Hindawi
Journal of Applied Mathematics
Volume 2023, Article ID 2997093, 6 pages
https://doi.org/10.1155/2023/2997093

Research Article

On Nonlocal Elliptic Problems of the Kirchhoff Type Involving the Hardy Potential and Critical Nonlinearity

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Received 21 March 2023; Revised 4 June 2023; Accepted 11 August 2023; Published 24 August 2023

Academic Editor: Sheng Zhang

In this article, we deal with the nonlocal elliptic problems of the Kirchhoff type involving the Hardy potential and critical nonlinearity on a bounded domain in R^3. Under an appropriate condition on the nonhomogeneous term and using variational methods, we obtain two distinct solutions.

1. Introduction and Main Results

Let $\Omega \subset \mathbb{R}^3$ be a regular bounded domain with a smooth boundary, α, β, λ, μ, and q positive constants. Given a function g specified later, we consider the following critical singular elliptic Kirchhoff type problem:

$$(\mathscr{P})\begin{cases} -\left[\alpha + \beta\int_\Omega \left(|\nabla u|^2 - \mu\dfrac{u}{|x|^2}\right)dx\right]\left(\Delta u - \mu\dfrac{u}{|x|^2}\right) = u^5 + \lambda\dfrac{u}{|x|^q} + g, & \text{in } \Omega, \\ u = 0, & \text{on } \dfrac{\mathbb{R}^3}{\Omega}. \end{cases} \tag{1}$$

It is pointed out that the appearance of the integral over the domain Ω implies that the equation is no longer a pointwise identity, and so the problem under consideration is nonlocal. This fact provokes some mathematical difficulties which make the study of this class of problems, particularly interesting. We refer to [1] and the references therein, for previous papers dealing with this subject. So, interested reader can find information on its historical development as well as the description of situations that can be modeled realistically, by the nonlocal problems, for example, physical and biological systems where the variable describes a process depending on the average of itself as population density. Then, the study of the solvability of nonlocal problems is motivated by its various applications.

For $\lambda = \mu = 0$, problem $(\mathscr{P})$ is related to the stationary analogue of the Kirchhoff model introduced by Kirchhoff himself [2] in 1883 as an extension of the classical D'Alembert wave equation; he take into account the changes in length of the strings produced by transverse vibrations.

Actually, despite the intense development on elliptic Kirchhoff type problems without singular term (i.e., for $\mu = \lambda = 0$), see, for example, [3] and the references therein, results on the multiplicity of solutions are still not very abundant.

In the regular case and without the Kirchhoff term, more precisely when $\alpha = \lambda = \mu = 0$, Tarantello [4] mainly imposed a suitable assumption on g and proved the existence of at least two solutions for $\beta = 1$. Benmansour and Bouchekif [5] extended the results obtained by Tarantello to the nonlocal case

$$\begin{cases} -\left[\alpha + \beta\int_\Omega |\nabla u|^2 dx\right]\Delta u = u^5 + g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^3$, α and β are positive constants, and g belongs to H^{-1}(H^{-1} is the topological dual of $H_0^1(\Omega)$) satisfying certain condition. They proved the existence of two weak solutions.

The same multiplicity result has been established by Sabri et al. in [6] when they considered under an appropriate condition on g, the following problem

$$\begin{cases} -\left[\alpha + \beta \int_\Omega |\nabla u|^2 dx\right] \Delta u = u^3 + g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a smooth bounded domain of $\mathbb{R}^4$ and α and β are positive constants.

Elliptic problems with singularity are widely studied in the literature, and there are many results dealing with this kind of problems; we refer interested readers to [7–10]. In particular, Kang and Deng [11] generalized the main result of [4] to the following singular problem:

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2_*-2}}{|x|^s} u + \lambda u + g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

with $\mu < \bar{\mu}, \bar{\mu} = (N-2)^2/4, 2_* = 2(N-s)/(N-2)$, $0 \le s < 2$, $0 < \lambda < \lambda_1(\mu)$, where $\lambda_1(\mu)$ is the first eigenvalue of the operator $L_\mu := -\Delta - \mu(1/|x|^2)$. Here, the authors imposed the presence of the term λu which provided them with the main tool to obtain the second solution if $N \ge 7$.

