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Research Article

Positive Solutions for Third-Order Boundary Value Problems with the Integral Boundary Conditions and Dependence on the First-Order Derivatives

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In this paper, by the use of a new fixed point theorem and the Green function of BVPs, the existence of at least one positive solution for the third-order boundary value problem with the integral boundary conditions is considered,where there is a nonnegative continuous function. Finally, an example which to illustrate the main conclusions of this paper is given.

1. Introduction

Third-order boundary value problems are originated in a variety of different fields of applied mathematics and physics, for example, deflection of a buckling beam with a fixed or varying cross-section, three-layer beams, electromagnetic waves, and flood tides from gravitational blowing. In recent year, researches on third-order nonlinear boundary value problems have received widespread attention, and many excellent results have been obtained, in references [1–10].

As we all know, boundary value problems with integral boundary conditions can describe many valuable phenomena more accurately. The study of many problems in the fields of heat conduction, chemical engineering, groundwater flow, thermoelasticity, plasma physics, etc. can be reduced to the study of boundary value problems with integral boundary conditions. However, in recent years, although the third-order boundary value problems with integral boundary conditions have received widespread attention, there are relatively few researches on the third-order boundary value problems with integral boundary conditions, in references [11–13].

By using the Guo-Krasnoselskii fixed point theorem, Zhao and Cunchen [11] investigated the existence and nonexistence of at least one or two monotone positive solutions for the following third-order boundary value problem with integral boundary conditions:

$$\begin{cases} u'''(t) + f\left(t, u(t), u'(t)\right) = 0, t \in (0, 1), \\ u(0) = u''(1) = 0, u'(0) = \int_0^1 g(t)u(t)dt. \end{cases} \quad (1)$$

By using the mixed monotone operator method, He and Xiaoling [13] proved the existence and uniqueness of positive solutions for the following third-order ordinary differential equations with integral boundary conditions:

$$\begin{cases} -u'''(t) = f(t, u(t), u(\xi t)) + g(t, u(t)), t \in (0, 1), \xi \in (0, 1), \\ u(0) = u''(0) = 0, u'(1) = \int_0^1 q(t)u'(t)dt. \end{cases} \quad (2)$$

All the above works were done under the assumption that derivative x' is not involved explicitly in the nonlinear term f. In this paper, we are concerned with the existence of positive solutions for the third-order boundary value problem with the integral boundary conditions:

$$\begin{cases} x'''(t) + f\left(t, x, x'\right) + g(t, x) = 0, 0 < t < 1, \\ x(0) = 0, x'(0) = \int_0^1 q(t)x'(t)dt, x''(1) = 0. \end{cases} \quad (3)$$

Throughout, we assume (H_1) $f : [0, 1] \times [0,+\infty)^2 \longrightarrow [0,+\infty)$ and $g : [0, 1] \times [0,+\infty) \longrightarrow [0,+\infty)$ are continuous; (H_2) $\mu = \int_0^1 q(t)dt, t \in [0, 1]$ and $\mu \neq 1, \sigma = \int_0^1 tq(t)dt, \sigma \in [0, 1]$.

2. Preliminary

Let $Y = C[0, 1]$ be the Banach space equipped with the norm $\|x\|_0 = \max_{t\in[0,1]} |x(t)|$.

Lemma 1 (see [12]). *Let $\mu \neq 1$. Then for any $y(t) \in C[0, 1]$, the problem*

$$\begin{cases} x'''(t) + y(t) = 0, 0 < t < 1, \\ x(0) = 0, x'(0) = \int_0^1 q(t)x'(t)dt, x''(1) = 0, \end{cases} \quad (4)$$

has a unique solution

$$x(t) = \int_0^1 \left[G_1(t, s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds, t \in [0, 1], \quad (5)$$

where,

$$\begin{gathered} G_1(t, s) = \frac{1}{2}\begin{cases} 2ts - s^2, & 0 \le s \le t \le 1, \\ t^2, & 0 \le t \le s \le 1, \end{cases} \\ G_2(t, s) = \frac{1}{2}\begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1, \end{cases} \\ G(t, s) = G_1(t, s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau. \end{gathered} \quad (6)$$

By (5), we get $x'(t) = \int_0^1 [G_2(t, s) + 1/(1-\mu)\int_0^1 G_2(\tau, s)q(\tau)d\tau]y(s)ds$.

Let $\mu < 1, \gamma \in (0, 1)$.

