,h}^2, \\ \frac{1}{2}kd\left(u_h^{n-1}, u_h^n, u_h^{n-1}\right) &\leq \frac{1}{2}c_dk\left\|u_h^{n-1}\right\|_h\left\|u_h^{n-1}\right\|_{1,h}\|u_h^n\|_{1,h} \\ &\leq \frac{\nu}{12}k\|u_h^n\|_{1,h}^2 + \frac{3}{4\nu}c_d^2k\left\|u_h^{n-1}\right\|_h^2\left\|u_h^{n-1}\right\|_{1,h}^2. \end{aligned} \tag{75}$$

Employing (40), we bound the last term of the left hand side of (72) by

$$-\frac{K_1^2}{2\nu}k\|f\|_\infty^2 \leq k\left(J(u_h^n) - J\left(u_h^{n-1}\right)\right) \leq \frac{K_1^2}{2\nu}k\|f\|_\infty^2. \tag{76}$$

Gathering (72)-(76) and recalling (39), we obtain

$$\begin{aligned} &\left\|u_h^n - u_h^{n-1}\right\|_h^2 + \frac{\nu}{4}k\|u_h^n\|_{1,h}^2 - \left(\nu + \frac{3}{2\nu}c_d^2M^2\right)k\left\|u_h^{n-1}\right\|_h^2 \\ &\quad\leq \frac{8K_1^2}{2\nu}k\|f\|_\infty^2, \end{aligned} \tag{77}$$

and hence,

$$k\|u_h^n\|_{1,h}^2 \leq 2\left(2 + \frac{3}{\nu^2}c_d^2M^2\right)k\left\|u_h^{n-1}\right\|_h^2 + \frac{16K_1^2}{\nu^2}k\|f\|_\infty^2, \tag{78}$$

from which we find, using (60),

$$2K_3M^2k\|u_h^n\|_{1,h}^2 \le \frac{2}{3} < 1. \tag{79}$$

(79) contradicts (70), and therefore, we obtain

$$\|u_h^n\|_{1,h}^2 \le \frac{1-\sqrt{\Delta_h^{n-1}}}{2K_3M^2k} = 2\frac{K_3M^2k\|\boldsymbol{u}_h^{n-1}\|_{1,h}^4 + \|\boldsymbol{u}_h^{n-1}\|_{1,h}^2 + (2/\nu)k\|\boldsymbol{f}\|_\infty^2 + C_1(2/\nu)k\|Ag\|_S^2}{1+\sqrt{1-x}}, \tag{80}$$

where $x = 4K_3M^2k(K_3M^2k\|u_h^{n-1}\|_{1,h}^4 + \|u_h^{n-1}\|_{1,h}^2 + (2/\nu)k \|f\|_\infty^2 + C_1(2/\nu)k\|Ag\|_S^2)$.

Since $x \le 4/5$ (by (60)) and

$$\frac{2}{1+\sqrt{1-x}} \le 1 + \frac{x}{2} \text{ if } 0 \le x \le \frac{4}{5}, \tag{81}$$

we obtain, using (59) and (60) and the fact that $M \ge K_1^2\sqrt{2}(\|f\|_\infty + C_1\|Ag\|_S)/\nu$,

$$\begin{aligned}\|u_h^n\|_{1,h}^2 &\le \left(K_3M^2k\|u_h^{n-1}\|_{1,h}^4 + \|u_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|f\|_\infty^2 + C_1\frac{2}{\nu}k\|Ag\|_S^2\right)\\ &\quad\times\left[1 + 2K_3M^2k\left(K_3M^2k\|u_h^{n-1}\|_{1,h}^4 + \|u_h^{n-1}\|_{1,h}^2\right.\right.\\ &\quad\left.\left.+\frac{2}{\nu}k\|f\|_\infty^2 + C_1\frac{2}{\nu}k\|Ag\|_S^2\right)\right] \le K_3M^2k\|u_h^{n-1}\|_{1,h}^4\\ &\quad+\|u_h^{n-1}\|_{1,h}^2 + \frac{2}{\nu}k\|f\|_\infty^2 + C_1\frac{2}{\nu}k\|Ag\|_S^2\\ &\quad+2K_3M^2k\left(L_1\|u_h^{n-1}\|_{1,h}^2 + \frac{\kappa_1}{\nu}\left(\|f\|_\infty^2 + C_1\|Ag\|_S^2\right)\right)^2\\ &\le \|u_h^{n-1}\|_{1,h}^2\left[1 + K_4M^2k\left(\|u_h^{n-1}\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right)\right]\\ &\quad+K_5k\left(\|f\|_\infty^2 + \|Ag\|_S^2\right),\end{aligned} \tag{82}$$

