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Research Article

Modified α-Parameterized Differential Transform Method for Solving Nonlinear Generalized Gardner Equation

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In this article, we present a novel enhancement to the α-parameterized differential transform method (PDTM) for solving nonlinear boundary value problems. The proposed method is applied to solve the generalized Gardner equation by utilizing genetic algorithms to obtain optimal parameter values. Our proposed approach extends the general differential transformation method, allowing for the use of various values for the coefficient α. Our solution procedure offers a distinct advantage by allowing the original differential transformation method to be divided into multiple steps, thereby illustrating specific solution properties for nonlinear boundary value problems. Additionally, possible alternative solutions based on varying parameter values are also explored and discussed. The results with those obtained through the DTM method and exact solutions are compared to confirm the accuracy of our method and its efficiency in reaching the exact solution quickly.

1. Introduction

Several methods were used in solving nonlinear partial differential equations, including the Adomian decomposition method, vibrational iteration method, and differential transform method [1, 2]. The differential transform method (DTM) is a potent approximate analytical technique for resolving nonlinear differential equations. Consequently, it offers a method that may be used broadly to create an analytical solution of differential equations. Real physical systems exhibit chaotic and nonlinear behaviour. Sometimes, it is impossible to solve the differential equations for these systems mathematically, so one must use specific procedures or methodologies to arrive at the analytical solutions. The DTM is one of the numerical techniques that allow us to obtain approximations of solutions to both linear and nonlinear differential equation systems. This method's key benefit is that it does not require linearization and may be used directly on nonlinear ODEs. The ease of use, accuracy of computations, and breadth of applications of DTM are its well-known benefits. Another significant benefit of this approach is its ability to drastically reduce the amount of computational labor required while still accurately delivering the series solution with a rapid convergence rate [3]. Many methods have been developed to solve nonlinear differential equations. One such method is the α-parameterized differential transform method (PDTM), which differs from the traditional DTM by calculating the coefficients of the Taylor polynomials differently [4]. Another variant is the reduced differential transform method (RDTM), which uses the DTM method and has been applied to solve two types of nonlinear partial differential equations [5, 6]. The generalized Gardner equation is as follows [7]:

$$u_t + \left(\rho + \beta u^n + \gamma u^{2n}\right) u_x + \delta u_{xxx} = 0. \tag{1}$$

With initial and boundary conditions,

$$\begin{aligned} &u(x, t_0) = f(x), t_0 \geq 0, -L \geq x \geq L, \\ &u(-L, t) = g_0(t), u(L, t) = g_1(t), \end{aligned} \tag{2}$$

where ρ, β, γ, and δ are arbitrary constants, with nonlinear terms of any order, and can take different values to derive known models. $f(x)$, $g_0(t)$, and $g_1(t)$ are known functions. The parameters in equation (1) will be determined based on plasma parameters at a later time. In the special case when $n=1, \beta \neq 0, \gamma = 0$, the generalized Gardner equation (1) becomes the KdV equation; when $n=1, \beta = 0, \gamma \neq 0$, it becomes the mKdV equation; when $n=1, \beta \neq 0, \gamma \neq 0$, equation (1) converts to the KdV-mKdV equation. Demiray and Bulut utilized the extended trial equation method to obtain exact solutions for the generalized Gardner equation [7]. Another group of researchers, new techniques for determining new solutions of partial differential equations were created by Ghanbari and Baleanu [8]. Mathanaranjan used the first integral method to solve the generalized Gardner which contains dual high-order nonlinear terms [9]. Taghizade and Neirameh constructed travelling wave solutions involving parameters of the Gardner equations [10]. Daghan and Donmez studied the travelling wave solution of the Gardner equation analytically using the two dependent expansion and direct integration approaches [11].

GA has been widely used by researchers to solve differential equations. One such example is the work by Firouzi et al. [12], who proposed the use of GA for estimating large synchronous generator parameters. X. Li and M. Yin [13] also applied GA along with other heuristic techniques to solve the boundary value problem. S. A. Malik et al. [14] proposed a crossbred algorithm that combined differential evolution with artificial bee colony, which was implemented for parameter assessment of chaotic systems. In another study, Zeeshan and Atlas [15] utilized GA to find the best solution of the integrodifferential equation. In this paper, we propose a generalization of the α-parameterized differential transform method (PDTM) and finding the parameter using genetic algorithm instead of using long steps to solve nonlinear systems to find them, which may take a long time, which is one of the disadvantages of the α-parameterized differential transform method (PDTM) [4].

