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Research Article

Computing some Laplacian Coefficients of Forests

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Let G be a finite simple graph with Laplacian polynomial $\psi(G,\lambda)=\sum_{k=0}^{n}(-1)^{n-k}c_k\lambda^k$. In an earlier paper, the coefficients c_{n-4} and c_{n-5} for forests with respect to some degree-based graph invariants were computed. The aim of this paper is to continue this work by giving an exact formula for the coefficient c_{n-6}.

1. Definitions and Notations

A simple undirected graph is a pair $G=(V,E)$ consisting of a set $V=V(G)$ of vertices and a set $E=E(G)$ of 2-element subsets of V. The elements of E are called edges, and the number of elements in V is called the order of G. The notations $n(G)$ and $m(G)$ denote the number of vertices and edges of G, respectively. There are two other graph notations worth mentioning now. The first one is $deg_G(v)$ which is the number of edges in G with one end point v, and the second one is $\mathrm{deg}_G(\mathrm{e})$ which is defined as the degree of vertex e in the line graph of G. Obviously, $\mathrm{deg}_G(\mathrm{e})=\mathrm{deg}_G(\mathrm{u})+\mathrm{deg}_G(\mathrm{v})-2$.

We use the notation uvw to denote the path of length two such that vertices u and w have degree one, and the vertex v has degree two. In a similar way, we use the notation $uvwx$ to denote a path of length three.

A graph G is said to be connected if for arbitrary vertices x and y in V, and there exists a sequence $x=x_0, x_1, \cdots, x_r=y$ of vertices such that $x_ix_{i+1}\in E$, $0\le i\le r-1$. The distance between two vertices u and v in a connected graph G, $d_G(u,v)$, is defined as the length of a shortest path connecting these vertices and the sum of such numbers is called the Wiener index of G, denoted by $W(G)$ [1]. The hyper-Wiener index is a generalization of the Wiener index. It was introduced for trees by Randić in 1993 [2] and for a general graph by Klein et al. [3]. This topological index is defined as $WW(G)=(1/2)\sum_{u,v\in V(G)}(d(u,v)+d^2(u,v))$.

A subgraph H of a graph G is a graph with vertex set $V(H)$ and edge set $E(H)$, such that $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. We use the notation $H\le G$ to denote that H is subgraph of G. If $Z\subseteq V$, and then the induced subgraph $G[Z]$ is the graph with vertex set Z and the edge set $\{uv\in E|\{u,v\}\subseteq Z\}$, and if $H\le G$, then $G-H$ is a subgraph of G, with vertex set $V(G)/V(H)$ and edge set $E(G)/\{uv|\{u,v\}\cap V(H)\ne\varnothing\}$.

The subdivision graph S is a graph constructed from G by inserting a new vertex on each edge of G. It is clear that $n(S)=n(G)+m(G)$ and $m(S)=2m(G)$.

Suppose G is a graph containing two edges e and f. If the edges e and f have a common vertex u, then we write $u=e\cap f$. In the case that e and f do not have common vertex, we will say the edges e and f are independent. If $M\subseteq E(G)$ and all pair of edges in M are independent, then the set M is called a matching for G. A k-matching is a matching of size k, $1\le k\le n/2$, and the number of such matchings is denoted by m_k. The matching polynomial of G and α is defined by $\alpha(G,x)=\sum_{k=0}^{\lfloor(n/2)\rfloor}(-1)^k m_k x^{n-2k}$, where $x\ne 0$. By definition, $\alpha(G,0)=1$, see [4] for more details.

In 1972, Gutman and Trinajstic′ [5] introduced the first degree-based graph invariant applicable in chemistry. This invariant is the first Zagreb index and can defined by the formula $M_1^2(G)=\sum_{v\in V}deg_G(v)^2$. The second Zagreb index $M_2^1(G)=\sum_{uv\in E}deg_G(u)deg_G(v)$ was introduced by Gutman et al. [6] three years later in 1975. The complete history of these graph invariants together with the most important mathematical results about them are reported in [7–10].

The forgotten index of G is another variant of the Zagreb group indices defined as $M_1^3(G)=\sum_{v\in V}deg_G(v)^3=\sum_{e=uv\in E}[deg_G(u)^2+deg_G(v)^2]$ [11]. It can be see that $M_1^\alpha(G)=\sum_{u\in V}deg_G(u)^\alpha$, and $\mathbb{R}\ni\alpha\neq 0,1$ is the general form of the first Zagreb index. Zhang S. and Zhang H. [12] obtained the extremal values of the general Zagreb index in the class of all unicyclic graphs. Miličević et al. [13] reformulated the first and second Zagreb indices in terms of the edge-degrees instead of the vertex-degrees. These invariants were defined the first and second reformulated Zagreb indices defined as $EM_1(G)=\sum_{e\cap f\neq\varnothing}[deg_G(e)+deg_G(f)]=\sum_{e\in E}deg_G(e)^2$ and $EM_2(G)=\sum_{e\cap f\neq\varnothing}deg_G(e)deg_G(f)$, respectively.

A $\{0,1\}$-matrix is a matrix whose entries consist only of the numbers 0 and 1. Suppose G is a graph with vertex set $V=\{u_1,\cdots,u_n\}$. The adjacency matrix of G is a $\{0,1\}$-matrix $A(G)=(a_{ij})$ in which $a_{ij}=1$ if and only if $u_iu_j\in E$. It is easy to see that A is a real symmetric matrix of order n and so all of its eigenvalues are real. The matrices $D(G)=[d_{ij}]$ and $L(G)=D(G)-A(G)$ in which $d_{ii}=\deg(u_i)$ and $d_{ij}=0, i\neq j$, are called the diagonal and Laplacian matrices of G, respectively. It is well-known that all eigenvalues of $L(G)$ are nonnegative real numbers with 0 as the smallest eigenvalue.

The Laplacian polynomial of a graph G is one of the most important polynomial associated to a graph. If G is a graph, then the Laplacian polynomial of G is the characteristic polynomial of $L(G)$. The roots of this polynomial are called the Laplacian eigenvalues of G. Suppose $\psi(G,x)=\det(xI_n-L)=\sum_{k=0}^n(-1)^{n-k}c_kx^k$ denotes the Laplacian polynomial of G. Since the coefficients of the Laplacian polynomial have graph theoretical meaning, some authors took into account the coefficients of this polynomial.

Let f be a topological index and G be a graph. For simplifying our arguments, we usually write f as $f(G)$.

Lemma 1. *Suppose G is a graph. The following statements hold:*

(1) *(Merris [14] and Mohar [15])* $c_0(G)=0$, $c_1(G)=n\tau(G)$, $c_n(G)=1$ *and* $c_{n-1}(G)=2m$, *where* $\tau(G)$ *is the number of spanning trees of G;*

(2) *(Yan and Yeh [16])* $c_2(G)=W(G)$, *when G is a tree;*

(3) *(Gutman [17])* $c_3(G)=WW(G)$, *when G is a tree;*

(4) *(Oliveira et al. [18])* $c_{n-2}(G)=1/2[4m^2-2m-M_1^2]$ *and* $c_{n-3}(G)=(1/3!)[4m^2(2m-3)-6M_1^2m+6M_1^2+2M_1^3-12t(G)]$, *where t(G) is the number of triangles in G.*

In [19–21], we proved the following formulas for the coefficients $c_{n-4}(G)$ and $c_{n-5}(G)$, when G is a forest, respectively:

$$\begin{aligned}c_{n-4}(G)&=\frac{1}{4!}\left[4m\left(4m^3-12m^2+51m-6M_1^2m-33M_1^2+4M_1^3+3\right)\right.\\&\quad+3M_1^2\left(17M_1^2-20\right)+72M_1^3-54M_1^4-24M_2^1\left.\right]\\&\quad-16\sum_{\{u,v\}\subset V(G)}deg_G(u)_2deg_G(v)_2\\&=\frac{1}{4!}\left[4m\left(4m^3-12m^2+3m-6M_1^2m+15M_1^2+4M_1^3+3\right)\right.\\&\quad\left.+3\left(M_1^2-2\right)^2-24M_1^3-6M_1^4-24M_2^1-12\right],\end{aligned}\tag{1}$$

$$\begin{aligned}c_{n-5}(G)=\frac{1}{5!}&\left[2m\left(16m^4-80m^3+60m^2-40M_1^2m^2+60m\right.\right.\\&+180M_1^2m+40M_1^3m+15\left(M_1^2\right)^2-120M_1^2\\&\left.-140M_1^3-30M_1^4-120M_2^1\right)-20M_1^2\left(3M_1^2+M_1^3+6\right)\\&\left.+120M_1^3+120M_1^4+24M_1^5+240M_2^1+120\alpha_{1,2}\right].\end{aligned}\tag{2}$$