Lemma 2 (see [11]). *For any $(t, s) \in [0, 1] \times [0, 1]$, $G_1(t, s)$ and $G_2(t, s)$ have*

(i) $t^2 G_1(1, s) \le G_1(t, s) \le 2s - s^2/2$

(ii) $ts \le G_2(t, s) \le s$

Lemma 3. *If $y \in C[0, 1], y(t) \ge 0$, then the unique solution $x(t)$ of problem (1) satisfies*

$$x(t) \ge 0, t \in [0, 1], \min_{t\in[\gamma,1]} x(t) \ge \gamma^2 \|x\|_0. \quad (7)$$

Proof. By Lemma 1, we get

$$x(t) = \int_0^1 \left[G_1(t, s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds, t \in [0, 1]. \quad (8)$$

By Lemma 2, we get

$$0 \le x(t) \le \int_0^1 \left[\frac{2s - s^2}{2} + \frac{1}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds. \quad (9)$$

So,

$$\|x\|_0 \le \int_0^1 \left[\frac{2s - s^2}{2} + \frac{1}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds. \quad (10)$$

For $\gamma \in (0, 1)$, we have

$$\begin{aligned} \min_{t\in[\gamma,1]} x(t) &= \min_{t\in[\gamma,1]} \int_0^1 \left[G_1(t, s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds \\ &\ge \min_{t\in[\gamma,1]} \int_0^1 \left[t^2 G_1(t, s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds \\ &\ge \gamma^2 \int_0^1 \left[\frac{2s - s^2}{2} + \frac{1}{1-\mu}\int_0^1 G_2(\tau, s)q(\tau)d\tau\right]y(s)ds \ge \gamma^2\|x\|_0. \end{aligned} \quad (11)$$

The proof is completed. □

Let X be a Banach space and $K \subset X$ a cone. Suppose $\alpha, \beta : X \longrightarrow R^+$ are two continuous convex functionals satisfying $\alpha(\lambda x) = |\lambda|\alpha(x), \beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X, \lambda \in R$, and $\|x\| \le M \max\{\alpha(x), \beta(x)\}$, for $x \in X$ and $\alpha(x) \le \alpha(y)$ for $x, y \in K, x \le y$, where $M > 0$ is a constant.

Theorem 4 (see [14]). *Let $r_2 > r_1 > 0, L > 0$ be constants and $\Omega_i = \{x \in X : \alpha(x) < r_i, \beta(x) < L\}, i = 1, 2,(13)$ two bounded open sets in X. Set*

$$D_i = \{x \in X : \alpha(x) = r_i\}, i = 1, 2. \quad (12)$$

Assume $T : K \longrightarrow K$ is a completely continuous operator satisfying the following:

(A1) $\alpha(Tx) < r_1, x \in D_1 \bigcap K$; $\alpha(Tx) > r_2, x \in D_2 \bigcap K$

(A2) $\beta(Tx) < L, x \in K$

(A3) There is a $p \in (\Omega_2 \bigcap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x + \lambda p) \geq \alpha(x)$, for all $x \in K$ and $\lambda \geq 0$

Then, T has at least one fixed point in $(\Omega_2 \setminus \bar{\Omega}_1) \bigcap K$.

3. The Main Results

Let $X = C^1[0,1]$ be the Banach space equipped with the norm $\|x\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|x'(t)|$ and $K = \{x \in X : x(t) \geq 0, \min_{t\in[\gamma,1]} x(t) \geq \gamma^2\|x\|_0\}$ is a cone in X.

Define two continuous convex functionals $\alpha(x) = \max_{t\in[0,1]}|x(t)|$ and $\beta(x) = \max_{t\in[0,1]}|x'(t)|$, for each $x \in X$, and then $\|x\| \leq 2 \max\{\alpha(x), \beta(x)\}$ and $\alpha(\lambda x) = |\lambda|\alpha(x), \beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X, \lambda \in R$; $\alpha(x) \leq \alpha(y)$ for $x, y \in K, x \leq y$.