The structure of this paper is as follows: Section 2 provides an overview of the differential transform method, while Section 3 provides a brief introduction to GA. Section 4 outlines the proposed method, and Section 5 demonstrates the application of GA-MPDTM for solving a nonlinear general Gardner equation. Finally, Section 6 contains concluding remarks.

2. The Differential Transform Method

The first description is for some basic properties of the differential transform method.

Definition: if $w(x,t)$ is a continuously differentiated analytical function with respect to both x and t in the corresponding domain, then the function $W_i(x)$ is defined by [16]

$$W_i(x) = \frac{1}{i!}\left[\frac{\partial^i}{\partial t^i} w(x,t)\right]_{t=0}. \tag{3}$$

Definition: the inverse transform of the transformed function $W_i(x)$ can be defined as follows:

$$w(x,t) = \sum_{i=0}^{\infty} W_i(x) t^i. \tag{4}$$

Then,

$$w(x,t) = \sum_{i=0}^{\infty} \frac{1}{i!}\left[\frac{\partial^i}{\partial t^i} w(x,t)\right]_{i=0} t^i. \tag{5}$$

Consider the nonlinear partial differential equation

$$Lw(x,t) + Rw(x,t) + Nw(x,t) = g(x,t), \tag{6}$$

where $L = (\partial/\partial t)$, R is the linear, N is the nonlinear operator, and $g(x,t)$ is an inhomogeneous term. The recursive formula of (5) becomes

$$(i+1)\, W_{i+1}(x,t) = G_i(x) - \mathrm{RW}_i(x) - \mathrm{NW}_i(x), \tag{7}$$

where $W_i(x), \mathrm{RW}_i(x), \mathrm{NW}_i(x)$, and $G_i(x)$ are the transformed functions. Table 1 enumerates the fundamental alterations that the RDTM effectively carries out.

The initial condition (6) is defined as

$$W_0(x) = f(x). \tag{8}$$

By inserting (8) into (7) and performing iterative calculations, we can derive the $W_i(x)$ values. These values can be used to approximately solve the n-terms by taking the inverse transformation of the sequence $\{W_i(x)\}_{i=0}^{n}$ as

$$\widetilde{w_n}(x,t) = \sum_{i=0}^{n} W_i(x) t^i. \tag{9}$$

Consequently, the solution of (7) reads the regularity convergence of the series $w(x,t)$ [17]

$$w(x,t) = \lim_{n \longrightarrow \infty} \widetilde{w_n}(x,t). \tag{10}$$

3. Genetic Algorithm

A genetic algorithm (GA) is a type of algorithms that uses natural selection to optimize a solution. Introduced by John Holland in 1970 [18]. GA is a simple and effective optimization technique. The search space, which represents all feasible solutions, comprises points where each point represents one feasible solution. The quality of each solution is expressed as a fitness value. A fitness function is utilized to evaluate the performance of all individuals in the population. The GA employs three primary operations in this population: selection, crossover, and mutation, to reach the optimal solution. The GA usually generates a random population of individuals at the start using a special procedure to create a higher quality initial population. Each chromosome consists of N genes representing unknown coefficients to be optimized.

The outline of the basic GAs involves these fundamental concepts [19].

(1) *Start.* A random population of n chromosomes is generated (expected solutions).

