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Research Article

The Consecutive Substitution Method for Boundary Value Problems (BVPs) with Retarded Argument

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In this study, we applied an approximate solution method for solving the boundary value problems (BVPs) with retarded argument. The method is the consecutive substitution method. The consecutive substitution method was applied and an approximate solution was obtained. The numerical solution and the analytical solution are compared in the table. The solutions were found to be compatible.

1. Introduction

Boundary value problems (BVPs) with retarded arguments are

$$\mathbf{x}''(\mathbf{t}) + a(t)x(t-\tau(t)) = f(t), (0 \le t \le T), \tag{1}$$

$$x(t) = \varphi(t)\,(\lambda_0 \le t \le 0), x(T) = x(c)(0 \le c \le T), \tag{2}$$

where $a(t) \ge 0$, $f(t) \ge 0$, $\varphi(t)\,(\lambda_0 \le t \le 0)$, and $\tau(t) \ge 0$ $(0 \le t \le T)$ are given continuous functions. Differential equations with retarded arguments originated there in the 18th century. Until this century, there were no studies on differential equations with retarded arguments. Studies on differential equations with retarded arguments began after this century, at this time initial value problems do not have an exact formula. It was first made in the thesis of [1]. Then, the approximation method was used successfully and discussed in [2]. The integral equation method is the most widely used method for the analytical solution of the BVPs [3, 4]. For the boundary condition $x(T) = x_T$ in problem (1) studied for different values of $\tau(t)$ in [5–7]. Problems in the thesis [5] were solved with the ordinary successive approximation method, the modified two-sided approximations method, and the modified successive approximations method, and then converted to Padé approximations and compared in [8–10]. The results of the successive approximation method and modified successive approximation method were compared in [11]. In addition, the solution of BVP for the arbitrary continuous function $\tau(t)$ in problem (1) was investigated under the conditions specified in [12] and approximately calculated by the CAS Wavelet method in [13]. For problem (1), we applied the sequential substitution method. With this method, we obtained an integral equation equivalent to BVP (1), and the solution of this integral equation is equivalent to the solution of BVP. The equivalent integral equation is usually a Fredholm integral equation. In this study, we obtained a Fredholm-Volterra integral equation for problem (1).

2. An Equivalent Integral Equation

In problem (1), if we take $\lambda(t) = t - \tau(t)$, then $t_0 \in [0, T]$ is a point located at the left side of T such that conditions $\lambda(t_0) = 0$ and $\lambda(t) \le 0\,(0 \le t \le t_0)$ are satisfied.

Where $\lambda_0 = \min_{0 \le t \le t_0} \lambda(t)$, let us assume that $\lambda(t)$ is a non-decreasing function in the interval $[t_0, T]$ and the equation $\lambda(t) = \sigma$ has a continuously differentiable $t = \gamma(\sigma)$ solution for arbitrary $[0, \lambda(t)]$. If $x^*(t)$ is the solution to BVP (1), then it turns out that $x^*(t)$ is also the solution to equation

$$x^*(t) = \widehat{h}(t) + t\int_0^T \frac{T-s}{T-c} a(s)x(s-\tau(s))\mathrm{ds} - t\int_0^c \frac{c-s}{T-c} a(s)x(s-\tau(s))ds - \int_0^t (t-s)a(s)x(s-\tau(s))\mathrm{ds}. \tag{3}$$

Here,

$$\widehat{h}(t) = \varphi(0) - t\int_0^T \frac{T-s}{T-c} f(s)\mathrm{ds} + t\int_0^c \frac{c-s}{T-c} f(s)ds + \int_0^t (t-s)f(s)\mathrm{ds}. \tag{4}$$

Let $\sigma = s - \tau(s)$. So equation (3) can be written as

$$x^*(t) = h(t) + t\int_0^{\lambda(T)} \frac{T-\gamma(\sigma)}{T-c} a(\gamma(\sigma))\gamma'(\sigma)x(\sigma)d\sigma - t\int_0^{\lambda(c)} \frac{c-\gamma(\sigma)}{T-c} a(\gamma(\sigma))\gamma'(\sigma)x(\sigma)d\sigma - \int_0^{\lambda(t)} (t-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)x(\sigma)d\sigma, \tag{5}$$

where

$$h(t) = \widehat{h}(t) + \int_{\lambda_0}^0 \frac{T-\gamma(\sigma)}{T-c} a(\gamma(\sigma))\gamma'(\sigma)\varphi(\sigma)d\sigma - \int_{\lambda_0}^0 \frac{c-\gamma(\sigma)}{T-c} a(\gamma(\sigma))\gamma'(\sigma)\varphi(\sigma)d\sigma - \int_{\lambda_0}^0 (t-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)\varphi(\sigma)d\sigma. \tag{6}$$