Suppose λ and ξ are two arbitrary real numbers. We now define three invariants which is useful in simplifying formulas in our results. These are

$$\begin{aligned}\alpha_{\lambda,\xi}(G)&=\sum_{uv\in E}\left[deg_G(u)^\lambda\,deg_G(v)^\xi+deg_G(u)^\xi deg_G(v)^\lambda\right],\\\beta(G)&=\sum_{e\sim f}deg_G(e\cap f)(deg_G(e)+deg_G(f)),\\M_2^\lambda(G)&=\sum_{uv\in E}(deg_G(u)deg_G(v))^\lambda.\end{aligned}\tag{3}$$

Note that the second Zagreb index is just the case of $\lambda=1$ in M_2^λ.

Let G and H be graphs. Set $\mathcal{S}_H(G)=\{X|X\leq G\,\text{and}\,X\cong H\}$. If f and g are two degree-based graph invariants, and then we define two new degree-based topological indices fg and Hf as $fg(G)=f(G)\times g(G)$ and $Hf(G)=\sum_{X\in\mathcal{S}_H(G)}f(G-X)$.

Let P_n denote the path graph on n vertices. In a recent paper [22], Das et al. presented the following formula for the number of k-matchings, $1\leq k\leq\lfloor(n/2)\rfloor$, in a graph G as

$$m_k(G)=\frac{1}{k}P_2m_{k-1}(G).\tag{4}$$

They also proved the following two results:

Lemma 2. *Let G be a graph with n vertices and m edges. Then*

(1) $P_2m(G)=m^2+m-M_1^2$.

(2) $P_2m^2(G)=m^3+M_1^3-2mM_1^2+2M_2^1+2m^2-2M_1^2+m$.

(3) $P_2m^3(G)=m^4-3m^2M_1^2+3mM_1^3+6mM_2^1-M_1^4-3\alpha_{1,2}+3m^3-6mM_1^2+3M_1^3+6M_2^1+3m^2-3M_1^2+m$.

(4) $P_2m^4(G) = M_1^5 + 4\alpha_{1,3} - 4mM_1^4 + 6M_2^2 - 12m\alpha_{1,2} + 6m^2M_1^3 + 12m^2M_2^1 - 4m^3M_1^2 - 4M_1^4 - 12\alpha_{1,2} + 12mM_1^3 + 24mM_2^1 - 12m^2M_1^2 + 6M_1^3 + 12M_2^1 - 12mM_1^2 - 4M_1^2 + m^5 + 4m^4 + 6m^3 + 4m^2 + m$,

(5) $P_2m^5(G) = m^6 + 5m^5 + (10 - 5M_1^2)m^4 + (10 + 10M_1^3 + 20M_2^1 - 20M_1^2)m^3 + (5 + 60M_2^1 - 10M_1^4 - 30\alpha_{1,2} + 30M_1^3 - 30M_1^2)m^2 + (30M_2^2 - 60\alpha_{1,2} + 5M_1^5 + 20\alpha_{1,3} + 30M_1^3 - 20M_1^2 + 60M_2^1 - 20M_1^4 + 1)m + 20\alpha_{1,3} - 5\alpha_{1,4} - 10\alpha_{2,3} + 10M_1^3 - 5M_1^2 + 20M_2^1 - 10M_1^4 + 5M_1^5 - M_1^6 + 30M_2^2 - 30\alpha_{1,2}$.

Lemma 3. *Let G be a graph with n vertices, m edges and girth ≥5. Then*

(1) $P_2M_1^2(G) = (m+3)M_1^2 - M_1^3 - 4M_2^1 - 2m$.

(2) $P_2M_1^3(G) = (m+3)M_1^3 - M_1^4 - 3\alpha_{1,2} + 6M_2^1 - 4M_1^2 + 2m$.

(3) $P_2M_1^4(G) = (m+4)M_1^4 - M_1^5 + 5M_1^2 - 2m - 4\alpha_{1,3} + 6\alpha_{1,2} - 6M_1^3 - 8M_2^1$.

(4) $P_2M_2^1(G) = (m-9)M_2^1 - 2EM_2 - 5M_1^3 + 11M_1^2 + M_1^4 + \alpha_{1,2} - 8m$.

(5) $P_2(mM_1^2)(G) = (M_1^2 - 2)m^2 + (4M_1^2 - M_1^3 - 4M_2^1 - 18)m - 6M_1^3 + \alpha_{1,2} - 2\beta + 17M_1^2 - 6M_2^1 + 8EM_1 + 4EM_2 - (M_1)^2 + M_1^4$.

The following theorem is crucial in our main result [21].

Theorem 4. *Let G be a graph with m edges. Then*

(1) $m_5(S(G)) = 1/15m^2[4m^3 - 20m^2 + 15m + 15] + 1/12m[8M_1^3m - 8M_1^2m^2 + 3(M_1^2)^2 + 36M_1^2m - 28M_1^3 - 24M_1^2 - 6M_1^4 - 24M_2^1] + \alpha_{1,2} - 1/6M_1^2[3M_1^2 + M_1^3 + 6] + 2M_2^1 + 1/5M_1^5 + M_1^4 + M_1^3$.

(2) $M_1^2(S(G)) = M_1^2 + 4m, M_1^3(S(G)) = M_1^3 + 8m, M_1^4(S(G)) = M_1^4 + 16m, M_1^5(S(G)) = M_1^5 + 32m, \alpha_{1,2}(S(G)) = 4M_1^2 + 2M_1^3, \alpha_{1,3}(S(G)) = 8M_1^2 + 2M_1^4, \beta(S(G)) = 2M_1^2 + M_1^4 - M_1^3, M_2^1(S(G)) = 2M_1^2, M_2^2(S(G)) = 4M_1^3, EM_1(S(G)) = M_1^3, EM_2(S(G)) = M_2^1 + 1/2M_1^4 - 1/2M_1^3$.

(3) $P_2(M_1^2)^2(S(G)) = (2m - 10)(M_1^2)^2 + (16m^2 - 2M_1^3 - 40m)M_1^2 + 32m^3 - 8mM_1^3 + 13M_1^3 + 6M_1^4 + M_1^5 + 24M_2^1 + 4\alpha_{1,2}$.

(4) $P_2(m^2M_1^2)(S(G)) = 32m^4 + (8M_1^2 - 32)m^3 - (4M_1^3 + 44M_1^2 - 8)m^2 + (20M_1^3 - 4(M_1^2)^2 + 30M_1^2 + 4M_1^4 + 16M_2^1)m + M_1^3M_1^2 + 2(M_1^2)^2 - 7M_1^3 - 5M_1^2 - 5M_1^4 - M_1^5 - 8M_2^1 - 2\alpha_{1,2}$.

(5) $P_2(mM_1^3)(S(G)) = 32m^3 + (4M_1^3 - 24)m^2 - (8M_1^3 + 16M_1^2 + 2M_1^4)m + 4m - (M_1^2 - 10)M_1^3 + 6M_1^2 + M_1^4 + M_1^5 - 6M_2^1 + 3\alpha_{1,2}$.

(6) $P_2(mM_2^1)(S(G)) = (8M_1^2 + 8)m^2 - (4M_1^3 + 10M_1^2 + 4M_2^1 + 4)m - 2(M_1^2)^2 + 2M_1^3 + M_1^2 + 2M_1^4 + 8M_2^1 + \alpha_{1,2}$.

(7) $P_2EM_2(S(G)) = 1/2m(4M_2^1 - 2M_1^3 + 2M_1^4 + 4) + 11/2M_1^3 - 2\alpha_{1,2} - 7/2M_1^2 - 3/2M_1^4 - 1/2M_1^5$.