In the following, we denote

$$\begin{aligned}\eta_0 &= \frac{1}{3} + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)q(\tau)d\tau ds,\\ \eta_1 &= \int_\gamma^1\left[\frac{2s-s^2}{2} + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]ds,\\ \eta_2 &= \frac{1}{2} + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)q(\tau)d\tau ds.\end{aligned} \tag{13}$$

We will suppose that there are $L > b > \gamma^2 b > c > 0$ such that $f(t,x,y) + g(t,x)$ satisfies the following growth conditions: $(H_3) f(t,x,y) + g(t,x) < c/\eta_0$, for $(t,x,y) \in [0,1] \times [0,c] \times [-L, L]$, $(t,x) \in [0,1] \times [0,c]$; $(H_4) f(t,x,y) + g(t,x) \geq b/\gamma^2\eta_1$, for $(t,x,y) \in [\gamma,1] \times [\gamma^2 b, b] \times [-L,L]$, $(t,x) \in [\gamma,1] \times [\gamma^2 b, b]$; and $(H_5) f(t,x,y) + g(t,s) < L/\eta_2$, for $(t,x,y) \in [0,1] \times [0,b] \times [-L,L]$, $(t,x) \in [0,1] \times [0,b]$.

Let

$$\begin{aligned}f^*(t,x,y) &= \begin{cases} f(t,x,y), (t,x,y) \in [0,1] \times [0,b] \times (-\infty,\infty),\\ f(t,b,y), (t,x,y) \in [0,1] \times (b,\infty) \times (-\infty,\infty),\end{cases}\\ f_1(t,x,y) &= \begin{cases} f^*(t,x,y), (t,x,y) \in [0,1] \times [0,\infty) \times [-L,L],\\ f^*(t,x,-L), (t,x,y) \in [0,1] \times [0,\infty) \times (-\infty,-L],\\ f^*(t,x,L), (t,x,y) \in [0,1] \times [0,\infty) \times [L,\infty).\end{cases}\end{aligned} \tag{14}$$

Let

$$\begin{aligned}g^*(t,x) &= \begin{cases} g(t,x), (t,x) \in [0,1] \times [0,b],\\ g(t,b), (t,x) \in [0,1] \times (b,\infty),\end{cases}\\ g_1(t,x) &= \begin{cases} g^*(t,x), (t,x) \in [0,1] \times [0,\infty),\\ g^*(t,x), (t,x) \in [0,1] \times [0,\infty),\\ g^*(t,x), (t,x) \in [0,1] \times [0,\infty).\end{cases}\end{aligned} \tag{15}$$

We denote

$$\begin{aligned}(Tx)(t) &= \int_0^1\left[G_1(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\\ &\quad\cdot\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds,\\ (Tx)'(t) &= \int_0^1\left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\\ &\quad\cdot\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds.\end{aligned} \tag{16}$$

Lemma 5. *Suppose (H_1) hold. Then, $T : K \longrightarrow K$ is completely continuous.*

Proof. For $x \in K$, by Lemma 3, we have $Tx \geq 0$.

So, we can get $T(K) \subset K$.

In the following, we will show that $T : K \longrightarrow K$ is completely continuous.

At first we show that $T : K \longrightarrow K$ is continuous.

Let $x_n, x^* \in K$; it satisfies $\|x_n - x^*\| \longrightarrow 0, (n \longrightarrow \infty)$; then, there is a constant $M_0 > 0$, such that $\max_{t\in[0,1]}\{|x_n(t)|, |x^*(t)|, |x_n'(t)|, |x^{*'}(t)|\} \leq M_0$, then

$$\begin{aligned}&|(Tx_n)(t) - (Tx^*)(t)|\\ &\quad= \left|\int_0^1 G(t,s)\left(f_1\left(s,x_n,x_n'\right) + g_1(s,x_n)\right)ds\right.\\ &\qquad\left.- \int_0^1 G(t,s)\left(f_1\left(s,x^*,x^{*'}\right) + g_1(s,x^*)\right)ds\right|\\ &\quad\leq \int_0^1 |G(t,s)|\left|f_1\left(s,x_n,x_n'\right) + g_1(s,x_n) - \left(f_1\left(s,x^*,x^{*'}\right)\right.\right.\\ &\qquad + g_1(s,x^*))|ds,\end{aligned}$$

$$\begin{aligned}&\left|(Tx_n)'(t) - (Tx^*)(t)\right|\\ &\quad= \left|\int_0^1\left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\left(f_1\left(s,x,x_n'\right)\right.\right.\\ &\qquad + g_1(s,x_n))ds - \int_0^1\left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\\ &\qquad\left.\times\left(f_1\left(s,x^*,x^{*'}\right) + g_1(s,x^*)\right)ds\right|\\ &\quad\leq \int_0^1\left|G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right|\left|f_1\left(s,x,x_n'\right)\right.\\ &\qquad\left. + g_1(s,x_n) - \left(f_1\left(s,x^*,x^{*'}\right) + g_1(s,x^*)\right)\right|ds\\ &\quad< \int_0^1\left[s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\left|f_1\left(s,x,x_n'\right) + g_1(s,x_n)\right.\\ &\qquad\left. - \left(f_1\left(s,x^*,x^{*'}\right) + g_1(s,x^*)\right)\right|ds.\end{aligned} \tag{17}$$

If f is uniformly continuous on $[0,1] \times [-M_0, M_0] \times [-M_0, M_0]$, we get

$$\|Tx_n - Tx^*\| \longrightarrow 0, (n \longrightarrow \infty). \tag{18}$$

Next, we show that $T : K \longrightarrow K$ is compact.