Let

$$\begin{aligned} K_1(\sigma) &= \frac{T-\gamma(\sigma)}{T-c}\, a(\gamma(\sigma))\, a(\gamma(\sigma))\gamma'(\sigma), \\ K_2(\sigma) &= -\frac{c-\gamma(\sigma)}{T-c}\, a(\gamma(\sigma))\, a(\gamma(\sigma))\gamma'(\sigma), \\ K(t,\sigma) &= -(t-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma), \\ x^*(t) &= h(t) + t\int_0^{\lambda(T)} K_1(\sigma)x^*(\sigma)d\sigma + t\int_0^{\lambda(c)} K_2(\sigma)x^*(\sigma)d\sigma + \int_0^{\lambda(t)} K(t,\sigma)x^*(\sigma)d\sigma, \end{aligned} \tag{7}$$

or

$$x(t) = h(t) + tF_\lambda^T x + tF_\lambda^c x + V_\lambda x, \tag{8}$$

where

$$F_\lambda^T x \equiv \int_0^{\lambda(T)} K_1(\sigma)x(\sigma)d\sigma, \quad F_\lambda^c x \equiv \int_0^{\lambda(T)} K_2(\sigma)x(\sigma)d\sigma \tag{9}$$

is the Fredholm operator, and

$$V_\lambda x \equiv \int_0^{\lambda(t)} K(t,\sigma)x(\sigma)d(\sigma) \tag{10}$$

is the Volterra operator. Problem (1) is equivalent to equation (8); it is a Fredholm-Volterra integral equation.

3. The Consecutive Substitution Method

In equation (8), if we replace x in the $V_\lambda x$ operator with the term on the right side of equation (8), then we obtain

$$x(t) = h(t) + V_\lambda h(t) + (t + V_\lambda t)F_\lambda^T x + (t + V_\lambda t)F_\lambda^c x + V_\lambda^2 x. \tag{11}$$

In equation (11), if we replace x in the $V_\lambda^2 x$ operator with the term on the right side of equation (8), then we obtain

$$x(t) = h(t) + V_\lambda h + V_\lambda^2 h + \left(t + V_\lambda t + V_l^2 t\right)F_\lambda^T x + \left(t + V_\lambda + V_\lambda^2 t\right)F_\lambda^c x + V_\lambda^3 x. \tag{12}$$

If this process is repeated n times, we have

$$x(t) = \sum_{i=0}^n V_\lambda^i h + \left(\sum_{i=0}^n V_\lambda^i t\right)F_\lambda^T x + \left(\sum_{i=0}^n V_\lambda^i t\right)F_\lambda^c x + V_\lambda^{n+1} x. \tag{13}$$

If we choose $h_n(t) = \sum_{i=0}^n V_\lambda^i h$ and $a_n(t) = \sum_{i=0}^n V_\lambda^i t$

$$x(t) = h_n(t) + a_n(t)\left(F_\lambda^T x + F_\lambda^c x\right) + V_\lambda^{n+1} x. \tag{14}$$

Now, we can prove that the formula is correct

$$|V_\lambda^n x| \le \frac{[K\lambda(T)]^n}{n!}\ (n \in N), \tag{15}$$

for the operator

$$V_\lambda^n x \equiv \int_0^{\lambda(t)} K(t,\sigma)x(\sigma)d(\sigma)(0 \le t \le T). \tag{16}$$

Consequently, for n which is large sufficient, we neglect the $|V_\lambda^n x|$ operator in equation (14). Then, the consecutive approximations are created by taking into account the Volterra operator.

$$x_n(t) = h_n(t) + a_n(t)\left(F_\lambda^T x_n + F_\lambda^c x_n\right). \tag{17}$$