2. Laplacian Coefficients and Degree-Based Invariants

The aim of this section is to present an exact formula for the coefficient c_{n-6} of the Laplacian polynomial in terms of some degree-based graphs invariants.

Lemma 5. *Let G be a graph with n vertices and m edges. Then,* $M_1^6(S(G)) = M_1^6 + 64m$, $\alpha_{1,4}(S(G)) = 2M_1^5 + 16M_1^2$, $\alpha_{2,3}(S(G)) = 4M_1^4 + 8M_1^3$.

Proof. Apply definition of $S(G)$, to prove that $M_1^6(S(G)) = M_1^6 + 64m$, $\alpha_{1,4}(S(G)) = \sum_{v\in V}\sum_{uv\in E}[2deg_G(v)^4 + 16deg_G(v)] = \sum_{v\in V}[2deg_G(v)^5 + 16deg_G(v)^2] = 2M_1^5 + 16M_1^2$ and $\alpha_{2,3}(S(G)) = \sum_{v\in V}\sum_{uv\in E}[4deg_G(v)^3 + 8deg_G(v)^2] = \sum_{v\in V}[4deg_G(v)^4 + 8deg_G(v)^3] = 4M_1^4 + 8M_1^3$. □

The next lemma is a direct consequence of Lemmas 2 and 5 and Theorem 4 (2).

Lemma 6. *Let G be a graph with n vertices and m edges. Then,*

(1) $P_2m(S(G)) = 4m^2 - 2m - M_1^2$.

(2) $P_2m^2(S(G)) = M_1^3 + 2M_1^2 - 2m(2M_1^2 - 4m^2 + 4m - 1).)$

(3) $P_2m^3(S(G)) = 16m^4 - 24m^3 - (12M_1^2 - 12)m^2 + (6M_1^3 + 12M_1^2 - 2)m - 3M_1^3 - 3M_1^2 - M_1^4$.

(4) $P_2m^4(S(G)) = 32m^5 - 64m^4 - (32M_1^2 - 48)m^3 + (24M_1^3 + 48M_1^2 - 16)m^2 - (24M_1^3 + 24M_1^2 + 8M_1^4 - 2)m + 6M_1^3 + 4M_1^2 + 4M_1^4 + M_1^5$.

(5) $P_2m^5(S(G)) = 64m^6 - 160m^5 - (80M_1^2 - 160)m^4 + (80M_1^3 + 160M_1^2 - 80)m^3 - (120M_1^3 + 120M_1^2 + 40M_1^4 - 20)m^2 + (60M_1^3 + 40M_1^2 + 40M_1^4 + 10M_1^5 - 2)m - 10M_1^3 - 5M_1^2 - 10M_1^4 - 5M_1^5 - M_1^6$.

It is easy to see that girth (G)≥6. Therefore, Lemma 3 and Theorem 4 (2) imply the following lemma:

Lemma 7. *Let G be a graph on n vertices and m edges. Then,*

(1) $P_2M_1^2(S(G)) = (2m-5)M_1^2 + 8m^2 - M_1^3$.

(2) $P_2M_1^3(S(G)) = 16m^2 + (2M_1^3 - 4)m - 3M_1^3 - 4M_1^2 - M_1^4$.

(3) $P_2M_1^4(S(G)) = (2m-4)M_1^4 + 32m^2 + 6M_1^3 - 19M_1^2 - M_1^5$.

(4) $P_2M_2^1(S(G)) = 4mM_1^2 - 2M_1^3 - 3M_1^2 - 2M_2^1 + 4m$.

(5) $P_2(mM_1^2)(S(G)) = 16m^3 + (4M_1^2 - 8)m^2 - (2M_1^3 + 16M_1^2)m - (M_1^2)^2 + 4M_1^3 + 5M_1^2 + 4M_2^1 + M_1^4$.

Lemma 8. *Let G be a graph with n vertices and m edges. Then,* $P_2M_1^5(S(G)) = 2mM_1^5 + 64m^2 - M_1^6 - 4m - 5M_1^5 + 10M_1^4 - 10M_1^3 - 26M_1^2$.

Proof. Apply definition of $P_2M_1^5(S(G))$ to show that $P_2M_1^5(S(G)) = 2mM_1^5(S(G)) - \sum_{v\in V}\sum_{uv}[deg_G(v)^5 + 5deg_G(v)^4 - 10deg_G(v)^3 + 10deg_G(v)^2 + 26deg_G(v) + 2] = 2mM_1^5(S(G)) - \sum_{v\in V}[deg_G(v)^6 + 5deg_G(v)^5 - 10deg_G(v)^4 + 10deg_G(v)^3 + 26deg_G(v)^2 + 2deg_G(v)]$. Now the proof follows from Theorem 4 (2) and simple calculations. □

Lemma 9. *Let G be a graph with n vertices and m edges. Then,* $P_2(m^3M_1^2)(S(G)) = 64m^5 + (16M_1^2 - 96)m^4 + (-8M_1^3 - 112M_1^2 + 48)m^3 + (-12(M_1^2)^2 + 72M_1^3 + 120M_1^2 + 12M_1^4 + 48M_2^1 - 8)m^2 + (12(M_1^2)^2 + (6M_1^3 - 44)M_1^2 - 54M_1^3 - 48M_2^1 - 34M_1^4 - 6M_1^5 - 12\alpha_{1,2})m - 3(M_1^2)^2 + (-3M_1^3 - M_1^4 + 5)M_1^2 + 10M_1^3 + 12M_2^1 + 12M_1^4 + 6M_1^5 + M_1^6 + 6\alpha_{1,2} + 2\alpha_{1,3}$.

Proof. By definition of $S(G)$, $P_2(m^3M_1^2)(S(G)) = \sum_{v\in V}\sum_{uv\in E}[2m - deg_G(v) - 1]^3[M_1^2(S(G)) - deg_G(v)^2 - 3deg_G(v) - 2deg_G(u)]$. Suppose $X = 16m^4M_1^2(S(G)) + (-8M_1^3 - 24M_1^2 - 24M_1^2(S(G)))m^3 + ((-12M_1^2 + 12)M_1^2(S(G)) + 36M_1^2 + 12M_1^4)m^2 + ((6M_1^3 + 12M_1^2 - 2)M_1^2(S(G)) - 42M_1^3 - 18M_1^2 - 6M_1^5 - 30M_1^4)m + (-3M_1^3 - 3M_1^2 - M_1^4)M_1^2(S(G)) + 10M_1^3 + 3M_1^2 + 12M_1^4 + 6M_1^5 + M_1^6$. Then, $P_2(m^3M_1^2)(S(G)) = X + \sum_{v\in V}\sum_{uv\in E}[-16m^3deg_G(u) + 2deg_G(u) + 24m^2deg_G(u)deg_G(v) + 48m^2deg_G(v)^2 - 12mdeg_G(u)deg_G(v)^2 + 2deg_G(u)deg_G(v)^3 + 24mdeg_G(u)deg_G(v) + 6deg_G(u)deg_G(v)^2 - 12mdeg_G(u) + 6deg_G(u)deg_G(v)]$. We now replace $\sum_{v\in V}\sum_{uv\in E}$ by $\sum_{uv\in E}$ to show that $P_2(m^3M_1^2)(S(G)) = X - 16m^3M_1^2 + (48M_1^3 + 24M_1^2 + 48M_2^1)m^2 + (-12M_1^2 - 48M_2^1 - 12\alpha_{1,2})m + 2M_1^2 + 12M_2^1 + 6\alpha_{1,2} + 2\alpha_{1,3}$. The proof now follows from Theorem 4 (2). □

Lemma 10. *Let G be a graph with n vertices and m edges. Then,* $P_2(m^2M_1^3)(S(G)) = 64m^4 + (8M_1^3 - 80)m^3 + (-20M_1^3 - 48M_1^2 - 4M_1^4 + 32)m^2 + ((-4M_1^2 + 50)M_1^3 + 40M_1^2 - 24M_2^1 + 4M_1^4 + 4M_1^5 + 12\alpha_{1,2} - 4)m + (M_1^3)^2 + (2M_1^2 - 19)M_1^3 - 8M_1^2 + 12M_2^1 - 8M_1^4 - 2M_1^5 - M_1^6 - 6M_2^2 - 3\alpha_{1,2}$.

