Hindawi
Journal of Applied Mathematics
Volume 2023, Article ID 2341137, 11 pages
https://doi.org/10.1155/2023/2341137

Research Article

Vibration Analysis of Viscoelastic Timoshenko Cracked Beams with Massless Viscoelastic Rotational Spring Models

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Received 14 March 2023; Revised 14 June 2023; Accepted 21 July 2023; Published 31 July 2023

Academic Editor: Hui Shen Shen

Based on the equivalent bending stiffness of the viscoelastic cracked beam with open cracks, the corresponding complex frequency characteristic equations of a Timoshenko viscoelastic cracked beam are obtained by using the method of separation of variables and the Laplace transform. The vibration characteristics of a viscoelastic Timoshenko cracked beams with the standard linear solid model and Kelvin-Voigt model are investigated. By numerical examples, the effects of the crack location, crack number, crack depth, and slenderness ratio on the vibration characteristics of the viscoelastic cracked beams are revealed.

1. Introduction

Viscoelastic materials [1, 2] are widely used in civil, mechanical, and aerospace engineering, etc. In order to investigate the vibration fatigue characteristics of the viscoelastic structures, the modal analysis method can be used to derive the analytical solutions. By using the complex modal method and Laplace transform, Huang and Huang [3] studied the free vibration of Timoshenko viscoelastic beams satisfying the standard linear solid constitutive equations. Considering the axial forces, Chen et al. [4] studied the vibration characteristics of clamped-clamped Timoshenko viscoelastic beams constituted by the Kelvin-Voigt model. Peng [5] applied the complex modal method and differential quadrature method to analyze the transverse vibration characteristics of the elastic Euler-Bernoulli and Timoshenko beams resting on the viscoelastic foundation.

Additionally, the correspondence principle, integral transformation, finite element method (FEM), differential quadrature method [6, 7], and other numerical methods [8, 9] were applied to analyze the static and dynamic properties of the viscoelastic beam structures.

This paper is organized as follows. Firstly, the equivalent flexural stiffness of the viscoelastic cracked beam established in reference [10] is used to present the motion equation of a Timoshenko viscoelastic beam with open cracks. Then, the general explicit analytical expressions for solving the complex frequency of Timoshenko viscoelastic cracked beams are derived by using the separation of variables method and Laplace transform. Finally, by numerical examples, the effects of the crack location, crack number, crack depth, and slenderness ratio on the vibration characteristics of the viscoelastic cracked beams are investigated.

2. Formulation of the Problem

According to the constitutive equation of the standard linear solid model [1, 2], E_1 and E_2 are the elastic modulus of elastic elements, η is the viscous coefficient of a viscous element, and the relaxation modulus $Y(t)$ and shear modulus $G(t)$ are defined as follows:

$$Y(t) = q_0 + \left(\frac{q_1}{p_1} - q_0\right) e^{-(t/p_1)}, G(t) = \frac{Y(t)}{2(1+\nu)}, \quad (1)$$

where the Poisson's ratio ν is a constant, and

$$p_1 = \frac{\eta}{E_1 + E_2}, q_0 = \frac{E_1 E_2}{E_1 + E_2}, q_1 = \frac{E_1 \eta}{E_1 + E_2}. \quad (2)$$

Suppose that the superscript denotes the Laplace transform of the function with respect to the time t, and the Laplace transform of relaxation modulus and shear modulus are given as

$$\bar{Y}(s)=\frac{q_0+sq_1}{s(1+sp_1)},\ \bar{G}(s)=\lambda_1\frac{q_0+sq_1}{s(1+sp_1)}, \tag{3}$$

where the parameter is $\lambda_1=0.5(1+\nu)^{-1}$, and s is the Laplace transform parameter.

The physical model of the viscoelastic beam is given as shown in Figure 1. Let us consider a viscoelastic rectangular beam with length L (x -axis), width b (y -axis), and height h (z -axis). Here, $w(x,t)$ and $\phi(x,t)$ denote the transverse deflection of the axial line and rotation angle of the beam cross section subjected to the distributed transverse load $q(x,t)$, respectively. Assuming that the crack at the location $x=x_j\ (j=1,2,\cdots,N)$ is always open, and it can be equivalent as a massless viscoelastic rotational spring. Then, the equivalent flexural stiffness of viscoelastic cracked beams in time domain and the Laplace domain established in reference [10] are given as follows:

$$\begin{cases} M(x,t)=-\left[(EI)_e(x,0)\dfrac{\partial\phi(x,t)}{\partial x}+\left(\dot{EI}\right)_e(x,t)*\dfrac{\partial\phi(x,t)}{\partial x}\right],\\ \dfrac{1}{\left(\overline{EI}\right)_e(x,s)}=\dfrac{1}{\bar{Y}(s)I}+\displaystyle\sum_{j=1}^{N}\dfrac{1}{\bar{k}_j(s)}\delta\left(x-x_j\right), \end{cases} \tag{4}$$

where $(EI)_e(x,t)$ is the equivalent bending stiffness of a viscoelastic beam with open cracks. Here, $(E\dot{I})_e(x,t)$ is the first derivative of $(EI)_e(x,t)$ with respect to the time t, and the asterisk * denotes the convolution, i.e., $f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$.

And by using the expression for the rectangular cross section beams in references [11, 12], the equivalent stiffness of crack at the location $x=x_j$ with the crack depth d_j in time domain and Laplace domain are given as, respectively,

$$k_j(t)=\mu_j IY(t),\bar{k}_j(s)=\mu_j I\bar{Y}(s),\mu_j=\frac{(0.9/h)\left[\left(d_j/h\right)-1\right]^2}{\left\{\left(d_j/h\right)\left[2-\left(d_j/h\right)\right]\right\}}, \tag{5}$$

where the moment of inertia of the neutral axis is given as $I=\iint_\Omega y^2\mathrm{d}y\mathrm{d}z$..

The bending moment and shearing force of a Timoshenko viscoelastic cracked beam are written as follow, respectively,

$$\begin{cases} M(x,t)=-\left[(EI)_e(x,0)\dfrac{\partial\phi(x,t)}{\partial x}+\left(\dot{EI}\right)_e(x,t)*\dfrac{\partial\phi(x,t)}{\partial x}\right],\\ F_s(x,t)=\kappa\iint_A\left\{G(0)\left[-\phi(x,t)+\dfrac{\partial w(x,t)}{\partial x}\right]+\dot{G}(t)*\left[-\phi(x,t)+\dfrac{\partial w(x,t)}{\partial x}\right]\right\}\mathrm{d}y\mathrm{d}z, \end{cases} \tag{6}$$

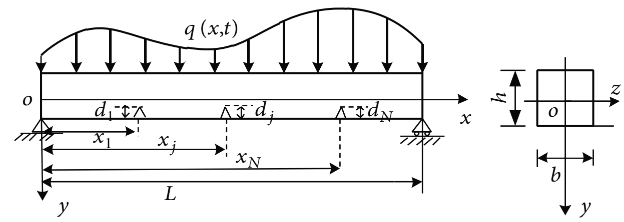


Figure 1: Geometric parameters of a viscoelastic Timoshenko cracked beam.

where κ is the shear correction factor of a Timoshenko beam, and the cross-section is given as $A=\iint_\Omega \mathrm{d}y\mathrm{d}z$.