In this paper, we would like to consider the nonlocal elliptic singular operator and a critical inhomogeneous nonlinearity also containing a Hardy term which is different than the previous works in the literature. Our main purpose is to give a multiplicity result. To our best knowledge, this kind of problems has not been considered before.

This problem is related to the following well-known weighted Hardy inequality [11]:

$$\int_\Omega |x|^{-2} u^2 dx \le 4 \int_\Omega |\nabla u|^2 dx, \text{for all } u \in C_0^\infty(\Omega). \tag{5}$$

Let $H_\mu(\Omega)$ for $0 \le \mu < \bar{\mu} = 1/4$ be the Sobolev weighted space endowed with the norm $\|u\|_\mu^2 := \int_\Omega (|x|^{-2}|\nabla u|^2 - \mu |x|^{-2} u^2) dx$, which is equivalent to the norm $\|\cdot\|_0$. From the work of Chaudhuri and Ramaswamy [9], we know that

$$\lambda_1(\mu) = \inf_{u \in H_u(\Omega)\setminus\{0\}} \frac{\|u\|_\mu^2}{\int_\Omega u^2/|x|^q} > 0. \tag{6}$$

We denote by $\|u\|_p$ the usual L^p-norm, and $\|u\|_-$ is the norm in H_μ^{-1}, the topological dual of $H_\mu(\Omega)$.

Our main result reads as follows.

Theorem 1. *Let $\beta > 0$, $0 \le \mu < 1/4$, $0 < \lambda < b\lambda_1(\mu)$, $0 < q < (1/2)\sqrt{1-4\mu}$, and $g \in H_\mu^{-1}$ satisfy the condition*

$$(\mathscr{H}_\beta) \left| \int_\Omega gv \right| < \frac{B^{1/2}(v)}{10^{3/2}} \left[\beta \|v\|_\mu^4 B(v) + 10\left(\alpha \|v\|_\mu^2 - \lambda \int_\Omega \frac{v^2}{|x|^q}\right) - \frac{B^2(v)}{10}\right], \text{for } \|v\|_6 = 1, \tag{7}$$

where

$$B(v) = 3\beta \|v\|_\mu^4 + \left[9\beta^2 \|v\|_\mu^8 + 20\|v\|_6^6 \left(\alpha \|v\|_\mu^2 - \lambda \int_\Omega \frac{v^2}{|x|^q}\right)\right]^{1/2}, \text{for all } v \in H_\mu(\Omega). \tag{8}$$

Then, the problem $(\mathscr{P})$ admits at least two solutions in $H_\mu(\Omega)$.

This work is organized as follows: Section 2 is devoted to some preliminary results which we will use later. In Section 3, we give the definition of the Palais-Smale condition and the proof of our main result.

2. Some Preliminary Results

Seeking a weak solution to the problem $(\mathscr{P})$ is equivalent to find a critical point to the C^1 energy functional J given by

$$J(u) = \frac{\beta}{4}\|u\|_\mu^4 + \frac{\alpha}{2}\|u\|_\mu^2 - \frac{1}{6}\|u\|_6^6 - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|x|^q} - \int_\Omega gu, \text{for all } u \in H_\mu(\Omega). \tag{9}$$

It means that $u \in H_\mu(\Omega)$ is said to be a weak solution of $(\mathscr{P})$ if it satisfies

$$\left(\beta\|u\|_\mu^2 + \alpha\right)\left(\int_\Omega \nabla u \nabla v - \mu \frac{uv}{|x|^2}\right) - \int_\Omega u^5 v - \lambda \int_\Omega \frac{uv}{|x|^q} - \int_\Omega gv = 0, \text{for all } v \in H_\mu(\Omega). \tag{10}$$

