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Research Article

Chromatic Schultz and Gutman Polynomials of Jahangir Graphs $J_{2,m}$ and $J_{3,m}$

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Topological polynomial and indices based on the distance between the vertices of a connected graph are widely used in the chemistry to establish relation between the structure and the properties of molecules. In a similar way, chromatic versions of certain topological indices and the related polynomial have also been discussed in the recent literature. In this paper, we present the chromatic Schultz and Gutman polynomials and the expanded form of the Hosoya polynomial and chromatic Schultz and Gutman polynomials, and then we derive these polynomials for special cases of Jahangir graphs.

1. Introduction

Suppose that $G(V, E)$ is a connected undirected graph with vertices set V and the edge set E. Then the distance between two vertices u, v is denoted by $d(u, v)$, which is defined as the length of shortest path between u and v. The degree of vertex u in G is denoted by $d(u)$, which is defined as the number of edges incident to u. The diameter of G is denoted by $D(G)$, which is defined as $D(G) = \mathrm{Max}_{u \in V}\{d(u, v): v \in V\}$.

Having a molecule, if we represent atoms by vertices and bonds by edges, we obtain a molecular graph [1, 2]. Algebraic polynomial plays a significant part in chemistry because of its important applications for molecular compounds.

Topological indices in biology and chemistry were used for the first time in 1947 when chemist Wiener [3] introduced Wiener's index which is the oldest topological index studied to demonstrate correlations between physicochemical properties of organic compounds of molecular graphs. At first, Wiener's index was used to predict the boiling of paraffin [4]. Recently, there is a proposal to use Wiener's index to predict conformational switching in RNA structures [5].

In chemistry, drug discovery commonly relies on the topological indices. If we compute topological indices of drug molecular structures, medical and pharmaceutical researchers will be able to understand their therapeutic properties which can compensate for the shortcomings of medicine and chemical experiments.

The Hosoya polynomial [6] is one of the known examples which determines distance-based topological indices, and through it, we can obtain the Wiener index. M-polynomial [7] introduced in 2015 plays the same role in determining closed forms of many degree-based topological indices [8, 9]. In [10], we see M-polynomial and some indices of Jahangir graph $J_{n,m}$. In [11], we see M-polynomial and some indices of polyhex nanotubes.

The Hosoya polynomial of graph is defined as $H(G, x) = (1/2)\sum_{u \in V}\sum_{v \in V} x^{d(u,v)}$ was introduced by Haruo Hosoya in 1988 as a counting polynomial. It actually counts the number of distances of path of different lengths in a molecular graph [12]. The Hosoya polynomial is very well studied. In 1993, Gutman introduced the Hosoya polynomial for a vertex of a graph, and these polynomials are correlated. The most interesting application of the Hosoya polynomial is the almost all distance-based graph invariants, which are used to predict physical, chemical, and pharmacological properties of organic molecules, which can be recovered from the Hosoya polynomial [13].

The Schultz index of graph G was introduced by Schultz [14] in 1989 as follows:

$$\mathrm{Sc}(G) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(d_u + d_v)d(u,v). \quad (1)$$

The Gutman index of graph G was introduced by Klavẑar and Gutman [15] in 1996 as follows:

$$\mathrm{Gut}(G) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(d_u \times d_v)d(u,v). \quad (2)$$

The recent update in the computational techniques of the Schultz and Gutman indices can be found in [16, 17] and implemented with relativistic parameters in [18, 19].

Also, the Schultz and Gutman polynomials of G are defined as $\mathrm{Sc}(G,x) = (1/2)\sum_{u\in V}\sum_{v\in V}(d_u + d_v)x^{d(u,v)}$ and $\mathrm{Gut}(G,x) = (1/2)\sum_{u\in V}\sum_{v\in V}(d_u + d_v)\,x^{d(u,v)}$, respectively, and the relationship between them as follows: $\mathrm{Gut}(G) = (\partial/\partial x)Gut(G,x)|_{x=1}, Sc(G) = (\partial/\partial x)Sc(G,x)|_{x=1}$ [20].