Proof. Apply definition of $S(G)$ to show that $P_2(m^2M_1^3)(S(G)) = \sum_{v\in V}\sum_{uv\in E}[2m - deg_G(v) - 1]^2[M_1^3(S(G)) - deg_G(v)^3 - 3deg_G(u)^2 + 3deg_G(u) - 7deg_G(v) - 2]$. Now a similar argument as Lemma 9 completes the proof. □

Lemma 11. *Let G be a graph with n vertices and m edges. Then,* $P_2(mM_1^4)(S(G)) = (4M_1^4 + 64m)m^2 - 32m^2 + (12M_1^3 - 54M_1^2 - 10M_1^4 - 2M_1^5)m + (4 - M_1^2)M_1^4 + 9M_1^3 + 19M_1^2 + 8M_2^1 + M_1^5 + M_1^6 - 6\alpha_{1,2} + 4\alpha_{1,3}$.

Proof. Apply definitions of $S(G)$ and $P_2(mM_1^4)$ to write the form $P_2(mM_1^4)(S(G)) = \sum_{v\in V}\sum_{uv\in E}[2m - deg_G(v) - 1][M_1^4(S(G)) - deg_G(v)^4 - 4deg_G(u)^3 + 6deg_G(u)^2 - 4deg_G(u) - 15deg_G(v)]$. Now a similar argument as Lemma 9 gives the proof. □

Lemma 12. *Let G be a graph with n vertices and m edges. Then,* $P_2\alpha_{1,2}(S(G)) = 2(4M_1^2 + 2M_1^3)m - 9M_1^3 + M_1^2 - 2M_1^4 - 6M_2^1 - \alpha_{1,2} + 4m$.

Proof. By definitions of $S(G)$ and $\alpha_{1,2}$, we can write

$$\begin{aligned} P_2\alpha_{1,2}(S(G)) = 2m\alpha_{1,2}(S(G)) - \sum_{v\in V}\Big[&\big(2deg_G(v)^2 + 4deg_G(v)\big) \\ &\cdot\big(deg_G(v)^2 + deg_G(v)\big) + \big(2deg_G(v)^2 \\ &+ 4deg_G(v)\big)\big(deg_G(v)^2 - deg_G(v)\big) \\ &- \big(2(deg_G(v)-1)^2 + 4(deg_G(v)-1)\big)\big(deg_G(v)^2 \\ &- deg_G(v)\big)\Big] - \sum_{v\in V}\sum_{uv\in E}\big(\big(2deg_G(v)^2 + 4deg_G(v)\big) \\ &- \big(deg_G(v)^2 + deg_G(v)\big)\big)\big(deg_G(u) - 1\big). \end{aligned} \tag{5}$$

Now by simple calculations, we obtain

$$\begin{aligned} P_2\alpha_{1,2}(S(G)) = 2m\alpha_{1,2}(S(G)) &- 10M_1^3 - 2M_1^2 - 2M_1^4 + 4m \\ &- \sum_{uv\in E}\big(deg_G(u)deg_G(v)^2 + deg_G(u)^2deg_G(v) \\ &+ 6deg_G(u)deg_G(v) - deg_G(u)^2 - deg_G(v)^2 \\ &- 3deg_G(u) - 3deg_G(v)\big) = 2m\alpha_{1,2}(S(G)) \\ &- 10M_1^3 - 2M_1^2 - 2M_1^4 + 4m - \alpha_{1,2} - 6M_2^1 \\ &+ M_1^3 + 3M_1^2, \end{aligned} \tag{6}$$

and Theorem 4 (2) gives the result. □

Lemma 13. *Let G be a graph with n vertices and m edges. Then,* $P_2(m(M_1^2)^2)(S(G)) = 64m^4 + (32M_1^2 - 32)m^3 + (4(M_1^2)^2 - 16M_1^3 - 112M_1^2)m^2 + (-30(M_1^2)^2 + (-4M_1^3 + 40)M_1^2 + 58M_1^3 + 80M_2^1 + 20M_1^4 + 2M_1^5 + 8\alpha_{1,2})m - (M_1^2)^3 + 10(M_1^2)^2 + (8M_1^3 + 8M_2^1 + 2M_1^4)M_1^2 - 13M_1^3 - 24M_2^1 - 15M_1^4 - 7M_1^5 - M_1^6 - 20\alpha_{1,2} - 4\alpha_{1,3}$.

Proof. By definition of $S(G)$, $P_2(m(M_1^2)^2)(S(G)) = \sum_{v\in V}\sum_{uv\in E}[2m - deg_G(v) - 1][M_1^2(S(G)) - deg_G(v)^2 - 3deg_G(v) - 2deg_G(u)]^2$. Now the proof follows from a similar argument as Lemma 9. □

Lemma 14. *Let G be a graph with n vertices and m edges. Then,* $P_2(M_1^2M_1^3)(S(G)) = 64m^3 + (8M_1^3 + 16M_1^2 - 16)m^2 + ((2M_1^3 - 60)M_1^2 - 20M_1^3 - 4M_1^4)m - 4(M_1^2)^2 + (-8M_1^3 - M_1^4 + 10)M_1^2 - (M_1^3)^2 + 17M_1^3 + 2\alpha_{1,3} + 10M_2^1 + 13M_1^4 + 3M_1^5 + M_1^6 + 6M_2^2 + 6\alpha_{1,2}$.

Proof. By definition of $S(G)$, $P_2(M_1^2M_1^3)(S(G)) = \sum_{v\in V}\sum_{uv\in E}[M_1^2(S(G)) - deg_G(v)^2 - 3deg_G(v) - 2deg_G(u)][M_1^3(S(G)) - deg_G(v)^3 - 3deg_G(u)^2 + 3deg_G(u) - 7deg_G(v) - 2]$. Now the proof can be completed in a similar way as Lemma 9. □

Define nine graph invariants as

$$\begin{aligned}
\Theta_1(G) &= \sum_{uvw\in\mathcal{S}_{P_3}(G)} [deg_G(u)deg_G(v)deg_G(w)], \\
\Theta_2(G) &= \sum_{uvw\in\mathcal{S}_{P_3}(G)} [deg_G(u)deg_G(w)], \\
\Theta_3(G) &= \sum_{uvwx\in\mathcal{S}_{P_4}(G)} [deg_G(u)deg_G(x)], \\
\Theta_4(G) &= \sum_{uvw\in\mathcal{S}_{P_3}(G)} \left[deg_G(u)^2deg_G(w) + deg_G(u)deg_G(w)^2\right], \\
\Theta_5(G) &= \sum_{uvw\in\mathcal{S}_{P_3}(G)} deg_G(v)^2[deg_G(u) + deg_G(w)], \\
\Theta_6(G) &= \sum_{uvwx\in\mathcal{S}_{P_4}(G)} [deg_G(u)deg_G(v) + deg_G(w)deg_G(x)], \\
\Theta_2^+(G) &= \sum_{uvw\in\mathcal{S}_{P_3}(G)} [deg_G(u) + deg_G(w)], \\
\Theta_3^+(G) &= \sum_{uvwx\in\mathcal{S}_{P_4}(G)} [deg_G(u) + deg_G(x)], \\
\Theta_3^{+,2}(G) &= \sum_{uvwx\in\mathcal{S}_{P_4}(G)} \left[deg_G(u)^2 + deg_G(x)^2\right].
\end{aligned} \tag{7}$$

Lemma 15. *Let G be a graph with n vertices and m edges. Then,* $P_2(M_1^2)^2(G) = (m+6)(M_1^2)^2 + (-2M_1^3 - 4m - 8M(1/2) - 12)M_1^2 + M_1^3 + M_1^5 + 2M_2^2 - 6\alpha_{1,2} + 4\alpha_{1,3} + 4m + 26M_2^1 - 2M_1^4 + 8\Theta_1 - 24\Theta_2 + 8\Theta_3 + 4\Theta_4$.