Utilizing the Laplace transform, one obtain

$$\begin{aligned} \bar{M}(x,s)&=-s\left(\overline{EI}\right)_e(x,s)\frac{\partial\bar{\phi}(x,s)}{\partial x},\\ \bar{F}_s(x,s)&=s\bar{G}(s)\kappa A\left[-\bar{\phi}(x,s)+\frac{\partial\bar{w}(x,s)}{\partial x}\right]. \end{aligned} \tag{7}$$

Substituting equations (3), (5), and (6) into equations (7) and utilizing the inverse Laplace transform, one obtain

$$\begin{cases} \left(1+p_1\dfrac{\partial}{\partial t}\right)M(x,t)=-I\left[1+\displaystyle\sum_{j=1}^{N}\dfrac{1}{\mu_j}\delta\left(x-x_j\right)\right]^{-1}\left(q_0+q_1\dfrac{\partial}{\partial t}\right)\dfrac{\partial\phi(x,t)}{\partial x},\\ \left(1+p_1\dfrac{\partial}{\partial t}\right)F_s(x,t)=\lambda_1\kappa A\left(q_0+q_1\dfrac{\partial}{\partial t}\right)\left[-\phi(x,t)+\dfrac{\partial w(x,t)}{\partial x}\right]. \end{cases} \tag{8}$$

The free vibration equations of the Timoshenko viscoelastic beam [4] are

$$\begin{cases} \rho A\dfrac{\partial^2 w(x,t)}{\partial t^2}-\dfrac{\partial F_s(x,t)}{\partial x}=0,\\ \rho I\dfrac{\partial^2\phi(x,t)}{\partial t^2}+\dfrac{\partial M(x,t)}{\partial x}-F_s(x,t)=0. \end{cases} \tag{9}$$

3. Solutions

Introduce the following dimensionless variables and parameters

$$\begin{cases} w^*=\dfrac{w}{L},\phi^*=\phi,\xi=\dfrac{x}{L},\xi_j=\dfrac{x_j}{L},\mu_j^*=\mu_jL,t^*=\dfrac{t}{T},\\ I^*=\dfrac{I}{L^4},A^*=\dfrac{A}{L^2},\rho^*=\dfrac{\rho L^2}{E_1T^2},m^*=\dfrac{M}{E_1L^3},V^*=\dfrac{F_s}{E_1L^2},\\ E_2^*=\dfrac{E_2}{E_1},\eta^*=\dfrac{\eta}{E_1T},p_1^*=\dfrac{\eta_2^*}{1+E_2^*},q_0^*=\dfrac{E_2^*}{1+E_2^*},q_1^*=\dfrac{\eta^*}{1+E_2^*}. \end{cases} \tag{10}$$

The dimensionless forms of equations (8) and (9) are given as follows

$$\begin{cases} \left(1+p_1^*\frac{\partial}{\partial t^*}\right)m^*(\xi,t^*) = -I^*\left[1+\sum_{j=1}^{N}\frac{1}{\mu_j^*}\delta(\xi-\xi_j)\right]^{-1}\left(q_0^*+q_1^*\frac{\partial}{\partial t^*}\right)\frac{\partial\phi^*(\xi,t^*)}{\partial\xi}, \\ \left(1+p_1^*\frac{\partial}{\partial t^*}\right)V^*(\xi,t^*) = \lambda_1\kappa A^*\left(q_0^*+q_1^*\frac{\partial}{\partial t^*}\right)\left[-\phi^*(\xi,t^*)+\frac{\partial w^*(\xi,t^*)}{\partial\xi}\right], \end{cases} \tag{11}$$

$$\begin{cases} \rho^* A^*\frac{\partial^2 w^*(\xi,t^*)}{\partial t^{*2}} - \frac{\partial V^*(\xi,t^*)}{\partial\xi} = 0, \\ \rho^* I^*\frac{\partial^2\phi^*(\xi,t^*)}{\partial t^{*2}} + \frac{\partial m^*(\xi,t^*)}{\partial\xi} - V^*(\xi,t^*) = 0. \end{cases} \tag{12}$$

Based on the separation of variables method [13], the vibration solutions [14] can be assumed as

$$\begin{cases} w^*(\xi,t^*) = W^*(\xi)e^{i\omega t^*}, \phi^*(\xi,t^*) = \Phi^*(\xi)e^{i\omega t^*}, \\ m^*(\xi,t^*) = M^*(\xi)e^{i\omega t^*}, V^*(\xi,t^*) = F_s^*(\xi)e^{i\omega t^*}. \end{cases} \tag{13}$$

Here in $i=\sqrt{-1}$, ω is the complex eigenfrequency, and the real part and imaginary part of ω are the natural frequency and decrement coefficient [15, 16], respectively. $W^*(\xi)$, $\Phi^*(\xi)$, $M^*(\xi)$, and $F_s^*(\xi)$ are the dimensionless mode functions of the transverse displacement, rotation angle, bending moment and shearing force for the cracked beam.

Substituting equation (13) into equations (11) and (12),

$$\begin{cases} M^*(\xi) = -I^*\frac{q_0^*+i\omega q_1^*}{1+i\omega p_1^*}Z^*(\xi), \\ F_s^*(\xi) = \lambda_1\kappa A^*\frac{q_0^*+i\omega q_1^*}{1+i\omega p_1^*}\left[-\Phi^*(\xi)+\frac{dW^*(\xi)}{d\xi}\right], \end{cases} \tag{14}$$

$$\begin{cases} c\frac{d}{d\xi}\left[\frac{dW^*(\xi)}{d\xi}-\Phi^*(\xi)\right]+aW^*(\xi)=0, \\ c\left[\frac{dW^*(\xi)}{d\xi}-\Phi^*(\xi)\right]+ar\Phi^*(\xi)+\frac{dZ^*(\xi)}{d\xi}=0. \end{cases} \tag{15}$$

Here,

$$Z^*(\xi) = \left[1+\sum_{j=1}^{N}\frac{1}{\mu_j^*}\delta(\xi-\xi_j)\right]^{-1}\frac{d\Phi^*(\xi)}{d\xi}, \tag{16}$$

$$r=\frac{I^*}{A^*}, a=-\frac{\rho^*}{r}(i\omega)^2\frac{1+i\omega p_1^*}{q_0^*+i\omega q_1^*}, c=\frac{\lambda_1\kappa}{r}. \tag{17}$$