A proper $k-$ coloring, or simply $k-$ coloring of graph $G(V,E)$ is a function $\varphi: V \longrightarrow C\,; C = \{c_1, c_2, c_3, \cdots, c_k\}$ such that for each $uv \in E$; $\varphi(u) \neq \varphi(v)$. A graph G is $k-$ coloring if there exists $k-$ coloring of G. The chromatic number $\chi(G)$ of graph G is the smallest k such that G is C.

An independent set [21] is a subset of vertices $S \subseteq V$ such that no two vertices in S are adjacent. A legal $k-$ coloring of G corresponds to partition of V in to k independent set. We can also define a function $\xi : V \longrightarrow \{1, 2, 3, \cdots, k\}$ such that $\xi(v_i) = s$ if $\varphi(v_i) = c_s\,; c_s \in C$.

In order to get φ^+ (maximal parameter coloring), we give the largest independent set, color c_k, then any vertex in this set will be $\xi_{\varphi^+(u)} = k$, and if we give the second largest independent set, color c_{k-1}, then any vertex in this set will be $\xi_{\varphi^+(u)} = k-1$. So, on till give the smallest independent set color c_1 then any vertex in this set will be $\xi_{\varphi^+(u)} = 1$.

In order to get φ^- (minimal parameter coloring), we give the largest independent set, color c_1, then any vertex in this set will be $\xi_{\varphi^-(u)} = 1$, and if we give the second largest independent set, color c_2, then any vertex in this set will be $\xi_{\varphi^-(u)} = 2$. So, to give the smallest independent set, color c_k, then any vertex in this set will be $\xi_{\varphi^-(u)} = k$.

Definition 1 (See [22]). Let G be a connected graph with chromatic number $\chi(G)$. Then the chromatic Schultz polynomial of G denoted by $S_\chi(G,x)$ is defined as

$$S_\chi(G,x) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(\xi(u) + \xi(v))x^{d(u,v)}. \quad (3)$$

Definition 2 (See [22]). Let G be a connected graph with chromatic number $\chi(G)$. Then the chromatic Gutman polynomial of G denoted by $Gut_\chi(G,x)$ is defined as

$$\mathrm{Gut}_\chi(G,x) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}(\xi(u) \times \xi(v))x^{d(u,v)}. \quad (4)$$

Definition 3 (See [22]). Let G be a connected graph with chromatic number φ^- and φ^+ be the minimal and maximal parameter coloring of G. Then,

(1) The $\chi^+ -$ chromatic Schultz polynomial of G, denoted by S_{χ^+}, is defined as $S_{\chi^+}(G,x) = (1/2)\sum_{u\in V}\sum_{v\in V}(\xi_{\varphi^+}(u) + \xi_{\varphi^+}(v))x^{d(u,v)}$

(2) The $\chi^+ -$ chromatic Gutman polynomial of G, denoted by Gut_{χ^+}, is defined as $\mathrm{Gut}_{\chi^+}(G,x) = (1/2)\sum_{u\in V}\sum_{v\in V}(\xi_{\varphi^+}(u) \times \xi_{\varphi^+}(v))x^{d(u,v)}$

(3) The $\chi^- -$ chromatic Schultz polynomial of G, denoted by S_{χ^-}, is defined as: $S_{\chi^-}(G,x) = (1/2)\sum_{u\in V}\sum_{v\in V}(\xi_{\varphi^-}(u) + \xi_{\varphi^-}(v))x^{d(u,v)}$

(4) The $\chi^- -$ chromatic Gutman polynomial of G, denoted by Gut_{χ^-}, is defined as $\mathrm{Gut}_{\chi^-}(G,x) = (1/2)\sum_{u\in V}\sum_{v\in V}(\xi_{\varphi^-}(u) \times \xi_{\varphi^-}(v))x^{d(u,v)}$

Note: when calculating the Hosoya polynomial, we do not calculate the distance between the vertex and itself. When calculating a chromatic Schultz and Gutman polynomials, we calculate the distance of the vertex from itself.