Proof. By definition of M_1^2 and some tedious calculations, one can see that

$$\begin{aligned}
&P_2\left(\left(M_1^2\right)^2\right)(G) \\
&= \sum_{uv\in E}\Bigg[M_1^2 - deg_G(u)^2 - deg_G(v)^2 - \sum_{xu\in E(G-uv)}[2deg_G(x) - 1] - \sum_{yv\in E(G-uv)}[2deg_G(y) - 1]\Bigg]^2 \\
&= (m+2)\left(M_1^2\right)^2 + \left(-2M_1^3 - 4m - 4\right)M_1^2 + 5M_1^3 - 2M_1^4 \\
&\quad + M_1^5 + 2M_2^1 + 2M_2^2 - 2\alpha_{1,2} + 4m + \sum_{uv\in E}\Bigg[4deg_G(u)^2 \\
&\quad \cdot \sum_{xu\in E(G-uv)} deg_G(x) + 4deg_G(u)^2 \sum_{yv\in E(G-uv)} deg_G(y) \\
&\quad + 4deg_G(v)^2 \sum_{xu\in E(G-uv)} deg_G(x) + 4deg_G(v)^2 \\
&\quad \cdot \sum_{yv\in E(G-uv)} deg_G(y) - \left(4M_1^2 - 8\right) \sum_{xu\in E(G-uv)} deg_G(x) \\
&\quad - \left(4M_1^2 - 8\right) \sum_{yv\in E(G-uv)} deg_G(y) - 4deg_G(u) \\
&\quad \cdot \sum_{xu\in E(G-uv)} deg_G(x) - 4deg_G(u) \sum_{yv\in E(G-uv)} deg_G(y) \\
&\quad - 4deg_G(v) \sum_{xu\in E(G-uv)} deg_G(x) - 4deg_G(v) \\
&\quad \cdot \sum_{yv\in E(G-uv)} deg_G(y) + \left(\sum_{xu\in E(G-uv)} 2deg_G(x)\right)^2 \\
&\quad + \left(\sum_{yv\in E(G-uv)} 2deg_G(y)\right)^2 + 2\left(\sum_{xu\in E(G-uv)} 2deg_G(x)\right) \\
&\quad \cdot \left(\sum_{vy\in E(G-uv)} 2deg_G(y)\right)\Bigg].
\end{aligned} \tag{8}$$

Therefore, $P_2(M_1^2)^2(G) = (m+6)(M_1^2)^2 + (-2M_1^3 - 4m - 8M_2^1 - 12)M_1^2 + M_1^3 + M_1^5 + 2M_2^2 - 6\alpha_{1,2} + 4\alpha_{1,3} + 4m + 26M_2^1 - 2M_1^4 + 8\Theta_1 - 24\Theta_2 + 8\Theta_3 + 4\Theta_4$, proving the lemma. □

Lemma 16. *Let G be a graph with n vertices and m edges. Then,* $P_2(mM_1^3)(G) = (M_1^3 + 2)m^2 + (4M_1^3 - 3\alpha_{1,2} - 4M_1^2 + 6M_2^1 - M_1^4 + 2)m + (-M_1^2 + 4)M_1^3 - 9\alpha_{1,2} + \alpha_{1,3} - 6M_1^2 + 14M_2^1 - 6\Theta_2 + 3\Theta_4 - M_1^4 + M_1^5 + 6M_2^2$.

Proof. Choose $uv \in E$ and set $A(uv) = \sum_{xu\in E(G-uv)}[3deg_G(x)^2 - 3deg_G(x) + 1]$ and $B(uv) = \sum_{yv\in E(G-uv)}[3deg_G(y)^2 - 3deg_G(y) + 1]$. Apply definition of $P_2(mM_1^3)$ to prove that $P_2(mM_1^3)(G) = \sum_{uv\in E}[m - deg_G(u) - deg_G(v) + 1][M_1^3 - deg_G(u)^3 - deg_G(v)^3 - A(uv) - B(uv)]$. Therefore

$$\begin{aligned}
P_2\left(mM_1^3\right)(G) &= m^2M_1^3 - M_1^2M_1^3 + mM_1^3 - mM_1^2 - mM_1^4 + 2m^2 \\
&\quad + M_1^3 - 3M_1^2 - M_1^4 + M_1^5 + 2M_2^1 + \alpha_{1,3} + 2m \\
&\quad - 3\sum_{uv\in E}\Bigg[(m - deg_G(u) - deg_G(v) + 1) \\
&\quad \cdot \left(\sum_{xu\in E(G-uv)}\left[deg_G(x)^2 - deg_G(x)\right] + \sum_{yv\in E(G-uv)}\left[deg_G(y)^2 - deg_G(y)\right]\right)\Bigg].
\end{aligned} \tag{9}$$

Now, a similar argument as Lemma 15 gives our result. □

Lemma 17. *Let G be a graph with n vertices and m edges. Then,* $P_2(m^2M_1^2)(G) = (M_1^2 - 2)m^3 + (-M_1^3 + 5M_1^2 - 4M_2^1 - 4)m^2 + (-2(M_1^2)^2 - 4M_1^3 + 11M_1^2 + 2M_1^4 - 20M_2^1 + 6\alpha_{1,2} + 8\Theta_2 - 2)m - 2(M_1^2)^2 + (M_1^3 + 2M_2^1 + 7)M_1^2 - 5M_1^3 + 11\alpha_{1,2} - 4\alpha_{1,3} - 2\Theta_4 - 20M_2^1 - 8\Theta_1 + 8\Theta_2 + 3M_1^4 - M_1^5 - 2M_2^2$.

Proof. Apply definition of $P_2(m^2M_1^2)$ and some tedious calculations to show that

$$\begin{aligned} P_2(m^2M_1^2)(G) &= (M_1^2 - 2)m^3 + (-M_1^3 + 3M_1^2 - 4)m^2 \\ &\quad + \left(-2(M_1^2)^2 - 4M_1^3 + 7M_1^2 + 2M_1^4 - 4M_2^1 + 2\alpha_{1,2} - 2\right)m \\ &\quad - 2(M_1^2)^2 + (M_1^3 + 2M_2^1 + 5)M_1^2 - 5M_1^3 \\ &\quad - 8M_2^1 + 3M_1^4 - M_1^5 - 2M_2^2 + 5\alpha_{1,2} - 2\alpha_{1,3} \\ &\quad - 2\sum_{uv\in E}[m - deg_G(u) - deg_G(v) + 1]^2 \\ &\quad \cdot \left[\sum_{xu\in E(G-uv)} deg_G(x) + \sum_{yv\in E(G-uv)} deg_G(y)\right]. \end{aligned} \tag{10}$$

Now, a similar argument as Lemma 15 gives the proof. □

Lemma 18. *Let G be a graph with n vertices and m edges. Then,* $EM_2(G) = \alpha_{1,2} - 6M_2^1 + (1/2)M_1^4 - (5/2)M_1^3 + 6M_1^2 - 4m + \Theta_2$.

Proof. By definition of EM_2, $EM_2(G) = \sum_{uvw\in \mathcal{S}_{P_3}(G)}[deg_G(u) + deg_G(v) - 2][deg_G(v) + deg_G(w) - 2] = \alpha_{1,2} - 6M_2^1 + (1/2)M_1^4 - (5/2)M_1^3 + 6M_1^2 - 4m + \Theta_2$. □

Lemma 19. *Let G be a graph with n vertices and m edges. Then,* $P_2\Theta_2(G) = (m+2)\Theta_2 - \Theta_1 - \Theta_4 - 2\Theta_3 + \Theta_3^+$.