(2) *Fitness Functions.* The fitness of each chromosome is evaluated using a fitness function

   (i) *New Population.* A new population is created through a sequence of steps

   (ii) *Selection (Reproduction).* Select two parent (two chromosomes) from a solution according to their fitness

   (iii) *Crossover.* The first is selection, where two chromosomes are chosen as parents based on their fitness level. These two individuals then undergo a crossover operation, which combines their genetic information to create new offspring

   (iv) *Mutation.* Mutations are also introduced to maintain genetic diversity and explore new potential solutions

   (v) *Acceptable.* Once a new offspring has been generated and evaluated, it is added to the new population

(3) *Replace.* This process continues until a new population is formed. At this point, this population replaces the previous one, and the process continues to a condition is met

(4) *Termination.* If the condition is satisfied, the best solution in the current population is returned, and the entire process can begin again from step one for further optimization

(5) *Loop.* Go to step 2

4. Proposed Method (GA-MPDTM)

The main idea of the proposed method depends on using the boundary conditions around an interval $[a, b]$ of the nonlinear differential equation in the differential transformation method (DTM) instead of relying on the initial condition only. If the differential equation contains the following boundary conditions,

$$w(x, a) = f(x), w(x, b) = g(x). \tag{11}$$

The solution is divided into the following steps:

First: finding the solution by differential transformation method (DTM) about the beginning of the interval.

$$\widetilde{w_l}(x, t) = \sum_{i=0}^{n} W_i(x)(t-a)^i. \tag{12}$$

Secondly: finding the solution by differential transformation method (DTM) around the end of the interval.

$$\widetilde{w_u}(x, t) = \sum_{i=0}^{n} W_i(x)(t-b)^i. \tag{13}$$

Third: calculation of the rate using the convexity function to get a general solution about the interval $t \in [a, b]$.

$$\widetilde{w_{num}}(x, t) = \alpha \widetilde{w_l}(x, t) + (1-\alpha)\widetilde{w_u}(x, t). \tag{14}$$

Once the MPDTM solution series (14) has been obtained, it can be used as a basis for formulating the fitness function in a genetic algorithm, which will then be employed to search for the optimal parameters of a nonlinear differential equation. This involves utilizing a set of equations to define the fitness function within the algorithm.

$$W(x, t, \alpha) = \frac{1}{\mathrm{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(w(x_i, t_j) - \widehat{w}(x_i, t_j)\right)^2, \tag{15}$$

where n and m represent the total numbers of steps. W represents the fitness function (mean square error).

5. Application

In this paper, the general Gardner equation (1) will be solved by the proposed method with boundary conditions [9]:

Case 1. Let

$$u(x, a) = \left(\frac{1}{2}\sqrt{\frac{(1+n)(1+2n)(-c+\rho)}{\gamma}} \cdot \left(1 + \tan\left(\frac{n}{2k}\sqrt{\frac{-c+\rho}{\delta}}(k(x-\mathrm{ca})+v)\right)\right)\right)^{1/n}, \tag{16}$$

Table 1: The transformations by using the RDTM.

Function	Transformed function
$h(x,t)=w(x,t)\mp v(x,t)$	$H_i(x)=W_i(x)\mp V_i(x)$
$h(x,t)=\alpha w(x,t)$	$H_i(x)=\alpha W_i(x)$ (α is a constant)
$h(x,t)=x^m t^n$	$H_i(x)=x^m\delta(i-n)$
$h(x,t)=x^m t^n w(x,t)$	$H_i(x)=x^m W_{i-n}(x)$
$h(x,t)=w(x,t)v(x,t)$	$H_i(x)=\sum_{k=0}^{i}V_k(x)W_{i-k}(x)=\sum_{k=0}^{i}W_k(x)V_{i-k}(x)$
$h(x,t)=\dfrac{\partial^i}{\partial t^i}w(x,t)$	$H_i(x)=(i+1)\cdots(i+k)W_{i+k}(x)=\dfrac{(i+k)!}{i!}W_{i+k}(x)$
$h(x,t)=\dfrac{\partial}{\partial x}w(x,t)$	$H_i(x)=\dfrac{\partial}{\partial x}W_i(x)$

$$u(x,b)=\left(\frac{1}{2}\sqrt{\frac{(1+n)(1+2n)(-c+\rho)}{\gamma}}\cdot\left(1+\tan\left(\frac{n}{2k}\sqrt{\frac{-c+\rho}{\delta}}(k(x-\mathrm{cb})+v)\right)\right)\right)^{1/n}. \tag{17}$$