The Hosoya polynomial gives only the number and lengths of paths between the vertices of the graph. If we have the term nx^m in the Hosoya polynomial, this will indicate that n is the number of paths of length m between the vertices in graph G. In this search, we will suggest new definitions which will be generalization for the Hosoya polynomial. The definitions will limit, in addition to the number and lengths of paths between the vertices of the graph, the colors of vertices in the beginning and the end of every path. As a result, the Hosoya polynomial becomes a special case from the suggested definitions. So, we can find the Hosoya polynomial from the new definitions, and these definitions are

Definition 4. Let G be a connected graph. Then, the expanded $H^+ -$ Hosoya polynomial denoted by $H^+(G,x,k)$ is defined as

$$H^+(G,x,k) = \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{u\in V_i^+}\sum_{v\in V_i^+} {x_i}^{d(u,v)} + \sum_{i=1}^{k}\sum_{j=1:j\neq i}^{k}\sum_{u\in V_i^+}\sum_{v\in V_j^+} {x_{i,j}}^{d(u,v)}\right), \quad (5)$$

where $V_i^+ = \{v \in V : \xi_{\varphi^+(v)} = i\}$; $1 \le i \le k$, $V_j^+ = \{v \in V : \xi_{\varphi^+(v)} = j\}$; $1 \le j \le k$, $H(G,x) = H^+(G,x,k)|_{\substack{x_i=x \\ x_{i,j}=x}}$.

Definition 5. Let G be a connected graph. Then, the expanded H^- – Hosoya polynomial denoted by $H^-(G,x,k)$ is defined as

$$H^-(G,x,k) = \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{u\in V_i^-}\sum_{v\in V_i^-} {x_i}^{d(u,v)} + \sum_{i=1}^{k}\sum_{j=1:j\neq i}^{k}\sum_{u\in V_i^-}\sum_{v\in V_j^-} {x_{i,j}}^{d(u,v)}\right)$$
$$H(G,x) = H^-(G,x,k)|_{\substack{x_i=x \\ x_{i,j}=x}}. \tag{6}$$

In order to clarify these suggested new definitions, to determine the difference between it and the Hosoya polynomial, and to know the new things it adds, we will present the following example. In Figure 1, we have the graph P_5.

We will compute the Hosoya polynomial for P_5.

$$H(P_5,x) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V} x^{d(u,v)} = x^4 + 2x^3 + 3x^2 + 4x^1. \tag{7}$$

By the Hosoya polynomial of P_5, we can find out the number and length of the paths between the vertices of P_5. The paths are as follows:

Four paths of length one, three paths of length two, two paths of length three, and one paths of length four.

The path P_5 has two independent sets $\{v_1, v_3, v_5\}$ and $\{v_2, v_4\}$, thus $\chi(P_5) = 2$. In order to compute expanded H^+ – Hosoya polynomial, we will give the vertices of set $\{v_1, v_3, v_5\}$ the color c_2 and the vertices of set $\{v_2, v_4\}$ the color c_1, so we have $\varphi : V \longrightarrow \{c_1, c_2\}$: $\varphi(v_1) = c_2, \varphi(v_2) = c_1$, $\varphi(v_3) = c_2$, $\varphi(v_4) = c_1, \varphi(v_5) = c_2$. We also, have $\xi : V \longrightarrow \{1,2\}$: $\xi_{\varphi^+}(v_1) = 2$, $\xi_{\varphi^+}(v_2) = 1, \xi_{\varphi^+}(v_3) = 2, \xi_{\varphi^+}(v_4) = 1$, $\xi_{\varphi^+}(v_5) = 2$.

$$V_1{}^+ = \{v_2, v_4\}, V_2{}^+ = \{v_1, v_3, v_5\},$$
$$H^+(P_5,x,2) = \frac{1}{2}\left(\sum_{i=1}^{2}\sum_{u\in V_i^+}\sum_{v\in V_i^+} {x_i}^{d(u,v)} + \sum_{i=1}^{2}\sum_{j=1:j\neq i}^{2}\sum_{u\in V_i^+}\sum_{v\in V_j^+} {x_{i,j}}^{d(u,v)}\right)$$
$$= {x_2}^4 + 2{x_{1,2}}^3 + {x_1}^2 + 2{x_2}^2 + 4{x_{1,2}}^1. \tag{8}$$