By the Laplace transformation of equations (15) and (16), one obtain

$$\begin{cases} (a+cs^2)\bar{W}^*(s) = cs\bar{\Phi}^*(s)-cC_1+csC_3+cC_4, \\ (ar-c)\bar{\Phi}^*(s) = -s\bar{Z}^*(s)+C_2-cs\bar{W}^*(s)+cC_3, \end{cases} \tag{18}$$

$$s\bar{\Phi}^*(s)-C_1 = \bar{Z}^*(s)+\sum_{j=1}^{N}\frac{Z^*(\xi_j)}{\mu_j^*}e^{-s\xi_j}, \tag{19}$$

where $\bar{W}^*(s)$, $\bar{\Phi}^*(s)$, and $\bar{Z}^*(s)$ are the Laplace transformation functions of the dimensionless functions of $W^*(\xi)$, $\Phi^*(\xi)$, and $Z^*(\xi)$ for the cracked beam, $C_m (m=1,2,3,4)$ are the undetermined functions, and

$$C_1=\Phi^*(0), C_2=Z^*(0), C_3=W^*(0), C_4=\left.\frac{dW^*(\xi)}{d\xi}\right|_{\xi=0}. \tag{20}$$

Combining equations (18) and (19),

$$\begin{aligned} \bar{Z}^*(s) = {} & \frac{1}{(s^2-\beta_1^2)(s^2+\beta_2^2)} \\ & \cdot\left\{\left(s^2+\frac{a}{c}\right)(c-ar)C_1+s\left(\frac{a}{c}+s^2\right)C_2+saC_3-s^2cC_4\right. \\ & \left.+\sum_{j=1}^{N}\frac{Z^*(\xi_j)}{\mu_j^*}\left[-ars^2+a\left(1-\frac{ar}{c}\right)\right]e^{-s\xi_j}\right\}, \end{aligned} \tag{21}$$

$$\begin{aligned} \bar{W}^*(s) = {} & \frac{1}{(s^2-\beta_1^2)(s^2+\beta_2^2)} \\ & \cdot\left[-(ar-c)C_1+sC_2+s(s^2+ar)C_3\right. \\ & \left.+(s^2+ar-c)C_4+\sum_{j=1}^{N}\frac{Z^*(\xi_j)}{\mu_j^*}s^2e^{-s\xi_j}\right], \end{aligned} \tag{22}$$

$$\begin{aligned} \bar{\Phi}^*(s) = {} & \frac{1}{(s^2-\beta_1^2)(s^2+\beta_2^2)} \\ & \cdot\left[s\left(\frac{a}{c}+c+s^2\right)C_1+\left(\frac{a}{c}+s^2\right)C_2+aC_3-csC_4\right. \\ & \left.+\sum_{j=1}^{N}\frac{Z^*(\xi_j)}{\mu_j^*}s\left(\frac{a}{c}+s^2\right)e^{-s\xi_j}\right], \end{aligned} \tag{23}$$

where

$$\begin{aligned} \beta_1 &= \sqrt{\frac{1}{2}\left[-\left(ar+\frac{a}{c}\right)+\sqrt{\left(ar-\frac{a}{c}\right)^2+4a}\right]}, \\ \beta_2 &= \sqrt{\frac{1}{2}\left[\left(ar+\frac{a}{c}\right)+\sqrt{\left(ar-\frac{a}{c}\right)^2+4a}\right]}. \end{aligned} \tag{24}$$

By the Laplace transformation of equation (21), then, let

$\xi = \xi_m$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_j < \cdots < \xi_N < 1$, one obtain

$$\begin{aligned} Z^*(\xi_m) &= C_1 \frac{c-ar}{\beta_1^2+\beta_2^2}\left[\frac{\sinh(\beta_1\xi_m)}{\beta_1}\left(\frac{a}{c}+\beta_1^2\right) - \frac{\sin(\beta_2\xi_m)}{\beta_2}\right. \\ &\quad \left.\cdot\left(\frac{a}{c}-\beta_2^2\right)\right] + \frac{C_2}{\beta_1^2+\beta_2^2}\left[\cosh(\beta_1\xi_m)\left(\frac{a}{c}+\beta_1^2\right) - \cos(\beta_2\xi_m)\left(\frac{a}{c}-\beta_2^2\right)\right] \\ &\quad + C_3 a\frac{\cosh(\beta_1\xi_m)-\cos(\beta_2\xi_m)}{\beta_1^2+\beta_2^2} \\ &\quad - C_4 c\frac{\beta_1\sinh(\beta_1\xi_m)+\beta_2\sin(\beta_2\xi_m)}{\beta_1^2+\beta_2^2} \\ &\quad + \sum_{j=1}^{m-1}\frac{1}{\mu_j^*}\frac{Z^*(\xi_j)}{\beta_1^2+\beta_2^2}\left\{\frac{\sinh\left[\beta_1(\xi_m-\xi_j)\right]}{\beta_1}\left[\frac{a}{c}(c-ar)-ar\beta_1^2\right]\right. \\ &\quad \left. - \frac{\sin\left[\beta_2(\xi_m-\xi_j)\right]}{\beta_2}\left[\frac{a}{c}(c-ar)+ar\beta_2^2\right]\right\}. \end{aligned} \tag{25}$$

Then, one obtain

$$Z^*(\xi_m) = X_m C_1 + \Pi_m C_2 + \Lambda_m C_3 + \Gamma_m C_4. \quad (m = 1, 2, 3, \cdots, N), \tag{26}$$

where

$$\begin{cases} X_m = \dfrac{1}{\beta_1^2+\beta_2^2}\left\{(c-ar)\left[\dfrac{a}{c}\Omega_1(\xi_m)+\Omega_2(\xi_m)\right] + \displaystyle\sum_{j=1}^{m-1}\dfrac{X_j}{\mu_j^*}\Omega_5(\xi_m-\xi_j)\right\}, \\ \Pi_m = \dfrac{1}{\beta_1^2+\beta_2^2}\left[\dfrac{a}{c}\Omega_4(\xi_m)+\Omega_3(\xi_m) + \displaystyle\sum_{j=1}^{m-1}\dfrac{\Pi_j}{\mu_j^*}\Omega_5(\xi_m-\xi_j)\right], \\ \Lambda_m = \dfrac{1}{\beta_1^2+\beta_2^2}\left[a\Omega_4(\xi_m) + \displaystyle\sum_{j=1}^{m-1}\dfrac{\Lambda_j}{\mu_j^*}\Omega_5(\xi_m-\xi_j)\right], \\ \Gamma_m = \dfrac{1}{\beta_1^2+\beta_2^2}\left[-c\Omega_2(\xi_m) + \displaystyle\sum_{j=1}^{m-1}\dfrac{\Gamma_j}{\mu_j^*}\Omega_5(\xi_m-\xi_j)\right], \end{cases} \tag{27}$$

$$\begin{cases} \Omega_1(\xi) = \dfrac{\sinh(\beta_1\xi)}{\beta_1} - \dfrac{\sin(\beta_2\xi)}{\beta_2}, \Omega_5(\xi) = a\left[\dfrac{c-ar}{c}\Omega_1(\xi) - r\Omega_2(\xi)\right]. \\ \Omega_2(\xi) = \beta_1\sinh(\beta_1\xi)+\beta_2\sin(\beta_2\xi), \Omega_4(\xi) = \cosh(\beta_1\xi)-\cos(\beta_2\xi), \\ \Omega_3(\xi) = \beta_1^2\cosh(\beta_1\xi)+\beta_2^2\cos(\beta_2\xi). \end{cases} \tag{28}$$