Clearly, J is not bounded from below on $H_\mu(\Omega)$, so we introduce the following appropriate subset of $H_\mu(\Omega)$:

$$\mathscr{A} = \left\{u \in H_\mu(\Omega) \setminus \{0\} : \left\langle J'(u), u \right\rangle = 0\right\}, \tag{11}$$

which we split into three subsets corresponding to local minima, points of inflection, and local maxima of J, respectively (for more details, see [12]).

$$\begin{aligned} \mathscr{A}^+ &:= \{u \in \mathscr{A} : L(u) > 0\}, \\ \mathscr{A}^0 &:= \{u \in \mathscr{A} : L(u) = 0\}, \\ \mathscr{A}^- &:= \{u \in \mathscr{A} : L(u) < 0\}, \end{aligned} \tag{12}$$

where $L(u) = 3\beta\|u\|_\mu^4 + \alpha\|u\|_\mu^2 - 5\|u\|_6^6 - \lambda\int_\Omega u^2/|x|^q$.

Let $f_u(t) = J(tu)$ for $t \in \mathbb{R}^*$ and $u \in H_\mu(\Omega) \setminus \{0\}$. Put $f'_u(t) = F_u(t) - \int gu$ where

$$F_u(t) = \beta\|u\|_\mu^4 t^3 + \left(\alpha\|u\|_\mu^2 - \lambda\int\frac{u^2}{|x|^q}\right)t - \|u\|_6^6 t^5. \quad (13)$$

The function $F_u(t)$ attains its maximum $A_\beta(u)$ at the point $t^u_{\max}$ where

$$\widetilde{A_\beta}(u) := \frac{B^{1/2}(u)}{10^{1/2}\|u\|_6^3}\left[\frac{\beta\|u\|_\mu^4 B(u)}{10\|u\|_6^6} + \left(\alpha\|u\|_\mu^2 - \lambda\int_\Omega\frac{u^2}{|x|^q}\right) - \frac{B^2(u)}{10^2\|u\|_6^6}\right], \quad (14)$$

with

$$B(v) = 3\beta\|v\|_\mu^4 + \left[9\beta^2\|v\|_\mu^8 + 20\|v\|_6^6\left(\alpha\|v\|_\mu^2 - \lambda\int_\Omega\frac{v^2}{|x|^q}\right)\right]^{1/2} \text{for all } v \in H_\mu(\Omega),$$

$$t^u_{\max} = 10^{-1/2}\left(3\beta\|u\|_\mu^4 + \left[9\beta^2\|u\|_\mu^8 + 20\|u\|_6^6\left(\alpha\|u\|_\mu^2 - \lambda\int_\Omega\frac{u^2}{|x|^q}\right)\right]^{1/2}\right)^{1/2}\|u\|_6^{-3}. \quad (15)$$

The following lemmas play crucial roles in the sequel of this work.

Lemma 2. *The functional J is coercive and bounded below on $\mathcal{A}$.*

Proof. We know that $\beta\|u\|_\mu^4 + \alpha\|u\|_\mu^2 = \|u\|_6^6 + \lambda\int_\Omega(u^2/|x|^q) + \lambda\int_\Omega gu$ (since $u \in \mathcal{A}$). Therefore, from the Hardy inequality, we get

$$\begin{aligned} J(u) &= \frac{\beta}{12}\|u\|_\mu^4 + \frac{\alpha}{3}\|u\|_\mu^2 - \frac{\lambda}{3}\int_\Omega\frac{u^2}{|x|^q} - \frac{5}{6}\int_\Omega gu \\ &\ge \frac{\alpha-4\lambda}{3}\|u\|_\mu^2 - \frac{5}{6}\|g\|_-\|u\|_\mu, \\ &\ge \frac{-25}{48(\alpha-4\lambda)}\|g\|_-^2, \end{aligned} \quad (16)$$

in particular, $c_0 \ge (-25/48(\alpha-4\lambda))\|g\|_-^2$, where $c_0 = \inf_{u\in\mathcal{A}} J(u)$.