By expanded H^+ – Hosoya polynomial of P_5, we can find that there is one path of length four has this path starting and ending with vertex having the color c_2, and two paths of length three where each path starting with vertex having the color c_1 and ending with vertex having the color c_2. One path of length two has this path starting and ending with vertex having the color c_1, two paths of length two have each path starting and ending with vertex have the color c_2, and four paths of length one each path

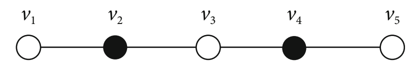

Figure 1: The graph P_5.

starting with vertex having the color c_1 and ending with vertex has the color c_2.

$$H(P_5,x) = H^+(P_5,x,2)|_{\substack{x_1=x \\ x_2=x \\ x_{1,2}=x}} = x^4 + 2x^3 + 3x^2 + 4x^1. \tag{9}$$

To compute expanded H^- – Hosoya polynomial of P_5, we will give the vertices of set $\{v_1, v_3, v_5\}$ the color c_1 and the vertices of set $\{v_2, v_4\}$ the color c_2 so we have

$$\varphi : V \longrightarrow \{c_1, c_2\}: \varphi(v_1) = c_1, \varphi(v_2) = c_2, \varphi(v_3) = c_1,$$
$$\varphi(v_4) = c_2, \varphi(v_5) = c_1$$
$$\xi : V \longrightarrow \{1,2\}: \xi_{\varphi^-}(v_1) = 1, \xi_{\varphi^-}(v_2) = 2, \xi_{\varphi^-}(v_3) = 1,$$
$$\xi_{\varphi^-}(v_4) = 2, \xi_{v^-}(v_5) = 1,$$
$$V_1{}^- = \{v_1, v_3, v_5\}, V_2{}^- = \{v_2, v_4\},$$
$$H^-(P_5,x,2) = \frac{1}{2}\left(\sum_{i=1}^{2}\sum_{u\in V_i^-}\sum_{v\in V_i^-} {x_i}^{d(u,v)} + \sum_{i=1}^{2}\sum_{j=1:j\neq i}^{2}\sum_{u\in V_i^-}\sum_{v\in V_j^-} {x_{i,j}}^{d(u,v)}\right)$$
$$= {x_1}^4 + 2{x_{1,2}}^3 + 2{x_1}^2 + {x_2}^2 + 4{x_{1,2}}^1,$$
$$H(P_5,x) = H^-(P_5,x,2)|_{\substack{x_1=x \\ x_2=x \\ x_{1,2}=x}}$$
$$= x^4 + 2x^3 + 3x^2 + 4x^1. \tag{10}$$

In this search we also suggest new definitions which will be generalization for the chromatic Schultz and Gutman polynomials where the new definitions will limit the colors of vertices in the beginning and the end of every path in the graph and these definitions are

Definition 6. Let G be a connected graph. Then, the expanded χ^+ – chromatic Schultz polynomial of G, denoted by $S_{\chi^+}(G,x,k)$, is defined as

$$S_{\chi^+}(G,x,k) = \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{u\in V_i^+}\sum_{v\in V_i^+}\left(\xi_{\varphi^+}(u) + \xi_{\varphi^+}(v)\right){x_i}^{d(u,v)} + \sum_{i=1}^{k}\sum_{j=1:j\neq i}^{k}\sum_{u\in V_i^+}\sum_{v\in V_j^+}\left(\xi_{\varphi^+}(u) + \xi_{\varphi^+}(v)\right){x_{i,j}}^{d(u,v)}\right),$$

$$S_{\chi^+}(G,x) = S_{\chi^+}(G,x,k)\Big|_{\substack{x_i=x\\ x_{i,j}=x}}. \tag{11}$$