Theorem 1. *Let $\tau(t) \geq 0, f(t)\ (0 \leq t \leq T)$ and $a(t)$ be known functions in problem (1) and $\alpha_n = 1 - F_\lambda^T a_n - F_\lambda^c a_n \neq 0$ also $\lim_{n\longrightarrow\infty} A_n([K\lambda(T)]^n/n!) = 0$ such that*

$$A_n = 1 + \frac{\|a_n(t)\|}{|\alpha_n|}\left(\int_0^{\lambda(T)} K_1(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)d\sigma\right). \tag{18}$$

Thus, the limit of approximations is

$$x_n(t) = h_n(t) + \frac{a_n(t)}{\alpha_n}\left(\int_0^{\lambda(T)} K_1(\sigma)h_n(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)h_n(\sigma)d\sigma\right). \tag{19}$$

This converges to the solution of problem (1), and the convergence speed is

$$|x_n(t) - x(t)| \leq A_n \frac{[K\lambda(T)]^n}{n!}. \tag{20}$$

Proof. Equation (17) with the degenerate kernel is Fredholm integral equation. The solution of

$$x_n(t) = h_n(t) + a_n(t)\left(\int_0^{\lambda(T)} K_1(\sigma)h_n(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)h_n(\sigma)d\sigma\right) \tag{21}$$

is the same as the solution of equation (14) and problem (1). Then, let us try to find the solution to equation (21).

For this, we use assistant equation $y(t) = h_n(t) + a_n(t)(F_\lambda^T y + F_\lambda^c y)$ where $c_n = F_\lambda^T y + F_\lambda^c y$. Thus $y(t)$ is like that

$$y(t) = h_n(t) + a_n(t)c_n. \tag{22}$$

Therefore,

$$c_n = \int_0^{\lambda(T)} K_1(\sigma)h_n(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)h_n(\sigma)d\sigma + c_n\left(\int_0^{\lambda(T)} K_1(\sigma)a_n(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)a_n(\sigma)d\sigma\right). \tag{23}$$

When $\alpha_n = 1 - F_\lambda^T a_n - F_\lambda^c a_n \neq 0$ is given, c_n is in the form of

$$c_n = \frac{1}{\alpha_n}\left(\int_0^{\lambda(T)} K_1(\sigma)h_n(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)h_n(\sigma)d\sigma\right). \tag{24}$$

If we use equation (24) in (22) for $n = 1, 2, \cdots$ then

$$x_n(t) = h_n(t) + \frac{a_n(t)}{\alpha_n}\left(\int_0^{\lambda(T)} K_1(\sigma)h_n(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)h_n(\sigma)d\sigma\right). \tag{25}$$

We obtained the approximate solution to problem (1) with these operations. Thus, the limit of $x_n(t)$ converges to the solution of the problem (1). Now, let us determine the error of equation (25), which is the approximate solution to problem (1). Using equation (14) and equation (17), we reached

$$x - x_n = a_n(t)\left(F_\lambda^T(x - x_n) + F_\lambda^c(x - x_n)\right) + V_\lambda^{n+1}x. \tag{26}$$

Assuming ϵ is $\epsilon = x - x_n$, then we get the Fredholm integral equation with a degenerated kernel

$$\epsilon = a_n(t)\left(F_\lambda^T\epsilon + F_\lambda^c\epsilon\right) + V_\lambda^{n+1}x. \tag{27}$$

It has been proven that the solution to equation (27) is found using the following formula:

$$\epsilon = V_\lambda^{n+1}x + \frac{a_n(t)}{\alpha_n}\left(\int_0^{\lambda(T)} K_1(\sigma)V_\lambda^{n+1}x(\sigma)d\sigma + \int_0^{\lambda(c)} K_2(\sigma)V_\lambda^{n+1}x(\sigma)d\sigma\right). \tag{28}$$

Thus, we write

$$|\epsilon| \leq \left[1 + \frac{\|a_n\|}{|\alpha_n|}\left(\int_0^{\lambda(T)} |K_1(\sigma)|d\sigma + \int_0^{\lambda(c)} |K_2(\sigma)|d\sigma\right)\right] \cdot \left\|V_\lambda^{n+1}x(\sigma)\right\|. \tag{29}$$