Using equation (12) and Table 1 when $n=1$, the solution about boundary condition (16) for equation (1) is

$$(i+1)U_{i+1}(x)+\rho\frac{\partial U_i(x)}{\partial x}+\beta\sum_{k=0}^{i}U_k(x)\frac{\partial U_{i-k}(x)}{\partial x}+\gamma\sum_{k=0}^{i}\sum_{s=0}^{k}U_{i-k}(x)U_{k-s}(x)\frac{\partial U_s(x)}{\partial x}+\delta\frac{\partial^3U_i(x)}{\partial x^3}=0. \tag{18}$$

When $i=0,1,2$ and $c=1,k=1,v=1,\gamma=1,\rho=2\gamma+1$, $\beta=-\sqrt{6}\gamma,\delta=-\gamma$,

$$\begin{aligned}
U_1(x)&=-\frac{1}{4}\frac{\sqrt{6}}{\cos\left(-(1/2)x+(1/2)a-(1/2)\right)^2},\\
U_2(x)&=-\frac{1}{8}\frac{\sqrt{6}\sin\left((1/2)x+(1/2)a-(1/2)\right)}{\cos\left(-(1/2)x+(1/2)a-(1/2)\right)^3},\\
U_3(x)&=\frac{1}{48}\frac{\sqrt{6}\left(-3+2\cos\left(-(1/2)x+(1/2)a-(1/2)\right)^2\right)}{\cos\left(-(1/2)x+(1/2)a-(1/2)\right)^4}.\\
&\vdots
\end{aligned} \tag{19}$$

Then,

$$U_L(x,t)=\sum_{k=0}^{m}U_k(x)(t-a)^k. \tag{20}$$

For $m=5$ and $a=0$,

$$\begin{aligned}
U_L(x,t)=\frac{1}{960}\frac{1}{\cos\left((1/2)x+(1/2)\right)^6}\cdot\Bigg(\sqrt{6}\Bigg(&480\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^6\\
&+480\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^5\sin\left(\frac{1}{2}x+\frac{1}{2}\right)\\
&-240t\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^4\\
&+120\sin\left(\frac{1}{2}x+\frac{1}{2}\right)t^2\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^3\\
&-60t^3\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^2+40t^3\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^4\\
&+30\sin\left(\frac{1}{2}x+\frac{1}{2}\right)t^4\cos\left(\frac{1}{2}x+\frac{1}{2}\right)\\
&-10\sin\left(\frac{1}{2}x+\frac{1}{2}\right)t^4\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^3\\
&+15t^5\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^2\\
&-2t^5\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^4-15t^5\Bigg)\Bigg).
\end{aligned} \tag{21}$$

In the same way, we find the solution about the boundary condition (17)

$$
\begin{aligned}
U_u(x,t) &= \sum_{k=0}^{\infty} U_k(x)(t-b)^k,\\
U_1(x) &= -\frac{1}{4}\,\frac{\sqrt{6}}{\cos\left(-(1/2)x+(1/2)b-(1/2)\right)^2},\\
U_2(x) &= -\frac{1}{8}\,\frac{\sqrt{6}\sin\left((1/2)x+(1/2)b-(1/2)\right)}{\cos\left(-(1/2)x+(1/2)b-(1/2)\right)^3},\\
U_3(x) &= \frac{1}{48}\,\frac{\sqrt{6}\left(-3+2\cos\left(-(1/2)x+(1/2)b-(1/2)\right)^2\right)}{\cos\left(-(1/2)x+(1/2)b-(1/2)\right)^4}.\\
&\vdots
\end{aligned}
\tag{22}
$$