Definition 7. Let G be a connected graph. Then, the expanded χ^+- chromatic Gutman polynomial of G, denoted by $Gut_{\chi^+}(G,x,k)$, is defined as

$$\begin{aligned}\mathrm{Gut}_{\chi^+}(G,x,k) &= \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{u\in V_i^+}\sum_{v\in V_i^+}\left(\xi_{\varphi^+}(u)\times\xi_{\varphi^+}(v)\right)x_i^{d(u,v)}\right.\\ &\quad\left.+\sum_{i=1}^{k}\sum_{j=1:j\neq i}^{k}\sum_{u\in V_i^+}\sum_{v\in V_j^+}\left(\xi_{\varphi^+}(u)\times\xi_{\varphi^+}(v)\right)x_{i,j}^{d(u,v)}\right)\\ &\quad\cdot \mathrm{Gut}_{\chi^+}(G,x) = \mathrm{Gut}_{\chi^+}(G,x,k)\Big|_{\substack{x_i=x\\ x_{i,j}=x}}.\end{aligned} \tag{12}$$

Definition 8. Let G be a connected graph. Then, the expanded χ^- $-$ chromatic Schultz polynomial of G, denoted by $S_{\chi^-}(G,x,k)$, is defined as

$$\begin{aligned}S_{\chi^-}(G,x,k) &= \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{u\in V_i^-}\sum_{v\in V_i^-}\left(\xi_{\varphi^-}(u)+\xi_{\varphi^-}(v)\right)x_i^{d(u,v)}\right.\\ &\quad\left.+\sum_{i=1}^{k}\sum_{j=1:j\neq i}^{k}\sum_{u\in V_i^-}\sum_{v\in V_j^-}\left(\xi_{\varphi^-}(u)+\xi_{\varphi^-}(v)\right)x_{i,j}^{d(u,v)}\right)\\ &\quad\cdot S_{\chi^-}(G,x) = S_{\chi^-}(G,x,k)\Big|_{\substack{x_i=x\\ x_{i,j}=x}}.\end{aligned} \tag{13}$$

Definition 9. Let G be a connected graph. Then, the expanded χ^- $-$ chromatic Gutman polynomial of G, denoted by $\mathrm{Gut}_{\chi^-}(G,x,k)$, is defined as

$$\begin{aligned}\mathrm{Gut}_{\chi^-}(G,x,k) &= \frac{1}{2}\left(\sum_{i=1}^{k}\sum_{u\in V_i^-}\sum_{v\in V_i^-}\left(\xi_{\varphi^-}(u)\times\xi_{\varphi^-}(v)\right)x_i^{d(u,v)}\right.\\ &\quad\left.+\sum_{i=1}^{k}\sum_{j=1:j\neq i}^{k}\sum_{u\in V_i^-}\sum_{v\in V_j^-}\left(\xi_{\varphi^-}(u)\times\xi_{\varphi^-}(v)\right)x_{i,j}^{d(u,v)}\right)\\ &\quad\cdot \mathrm{Gut}_{\chi^-}(G,x)\\ &= \mathrm{Gut}_{\chi^-}(G,x,k)\Big|_{\substack{x_i=x\\ x_{i,j}=x}}.\end{aligned} \tag{14}$$

The Jahangir graph class of graphs is named after the figure which appears on the tomb of Noureddine Muhammad Salim, known by his imperial name, Jahangir corresponding to $J_{2,8}$. Jahangir was the fourth Mughal Emperor who ruled from 1605 to 1627. His tomb is located 5 kilometer northwest of Lahore, Pakistan along the banks of the River Ravi. The Jahangir graph $J_{n,m}$ [23] is a graph on $nm+1$ vertices and $m(n+1)$ edges for $n\geq 2$ and $m\geq 3$. Jahangir graph $J_{n,m}$ consists of cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at the distance to each other.

In [22], we see the chromatic Gutman polynomial of some cycle related graphs. In [24–29], we see the Wiener index and the Hosoya polynomial of $J_{n,m}$ for $n=2,3,4,5,6,7$. In [30–32], we see Schultz' polynomial, Gutman's polynomial, Schultz' index, and Gutman's index of $J_{n,m}$ for $n=2,3,4$.