Thus, J is coercive and bounded from below on $\mathcal{A}$. □

Lemma 3. *Under the condition ($\mathcal{H}_\beta$) and for all $u \in H_\mu(\Omega) \setminus \{0\}$, there exist unique $t_1^+ = t_1^+(u)$, $t^- = t^-(u) \neq 0$, and $t_2^+ = t_2^+(u)$ such that $t_1^+ < -t^u_{\max} < t^- < t^u_{\max} < t_2^+$, t_1^+u, $t_2^+u \in \mathcal{A}^-$, $t^-u \in \mathcal{A}^+$, $J(t_1^+u) = \max_{t\le -t^u_{\max}} J(tu)$, $J(t^-u) = \min_{|t|\le t^u_{\max}} J(tu)$, and $J(t_2^+u) = \max_{t\ge t^u_{\max}} J(tu)$.*

Proof. If ($\mathcal{H}_\beta$) is verified then, there exist unique $t_1^+ = t_1^+(u)$, $t^- = t^-(u)$, $t_2^+ = t_2^+(u)$, and $t_{1^+} < -t^u_{\max} < t^- < t^u_{\max} < t_2^+$ such that $F_u(t_1^+) = {}_\Omega gu$ and $F'_u(t_1^+) < 0$, which implies that $t_1^+u \in \mathcal{A}^-$ and $J(t_1^+u) \ge J(tu)$, for all $t \le -t^u_{\max}$; $F_u(t^-) = \int_\Omega gu$ and $F'_u(t^-) > 0$, which implies that $t^-u \in \mathcal{A}^+$ and $J(t^-u) \le J(tu)$, for all $|t| \le t^u_{\max}$; and $F_u(t_2^+) = \int_\Omega gu$ and $F'_u(t_2^+) < 0$, which implies that $t_2^+u \in \mathcal{A}^-$ and $J(t_2^+u) \ge J(tu)$, for all $t \ge t^u_{\max}$. □

For $0 \le \mu < 1/4$, we know that the best constant

$$S_\mu := \inf_{u\in H_\mu\setminus\{0\}}\frac{\|u\|_\mu^2}{\left(\int_\Omega u^6 dx\right)^{1/3}} \quad (17)$$

is attained when $\Omega = \mathbb{R}^3$ by the functions

$$u_\varepsilon(x) = \frac{(3\varepsilon(1-4\mu))^{1/4}}{\left(\varepsilon|x|^{\left(1-\sqrt{1-4\mu}\right)} + |x|^{\left(1+\sqrt{1-4\mu}\right)}\right)^{1/2}}, \quad (18)$$

(see [13, 14]). Let R be a positive constant and set $\phi \in C_0^\infty(\Omega)$ such that $0 \le \phi(x) \le 1$ for $|x| \le R$ and $\phi(x) \equiv 1$ for $|x| \le R/2$ and $B_R(0) \subset \Omega$. Set $U_\varepsilon(x) = \phi(x)u_\varepsilon(x)$ and $v_\varepsilon = U_\varepsilon(x)/\left(\int_\Omega(U_\varepsilon(x))^6\right)^{1/6}$; then, by [15], we have the following lemma.