with appropriate choice of constants K_4 and K_5. □

To prove that scheme (35) is conditionally stable on a finite interval of time, we need the following discrete Gronwall lemma [22].

Lemma 5. *Discrete Gronwall Lemma.*

Given $k > 0$, an integer $n_\star > 0$, and positive sequences α_n, β_n, and γ_n such that

$$\alpha_n \le \alpha_{n-1}(1 + k\beta_{n-1}) + k\gamma_n, \quad \text{for all } n = 1, \cdots, n_\star, \tag{83}$$

we have

$$\alpha_n \le \alpha_0 \exp\left(k\sum_{i=0}^{n-1}\beta_i\right) + \sum_{i=1}^{n-1}k\gamma_i \exp\left(k\sum_{j=i}^{n-1}\beta_j\right) + k\gamma_n, \quad \text{for all } n = 2, \cdots, n_\star. \tag{84}$$

Proof. Using recursively (83), we derive

$$\alpha_n \le \alpha_0\prod_{i=0}^{n-1}(1 + k\beta_i) + \sum_{i=1}^{n-1}k\gamma_i\prod_{j=i}^{n-1}(1 + k\beta_i) + k\gamma_n, \tag{85}$$

and since $1 + x \le \exp x$, for all $x \in \mathbb{R}$, the conclusion of the lemma follows. □

Proposition 6. *Estimates on a finite interval of time.*

Let $T > 0$ and $M \ge K_1^2\sqrt{2}(\|f\|_\infty + C_1\|Ag\|_S)/\nu$ be fixed, and let $\|u_0\| \le M$. Assume that, besides the CFL-condition (37), k also satisfies

$$k \le \min\left\{\kappa_1, \kappa_2(M, \|f\|_\infty, \|Ag\|_S), \kappa_3\left(M, \|u_h^0\|_{1,h}, \|f\|_\infty, \|Ag\|_S, T\right)\right\}, \tag{86}$$

where

$$\begin{aligned}\kappa_2(M, \|f\|_\infty) &= \frac{1}{12K_3K_6M^2\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)},\\ \kappa_3\left(M, \|u_h^0\|_{1,h}, \|f\|_\infty, T\right) &= \frac{1}{12K_3M^2L_1L_2\left(M, \|u_h^0\|_{1,h}, \|f\|_\infty, \|Ag\|_S, T\right)}.\end{aligned} \tag{87}$$

$L_2(\cdot,\cdot,\cdot,\cdot)$ is a monotonically increasing function in all its arguments and is given in (95) below and $K_6 = 8K_1^2/\nu^2$.

Then,

(a) Relation (58) holds for all $n = 1, \cdots, N = \lfloor T/k\rfloor$ (integer part of T/k)

(b) $\|u_h^n\|_{1,h}^2 \le L_2(M, \|u_h^0\|_{1,h}, \|f\|_\infty, \|Ag\|_S, nk)$, for all $n = 1, \cdots, N = \lfloor T/k\rfloor$

Proof. Let $T > 0$ and let h, k be such that (37) and (86) are satisfied.

We will use induction on n. If $n = 1$, assumption (86) implies

$$K_3M^2k\left(L_1\|u_h^0\|_{1,h}^2 + \frac{2\kappa_1}{\nu}\left(\|f\|_\infty^2 + C_1\|Ag\|_S^2\right)\right) \le \frac{1}{6}. \tag{88}$$