Proof. By definition of $P_2\Theta_2$,

$$\begin{aligned} P_2\Theta_2(G) &= m\Theta_2 - \sum_{uvw\in\mathcal{S}_{P_3}(G)} deg_G(u)deg_G(w)\big(deg_G(u) \\ &\quad + deg_G(v) + deg_G(w) - 2\big) - \sum_{xyab\in\mathcal{S}_{P_4}(G)} \\ &\quad \cdot (2deg_G(x)deg_G(b) - deg_G(x) - deg_G(b)) \\ &= (m+2)\Theta_2 - \Theta_1 - \Theta_4 - 2\Theta_3 + \Theta_3^+, \end{aligned} \tag{11}$$

as desired. □

By a similar arguments as Lemma 15, we can prove the following two lemmas:

Lemma 20. *Let G be a graph with n vertices and m edges. Then,* $P_2(mM_2^1)(G) = m^2M_2^1 + (2M_2^1 - 2\Theta_2 - \alpha_{1,2} + \Theta_2^+)m - (M_1^2 - 1)M_2^1 + \Theta_2^+ + 2\Theta_1 - 4\Theta_2 + 2\Theta_3 + \Theta_4 + \Theta_5$.

Lemma 21. *Let G be a graph with n vertices and m edges. Then,* $P_2\alpha_{1,2}(G) = (m+1)\alpha_{1,2} - 2M_2^2 - \alpha_{1,3} + \Theta_3^+ - \Theta_3^{+,2} - 2\Theta_6$.

For the sake of completeness, we mention here two results which are useful in our next calculations.

Lemma 22 (see [19]). *Let G be a graph. Then,* $\beta(G) = \alpha + M_1^4 - 3M_1^3 + 2M_1^2 - 2M_2^1$.

Theorem 23 (see [4]). *Let G be a graph with m edges and girth ≥5. Then,* $m_5(G) = (1/5!)[m(m^4 + 10m^3 + 43m^2 + 54m - 328) + 30(M_1^2)^2 - 12\alpha_{1,2}(m-7) - 20\alpha_{1,3} - 2M_1^2(2m^3 + 30m^2 + 61m - 225) + 12\beta + 2M_2^1(6m^2 + 66m - 239) + M_1^3(6m^2 + 24m - 149) + 2M_1^4(m+10) + 6M_2^2 - EM_2 - 5M_1^5 + 3P_2(M_1^2)^2 + 8P_2(mM_1^3) - 6P_2(m^2M_1^2) - P_2EM_2 + P_2(mM_2^1)]$.

Now, Lemmas 2 (1), 3, 15, 16, 17, 18, 19, 20, 21,and 22 and Theorem 23 give the following theorem:

Theorem 24. *Let G be a graph with m edges and girth ≥5. Then,* $m_5(G) = (1/5!)[m^5 + 10m^4 - (10M_1^2 - 55)m^3 + (60M_2^1 - 90M_1^2 + 20M_1^3 + 190)m^2 + (15(M_1^2)^2 + 140M_1^3 - 376M_1^2 - 30M_1^4 + 492M_2^1 - 120\alpha_{1,2} - 120\Theta_2 + 24\Theta_2^+ + 336)m + 60(M_1^2)^2 - (60M_2^1 + 20M_1^3 + 768)M_1^2 - 120M_1^4 + 24M_1^5 + 120M_2^2 - 504\alpha_{1,2} + 96\alpha_{1,3} + 24\Theta_2^+ - 48\Theta_3^+ + 96\Theta_4 + 24\Theta_3^{+,2} + 336M_1^3 + 1440M_2^1 + 132\Theta_1 - 600\Theta_2 + 120\Theta_3 + 24\Theta_5 + 48\Theta_6]$.

Lemma 25. *Let G be a graph with n vertices and m edges. Then,* $\Theta_1(S(G)) = 2[M_2^1 + M_1^3 - M_1^2]$, $\Theta_2(S(G)) = M_2^1 + 2M_1^2 - 4m$, $\Theta_2^+(S(G)) = 3M_1^2 - 4m$, $\Theta_3(S(G)) = 4M_2^1 - 2M_1^2$, $\Theta_3^+(S(G)) = 2M_2^1 + M_1^2 - 4m$, $\Theta_3^{+,2}(S(G)) = \alpha_{1,2} + 4M_1^2 - M_1^3 - 8m$, $\Theta_4(S(G)) = \alpha_{1,2} + 8M_1^2 - 16m$, $\Theta_5(S(G)) = 4M_1^2 + 2M_1^4 - 2M_1^3$, *and* $\Theta_6(S(G)) = 4M_2^1 + 2M_1^3 - 4M_1^2$.

Proof. The proof is straightforward and so it is omitted. □

Lemma 26. *Let G be a graph with n vertices and m edges. Then,* $P_2\Theta_2(S(G)) = (4M_1^2 + 2M_2^1 + 4)m - 8m^2 - 2M_1^3 + 3M_1^2 - 6M_2^1 - \alpha_{1,2}$.

Proof. Lemmas 19, 25, and 6 (1) give our result. □

Lemma 27. *Let G be a graph with n vertices and m edges. Then,*

(1) $P_2\Theta_1(S(G)) = (4M_1^3 - 4M_1^2 + 4M_2^1 - 8)m - 2M_1^3 + 10M_1^2 - 2M_1^4 - 4\alpha_{1,2}$,

(2) $P_2\Theta_2^+(S(G)) = (6M_1^2 + 8)m - 8m^2 - 3M_1^3 - M_1^2 - 4M_2^1$,

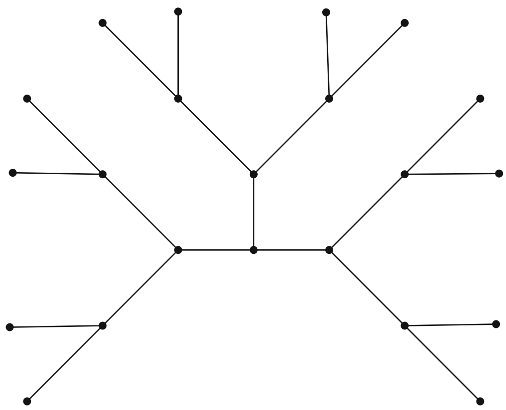

Figure 1: The rooted tree $T(3,3)$.

Table 1: The Laplacian coefficients $c_{n-x}(T(3,t))$, where $x, t \in \{2, 3, 4, 5, 6\}$.

x	t = 2	3	4	5	6
2	132	810	3894	16974	70782
3	512	9520	107888	1013104	8754032
4	1146	76329	2151219	44481015	804407871
5	1524	442926	32892762	1532049426	58577653506
6	1196	1926456	401303300	43109506572	3521109479132

(3) $P_2\Theta_3(S(G)) = (8M_2^1 - 4M_1^2 - 4)m + 2M_1^3 + 6M_1^2 - 4M_2^1 - 4\alpha_{1,2} - 2\Theta_2 + \Theta_2^+$,

(4) $P_2\Theta_3^+(S(G)) = 2m(2M_2^1 + M_1^2 - 4m) - M_1^3 + 6M_1^2 - 6M_2^1 - 2\alpha_{1,2} - \Theta_2^+$,

(5) $P_2\Theta_3^{+,2}(S(G)) = (8M_1^2 - 2M_1^3 + 2\alpha_{1,2} + 4)m - 16m^2 - M_1^3 + 3M_1^2 - 2M_2^1 + M_1^4 - 2M_2^2 - 2\alpha_{1,2} - \alpha_{1,3} - 3\Theta_2^+$,

(6) $P_2\Theta_4(S(G)) = (16M_1^2 + 2\alpha_{1,2} + 12)m - 32m^2 - 7M_1^3 + 11M_1^2 - 14M_2^1 - 2M_2^2 - 3\alpha_{1,2} - \alpha_{1,3}$,

(7) $P_2\Theta_5(S(G)) = (4M_1^4 - 4M_1^3 + 8M_1^2 + 16)m + 13M_1^3 - 18M_1^2 - 8M_2^1 - 5M_1^4 - 2M_1^5 + \alpha_{1,2} - \alpha_{1,3}$,

(8) $P_2\Theta_6(S(G)) = (4M_1^3 - 8M_1^2 + 8M_2^1 - 12)m + M_1^3 + 13M_1^2 + 2M_2^1 - 2M_1^4 - 7\alpha_{1,2}$,

(9) $P_2(m\Theta_2)(S(G)) = (8M_1^2 + 4M_2^1 + 16)m^2 - 16m^3 + (4M_1^2 - 4M_1^3 - 10M_2^1 - 2\alpha_{1,2} - 2\Theta_2^+ - 4)m - 2(M_1^2)^2 - (M_2^1 + 4)M_1^2 - 2M_1^3 + 8M_2^1 + 2\Theta_2 + 2M_1^4 + 3\alpha_{1,2} + \alpha_{1,3} + \Theta_2^+$,

(10) $P_2(m\Theta_2^+)(S(G)) = (12M_1^2 + 24)m^2 - 16m^3 - (6M_1^3 + 4M_1^2 + 8M_2^1 + 8)m - 3(M_1^2)^2 - 3M_1^2 + 12M_2^1 + 3M_1^4 + 2\alpha_{1,2}$.