Lemma 4. *Let u be a solution of ($\mathcal{P}$); then, we get*

$$\begin{aligned} &\|v_\varepsilon\|_\mu^2 = S_\mu + O\left(\varepsilon^{1/2}\right), \int_\Omega\frac{|v_\varepsilon|^2}{|x|^q}dx \\ &\quad = O\left(\varepsilon^{q/\left(2\sqrt{1-4\mu}\right)}\right) \text{ when } 0 < q < \sqrt{1-4\mu}, \\ &\int_\Omega v_\varepsilon\, dx = O\left(\varepsilon^{1/4}\right), \\ &\int_\Omega |v_\varepsilon|^5 u\, dx = O\left(\varepsilon^{1/4}\right), \\ &\int_\Omega |u|^5 v_\varepsilon dx = O\left(\varepsilon^{1/4}\right). \end{aligned} \quad (19)$$

Lemma 5. *Let $0 < q < (1/2)\sqrt{1-4\mu}$ and assume that the condition ($\mathcal{H}_\beta$) is verified; then, for ε_0 small enough, we have*

$$\sup_{t\ge 0} J(tv_\varepsilon) < c^*, \quad (20)$$

where

$$c^* = \frac{\alpha\beta}{4}S_\mu^3 + \frac{\beta^3}{24}S_\mu^6 + \left(\beta^2 S_\mu^4 + 4\alpha S_\mu\right)^{1/2}\left[\frac{\alpha}{6}S_\mu + \frac{\beta^2}{24}S_\mu^4\right], \quad (21)$$

for every $0 < \varepsilon < \varepsilon_0$.

3. Proof of Theorem 1

3.1. Existence of Solution in $\mathcal{A}^+$

Proposition 6. *If ($\mathcal{H}_\beta$) holds, then $c_0 < 0$, and there is a critical point u_0 of J such that $J(u_0) = c_0 = \inf_{u\in\mathcal{A}} J(u)$ and u_0 is a local minimizer for J.*

Proof. The proof is exactly the same as the one given in the proof of Theorem 1.1 in [5]. We omit the details here. □

3.2. Existence of Solution in $\mathscr{A}^-$. In this part, we prove the existence of a second solution u_1 such that $J(u_1) = c_1 = \inf_{u\in\mathscr{A}^-} J(u)$.

Proposition 7. *If $(\mathscr{H}_\beta)$ holds, then $0 < c_1 < c_*$, and there is a critical point u_1 of J such that $J(u_1) = c_1$.*

Due to the presence of the critical Sobolev exponent, a loss of compactness occurs, so we need the concept of the Palais-Smale condition.

A sequence (u_n) is said to be a Palais-Smale sequence at level c ($(\mathrm{P}-\mathrm{S})_c$ in short) for J in $H_\mu(\Omega)$ if

$$J(u_n) = c + o_n(1), J'(u_n) = o_n(1) \text{ in } H_\mu^{-1}. \quad (22)$$

When we say that a functional J verifies $(\mathrm{P}-\mathrm{S})$ condition at level c, we mean that any $(\mathrm{P}-\mathrm{S})_c$ sequence for J has a convergent subsequence in $H_\mu(\Omega)$.

In order to obtain the second solution, we give the following important lemma.

Lemma 8. *Let g verifying $(\mathscr{H}_\beta)$, then J satisfies the $(P-S)_c$ condition for*

$$c < c^* = \frac{\alpha\beta}{4} S_\mu^3 + \frac{\beta^3}{24} S_\mu^6 + \frac{\alpha}{6} S_\mu E_1 + \frac{\beta^2}{24} S_\mu^4 E_1 + c_0, \quad (23)$$

where $E_1 = (\beta^2 S_\mu^4 + 4\alpha S_\mu)^{1/2}$.

Proof. Let (u_n) be a $(P-S)_c$ sequence with $c < c^*$; then, (u_n) is a bounded sequence in $H_\mu(\Omega)$. Thus, by the compact embedding theorem, it has a subsequence still denoted (u_n) such that $u_n \longrightarrow u$ weakly in $H_\mu(\Omega)$, $u_n \longrightarrow u$ a.e in Ω, $u_n/|x|^{q/2} \longrightarrow u/|x|^{q/2}$ in $L^2(\Omega)$, and $u_n \longrightarrow u$ in $L^s(\Omega)$ for all $1 \le s < 6$.