For $m=5$ and $b=1$,

$$
\begin{aligned}
U_u(x,t) = \frac{1}{960}\,\frac{1}{\cos\left((1/2)x\right)^6}\Bigg(\sqrt{6}\Bigg(&15+150t^2-150t^3+75t^4\\
&-15t^5+60\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t^2\\
&-200\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t\\
&+30\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t^4\\
&-120\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t^3\\
&+180\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t^2\\
&-120\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t+45\cos\left(\frac{1}{2}x\right)^2\\
&+202\cos\left(\frac{1}{2}x\right)^4+480\cos\left(\frac{1}{2}x\right)^6\\
&+110\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3-130\cos\left(\frac{1}{2}x\right)^4 t\\
&+90\cos\left(\frac{1}{2}x\right)^2 t^3+30\cos\left(\frac{1}{2}x\right)^2 t^2\\
&-105\cos\left(\frac{1}{2}x\right)^2 t+20\cos\left(\frac{1}{2}x\right)^4 t^3\\
&-100\cos\left(\frac{1}{2}x\right)^4 t^2-75\cos\left(\frac{1}{2}x\right)^2 t^4\\
&+15\cos\left(\frac{1}{2}x\right)^2 t^5-2\cos\left(\frac{1}{2}x\right)^4 t^5\\
&+10\cos\left(\frac{1}{2}x\right)^4 t^4+30\sin\left(\frac{1}{2}x\right)\cos\frac{1}{2}x\\
&-10\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t^4\\
&+40\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t^3\\
&+480\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^5-75t\Bigg)\Bigg).
\end{aligned}
\tag{23}
$$

Consequently, the final solution of equation (1) via equation (14) is as follows:

$$
\begin{aligned}
U(x,t) &= \alpha U_L(x,t)+(1-\alpha)U_u(x,t)\\
&= \frac{1}{960}\,\frac{1}{\cos\left((1/2)x+(1/2)\right)^6}\Bigg(\alpha\sqrt{6}\Bigg(480\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^6\\
&\quad+480\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^5\sin\left(\frac{1}{2}x+\frac{1}{2}\right)\\
&\quad-240t\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^4\\
&\quad+120\sin\left(\frac{1}{2}x+\frac{1}{2}\right)t^2\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^3\\
&\quad-60t^3\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^2+40t^3\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^4\\
&\quad+30\sin\left(\frac{1}{2}x+\frac{1}{2}\right)t^4\cos\left(\frac{1}{2}x+\frac{1}{2}\right)\\
&\quad-10\sin\left(\frac{1}{2}x+\frac{1}{2}\right)t^4\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^3\\
&\quad+15t^5\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^2-2t^5\cos\left(\frac{1}{2}x+\frac{1}{2}\right)^4-15t^5\Bigg)\Bigg)\\
&\quad+\frac{1}{960}\,\frac{1}{\cos\left((1/2)x\right)^6}\Bigg((1-\alpha)\sqrt{6}\Bigg(15+150t^2-150t^3\\
&\quad+75t^4-15t^5+60\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t^2\\
&\quad-200\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t\\
&\quad+30\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t^4\\
&\quad-120\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t^3\\
&\quad+180\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t^2\\
&\quad-120\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)t+45\cos\left(\frac{1}{2}x\right)^2\\
&\quad+202\cos\left(\frac{1}{2}x\right)^4+480\cos\left(\frac{1}{2}x\right)^6\\
&\quad+110\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3-130\cos\left(\frac{1}{2}x\right)^4 t\\
&\quad+90\cos\left(\frac{1}{2}x\right)^2 t^3+30\cos\left(\frac{1}{2}x\right)^2 t^2\\
&\quad-105\cos\left(\frac{1}{2}x\right)^2 t+20\cos\left(\frac{1}{2}x\right)^4 t^3\\
&\quad-100\cos\left(\frac{1}{2}x\right)^4 t^2-75\cos\left(\frac{1}{2}x\right)^2 t^4\\
&\quad+15\cos\left(\frac{1}{2}x\right)^2 t^5-2\cos\left(\frac{1}{2}x\right)^4 t^5\\
&\quad+10\cos\left(\frac{1}{2}x\right)^4 t^4+30\sin\left(\frac{1}{2}x\right)\cos\frac{1}{2}x\\
&\quad-10\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t^4+40\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^3 t^3\\
&\quad+480\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)^5-75t\Bigg)\Bigg).
\end{aligned}
\tag{24}
$$

Table 2: Compare the approximate solution by DTM and modified α-PDTM for different values α when $n=1, c=1, k=1, \nu=1, \gamma=1, \rho=3, \beta=-\sqrt{6}, \delta=-1, t=0.5$.