Proof. The proof has tedious calculations similar to Lemma 15. □

Lemma 28. *Let G be a graph with n vertices and m edges. Then,*

(1) $P_2\alpha_{1,3}(S(G)) = (16M_1^2 + 4M_1^4 + 20)m + 4M_1^3 - 17M_1^2 - 14M_2^1 - 7M_1^4 - 2M_1^5 - \alpha_{1,3}$.

(2) $P_2(m\alpha_{1,2})(S(G)) = (8M_1^3 + 16M_1^2 + 8)m^2 - (22M_1^3 + 6M_1^2 + 12M_2^1 + 4M_1^4 + 2\alpha_{1,2} + 4)m - 4(M_1^2)^2 - (2M_1^3 + 3)M_1^2 + 9M_1^3 + 4M_2^1 + 6M_1^4 + 2M_1^5 + 2M_2^2 + 9\alpha_{1,2}$.

(3) $P_2(M_1^2M_2^1)(S(G)) = (4m - 13)(M_1^2)^2 + (16m^2 - 4M_1^3 - 2M_2^1 - 8m - 10)M_1^2 + 16m^2 - (8M_1^3 + 8M_2^1)m + 6M_1^3 + 18M_2^1 + 4\Theta_2 + 6M_1^4 + 2M_1^5 + 10\alpha_{1,2} + \alpha_{1,3}$.

(4) $P_2(m^2M_2^1)(S(G)) = (16M_1^2 + 16)m^3 - (8M_1^3 + 28M_1^2 + 8M_2^1 + 16)m^2 + (8M_1^3 - 8(M_1^2)^2 + 8M_1^2 + 8M_1^4 + 32M_2^1 + 4\alpha_{1,2} + 4)m + 4(M_1^2)^2 + (2M_1^3 + 1)M_1^2 - 14M_2^1 - 4M_1^4 - 2M_1^5 - 5\alpha_{1,2} - \alpha_{1,3}$.

(5) $P_2M_2^2(S(G)) = (8m - 9)M_1^3 - 8m + 12M_1^2 - 4M_1^4 - 3\alpha_{1,2}$.

Proof. By definitions of $S(G)$, we have

$$\begin{aligned}
&P_2\alpha_{1,3}(S(G)) \\
&= 2m\alpha_{1,3}(S(G)) - \sum_{v\in V}\sum_{uv\in E}\Bigg[\big(2deg_G(v)^3 + 8deg_G(v)\big)deg_G(v) + 2deg_G(u)^3 + 8deg_G(u) \\
&\quad + \big(2deg_G(u)^3 + 8deg_G(u)\big)(deg_G(u) - 1) - \big(2(deg_G(u) - 1)^3 + 8(deg_G(u) - 1)\big) \\
&\quad \cdot (deg_G(u) - 1) + \sum_{xv\in E(G-uv)}\big(deg_G(x)^3 + 7deg_G(x)\big)\Bigg] \\
&= \big(16M_1^2 + 4M_1^4 + 20\big)m + 4M_1^3 - 17M_1^2 - 14M_2^1 - 7M_1^4 - 2M_1^5 - \alpha_{1,3}, P_2(m\alpha_{1,2})(S(G)) \\
&= \sum_{v\in V}\sum_{uv\in E}[2m - deg_G(v) - 1]\Bigg[\alpha_{1,2}(S(G)) - \big(2deg_G(v)^2 + 4deg_G(v)\big)deg_G(v) - 2deg_G(u)^2 \\
&\quad - 4deg_G(u) - \big(2deg_G(u)^2 + 4deg_G(u)\big) \cdot (deg_G(u) - 1) + \big(2(deg_G(u) - 1)^2 - 4(deg_G(u) - 1)\big) \\
&\quad \cdot (deg_G(u) - 1) - \sum_{xv\in E(G-uv)}\big(deg_G(x)^2 + 3deg_G(x)\big)\Bigg] \\
&= \big(8M_1^3 + 16M_1^2 + 8\big)m^2 - \big(22M_1^3 + 6M_1^2 + 12M_2^1 + 4M_1^4 + 2\alpha_{1,2} + 4\big)m - 4\big(M_1^2\big)^2 - \big(2M_1^3 + 3\big)M_1^2 + 9M_1^3 + 4M_2^1 \\
&\quad + 6M_1^4 + 2M_1^5 + 2M_2^2 + 9\alpha_{1,2}, P_2\big(M_1^2M_2^1\big)(S(G)) \\
&= \sum_{v\in V}\sum_{uv\in E}\big[M_1^2(S(G)) - deg_G(v)^2 - 3deg_G(v) - 2deg_G(u)\big]
\end{aligned}$$

Table 2: The Laplacian coefficients $c_{n-x}(T(4,t))$, where $x, t \in \{2,3,4,5,6\}$.

x	$t=2$	3	4	5	6
2	450	5202	50562	466578	4234050
3	3680	166736	5259296	149307536	4098568160
4	19549	3849829	405115261	35685894085	2971474597789
5	71496	68251680	24647441832	6795068311872	1721091168665352
6	186394	967057330	1233678403066	1073738466435154	829575812820551386

$$
\begin{aligned}
&\cdot \left[M_2^1(S(G)) - 2deg_G(v)^2 - 2deg_G(u) - 2deg_G(u) \right. \\
&\left. \cdot (deg_G(u)-1) + 2(deg_G(u)-1)^2 - \sum_{xv\in E(G-uv)} deg_G(x) \right] \\
&= (4m-13)\left(M_1^2\right)^2 + \left(16m^2 - 4M_1^3 - 2M_2^1 - 8m - 10\right)M_1^2 \\
&\quad + 16m^2 - \left(8M_1^3 + 8M_2^1\right)m + 6M_1^3 + 18M_2^1 + 4\Theta_2 \\
&\quad + 6M_1^4 + 2M_1^5 + 10\alpha_{1,2} + \alpha_{1,3}, \\
&P_2\left(m^2M_2^1\right)(S(G)) \\
&= \sum_{v\in V}\sum_{uv\in E} [2m - deg_G(v) - 1]^2 \left[M_2^1(S(G)) \right. \\
&\quad - 2deg_G(v)^2 - 2deg_G(u) - 2deg_G(u)(deg_G(u)-1) \\
&\quad \left. + 2(deg_G(u)-1)^2 - \sum_{xv\in E(G-uv)} deg_G(x) \right] \\
&= \left(16M_1^2 + 16\right)m^3 - \left(8M_1^3 + 28M_1^2 + 8M_2^1 + 16\right)m^2 \\
&\quad + \left(8M_1^3 - 8\left(M_1^2\right)^2 + 8M_1^2 + 8M_1^4 + 32M_2^1 \right. \\
&\quad \left. + 4\alpha_{1,2} + 4\right)m + 4\left(M_1^2\right)^2 + \left(2M_1^3 + 1\right)M_1^2 \\
&\quad - 14M_2^1 - 4M_1^4 - 2M_1^5 - 5\alpha_{1,2} - \alpha_{1,3}, \\
&P_2M_2^2(S(G)) \\
&= \sum_{v\in V}\sum_{uv\in E} \left[M_2^1(S(G)) - 4deg_G(v)^3 - 4deg_G(u)^2 \right. \\
&\quad - 4deg_G(u)^2(deg_G(u)-1) + 4(deg_G(u)-1)^3 \\
&\quad \left. - \sum_{xv\in E(G-uv)} 3deg_G(x)^2 \right] \\
&= (8m-9)M_1^3 - 8m + 12M_1^2 - 4M_1^4 - 3\alpha_{1,2},
\end{aligned} \tag{12}
$$