Substituting equation (26) into the Laplace transformation of equations (21), (22), and (23), respectively, the dimensionless functions of $Z^*(\xi)$, $W^*(\xi)$, and $F^*(\xi)$ are expressed as

$$\begin{aligned} Z^*(\xi) &= \frac{C_1}{\beta_1^2+\beta_2^2}\left\{(c-ar)\left[\frac{a}{c}\Omega_1(\xi)+\Omega_2(\xi)\right] + \sum_{j=1}^{N}\frac{X_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right\} \\ &\quad + \frac{C_2}{\beta_1^2+\beta_2^2}\left[\frac{a}{c}\Omega_4(\xi)+\Omega_3(\xi) + \sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right] \\ &\quad + \frac{C_3}{\beta_1^2+\beta_2^2}\left[a\Omega_4(\xi) + \sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right] \\ &\quad + \frac{C_4}{\beta_1^2+\beta_2^2}\left[-c\Omega_2(\xi) + \sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right], \end{aligned} \tag{29}$$

$$\begin{aligned} W^*(\xi) &= \frac{C_1}{\beta_1^2+\beta_2^2}\left[(c-ar)\Omega_1(\xi) + \sum_{j=1}^{N}\frac{X_j}{\mu_j^*}H(\xi-\xi_j)\Omega_2(\xi-\xi_j)\right] \\ &\quad + \frac{C_2}{\beta_1^2+\beta_2^2}\left[\Omega_4(\xi) + \sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}H(\xi-\xi_j)\Omega_2(\xi-\xi_j)\right] \\ &\quad + \frac{C_3}{\beta_1^2+\beta_2^2}\left[ar\Omega_4(\xi)+\Omega_3(\xi) + \sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}H(\xi-\xi_j)\Omega_2(\xi-\xi_j)\right] \\ &\quad + \frac{C_4}{\beta_1^2+\beta_2^2}\left[(ar-c)\Omega_1(\xi)+\Omega_2(\xi) + \sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}H(\xi-\xi_j)\Omega_2(\xi-\xi_j)\right], \end{aligned} \tag{30}$$

$$\begin{aligned} \Phi^*(\xi) &= \frac{C_1}{\beta_1^2+\beta_2^2}\left[\left(\frac{a}{c}+c\right)\Omega_4(\xi)+\Omega_3(\xi) + \sum_{j=1}^{N}\frac{X_j}{\mu_j^*}H(\xi-\xi_j)\Omega_6(\xi-\xi_j)\right] \\ &\quad + \frac{C_2}{\beta_1^2+\beta_2^2}\left[\frac{a}{c}\Omega_1(\xi)+\Omega_2(\xi) + \sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}H(\xi-\xi_j)\Omega_6(\xi-\xi_j)\right] \\ &\quad + \frac{C_3}{\beta_1^2+\beta_2^2}\left[a\Omega_1(\xi) + \sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}H(\xi-\xi_j)\Omega_6(\xi-\xi_j)\right] \\ &\quad + \frac{C_4}{\beta_1^2+\beta_2^2}\left[-c\Omega_4(\xi) + \sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}H(\xi-\xi_j)\Omega_6(\xi-\xi_j)\right], \end{aligned} \tag{31}$$

where

$$\Omega_6(\xi) = \frac{a}{c}\Omega_4(\xi) + \Omega_3(\xi). \tag{32}$$

Substituting equations (29), (30), and (31) into equations (14), the dimensionless functions $M^*(\xi)$ and $F_s{}^*(\xi)$ are expressed as

$$\begin{aligned} M^*(\xi) &= -\frac{q_0^*+i\omega q_1^*}{1+i\omega p_1^*}\frac{I^*}{\beta_1^2+\beta_2^2}\left\langle C_4\left[-c\Omega_2(\xi) + \sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right]\right. \\ &\quad + C_1\left\{(c-ar)\left[\frac{a}{c}\Omega_1(\xi)+\Omega_2(\xi)\right] + \sum_{j=1}^{N}\frac{X_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right\} \\ &\quad + C_2\left[\frac{a}{c}\Omega_4(\xi)+\Omega_3(\xi) + \sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right] \\ &\quad \left. + C_3\left[a\Omega_4(\xi) + \sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}H(\xi-\xi_j)\Omega_5(\xi-\xi_j)\right]\right\rangle. \end{aligned} \tag{33}$$

$$\begin{aligned} F_s^*(\xi) &= -\frac{q_0^*+i\omega q_1^*}{1+i\omega p_1^*}\frac{\lambda_1\kappa A^*}{\beta_1^2+\beta_2^2} \\ &\quad \cdot\left\{-C_3\left[ar\Omega_2(\xi)+\beta_1^3\sinh(\beta_1\xi)-\beta_2^3\sin(\beta_2\xi)\right.\right. \\ &\quad \left. - a\Omega_1(\xi) - \frac{a}{c}\sum_{j=1}^{N}\frac{\Lambda_j}{\mu_j^*}H(\xi-\xi_j)\Omega_4(\xi-\xi_j)\right] \\ &\quad + C_1\left[\left(\frac{a}{c}+ar\right)\Omega_4(\xi)+\Omega_3(\xi) + \frac{a}{c}\sum_{j=1}^{N}\frac{X_j}{\mu_j^*}H(\xi-\xi_j)\Omega_4(\xi-\xi_j)\right] \\ &\quad + C_2\frac{a}{c}\left[\Omega_1(\xi) + \sum_{j=1}^{N}\frac{\Pi_j}{\mu_j^*}H(\xi-\xi_j)\Omega_4(\xi-\xi_j)\right] \\ &\quad \left. - C_4\left[ar\Omega_4(\xi)+\Omega_3(\xi) - \frac{a}{c}\sum_{j=1}^{N}\frac{\Gamma_j}{\mu_j^*}H(\xi-\xi_j)\Omega_4(\xi-\xi_j)\right]\right\}. \end{aligned} \tag{34}$$

By the boundary conditions, the set of linear equations is derived to determine the functions $\{\mathbf{C}\}$

$$[\mathbf{A}]\{\mathbf{C}\} = \mathbf{0}. \tag{35}$$

Here, $[\mathbf{A}]$ is a 4×4 coefficient vector, and $\{\mathbf{C}\} = \{C_1, C_2, C_3, C_4\}^{\mathrm{T}}$.

If there exists a nonzero solution of $\{\mathbf{C}\}$, the determinant of the coefficients vector is zero, i.e.,

$$\det\,[\mathbf{A}] = 0. \tag{36}$$

By utilizing MATLAB programs, the complex eigenfrequency ω can be obtained with the different boundary conditions.