Let $w_n = u_n - u$. It follows that $w_n \longrightarrow 0$ weakly in $H_\mu(\Omega)$

$$\begin{aligned} \|u_n\|_\mu^2 &= \|w_n + u\|_\mu^2 = \|w_n\|_\mu^2 + \|u\|_\mu^2 + o_n(1), \\ \|u_n\|_\mu^4 &= \|w_n\|_\mu^4 + 2\|w_n\|_\mu^2\|u\|_\mu^2 + \|u\|_\mu^4 + o_n(1), \end{aligned} \quad (24)$$

and from the Brézis-Lieb lemma [16], we obtain

$$\|u_n\|_6^6 = \|w_n\|_6^6 + \|u\|_6^6 + o_n(1). \quad (25)$$

Since $J(u_n) = c + o_n(1)$, we get

$$\begin{aligned} &\frac{\beta}{4}\|w_n\|_\mu^4 + \frac{\alpha}{2}\|w_n\|_\mu^2 + \frac{\beta}{2}\|w_n\|_\mu^2\|u\|_\mu^2 - \frac{1}{6}\|w_n\|_6^6 \\ &\quad = J(u_n) - J(u) = c - J(u) + o_n(1). \end{aligned} \quad (26)$$

By the fact that $J'(u_n) = o_n(1)$ and $\langle J'(u), u\rangle = 0$, we have

$$\begin{aligned} \left\langle J'(u_n), u_n \right\rangle &= \beta\|u_n\|_\mu^4 + \alpha\|u_n\|_\mu^2 - \|u_n\|_6^6 - \int_\Omega g u_n \\ &= \beta\left(\|w_n\|_\mu^4 + 2\|w_n\|_\mu^2\|u\|_\mu^2 + \|u\|_\mu^4\right) \\ &\quad + \alpha\left(\|w_n\|_\mu^2 + +\|u\|_\mu^2\right) - \|w_n\|_6^6 - \|u\|_6^6 - \int_\Omega gu \\ &= \beta\|w_n\|_\mu^4 + \alpha\|w_n\|_\mu^2 + 2\beta\|w_n\|_\mu^2\|u\|_\mu^2 \\ &\quad - \|w_n\|_6^6 + \left\langle J'(u), u \right\rangle = o_n(1). \end{aligned} \quad (27)$$

So, we get

$$\beta\|w_n\|_\mu^4 + \alpha\|w_n\|_\mu^2 + 2\beta\|w_n\|_\mu^2\|u\|_\mu^2 - \|w_n\|_6^6 = o_n(1). \quad (28)$$

Assume that $\|w_n\|_\mu \longrightarrow l$ with $l > 0$, it follows that

$$\|w_n\|_6^6 = \beta l^4 + \alpha l^2 + 2\beta l^2\|u\|_\mu^2 \ge \beta l^4 + \alpha l^2 + 2\beta l^2\|u\|_\mu^2. \quad (29)$$

From the definition of S_μ, we lead to

$$\|w_n\|_\mu^2 \ge S_\mu\|w_n\|_6^2, \text{ for all } n. \quad (30)$$

As $n \longrightarrow +\infty$, we deduce that

$$l^2 \ge \frac{\beta}{2} S_\mu^3 + \frac{1}{2} S_\mu\left(\beta^2 S_\mu^4 + 4S_\mu\left(\alpha + 2\beta\|u\|_\mu^2\right)\right)^{1/2}. \quad (31)$$

Consequently, we obtain

$$\begin{aligned} c &= \frac{\beta}{12} l^4 + \frac{\alpha}{3} l^2 + \frac{\beta}{6} l^2\|u\|^2 + J(u) \ge \frac{\beta}{12} l^4 + \frac{\alpha}{3} l^2 + c_0 \\ &\ge \frac{\alpha\beta}{4} S_\mu^3 + \frac{\beta^3}{24} S_\mu^6 + \frac{b}{6} S_\mu E_1 + \frac{\beta^2}{24} S_\mu^4 E_1 + c_0 = c^*, \end{aligned} \quad (32)$$

which is a contradiction. Therefore, $l = 0$; then, $u_n \longrightarrow u$ strongly in $H_\mu(\Omega)$. □