proving the lemma. □

Theorem 29. *Let G be a graph with m edges. Then,* $m_6(S(G)) = (1/6!)[64m^6 - 480m^5 + 720m^4 + 600m^3 - 360m^2 - 480m - 720M_1^5 - 2160\alpha_{1,2} - 720\alpha_{1,3} + 540(M_1^2)^2 - 2340m^2M_1^2 + 2160mM_1^3 - 1080M_1^4 + 720M_2^1 - 240M_1^2M_1^3m - 120M_1^6 - 720M_2^2 + 600M_1^2M_1^3 + 1680M_1^2m^3 - 810(M_1^2)^2m - 1920M_1^3m^2 + 1620M_1^4m + 3600M_2^1m - 720\Theta_2 - 1260mM_1^2 + 720M_1^2 + 480M_1^3 - 240m^4M_1^2 + 320m^3M_1^3 + 360M_1^2M_2^1 + 90M_1^2M_1^4 + 180(M_1^2)^2m^2 - 360M_1^4m^2 - 1440M_2^1m^2 + 288M_1^5m + 1440\alpha_{1,2}m + 40(M_1^3)^2 - 15(M_1^2)^3]$.

Proof. The proof follows from Theorems 24 and 4, Equation (4), and Lemmas 6, 7, 8, 9, 10, 11, 12, 13, 14, 26, 27, and 28. □

We mention here a useful result of Zhou and Gutman [23].

Theorem 30. *Let G be an n-vertex forest. Then,* $c_{n-k}(G) = m_k(S(G))$ *, for* $0 \le k \le n$.

We are now ready to prove the main result of this section.

Theorem 31. *Let G be a forest with m edges. Then,* $c_{n-6}(G) = (1/6!)[64m^6 - 480m^5 + 720m^4 + 600m^3 - 360m^2 - 480m - 720M_1^5 - 2160\alpha_{1,2} - 720\alpha_{1,3} + 540(M_1^2)^2 - 2340m^2M_1^2 + 2160mM_1^3 - 1080M_1^4 + 720M_2^1 - 240M_1^2M_1^3m - 120M_1^6 - 720M_2^2 + 600M_1^2M_1^3 + 1680M_1^2m^3 - 810(M_1^2)^2m - 1920M_1^3m^2 + 1620M_1^4m + 3600M_2^1m - 720\Theta_2 - 1260mM_1^2 + 720M_1^2 + 480M_1^3 - 240m^4M_1^2 + 320m^3M_1^3 + 360M_1^2M_2^1 + 90M_1^2M_1^4 + 180(M_1^2)^2m^2 - 360M_1^4m^2 - 1440M_2^1m^2 + 288M_1^5m + 1440\alpha_{1,2}m + 40(M_1^3)^2 - 15(M_1^2)^3]$.

Proof. The proof follows from Theorems 29 and 30. □

If T is an n-vertex tree, then $m = n - 1$. Therefore by previous theorem we have the following corollary:

Corollary 32. *Let T be a tree on n vertices. Then,* $c_{n-6}(T) = (1/6!)[(8(n-1)(8n^5 - 100n^4 + 410n^3 - 635n^2 + 355n - 98) - 240M_1^2M_1^3n + 1440\alpha_{1,2}n + 288M_1^5n + 9420M_1^2n - 1170(M_1^2)^2n + 2340M_1^4n + 6480M_2^1n + 6960M_1^3n - 1440M_2^1n^2 + 180(M_1^2)^2n^2 - 360M_1^4n^2 + 320M_1^3n^3 - 2880M_1^3n^2 - 240M_1^2n^4 + 2640M_1^2n^3 - 8820M_1^2n^2 + 360M_1^2M_2^1 + 90M_1^2M_1^4 + 840M_1^2M_1^3 - 15(M_1^2)^3 + 40(M_1^3)^2 + 1530(M_1^2)^2 - 720\Theta_2 - 120M_1^6 -$

$720M_2^2 - 3920M_1^3 - 3060M_1^4 - 4320M_2^1 - 2280M_1^2 - 720\alpha_{1,2} - 1008M_1^5 - 3600\alpha_{1,2}]$.

3. Applications

This section aims is to apply our results in Section 3 for computing the Laplacian coefficients c_{n-k}, $k = 2, 3, 4, 5, 6$, when G is a certain tree. Let G be a graph. The number of edges connecting vertices of degree i and j in a graph G is denoted by $m_{i,j}$. Let $\mathscr{P}_3^{i,j} = |\{uvw \in \mathscr{S}_{P_3}(G) | \{deg_G(u), deg_G(w)\} = \{i, j\}\}|$. As [21], we first assume that $T(k, t)$ be a rooted tree with degree sequence $k, k, \cdots, k, 1, 1, \cdots, 1$, and t is the distance between the center and any pendant vertex, as shown in Figure 1. By definition of $T(k, t)$, we have $n(T(k, t)) = ((k(k-1)^t - 2)/(k-2))$, $m_{1,k}(T(k, t)) = k(k-1)^{t-1}$, $m_{k,k}(T(k, t)) = n(T(k, t)) - 1 - m_{1,k}(T(k, t))$, $\mathscr{P}_3^{1,k} = m_{1,k}(T(k, t))$, $\mathscr{P}_3^{1,1} = k(k-1)^{t-2}k - 1_2$, and $\mathscr{P}_3^{k,k} = (1/2)M_1(T(k, t)) - n(T(k, t)) + 1 - \mathscr{P}_3^{1,1} - \mathscr{P}_3^{1,k}$. Therefore by Lemma 1, Equations (1) and (2), Corollary 32, and the others simple calculations, we have

$$\begin{aligned}
c_{n-1}(T(3,t)) &= 6 \times 2^t - 6,\\
c_{n-2}(T(3,t)) &= 18 \times 2^{2t} - \frac{93}{2}2^t + 30,\\
c_{n-3}(T(3,t)) &= 36 \times 2^{3t} - 171 \times 2^{2t} + 272 \times 2^t - 144,\\
c_{n-4}(T(3,t)) &= 54 \times 2^{4t} - 405 \times 2^{3t} + \frac{9177}{8}2^{2t}\\
&\quad - \frac{5799}{4}2^t + 687,\\
c_{n-5}(T(3,t)) &= \frac{324}{5}2^{5t} - 702 \times 2^{4t} + \frac{12267}{4}2^{3t}\\
&\quad - \frac{26967}{4}2^{2t} + \frac{74427}{10}2^t - 3294,\\
c_{n-6}(T(3,t)) &= \frac{324}{5}2^{6t} - \frac{4779}{5}2^{5t} + \frac{23697}{4}2^{4t} - \frac{315711}{16}2^{3t}\\
&\quad + \frac{1488293}{40}2^{2t} - \frac{376247}{10}2^t + 15932,\\
c_{n-1}(T(4,t)) &= 4 \times 3^t - 4,\\
c_{n-2}(T(4,t)) &= 8 \times 3^{2t} - 24 \times 3^t + 18,\\
c_{n-3}(T(4,t)) &= \frac{32}{3}3^{3t} - 64 \times 3^{2t} + \frac{392}{3}3^t - 88,\\
c_{n-4}(T(4,t)) &= \frac{32}{3}3^{4t} - \frac{320}{3}3^{3t} + \frac{1232}{3}3^{2t} - \frac{2132}{3}3^t + 457,\\
c_{n-5}(T(4,t)) &= \frac{128}{15}3^{5t} - 128 \times 3^{4t} + \frac{2368}{3}3^{3t}\\
&\quad - 2480 \times 3^{2t} + \frac{19644}{5}3^t - 2484,\\
c_{n-6}(T(4,t)) &= \frac{256}{45}3^{6t} - \frac{1792}{15}3^{5t} + \frac{9664}{9}3^{4t} - \frac{15776}{3}3^{3t}\\
&\quad + \frac{661864}{45}3^{2t} - \frac{110756}{5}3^t + 13990.
\end{aligned} \tag{13}$$

For example, see Tables 1 and 2.

Data Availability

All data generated or analyzed during this study are included in this published article. There are no experimental data in this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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