Then, by the hypothesis, $A_n = 1 + (\|a_n\|/|\alpha_n|)(\int_0^{\lambda(T)}|K_1(\sigma)|d\sigma + \int_0^{\lambda(c)}|K_2(\sigma)|d\sigma)$ and we have

$$|x_n(t) - x(t)| \leq A_n \frac{[K\lambda(T)]^n}{n!}\|x\|. \tag{30}$$

□

Example. Let us consider BVP

$$\mathbf{x}''(\mathbf{t}) + tx\left(t - \frac{1}{2}\sqrt{t}\right) = -2t^3 + 2t^{5/2} + \frac{5}{2}t^2 - \frac{3}{2}t^{3/2} - 4\ (0 \leq t \leq 1), \tag{31}$$

$$x(t) = 0(-1/16 \leq t \leq 0)\, x(1) = x\left(\frac{1}{2}\right)\left(0 \leq \frac{1}{2} \leq 1\right). \tag{32}$$

Table 1: Values at some point in the interval $[0, 1]$.

t_i	$x(t_i)$	$x^*(t_i)$	$\epsilon(t_i)$
0.000000	0.00	0.000000	0.000000
0.250000	0.6250	0.632407	0.007407
0.500000	1.00	1.068917	0.068917
1.000000	1.00	1.167506	0.167506

Using the method given above, this equation can be written as the Fredholm-Volterra integral equation as follows:

$$\begin{aligned} x^*(t) &= 2.872933230t - 2t^2 - 0.1714285714t^{7/2} \\ &\quad + 0.2083333333t^4 + 0.1269841270t^{9/2} - 0.1t^5 \\ &\quad + \frac{t}{8}\int_0^{1/2}\left[3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma \\ &\quad - \frac{t}{8}\int_0^{1/2-(\sqrt{2}/4)}\left[1 - 4\sigma - 16\sigma^2 + \frac{1 + 4\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma \\ &\quad - \frac{1}{16}\int_0^{t-(\sqrt{t}/2)}\left[(4t-1) + (16t-12)\sigma - 16\sigma^2\right. \\ &\quad \left. + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma. \end{aligned} \tag{33}$$

Let

$$\begin{aligned} h(t) &= 2.872933230t - 2t^2 - 0.1714285714t^{7/2} \\ &\quad + 0.2083333333t^4 + 0.1269841270t^{9/2} - 0.1t^5, \\ K_1(\sigma) &= \frac{1}{8}\left[3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right], \\ K_2(\sigma) &= \frac{1}{8}\left[1 - 4\sigma - 16\sigma^2 + \frac{1 + 4\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right], \\ K(t,\sigma) &= -\frac{1}{16}\left[(4t-1) + (16t-12)\sigma - 16\sigma^2\right. \\ &\quad \left. + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right], \\ F_\lambda^T x &\equiv \int_0^{1/2} K_1(\sigma)x(\sigma)d\sigma, \\ F_\lambda^c x &\equiv \int_0^{(1/2)-(\sqrt{2}/4)} K_2(\sigma)x(\sigma)d\sigma, \\ V_\lambda x &\equiv -\int_0^{t-(\sqrt{t}/2)} K(t,\sigma)x(\sigma)d(\sigma). \end{aligned} \tag{34}$$

Therefore, the integral equation (33) can be written as

$$x^*(t) = h(t) + tF_\lambda^T x + tF_\lambda^c x + V_\lambda x, \tag{35}$$

and this equation is equivalent to problem (31). Some values of the solution of this equation are obtained by using the method of the consecutive substitution method of third which are given in Table 1, where the first approximation is $x_0(t) = 2.872933230t$ and analytical solution is $x(t) = -2t^2 + 3t$.

4. Conclusion

In this article, a suitable approximation method is applied to the solution of a differential equation with retarded argument. An equivalent integral equation was obtained to find the solution of BVP (1). This equation is the Fredholm-Volterra integral equation. After obtaining the integral equation, the consecutive substitution method was applied, and an approximate solution was obtained. The approximate solutions calculated for problem (1) are compared with the analytical solution in Table 1 for some values of t. The obtained results were found to be compatible. Calculations related to the above-mentioned example were made using Maple.

Data Availability

No.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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