In the search of our second solution, it is natural to show that $c_1 = \inf_{u\in\mathscr{A}^-} J(u) < c^*$. Set $\Gamma = \{k : [0,1] \longrightarrow H_\mu(\Omega)$ continuous, $k(0) = u_0, k(1) = u_0 + t_0 u_\varepsilon\}$.It is obvious that $k : [0,1] \longrightarrow H_\mu(\Omega)$ given by $k(t) = u_0 + t(t_0 u_\varepsilon)$ belongs to Γ.

Lemma 9. *Let $0 < q < (1/2)\sqrt{1-4\mu}$ and assume that the condition $(\mathscr{H}_\beta)$ is verified for β a small enough positive number. Then, for every $t > 0$, there exists $\varepsilon_0 = \varepsilon_0(t,\beta)$ such that $J(u_0 + tu_\varepsilon) < c^*$ for every $0 < \varepsilon < \varepsilon_0$.*

Proof. Let us consider the functional $I_1 : H_\mu \longrightarrow \mathbb{R}$ defined by

$$I_1(u) = \frac{\beta}{4}\|u\|_\mu^4 + \frac{\alpha}{2}\|u\|_\mu^2 - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|x|^q} - \frac{1}{6}\|u\|_6^6. \quad (33)$$

We know that $\lambda < b\lambda_1(\mu)$, $\lim_{t\longrightarrow+\infty} I_1(tu_\varepsilon) = -\infty$,$I(0) = 0$, and $I_1(tu_\varepsilon) > 0$ for t near 0^+, so $\sup_{t\geq 0} I_1(tu_\varepsilon) = I_1(t_\varepsilon u_\varepsilon)$, where

$$t_\varepsilon = \left[\frac{1}{2}\left(\beta\|u_\varepsilon\|_\mu^4 + \left(\beta^2\|u_\varepsilon\|_\mu^8 + 4\alpha\|u_\varepsilon\|_\mu^6 - \lambda\int_\Omega \frac{u_\varepsilon^2}{|x|^q}\right)^{1/2}\right)\right]^{1/2}. \quad (34)$$

From Lemma 4 and as $0 < q < \sqrt{1-4\mu}$, we deduce that

$$\begin{aligned}\sup_{t\geq 0} I_1(tu_\varepsilon) = {} & \frac{\beta\alpha}{4}S_\mu^3 + O\left(\varepsilon^{1/2}\right) + \frac{\beta^3}{24}S_\mu^6 + O\left(\varepsilon^{1/2}\right) \\ & + \frac{\alpha}{6}S_\mu\left(\beta^2 S_\mu^4 + 4\alpha S_\mu\right)^{1/2} + O\left(\varepsilon^{1/2}\right) \\ & + \frac{\beta^2}{24}S_\mu^4\left(\beta^2 S_\mu^4 + 4\alpha S_\mu\right)^{1/2} + O\left(\varepsilon^{1/2}\right) \\ & - O\left(\varepsilon^{q/\left(2\sqrt{1-4\mu}\right)}\right) \leq c^* + O\left(\varepsilon^{1/2}\right) \\ & - O\left(\varepsilon^{q/\left(2\sqrt{1-4\mu}\right)}\right).\end{aligned} \quad (35)$$

On the other hand, we have

$$J(u_0 + tu_\varepsilon) = J(u_0) + I_1(tu_\varepsilon) + I_2(tu_\varepsilon), \quad (36)$$

where

$$\begin{aligned}I_2(tu_\varepsilon) = {} & \frac{\beta t^2}{4}\left[4\left(\int_\Omega \nabla u_0 \nabla u_\varepsilon\right)^2 + 2\|u_0\|^2\|u_\varepsilon\|^2 + 4t\|u_\varepsilon\|^2\int_\Omega \nabla u_0\nabla u_\varepsilon\right] \\ & + O\left(\varepsilon^{1/4}\right).\end{aligned} \quad (37)$$

Then, for β small enough, there exists $t^*_{\beta,\varepsilon} > 0$ also small enough such that $I_2(t) < O(\varepsilon^{1/4})$ for all $t > t^*_{\beta,\varepsilon}$.