4. Numerical Results and Discussion

To verify the correctness and applicability of the present exact analytical method (EAM), the numerical examples for comparisons have been provided. Let $d_1 \longrightarrow 0$, the present model is degenerated into the intact model of the standard linear solid mechanism. Huang and Huang [3] analyzed the vibration properties of the clamped-clamped beam. Based on the common physical and geometric parameters with reference [3], the analytical results of the first eigenfrequencies are shown in Table 1. It is noticed that the errors of the natural frequency and decrement coefficient are less than 2.5% and 6%, respectively.

Then, let $E_1 \longrightarrow \infty$ and $d_1 \longrightarrow 0$, the present model is degenerated into the Kelvin-Voigt intact model. Anderson and Simone [17] analyzed the vibration properties of the simple-supported and clamped-clamped beams. The geometric and physical parameters are $L = 0.5$ m, $A = 1.5625 \times 10^{-2}\mathrm{m}^2$, $I = 2.0345 \times 10^{-5}\mathrm{m}^4$, $\rho = 7850\mathrm{kg/m}^3$, $E_2 = 210 \times 10^9$ Pa, $G = 80.8 \times 10^9$Pa, and $\eta = 2 \times 10^{-8} E_2 \cdot \mathrm{h}$. The first two eigenfrequencies are shown in Table 2. As a whole, the analytical results are consistent with the results in Tables 1 and 2 to some extent.

For a standard linear solid beam under the simple-supported boundary conditions, it is supposed that the material and geometric parameters are $E_2 = 39.68$GPa, $E_1 = 14$GPa, $\rho = 500\mathrm{kg/m}^3$, $L = 1$ m, and $b = 0.1$ m. Additionally, Poisson's ratio $\nu = 0.3$, the uniform sudden load $Q_0 = 10^6\mathrm{N} \cdot \mathrm{m}^{-1}$, and the shear correction factor $\kappa = 10(1 + \nu)/(12 + 11\nu)$. In order to analyze the effect of viscous coefficient on the vibration properties of the viscoelastic beam, the viscous coefficient is taken as $\eta \in 6.9 \times [10^4, 10^{12}]$ GPa $\cdot$ h according to references [15, 16].

For the sake of simplicity, the k-th eigenfrequency of the viscoelastic Timoshenko and Euler-Bernoulli beams are defined by ω_{TB} and ω_{EB}, respectively. Additionally, the real part (natural frequency) and imaginary part (decrement coefficient) of the k-th eigenfrequency ω_{k} are defined by Re (ω_k) and Im (ω_k). To consider the effect of viscous coefficient η, let slenderness ratio $L/h = 50$ and crack location $x_1 = 0.4L$. The first two eigenfrequencies of the simply-supported standard linear solid viscoelastic Timoshenko and Euler-Bernoulli beams when the crack depth $d_1/h = 0.0001$ and $d_1/h = 0.4$ are analyzed, respectively. In Tables 3 and 4, it is shown that when the slenderness ratio is high, the decrement coefficient Im (ω_k) and natural frequency Re (ω_k) of the first two eigenfrequencies of the Timoshenko model are close to those of the Euler-Bernoulli model, which means that the transverse shear deformation and moment of inertia has less influence on dynamic characteristics of beam when the slenderness ratio is high.

Table 1: First eigenfrequency of the clamped-clamped viscoelastic intact beam.

	EAM	Ref. [3]
1st	21.6236-2.3659i	22.1745-2.2284i

Table 2: First two eigenfrequencies of the simple-supported and clamped-clamped viscoelastic beam.

	EAM	Ref. [17]
Simple-supported		
1st	6707.09-0.450i	6712.36-0.45i
2nd	22126.22-4.89i	21713-4.90i
Clamped-clamped		
1st	12279.1-1.507i	12279.91-1.51i
2nd	26895.14-7.227i	25212.07-7.42i

Then, let $L/h = 10$, the effects of the viscous coefficient η on the first three frequencies of the simply-supported viscoelastic intact beams with standard linear solid model (SLS) and Kelvin-Voigt model (KV) are analyzed in Tables 5 and 6, respectively. It is seen that, with the viscous coefficient increasing, the decrement coefficient Im (ω_k) of the first three frequencies first increase and then decrease. When $\eta \in 6.9 \times [10^4, 10^8]$GPa $\cdot$ h, the decrement coefficient Im (ω_k) increases with the order of mode function increasing. While $\eta \in 6.9 \times [10^9, 10^{12}]$GPa $\cdot$ h, the decrement coefficient tends to be a constant. There is a similar conclusion presented by Peng [5] based on the vibration properties of the Timoshenko elastic beam resting on the viscoelastic foundation.

Additionally, with the viscous coefficient η and order of mode function increasing, the natural frequency Re (ω_k) of the first three frequencies with the SLS model increases, and then it almost remains a constant when $\eta \geq 6.9 \times 10^8$ GPa $\cdot$ h. However, the natural frequency Re (ω_k) of the first three frequencies with KV model decreases first, and when $\eta = 6.9 \times 10^7$GPa $\cdot$ h, it reduces to zero [16].

$$\begin{aligned} E_{\mathrm{Re},k} &= \frac{\mathrm{Re}\,(\omega_{\mathrm{EB},k}) - \mathrm{Re}\,(\omega_{\mathrm{TB},k})}{\mathrm{Re}\,(\omega_{\mathrm{EB},k})} \times 100\%, \\ E_{\mathrm{Im},k} &= \frac{\mathrm{Im}\,(\omega_{\mathrm{EB},k}) - \mathrm{Im}\,(\omega_{\mathrm{TB},k})}{\mathrm{Im}\,(\omega_{\mathrm{EB},k})} \times 100\%. \end{aligned} \tag{37}$$

Let $\kappa \longrightarrow \infty$, equations (30)–(36) are degenerated into the analytical expressions of the viscoelastic Euler-Bernoulli

TABLE 3: The first two eigenfrequencies of the simply-supported standard linear solid viscoelastic Timoshenko and Euler-Bernoulli beams with different viscous coefficient η when $d_1/h = 0.0001$.

η	1st				2nd			
	Re ($\omega_{TB,01}$)	Im ($\omega_{TB,01}$)	Re ($\omega_{EB,01}$)	Im ($\omega_{EB,01}$)	Re ($\omega_{TB,02}$)	Im ($\omega_{TB,02}$)	Re ($\omega_{EB,02}$)	Im ($\omega_{EB,02}$)
6.9×10^4	259.06	0.0152	259.24	0.0152	1034.19	0.2422	1036.95	0.2435
6.9×10^5	259.07	0.1520	259.24	0.1522	1034.22	2.4215	1036.98	2.4344
6.9×10^6	259.11	1.5186	259.28	1.5206	1037.11	23.941	1039.89	24.067
6.9×10^7	263.50	14.090	263.69	14.107	1158.05	80.359	1161.33	80.459
6.9×10^8	299.26	9.7928	299.46	9.7932	1202.35	10.136	1205.56	10.137
6.9×10^9	301.30	1.0156	301.50	1.0156	1202.87	1.0159	1206.08	1.0159
6.9×10^{10}	301.32	0.1016	301.52	0.1016	1202.88	0.1016	1206.09	0.1016
6.9×10^{11}	301.32	0.0102	301.52	0.0102	1202.88	0.0102	1206.09	0.0102
6.9×10^{12}	301.32	0.0010	301.52	0.0010	1202.88	0.0010	1206.09	0.0010

TABLE 4: First two frequencies of the simply-supported standard linear solid viscoelastic Timoshenko and Euler-Bernoulli beams with a single crack for different viscous coefficient η when $d_1/h = 0.4$.