Thus, the conclusion (61) of Lemma 4 holds for $n = 1$. Now assume that (60) holds for $n = 1, \cdots, m$, for some $m \le N$. Hence, (61) holds for $n = 1, \cdots, m$. If we rewrite (61) as (83) with

$$\alpha_n = \|u_h^n\|_{1,h}^2, \quad \beta_n = K_4 M^2 \left(\|u_h^n\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right) \text{ and } \gamma_n = K_5\left(\|f\|_\infty^2 + \|Ag\|_S^2\right) \tag{89}$$

and noting that, using (41), we have

$$\begin{aligned} k\sum_{j=i}^{m-1}\beta_j &= K_4 M^2 k \sum_{j=i}^{m-1}\left(\left\|u_h^j\right\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right) \\ &\le 2K_7 M^2\left[M^2 + (m-i)k\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right], \end{aligned} \tag{90}$$

and therefore,

$$\begin{aligned} \sum_{i=1}^{m-1} k\gamma_i \exp\left(k\sum_{j=i}^{m-1}\beta_j\right) &\le K_5 k\left(\|f\|_\infty^2 + \|Ag\|_S^2\right) \\ &\times \sum_{i=1}^{m-1}\exp\left(2K_7M^2\left[M^2 \quad + (m-i)k\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right]\right) \\ &\le K_5\left(\|f\|_\infty^2 + \|Ag\|_S^2\right) mk \exp\left(2K_7M^4\right) \exp \\ &\quad \times \left(2K_7M^2 mk\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right). \end{aligned} \tag{91}$$

Similarly, for $i = 0$, we have

$$\begin{aligned} k\sum_{j=0}^{m-1}\beta_j &= K_4M^2k\sum_{j=0}^{m-1}\left(\left\|u_h^j\right\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right) \\ &\le 2K_7M^2\left(M^2 + mk\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right) \\ &\quad + K_4M^2k\left\|u_h^0\right\|_{1,h}^2. \end{aligned} \tag{92}$$

Using (86) and recalling that $L_1 \ge 2$, the last term of (83) can be bounded as

$$\begin{aligned} K_4M^2k\left\|u_h^0\right\|_{1,h}^2 &\le \frac{K_4\left\|u_h^0\right\|_{1,h}^2}{12K_3L_1L_2\left(M, \left\|u_h^0\right\|_{1,h}, \|f\|_\infty, \|Ag\|_S, T\right)} \\ &\le \frac{K_4}{24K_3}. \end{aligned} \tag{93}$$

Then, Lemma 5 and relations (90)-(93) imply

$$\|u_h^m\|_{1,h}^2 \le L_2\left(M, \left\|u_h^0\right\|_{1,h}, \|f\|_\infty, \|Ag\|_S, mk\right), \tag{94}$$

where

$$\begin{aligned} L_2&\left(M, \left\|u_h^0\right\|_{1,h}, \|f\|_\infty, \|Ag\|_S, mk\right) \\ &= \left\|u_h^0\right\|_{1,h}^2 \exp\left(2K_7M^4 + \frac{K_4}{24K_3}\right) \\ &\quad \times \exp\left(2K_7M^2mk\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right) \\ &\quad + 2K_5\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)mk\exp\left(2K_7M^2\right)\exp \\ &\quad \times \left(2K_7M^2mk\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right). \end{aligned} \tag{95}$$

Using (94) and recalling assumption (86), it is easily checked that condition (60) holds for $n - 1 = m$, and by the same Lemma 5, we have (61) that holds for $n = m + 1$. □

To prove the uniform bound of $\|u_h^n\|_{1,h}$ for all $n \ge 1$, we will repeatedly apply Proposition 6 on different intervals of time, considering different initial values, and we will need the following discrete uniform Gronwall lemma, a generalized version of the discrete uniform Gronwall lemma of Shen [22], whose proof can be found in [7].

Lemma 7. *Discrete uniform Gronwall lemma.*

Given $k > 0$, positive integers n_1, n_2, n_* such that $n_1 \le n_*$, $n_1 + n_2 + 1 \le n_*$, and positive sequences α_n, β_n and γ_n such that

$$\alpha_n \le \alpha_{n-1}(1 + k\beta_{n-1}) + k\gamma_n, \quad \text{for all} \quad n = 1, \cdots, n_*. \tag{96}$$