Therefore, as $0 < q < (1/2)\sqrt{1-4\mu}$, there exists ε_0 small enough such that we get

$$J(u_0 + tu_\varepsilon) < c^*, \text{ for every } 0 < \varepsilon < \varepsilon_0. \quad (38)$$

□

As in [4], we remark that under the condition $(\mathscr{H}_\beta)$, the manifold $\mathscr{A}^-$ disconnects $H_\mu(\Omega)$ in exactly two components V_1 and V_2. More precisely, $H_\mu(\Omega)\setminus\mathscr{A}^- = V_1 \cup V_2$ and $\mathscr{A}^+ \subset V_1$, where

$$\begin{aligned}V_1 &= \{0\} \cup \left\{\frac{u}{\|u\|_\mu} < t^+\left(\frac{u}{\|u\|_\mu}\right)\right\}, \\ V_2 &= \left\{\frac{u}{\|u\|_\mu} > t^+\left(\frac{u}{\|u\|_\mu}\right)\right\}.\end{aligned} \quad (39)$$

In particular, $u_0 \in V_1$.

From a direct computation, we deduce that

$$0 < t^+(u) < C_1, \text{ for all } u \text{ such that } \|u\|_\mu = 1. \quad (40)$$

Therefore, for $t_0 > 0$ sufficiently large, the estimate (38) holds for all $0 < \varepsilon < \varepsilon_0$.

Thus, we derive that $H_\mu(\Omega)$

$$\|u_0 + t_0 u_\varepsilon\|_\mu^2 > C_1^2 \geq \left[t^+\left(\frac{u_0 + t_0 u_\varepsilon}{\|u_0 + t_0 u_\varepsilon\|}\right)\right]^2, \text{ for all } \varepsilon > 0 \text{ small enough.} \quad (41)$$

Set $\Gamma = \{k : [0,1] \longrightarrow H_\mu(\Omega) \text{continuous}, k(0) = u_0, k(1) = u_0 + t_0 u_\varepsilon\}$.

It is obvious that $k : [0,1] \longrightarrow H_\mu(\Omega)$ given by $k(t) = u_0 + t(t_0 u_\varepsilon)$ belongs to Γ. We conclude that

$$c = \inf_{k\in\Gamma}\max_{t\in[0,1]} J(k(t)) < c^*. \quad (42)$$

As the range of any $k \in \Gamma$ intersects $\mathscr{A}^-$, one has

$$c \geq c_1 = \inf_{\mathscr{A}^-} J, \quad (43)$$

from this and applying another time the Ekeland variational principle, we can obtain a minimizing sequence $(u_n) \subset \mathscr{N}^-$ such that

$$\begin{aligned}J(u_n) &\longrightarrow c_1, \\ \left\|J'(u_n)\right\| &\longrightarrow 0.\end{aligned} \quad (44)$$

We also deduce that $c_1 < c^*$.

Consequently, we obtain a subsequence (u_{n_k}) of (u_n) and $u_1 \in H_\mu(\Omega)$ such that

$$u_{n_k} \longrightarrow u_1 \text{ strongly in } H_\mu(\Omega). \quad (45)$$

This implies that u_1 is a critical point for J,$u_1 \in \mathscr{A}^-$, and $J(u_1) = c_1$.

Therefore, according to Proposition 6 and Proposition 7, the proof of Theorem 1 is achieved.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by Qassim University, represented by the Deanship of Scientific Research and Algerian Ministry of Higher Education and Scientific Research.

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