Let $B \subset K$ be bounded, and then, there is $M > 0$, such that $\|x\| \leq M$. For $x \in B$, we have

$$\begin{aligned} |(Tx)(t)| &= \left|\int_0^1 \left[G_1(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\right. \\ &\quad \left.\cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &< \int_0^1 \left[\frac{2s-s^2}{2} + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]ds \times C^*, \end{aligned} \tag{19}$$

where $C^* = \max\{|f_1(t,x,x') + g_1(t,x)|\,; t \in [0,1], x \in B\}$.

$$\begin{aligned} \left|(Tx)'(t)\right| &= \left|\int_0^1 \left[G_2(t,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\right. \\ &\quad \left.\cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &< \left|\int_0^1 \left[s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]ds\right| \times C^*. \end{aligned} \tag{20}$$

It is clear that $T(B)$ is a bounded set in K; because $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for $\varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$, such that $|G_1(t_1,s) - G_1(t_2,s)| < \varepsilon$, $|G_2(t_1,s) - G_2(t_2,s)| < \varepsilon$ for $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$.

For $x \in B$, we have

$$\begin{aligned} &\left|(Tx)'(t_1) - (Tx)'(t_2)\right| = \left|\int_0^1 \frac{\partial G(t,s)}{\partial t}\right|_{t=t_1}\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds \\ &\quad \cdot - \int_0^1 \frac{\partial G(t,s)}{\partial t}\bigg|_{t=t_2}\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\bigg| \\ &\quad = \left|\int_0^1 \left[G_2(t_1,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right. \\ &\quad \left.- \int_0^1 \left[G_2(t_2,s) + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &\quad \leq \left|\int_0^{t_1} \left[s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right. \\ &\quad \left.- \int_0^{t_2} \left[s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &\quad \leq \frac{1}{2}|(t_1 - t_2)(t_1 + t_2)| \times C^* \leq \varepsilon C^*. \end{aligned} \tag{21}$$

Therefore $T(B)$ is equicontinuous. Using the Arzela-Ascoli theorem, a standard proof yields $T : K \longrightarrow K$ is completely continuous. □

Theorem 6. *Suppose (H_1)-(H_5) hold. Then, BVP (1) has at least one positive solution $x(t)$ satisfying*

$$c < \alpha(x) < b, \beta(x) < L. \tag{22}$$

Proof. Take $\Omega_1 = \{x \in X : |x(t)| < c, |x'(t)| < L\}$ and $\Omega_2 = \{x \in X : |x(t)| < b, |x'(t)| < L\}$, two bounded open sets in X, and $D_1 = \{x \in X : \alpha(x) = c\}$ and $D_2 = \{x \in X : \alpha(x) = b\}$.

By Lemma 5, $T : K \longrightarrow K$ is completely continuous, and there is a $p \in (\Omega_2 \bigcap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$, for all $u \in K$ and $\lambda \geq 0$.

By Lemma 2 and (H_3), for $x \in D_1 \cap K$, $\alpha(x) = c$, we get

$$\begin{aligned} \alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 \left[G_1(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\right. \\ &\quad \left.\cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &\leq \int_0^1 \left[\frac{2s-s^2}{2} + \frac{1}{1-\sigma}\int_0^1 G(\tau,s)g(\tau)d\tau\right] \\ &\quad \cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds \\ &= \left[\frac{1}{3} + \frac{1}{1-\mu}\int_0^1\int_0^1 (G_2(\tau,s)g(\tau)d\tau)ds\right] \times \frac{c}{\eta_0} = c. \end{aligned} \tag{23}$$

By Lemma 2, for $x \in D_2 \cap K$, $\alpha(x) = b$, there is $x(t) \geq \gamma^2 \alpha(x) = \gamma^2 b$, $t \in [\gamma, 1]$.