α	x	$U_{\text{exact}} - U_L$ DTM	$U_{\text{exact}} - U$ MPDTM
0.25	0.25	3.479760×10^{-4}	1.026830×10^{-4}
	0.5	9.122950×10^{-4}	2.645540×10^{-4}
	0.75	2.785476×10^{-3}	7.755900×10^{-4}
	1	1.053285×10^{-2}	2.813242×10^{-3}
0.5	0.25	3.479760×10^{-4}	1.844480×10^{-4}
	0.5	9.122950×10^{-4}	4.804680×10^{-4}
	0.75	2.785476×10^{-3}	1.445552×10^{-3}
	1	1.053285×10^{-2}	5.386446×10^{-3}
0.75	0.25	3.479760×10^{-4}	2.662120×10^{-4}
	0.5	9.122950×10^{-4}	6.963820×10^{-4}
	0.75	2.785476×10^{-3}	2.115514×10^{-3}
	1	1.053285×10^{-3}	7.959648×10^{-4}

The exact solution for equation (1) with the boundary conditions ((16)–(17)) is [9]

$$u(x,t) = \left(\frac{1}{2}\sqrt{\frac{(1+n)(1+2n)(-c+\rho)}{\gamma}} \cdot \left(1+\tan\left(\frac{n}{2k}\sqrt{\frac{-c+\rho}{\delta}}(k(x-ct)+\nu)\right)\right)\right)^{(1/n)}. \tag{25}$$

We note that in Table 2, the α-parameterized differential transform method gives better results than classic differential transform method and for different values for α. A genetic algorithm was used to obtain the best value.

In Figure 1, we observe the convergence of the approximate solution using MPDTM with the exact solution. The methodology suggests starting with a standard model structure in which certain parameters are unknown. The aim is to determine the optimal values of these parameters $\alpha, \rho, \gamma, \beta, \delta, \nu$ within the context of the nonlinear Gardner equation (1), with the goal of minimizing the error.

Case 2. Let us have the boundary conditions [9].

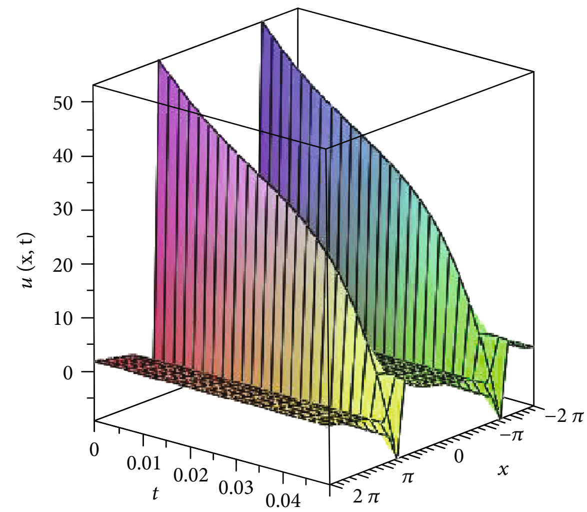
$$u(x,a) = \frac{d}{3f}\cdot\left(1-\tanh\left(\frac{d}{3k\sqrt{2f}}(k(x-ca)+\nu)\right)\right)^{(1/n)}, \tag{26}$$

$$u(x,b) = \frac{d}{3f}\cdot\left(1-\tanh\left(\frac{d}{3k\sqrt{2f}}(k(x-cb)+\nu)\right)\right)^{(1/n)}, \tag{27}$$

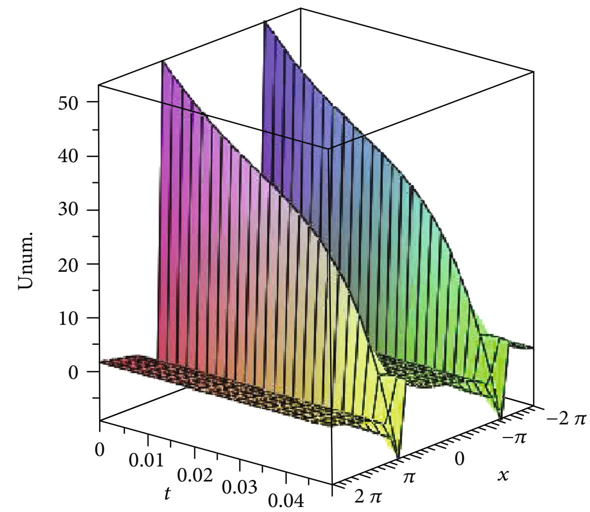
such that $b>0, \beta=2d, \gamma=-3f, c=2d^2/9f$ and ρ, δ arbitrary constants.