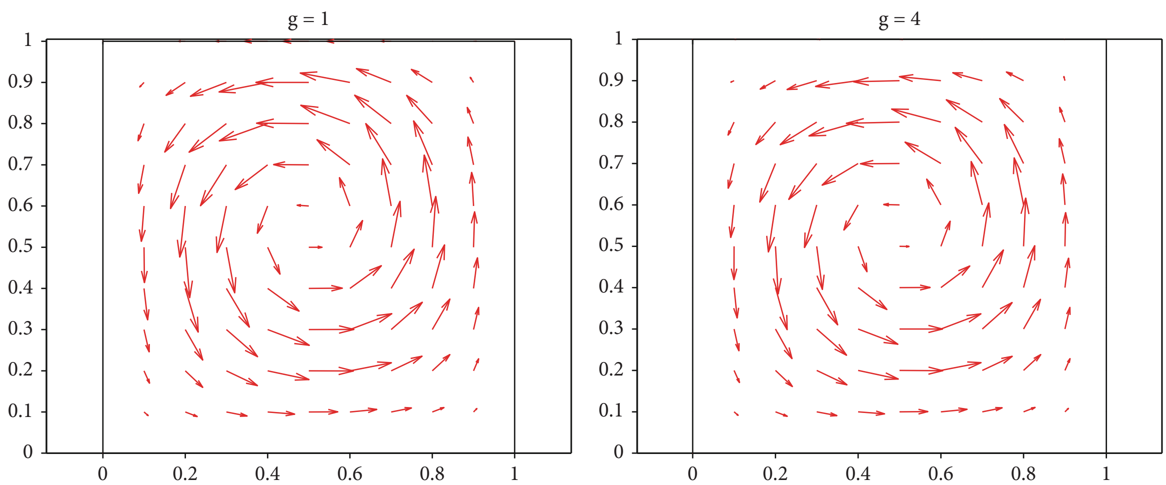
Assume also that for any n' satisfying $n_1 \le n' \le n_* - n_2$

$$\begin{aligned} \sum_{n=n}^{\prime n' + n_2} k\beta_n &\le C_1(n_1, n_*), \quad \sum_{n=n'}^{n'+n_2} k\alpha_n \\ &\le C_2(n_1, n_*), \quad \sum_{n=n'}^{n'+n_2} k\gamma_n \\ &\le C_3(n_1, n_*), \end{aligned} \tag{97}$$

then we have

$$\alpha_n \le \left(\frac{C_3(n_1, n_*)}{kn_2} + C_2(n_1, n_*)\right)\exp\left(C_1(n_1, n_*)\right), \quad \text{for any } n_1 + n_2 + 1 \le n \le n_*. \tag{98}$$

Theorem 8. Uniform bound of $\|u_h^n\|_{1,h}$ for all $n \ge 1$. *Let $u_0 \in V_\sigma \cap C^1(\bar{\Omega})^2$, $f \in L^\infty(\mathbb{R}^+; H)$, and assume that $\|u_0\| \le M$, where $M \ge K_1^2\sqrt{2}(\|f\|_\infty + C_1\|Ag\|_S)/\nu$. Also let $r \ge 4\kappa_1$ be arbitrarily fixed and assume that, besides the CFL-condition*



(a) (b)

Figure 1: Velocity field, respectively, for $g = 1$ and $g = 4$.

(37), k also satisfies

$$k \le \min \left\{ \kappa_1, \kappa_2\left(M, \|f\|_\infty, \|Ag\|_S\right), \kappa_3 \cdot \left(M, K_2 \|u_h^0\|_{C^1(\bar{\Omega})^2}, \|f\|_\infty, \|Ag\|_S, r\right), \kappa_3 \cdot \left(M, \rho_1, \|f\|_\infty, \|Ag\|_S, r\right) \right\}, \tag{99}$$

where $\kappa_1, \kappa_2, \kappa_3$ are defined above and ρ_1 is given in (107) below.

Then, we have

$$\|u_h^n\|_{1,h}^2 \le L_3\left(\|u_h^0\|_{C^1(\bar{\Omega})^2}, \|f\|_\infty, \|Ag\|_S\right), \quad \text{for all } n \ge 1, \tag{100}$$

where $L_3(\cdot,\cdot,\cdot)$ is a continuous function defined on $\mathbb{R}_+^3$, increasing.