So, by (H_4), we get

$$\begin{aligned} \alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 \left[G_1(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\right. \\ &\quad \left.\cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &\geq \int_0^1 \left[t^2 G_1(1,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right] \\ &\quad \cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds \\ &> \gamma^2\int_\gamma^1 \left[\frac{2s-s^2}{2} + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]ds \times \frac{b}{\gamma^2\eta_1} = b. \end{aligned} \tag{24}$$

By (H_5), for $x \in K$, we have

$$\begin{aligned} \beta(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 \left[G_2(t,s) + \frac{t}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\right. \\ &\quad \left.\cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &\leq \left|\int_0^1 \left[s + \frac{1}{1-\mu}\int_0^1 G_2(\tau,s)q(\tau)d\tau\right]\right. \\ &\quad \left.\cdot \left(f_1\left(s,x,x'\right) + g_1(s,x)\right)ds\right| \\ &< \left[\frac{1}{2} + \frac{1}{1-\mu}\int_0^1\int_0^1 (G_2(\tau,s)g(\tau)d\tau)ds\right] \times \frac{L}{\eta_2} = L. \end{aligned} \tag{25}$$

Theorem 4 implies there is $x \in (\Omega_2 \setminus \bar{\Omega}_1) \bigcap K$ such that $x = Tx$. So, $x(t)$ is a positive solution for BVP (1) satisfying

$$c < \alpha(x) < b, \beta(x) < L. \tag{26}$$

Thus, Theorem 6 is completed. □

4. Example

Example 1. Consider the following boundary value problem

$$\begin{cases} x^{('')}(t) + f\left(t, x, x'\right) + g(t, x) = 0, 0 < t < 1, \\ x(0) = 0, x'(0) = \int_0^1 q(t)x'(t)dt, x''(1) = 0, \end{cases} \tag{27}$$

where,

$$f(t,x,y)+g(t,x)=\begin{cases} \frac{t}{3}x + x + |\cos y|, \\ (t,x,y) \in [0,1]\times[0,2.2]\times[-60306,60306], (t,x)\in[0,1]\times[0,2.2] \\ \frac{2280t}{3}(x-2.2) + 420001(x-2.2) + \frac{11t}{15} + 2.2 + |\cos y|, \\ (t,x,y) \in [0,1]\times[2.2,2.3]\times[-60306,60306], (t,x)\in[0,1]\times[2.2,2.3] \\ \frac{t}{3}(231-x) + 180.039(x+231) + |\cos y|, \\ (t,x,y) \in [0,1]\times[2.3,231]\times[-60306,60306], (t,x)\in[0,1]\times[2.3,231]. \end{cases} \tag{28}$$

In this problem, we know that $q(t) = t^2$; then, we can get $\mu = \int 0^1 t^2 dt = 1/3$. Choose $\gamma = 1/10 \in (0, 1/2)$, then $\rho = 4\delta^2(1-\delta) = 7/128$.

Furthermore, we obtain

$$\eta_0 = \frac{67}{120}, \quad \eta_1 \approx 0.551, \quad \eta_2 = \frac{29}{40}. \tag{29}$$

If we take $c = 2.2$, $b = 231$, and $L = 60306$, then we get $\gamma^2 b \approx 2.31 > 2.3$.

$$f(t,x,y) + g(t,x) = \frac{t}{3}x + x + |\cos y| \le 2.17 < \frac{c}{\eta_0} \approx 3.94, \text{for } (t,x,y) \in [0,1]\times[0,2.2] \times[-60306,60306], (t,x)\in[0,1]\times[0,2.2],$$

$$f(t,x,y) + g(t,x) = \frac{t}{3}(231-x) + 180.039(x+231) + |\cos y| > 42011.7 > \frac{b}{\gamma^2\eta_1} \approx 42000, \text{for } (t,x,y) \in [0,1] \times[2.3,231]\times[-60306,60306], (t,x)\in[0,1]\times[2.3,231],$$

$$f(t,x,y) + g(t,x)\frac{t}{3}(231-x) + 180.039(x+231) + |\cos y| < 83179.02 < \frac{L}{\eta_2} \approx 83180.69, \text{for}(t,x,y) \in [0,1] \times[2.3,231]\times[-60306,60306], (t,x)\in[0,1]\times[2.3,231]. \tag{30}$$

Then, all the conditions of Theorem 6 are satisfied. Therefore, by Theorem 6 we know that boundary value problem (1) has at least one positive solution $x(t)$ satisfying

$$2.2 < \alpha(x) < 231, \beta(x) < 60306. \tag{31}$$

Data Availability

This paper is a basic theoretical study without data support.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

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