η	1st				2nd			
	Re ($\omega_{TB,1}$)	Im ($\omega_{TB,1}$)	Re ($\omega_{EB,1}$)	Im ($\omega_{EB,1}$)	Re ($\omega_{TB,2}$)	Im ($\omega_{TB,2}$)	Re ($\omega_{EB,2}$)	Im ($\omega_{EB,2}$)
6.9×10^4	250.26	0.0142	250.41	0.0142	1021.22	0.2361	1023.90	0.2374
6.9×10^5	250.26	0.1418	250.41	0.1420	1021.25	2.3612	1023.92	2.3735
6.9×10^6	250.31	1.4172	250.45	1.4189	1024.03	23.351	1026.73	23.472
6.9×10^7	254.28	13.218	254.43	13.233	1142.60	79.879	1145.78	79.979
6.9×10^8	288.95	9.7668	289.13	9.7673	1187.26	10.136	1190.37	10.136
6.9×10^9	291.06	1.0156	291.23	1.0156	1187.79	1.0159	1190.90	1.0159
6.9×10^{10}	291.08	0.1016	291.26	0.1016	1187.79	0.1016	1190.90	0.1016
6.9×10^{11}	291.08	0.0102	291.26	0.0102	1187.79	0.0102	1190.90	0.0102
6.9×10^{12}	291.08	0.0010	291.26	0.0010	1187.79	0.0010	1190.90	0.0010

TABLE 5: First three frequencies of the simply-supported standard linear solid viscoelastic beam with different viscous coefficient η.

η	1st		2nd		3rd	
	Re ($\omega_{TB,01}$)	Im ($\omega_{TB,01}$)	Re ($\omega_{TB,02}$)	Im ($\omega_{TB,02}$)	Re ($\omega_{TB,03}$)	Im ($\omega_{TB,03}$)
6.9×10^4	1275.20	0.3682	4876.05	5.3834	10281.3	23.931
6.9×10^5	1275.25	3.6813	4879.09	53.698	10309.7	236.63
6.9×10^6	1280.65	36.178	5134.38	403.76	11507.9	801.35
6.9×10^7	1444.61	87.109	5660.20	100.55	11952.6	101.36
6.9×10^8	1482.76	10.144	5671.23	10.159	11957.9	10.159
6.9×10^9	1483.19	1.0160	5671.34	1.0160	11957.9	1.016
6.9×10^{10}	1483.19	0.1016	5671.35	0.1016	11957.9	0.1016
6.9×10^{11}	1483.19	0.0102	5671.35	0.0102	11957.9	0.0102
6.9×10^{12}	1483.19	0.0010	5671.35	0.0010	11957.9	0.0010

cracked beam. The coefficients $E_{\mathrm{Im},k}$ and $E_{\mathrm{Re},k}$ are defined by equation (37) to consider the effects of the transverse shear deformation and moment of inertia. And based on the SLS model and KV model, the variations of the first frequency of the simply-supported viscoelastic beams with a single crack and a row of equal cracks distributed with equal spacing for different viscous coefficient η are analyzed in Tables 7–10, respectively. It is found that, when $\eta \in 6.9 \times [$

Table 6: First three frequencies of the simply-supported Kelvin-Voigt viscoelastic beam with different viscous coefficient η.

η	1st		2nd		3rd	
	Re ($\omega_{TB,01}$)	Im ($\omega_{TB,01}$)	Re ($\omega_{TB,02}$)	Im ($\omega_{TB,02}$)	Re ($\omega_{TB,03}$)	Im ($\omega_{TB,03}$)
6.9×10^4	2496.69	5.4121	9547.07	79.131	20127	351.79
6.9×10^5	2496.30	54.121	9514.59	791.31	19821	3517.96
6.9×10^6	2437.54	541.22	4519.79	7915.27	0	64116
6.9×10^7	0	10234	0	162289	0	819881
6.9×10^8	0	144434	910274	287925	1992351	287925
6.9×10^9	248022	28792	954308	28792	2012850	28792
6.9×10^{10}	249671	2879.2	954738	2879.2	2013054	2879.2
6.9×10^{11}	249688	287.92	954742	287.92	2013056	287.92
6.9×10^{12}	249688	28.792	954742	28.792	2013056	28.792

Table 7: Variations of the first frequency of the simply-supported SLS viscoelastic beam with a single crack for different viscous coefficient η and crack location ξ_1.

η	$\xi_1 = 0.1$		$\xi_1 = 0.2$		$\xi_1 = 0.3$		$\xi_1 = 0.4$		$\xi_1 = 0.5$	
	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$
6.9×10^4	1.580	3.139	1.475	2.937	1.388	2.724	1.278	2.610	1.249	2.570
6.9×10^5	1.580	3.128	1.483	2.926	1.388	2.731	1.278	2.601	1.258	2.548
6.9×10^6	1.605	3.075	1.444	2.878	1.383	2.693	1.328	2.566	1.291	2.515
6.9×10^7	1.659	0.503	1.523	0.516	1.416	0.531	1.416	0.544	1.339	0.546
6.9×10^8	1.562	0	1.452	0.010	1.380	0.010	1.316	0	1.263	0.010
6.9×10^9	1.595	0.010	1.480	0.010	1.417	0.010	1.346	0.010	1.301	0
6.9×10^{10}	1.595	0	1.480	0	1.417	0	1.346	0	1.301	0
6.9×10^{11}	1.595	0	1.480	0	1.417	0	1.346	0	1.301	0
6.9×10^{12}	1.595	0	1.480	0	1.417	0	1.346	0	1.301	0

Table 8: Variations of the first frequency of the simply-supported SLS viscoelastic beam with cracks for different viscous coefficient η and crack number N.