Moreover, there exists an $N > 0$ such that

$$\|u_h^n\|_{1,h}^2 \le L_4\left(\|f\|_\infty, \|Ag\|_S\right), \quad \text{for all } n \ge N. \tag{101}$$

Proof. In order to derive uniform bounds $\|u_h^n\|_{1,h}$ for all $n \ge 1$, we apply Proposition 6 on successive intervals of time, with different initial values. On each interval considered, we obtain a bound $L_2(\cdot,\cdot,\cdot,\cdot)$ which depends on the norm $\|u_h^0\|_{1,h}$ and on the length of the interval. Using the discrete uniform Gronwall lemma, we majorize the norm of the initial values $\|u_h^0\|_{1,h}$ by a constant ρ_1, and recalling the fact that L_2 is an increasing function of its arguments, we obtain a bound independent on the initial value considered.

First using (33) and (99) and since κ_3 is a decreasing function of its arguments, we can apply Proposition 6 with $T = r$ to obtain

$$\|u_h^n\|_{1,h}^2 \le \|u_h^{n-1}\|_{1,h}^2 \left[1 + K_4 M^2 k\left(\|u_h^{n-1}\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right)\right] + K_5 k\left(\|f\|_\infty^2 + \|Ag\|_S^2\right), \tag{102}$$

$$\|u_h^n\|_{1,h}^2 \le L_2\left(M, \|u_h^0\|_{1,h}, \|f\|_\infty, \|Ag\|_S, r\right), \quad \text{for all } n = 1, \cdots, N_r := \lfloor r/k \rfloor. \tag{103}$$

To extend the bound (103) to $n = N_r + 1, \cdots, 2N_r$, we apply again Proposition 6, namely, $L_2(M, \|u_h^{N_r}\|_{1,h}, \|f\|_\infty, \|Ag\|_S, r)$ depends on the discrete initial value; we want to bound $\|u_h^{N_r}\|_{1,h}$ independent of h and k.

Rewrite (102) in the form of (96) with $\alpha_n = \|u_h^n\|_{1,h}^2$, $\gamma_n = K_5(\|f\|_\infty^2 + \|Ag\|_S^2)$ and $\beta_n = K_4 M^2 (\|u_h^{n-1}\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2)$. Then, we apply Lemma 7 with $n_1 = 1$, $n_2 = N_r - 2$, $n_* = N_r$ to obtain the bound of $\|u_h^{N_r}\|_{1,h}$. For $n' = 1, 2$, using (41), we have

$$k \sum_{n=n}^{\prime n'+n_2} \beta_n = K_4 M^2 k \sum_{n=n}^{\prime n'+n_2} \left(\|u_h^n\|_{1,h}^2 + \|f\|_\infty^2 + \|Ag\|_S^2\right) \le K_8 M^2 \left(M^2 + r\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right), \tag{104}$$

$$k \sum_{n=n}^{\prime n'+n_2} \gamma_n = K_5 k \sum_{n=n}^{\prime n'+n_2} \left(\|f\|_\infty^2 + \|Ag\|_S^2\right) \le K_5 r\left(\|f\|_\infty^2 + \|Ag\|_S^2\right), \tag{105}$$

$$k \sum_{n=n}^{\prime n'+n_2} \alpha_n = k \sum_{n=n}^{\prime n'+n_2} \|u_h^n\|_{1,h}^2 \le K_9 \left(M^2 + r\|f\|_\infty^2\right). \tag{106}$$

Then, Lemma 7, together with the assumption $r \ge 4\kappa_1$, yields

$$\left\|u_h^{N_r}\right\|_{1,h}^2 \le \left[2K_9\left(M^2/r + \|f\|_\infty^2\right) + K_5 r\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right] \times \exp\left(K_8 M^2\left(M^2 + r\left(\|f\|_\infty^2 + \|Ag\|_S^2\right)\right)\right) \coloneqq \rho_1\left(M, \|f\|_\infty, \|Ag\|_S, r\right). \quad (107)$$

Taking into account assumption (99) on the time step k, relation (107), and the fact that $L_2(\cdot, \cdot, \cdot)$ is an increasing function of its arguments, we apply Proposition 6 with $T = r$ and initial data $u_h^{N_r}$. We obtain that the relation (61) holds for all $n = N_r + 1, \cdots, 2N_r$, and

$$\|u_h^n\|_{1,h}^2 \le L_2\left(M, \left\|u_h^{N_r}\right\|_{1,h}, \|f\|_\infty, \|Ag\|_S, r\right) \le L_2\left(M, \rho_1, \|f\|_\infty, \|Ag\|_S, r\right), \quad \text{for all} \quad n = N_r + 1, \cdots, 2N_r. \quad (108)$$