Using the proposed method, we get

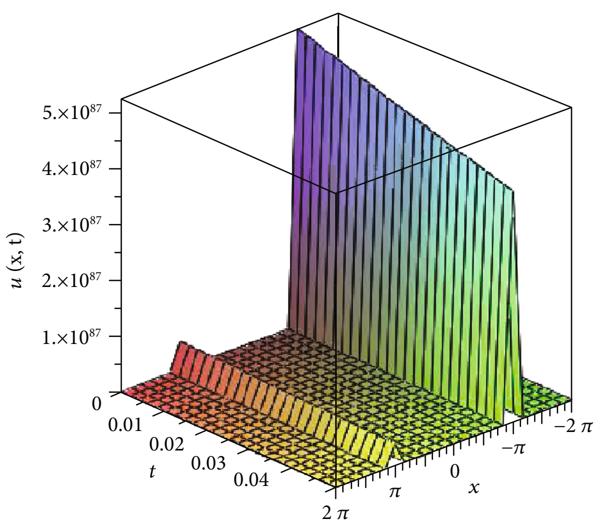
$$\begin{aligned} U(x,t) &= \alpha U_L(x,t) + (1-\alpha)U_u(x,t) \\ &= \frac{1}{1296}\frac{1}{\cosh((1/6)x+(1/6))^4} \\ &\quad\cdot\left(\alpha\left(216\cosh\left(\frac{1}{6}x+\frac{1}{6}\right)^4 - 216\sinh\right.\right. \\ &\quad\cdot\left(\frac{1}{6}x+\frac{1}{6}\right)^1\cosh\left(\frac{1}{6}x+\frac{1}{6}\right)^3 \\ &\quad + 36t\rho\cosh\left(\frac{1}{6}x+\frac{1}{6}\right)^2 + 6t\beta\cosh\left(\frac{1}{6}x+\frac{1}{6}\right)^2 \\ &\quad \left.\left. -\gamma t - 6\delta t + \cdots\right)\right) + \frac{1}{1296}\frac{1}{\cosh((1/6)x+(4/27))^4} \\ &\quad\cdot\left((1-\alpha)\left(216\cosh\left(\frac{1}{6}x+\frac{4}{27}\right)^4\right.\right. \\ &\quad - 216\sinh\left(\frac{1}{6}x+\frac{4}{27}\right)^1\cosh \\ &\quad\left.\left.\cdot\left(\frac{1}{6}x+\frac{4}{27}\right)^3\frac{1}{6}x+\frac{4}{27}\right)\right). \end{aligned} \tag{28}$$



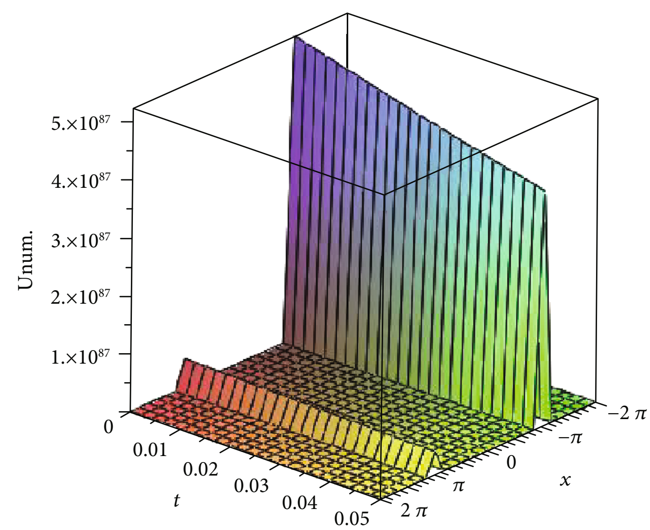
(a)



(b)



(c)



(d)

FIGURE 1: The numerical solutions using MPDTM when $n = 1$, $t = 0.025$: (a) the exact solution, (b) the numerical solution $U(x, t)$ with $\alpha = 1$, (c) the exact solution, and (d) the numerical solution $U(x, t)$ with $\alpha = 0$.