η	$N = 0$		$N = 1$		$N = 2$		$N = 4$		$N = 8$	
	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$	$E_{Re,1}$	$E_{Im,1}$
6.9×10^4	1.619	3.217	1.249	2.570	1.168	2.345	1.018	2.038	0.863	1.752
6.9×10^5	1.620	3.215	1.258	2.548	1.168	2.341	1.020	2.038	0.864	1.694
6.9×10^6	1.633	3.157	1.291	2.515	1.146	2.317	1.021	2.025	0.892	1.690
6.9×10^7	1.694	0.497	1.339	0.546	1.253	0.569	1.097	0.608	0.884	0.672
6.9×10^8	1.621	0.010	1.263	0.010	1.164	0.010	1.001	0	0.870	0.010
6.9×10^9	1.619	0	1.301	0	1.214	0	1.057	0	0.835	0
6.9×10^{10}	1.620	0	1.301	0	1.214	0	1.057	0	0.835	0
6.9×10^{11}	1.620	0	1.301	0	1.214	0	1.057	0	0.835	0
6.9×10^{12}	1.620	0	1.301	0	1.214	0	1.057	0	0.835	0

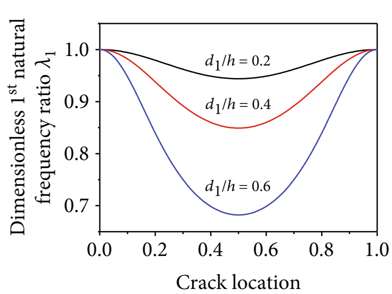
$10^4, 10^7]$GPa $\cdot$ h, the coefficient $E_{Im,1}$ of the SLS viscoelastic cracked beam decreases with the crack location ξ_1 decreasing when $\xi_1 \le 0.5$, and the crack number and viscous coefficient increasing. While $\eta \in 6.9 \times [10^8, 10^{12}]$GPa $\cdot$ h, the coefficient $E_{Im,1}$ tends to be zero. It is revealed that the viscous coefficient has little effect on the variations of the first

Table 9: Variations of the first frequency of the simply-supported KV viscoelastic beam with a single crack for different viscous coefficient η and crack location ξ_1.

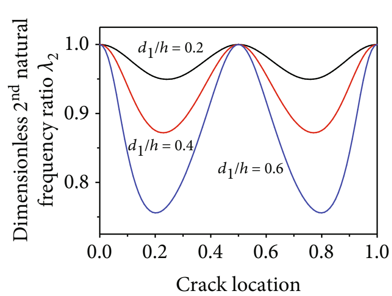
η	$\xi_1 = 0.1$		$\xi_1 = 0.2$		$\xi_1 = 0.3$		$\xi_1 = 0.4$		$\xi_1 = 0.5$	
	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$
6.9×10^4	1.582	3.129	1.481	2.926	1.373	2.733	1.310	2.599	1.289	2.548
6.9×10^5	1.578	3.129	1.473	2.926	1.373	2.733	1.310	2.599	1.285	2.548
6.9×10^6	1.501	3.129	1.410	2.926	1.324	2.733	1.259	2.601	1.237	2.548
6.9×10^7	—	3.330	—	3.141	—	2.960	—	2.837	—	2.788
6.9×10^8	—	4.579	—	3.994	—	3.531	—	3.258	—	3.164
6.9×10^9	1.598	0	1.496	0	1.398	0	1.332	0	1.306	0
6.9×10^{10}	1.577	0	1.474	0	1.376	0	1.309	0	1.283	0
6.9×10^{11}	1.577	0	1.473	0	1.376	0	1.309	0	1.282	0
6.9×10^{12}	1.577	0	1.473	0	1.376	0	1.309	0	1.282	0

Table 10: Variations of the first frequency of the simply-supported KV viscoelastic cracked beam for different viscous coefficient η and crack number N.

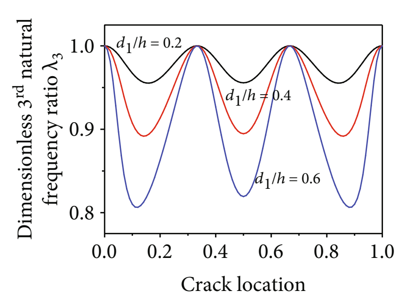
η	$N = 0$		$N = 1$		$N = 2$		$N = 4$		$N = 8$	
	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$	$E_{\mathrm{Re},1}$	$E_{\mathrm{Im},1}$
6.9×10^4	1.619	3.217	1.289	2.548	1.174	2.345	1.033	2.044	0.854	1.694
6.9×10^5	1.618	3.214	1.285	2.548	1.179	2.345	1.028	2.044	0.847	1.694
6.9×10^6	1.539	3.214	1.237	2.548	1.141	2.342	1.001	2.040	0.835	1.694
6.9×10^7	—	3.416	—	2.788	—	2.608	—	2.357	—	2.148
6.9×10^8	—	4.874	—	3.164	—	2.796	—	2.323	—	1.844
6.9×10^9	1.641	0	1.306	0	1.203	0	1.053	0	0.883	0
6.9×10^{10}	1.620	0	1.283	0	1.179	0	1.027	0	0.852	0
6.9×10^{11}	1.619	0	1.282	0	1.179	0	1.026	0	0.851	0
6.9×10^{12}	1.619	0	1.282	0	1.179	0	1.026	0	0.851	0



(a) First frequency ratio



(b) Second frequency ratio



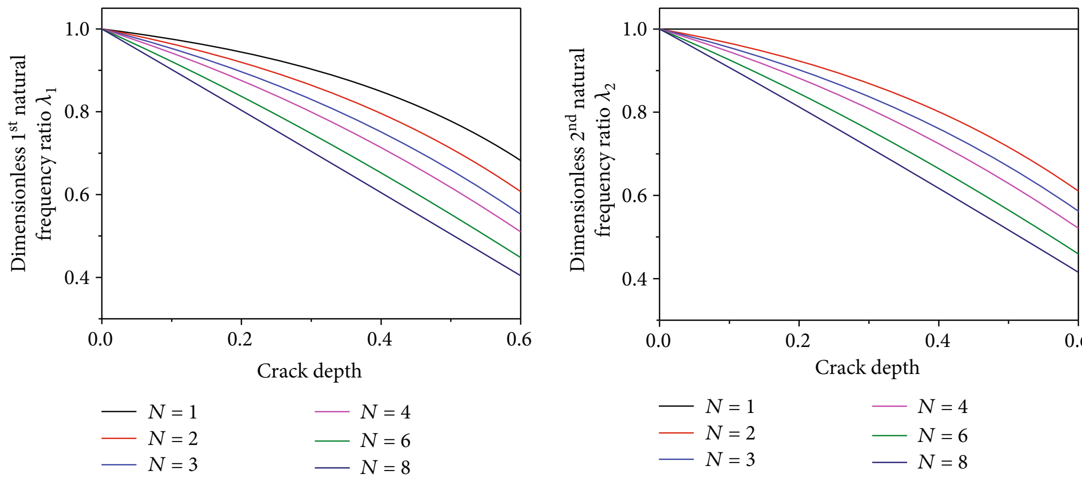
(c) Third frequency ratio

Figure 2: Variations of the first three frequency ratio versus crack location ξ_1 of the simply-supported cracked beam with different crack depth d_1/h.

frequency between the Timoshenko beam model and Euler beam model. The results of the coefficient $E_{\mathrm{Im},1}$ based on the KV model are similar with those of the SLS model.