Applying again Lemma 7 with $n_1 = N_r + 1$, $n_2 = N_r - 2$ and $n_\star = 2N_r$, we obtain

$$\left\|u_h^{2N_r}\right\|_{1,h}^2 \le \rho_1. \quad (109)$$

Iterating the above procedure, we find

$$\|u_h^n\|_{1,h}^2 \le L_2\left(M, \rho_1, \|f\|_\infty, \|Ag\|_S, r\right) \coloneqq L_3\left(\|f\|_\infty, \|Ag\|_S\right), \quad \text{for all } n \ge N_r, \quad (110)$$

and recalling (103), we conclude

$$\|u_h^n\|_{1,h}^2 \le \max\left\{L_2\left(M, \|u_h^0\|_{1,h}, \|f\|_\infty, \|Ag\|_S, r\right), L_3\left(\|f\|_\infty, \|Ag\|_S\right)\right\} \le \max\left\{L_2\left(M, K_2\|u_0\|_{C^1(\bar{\Omega})^2}, \|f\|_\infty, \|Ag\|_S, r\right), L_3\left(\|f\|_\infty, \|Ag\|_S\right)\right\} \text{ by } (32) \coloneqq L_4\left(K_2\|u_0\|_{C^1(\bar{\Omega})^2}, \|f\|_\infty, \|Ag\|_S\right), \quad \text{for all } n \ge 1. \quad (111)$$

As for the N beyond which $\|u_h^n\|_{1,h}$ is bounded independent of u_0, we can evidently take $N = N_r$ (see (110)). This completes the proof of the theorem. □

6. Numerical Experiments

Let us explain our numerical experiments. We assume $\Omega = (0,1)^2$, the boundary of which consists of two portions Γ and S given by

$$\Gamma = (0, y), 0 < y < 1U(x, 0), 0 < x < 1U(1, y), 0 < y1U(x, 1), 0 < x < 1,$$
$$S = \left\{\frac{(x, 1)}{0 < x < 1}\right\}. \quad (112)$$

The time interval is given by $[0, T]$ with $T = 1$. For the triangulation $\mathcal{T}_h$ of $\bar{\Omega}$, we employ a uniform $N \times N$ mesh, where N denotes the division number of each side of the domain. The implementation is done by extending the Matlab code developed in [23, 24]. In all the examples presented, the velocity and pressure will be approximated by $P2 - P1$ element. Let us consider

$$\begin{cases} u_1(t, x, y) = 20x^2(1 - x)^2 y(1 - 2y)\exp(-t), \\ u_2(t, x, y) = -20x(1 - x)(1 - 2x)(1 - y)^2 y^2 \exp(-t), \\ p(t, x, y) = (2x - 1)(2y - 1)\exp(-t). \end{cases} \quad (113)$$

The initial condition is given by $u_0(x, y) = (u_1(0, x, y), u_2(x, y))$.

The functions f and g are chosen such that (u, p) defined above is the solution of (1)-(5).

It is easy to verify that the solution u satisfies $u = 0$ on Γ, $u \cdot n = u_2 = 0$, $u_1 \ne 0$ on S. By direct computations, we have

$$\sigma_\tau = -60x^2(1 - x)^2 \exp(-t) \text{ on } S \times [0, T],$$
$$u_\tau = 20x^2(1 - x)^2 \exp(-t) \text{ on } S \times [0, T],$$
$$\max_S|\sigma_\tau| = 3.75\exp(-t), \quad \forall t \in [0, T]. \quad (114)$$