TABLE 3: Compare the approximate solution by DTM and modified α-PDTM when $n = 1$, $m = 3$, $d = 1$, $f = 2$, $k = 1$, $v = 1$, $\gamma = -6$, $\rho = 0.1$, $\beta = 2$, $\delta = 1$, $t = 0.5$.

x	$U_{\text{exact}} - U_L$ DTM	$U_{\text{exact}} - U$ MPDTM
0.1	1.349547×10^{-3}	1.936990×10^{-6}
0.2	1.341637×10^{-3}	2.105540×10^{-6}
0.3	1.333057×10^{-3}	2.269560×10^{-6}
0.4	1.323826×10^{-3}	2.429040×10^{-6}
0.5	1.313963×10^{-3}	2.583320×10^{-6}
0.6	1.303488×10^{-3}	2.732380×10^{-6}
0.7	1.292423×10^{-3}	2.875940×10^{-6}
0.8	1.280790×10^{-3}	3.013740×10^{-6}
0.9	1.268611×10^{-3}	3.145810×10^{-6}
1	1.255910×10^{-3}	3.271760×10^{-6}

TABLE 4: The values of the parameters used for GA.

Name	Values
Population (solutions) size	100
Max generations	500
Crossover fraction	0.5
Initial population range	(-10, 10)
Function tolerance	$1e-50$

The exact solution for equation (1) with the boundary conditions ((26)–(27)) is [9]

$$u(x, t) = \frac{d}{3f} \cdot \left(1 - \tanh\left(\frac{d}{3k\sqrt{2f}}(k(x - ct) + v)\right)\right)^{1/n}. \tag{29}$$

Table 5: Optimization parameter values for solving equation (1) by GA-MPDTM.

ρ	γ	β	δ	α	MSE
0	-6	2	1	0.881486	8.650360×10^{-19}
0	-0.6	0.2	0.5	0.850026	1.521751×10^{-10}
0.001	-6	2	1	0.849960	1.854641×10^{-10}
0.1	-6	1	1	0.849888	1.259790×10^{-7}
0.1	-15	4	0.5	0.849326	3.260217×10^{-6}

Note that in Table 3, the α-parameterized differential transform method gives better results than classic differential transform method and for different values for α. As listed in Table 3, the parameter values utilized in GA code are available within the commercial software MATLAB® R2021a.

We note that in Tables 4 and 5, the use of the genetic algorithm helped in understanding the behavior of parameters in solving a general Gardner equation where the mean square error decreases when $\rho \longrightarrow 0$ and the best value for α-parameterized in modified differential transform method in interval [0,1] is $\alpha \longrightarrow 0.8$.

6. Conclusion

This paper presents a solution for the general Gardner equation using modified α-parameterized differential transform method and genetic algorithm. The proposed method effectively generates approximate solutions for generalized Gardner equations, which, for specific parameter selections, reduce the numerical solutions to the exact solutions of classical Gardner. The α-parameterized differential transform method and genetic algorithm have been utilized to approximate solutions for the Gardner equation. These solutions can be applied to physical plasmas that contain electron acoustic waves in a nonextensive positron electron-ion configuration. The findings are significant in understanding the dynamic characteristics of solitons and shock waves within different astrophysical plasma systems. This research provides a more comprehensive understanding of electron acoustic solitons. In addition, the obtained solutions are modified through genetic algorithm to optimize parameters and describe their effects on the solutions. This method can be considered a beneficial approach for acquiring novel solutions of nonlinear partial differential equations. These newly derived, precise solutions may be invaluable in explaining various nonlinear physical phenomena. Using the program MAPLE and MATLAB, they are possible to secure the remaining components of the equation similarly.

Data Availability

The authors will provide the data used to support the findings of this study upon request.

Conflicts of Interest

The authors affirm that there are no conflicts of interest pertaining to the publication of this paper.

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