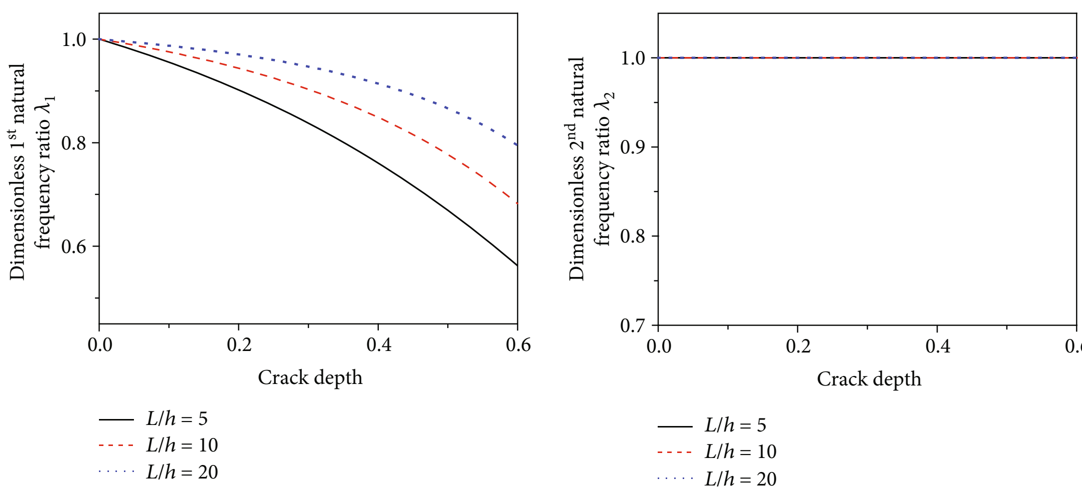
To sum up, for a higher value of η, the crack depth, crack number, and order of mode function has very less effect on the decrement coefficient Im (ω_k) of the viscoelastic beam. Therefore, the following analyses are mainly focused on the effects of crack depth, crack number, order of mode function, and slenderness ratio on the natural frequency Re (ω_k) of the viscoelastic beams.

Next, to consider the effect of a crack, it is supposed that ω_{0k} and ω_k are the k-th eigenfrequency of the viscoelastic intact and cracked beam, respectively, then the corresponding k-th natural frequency ratio is defined as $\lambda_k = \mathrm{Re}\ (\omega_k)/$



(a) First frequency ratio (b) Second frequency ratio

Figure 3: Variations of the first two frequency ratio versus crack depth d/h of the simply-supported cracked beam with different crack number N.



(a) First frequency ratio (b) Second frequency ratio

Figure 4: Variations of the first two frequency ratio versus crack depth d_1/h of the simply-supported beam with a single crack for different slenderness ratio L/h.

Re (ω_{0k}). In the case of a simple-supported Timoshenko viscoelastic beam with a single crack, the variations of the first three frequency ratio versus crack location ξ_1 of the cracked beam with different crack depth $d_1/h = 0.2$, 0.4, and 0.6 are presented in Figure 2. It is noticed that the crack location has a significant effect on the natural frequency ratio. When $\xi_1 = 0.5$ because of the midspan moment of the 2nd modal functions is null in Figure 2(b), the 2nd natural frequency ratio is $\lambda_2 = 1$. Similarly, when $\xi_1 = 1/3$ or $2/3$, the crack depth has no effect on the 3rd natural frequency ratio in Figure 2(c). It is concluded that, the k-th natural frequency ratio is independent with the crack depth when the crack is located at some critical position.

Then, to consider the effect of crack depth d/h, the variations of the first two frequency ratio of the simply-supported viscoelastic beam with the symmetrically distributed cracks N are presented in Figure 3. It is found that the first two natural frequencies decrease with the crack number N and crack depth d_j/h increasing. When $N = 1$,

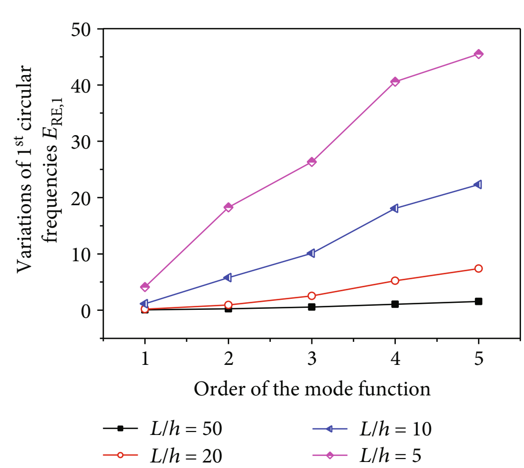


Figure 5: Variations of the first five circular frequencies of the simply-supported viscoelastic beam with a single crack for different slenderness ratio L/h.

the 2nd natural frequency ratio is independent with the crack depth in Figure 3(b).

Next, to consider the effect of slenderness ratio, the variations of the first two frequency ratio versus crack depth d_1/h of the simply-supported beam with a single crack ($\xi_1 = 0.5$) are analyzed in Figure 4. It is noticed that the 1st frequency ratio increases with the slenderness ratio increasing, while the 2nd frequency ratio is independent with the slenderness ratio.

Finally, the variations of the first five circular frequencies of the simply-supported Timoshenko viscoelastic beam with a single crack for different slenderness ratio L/h are presented in Figure 5. It is found that the variations of the first five circular frequencies between the Timoshenko and Euler cracked beam models increase with the order of mode function increasing and slenderness ratio decreasing. When $L/h = 50$, the variations of the 1st circular frequency is $E_{\text{Re},1} = 0.1\%$ and that of the 5th circular frequency is $E_{\text{Re},5} = 1.5\%$. While $L/h = 20$, 10, and 5, the corresponding variations of the 1st circular frequency are 0.2%, 1.2%, and 4.1% and those of the 5th circular frequency are 7.4%, 22.3%, and 45.5%. It is seen that the effects of the transverse shear deformation and moment of inertia on the variations of the first five circular frequencies is dependent with the slenderness ratio.

5. Conclusions

In this paper, the dynamic properties of a viscoelastic Timoshenko cracked beams based on the standard linear solid model and Kelvin-Voigt model are investigated. Based on the equivalent bending stiffness of the viscoelastic cracked beam with open cracks, the corresponding complex frequency characteristic equations of a Timoshenko viscoelastic cracked beam are obtained by using the method of separation of variables and the Laplace transform. Some conclusions arising from the numerical results can be summarized as follows:

(1) For a higher value of η, the crack depth, crack number, and order of mode function has very less effect on the decrement coefficient $\text{Im}\,(\omega_k)$ of the viscoelastic beam

(2) The transverse shear deformation and moment of inertia has less influence on dynamic characteristics of beam when the slenderness ratio is high, while the variations of the first five circular frequencies between the Timoshenko and Euler cracked beam models increase with the order of mode function increasing and slenderness ratio decreasing

(3) The first three natural frequencies of the viscoelastic cracked beam with the SLS model decrease with the crack number and crack depth increasing. And when the crack is located at some critical position, the k-th natural frequency is independent with the crack number, crack depth, and slenderness ratio

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

The author acknowledges the financial support of Nanhu Scholars Program for Young Scholars of XYNU.

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