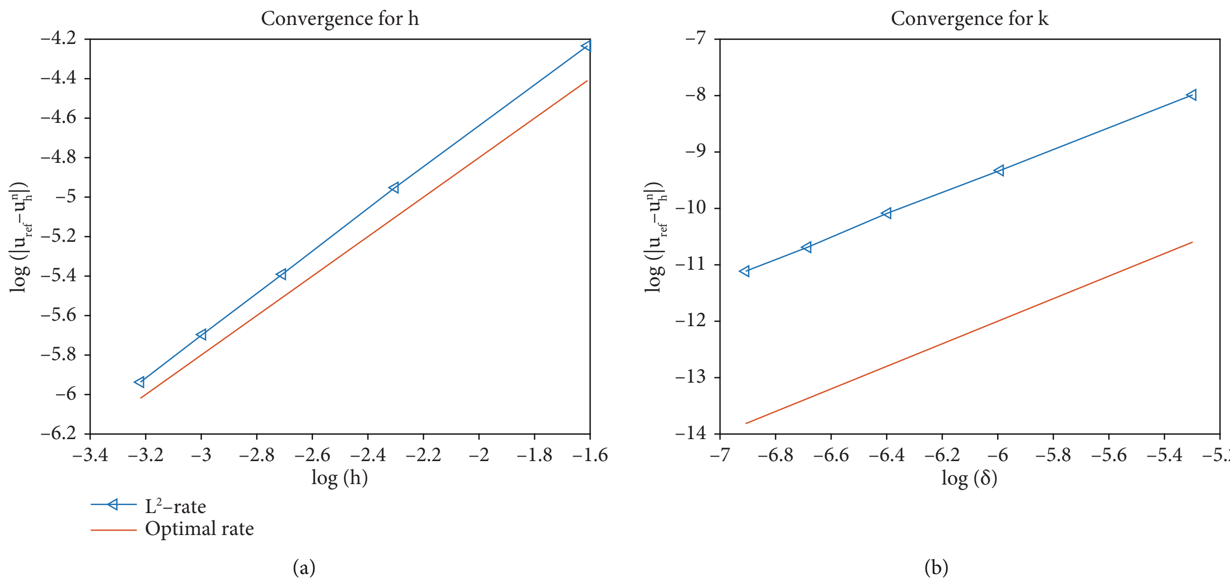


FIGURE 2: L^2-error estimate, respectively, for mesh size h and time step k.

On the other hand, from the slip boundary conditions (5), we have

$$|\sigma_\tau| \leq g \quad \text{on} \quad S \times [0, T], \tag{115}$$

then we find from (104) that for the given function g:

$$\begin{aligned} &g \geq 3.75 \exp(-t) \Rightarrow (113) \text{ remains a solution}, \\ &g < 3.75 \exp(-t) \Rightarrow (6.3) \text{ is no longer a solution and a non} \\ &\quad - \text{trivial slip occurs}. \end{aligned} \tag{116}$$

Indeed, it is observable in Figure 1, slip and non-slip condition on the boundary. In fact in Figure 1(a), $g < 3.75 \exp(-t)$ and we see the manifestation of the slip due to the adherence of the flow at the boundary, whereas in Figure 1(b), $g \geq 3.75 \exp(-t)$ and no slip occurs.

To analyze the convergence rate, we simulated the same problem. Since we do not know the explicit exact solution when $g = 1$, we employ the approximate solutions with $N = 60$ as the reference solutions $(u_{\text{ref}}, p_{\text{ref}})$, and we compute the L^2-norm for velocity of the difference of the reference solution and the approximate solution (u_h, p_h).

For the convergence with respect to the mesh size h, we choose $k = h^2$ and we solve problem (35) with different values of h $(h = 1/5\,; 1/10\,; 1/15\,; 1/20\,; 1/25)$. In Figure 2(a), we plot the log of L^2-errors against log (h).

For the convergence with respect to the time step k, h is fixed $(h = 0.01)$ and we solve problem (35) with different time steps $k = 0.1\,; 0.05\,; 0.025\,; 0.0125$. Figure 2(b) shows the plots of logL^2-error norm against log (k).

7. Conclusions

In this paper, we have proposed and analyzed the Crank-Nicolson scheme for the two-dimensional Navier-Stokes equations driven by slip boundary conditions of friction type. We established the well-posedness and stability of the numerical scheme in L^2-norm and H^1-norm for all positive time using the Crank-Nicholson scheme in time and the finite element method in space. We have proven that the numerical scheme is stable in L^2 and H^1-norms with the aid of different versions of discrete Grownwall lemmas, under a CFL-type condition.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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