2. Discussion and Results

In this paper we compute the chromatic Schultz and Gutman polynomials of Jahangir graph $J_{n,m}$ for $n=2,3$. Also, we suggest three new definitions; the first definition is the expanded Hosoya polynomial. The second definition is expanded chromatic Schultz polynomial. The third definition is expanded chromatic Gutman polynomial. We compute these definitions of Jahangir graph $J_{n,m}$ for $n=2,3$.

Theorem 10.

$$\begin{aligned}S_{\chi^+}(J_{2,m},x) &= \left(2m^2-6m\right)x^4+\left(3m^2-6m\right)x^3\\ &\quad+\left(m^2+7m\right)x^2+9mx^1+6m+4,\\ Gut_{\chi^+}(J_{2,m},x) &= \left(2m^2-6m\right)x^4+\left(2m^2-4m\right)x^3\\ &\quad+\left(\frac{1}{2}m^2+\frac{15}{2}m\right)x^2+6mx^1+5m+4,\\ H^+(J_{2,m},x,2) &= \frac{1}{2}(m(m-3))x_2^4+m(m-2)x_{1,2}^3\\ &\quad+\frac{1}{2}m(m-1)x_1^2+2mx_2^2+3mx_{1,2}^1,\\ S_{\chi^+}(J_{2,m},x,2) &= 2(m-3)x_2^4+3m(m-2)x_{1,2}^3\\ &\quad+m(m-1)x_1^2+8mx_2^2++9mx_{1,2}^1+6m+4,\\ Gut_{\chi^+}(J_{2,m},x,2) &= 2m(m-3)x_2^4+2m(m-2)x_{1,2}^3\\ &\quad+\frac{1}{2}m(m-1)x_1^2+8mx_2^2+6mx_{1,2}^1+5m+4.\end{aligned} \tag{15}$$

Proof. Let $V=\{v_1,v_2,v_3,\cdots,v_{2m},v_{2m+1}\}$ be the vertices of $J_{2,m}$. It is obvious that $\chi(J_{2,m})=2$ because C_{2m} contains an even number of vertices, therefore $\chi(C_{2m})=2$ and vertex v_{2m+1} taking the same color of v_{2i} for $1\leq \mathrm{i}\leq \mathrm{m}$. Let c_1,c_2 be the two colors we use for coloring $J_{2,m}$.

With respect to φ^+, the vertices $v_1,v_3,v_5,\cdots,v_{2m-1}$, get the color c_1 and the vertices $v_2,v_4,v_6,\cdots.v_{2m},v_{2m+1}$, get the color c_2 because the number of vertices $v_2,v_4,v_6,\cdots,v_{2m+1}$ is bigger than the number of vertices $v_1,v_3,v_5,\cdots,v_{2m-1}$. We have $V_1{}^+=\{v_1,v_3,v_5,\cdots,v_{2m-1}\}$, $V_2{}^+=\{v_2,v_4,v_6,\cdots,v_{2m},v_{2m+1}\}$. As we can see in Figure 2, the black vertices take the color which is c_1 and the white vertices take the color which is c_2. □

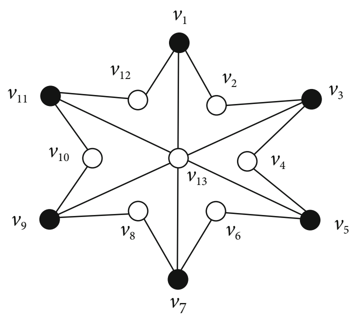

FIGURE 2: Jahangir graph $J_{2,6}$.

In Table 1, we will show in the first column the possible distance between different pairs of vertices $J_{2,m}$, in the second column, we will show the number of corresponding color pairs, and in the third column, we will show the number color pairs.

From the Table 1, we have

$$
\begin{aligned}
S_{\chi^+}(J_{2,m},x) &= 4\left(\frac{1}{2}m(m-3)\right)x^4 + 3m(m-2)x^3 \\
&\quad + \left(2\left(\frac{1}{2}m(m-1)\right) + 4(2m)\right)x^2 \\
&\quad + 3(3m)x^1 + 2m + 4(m+1) \\
&= \left(2m^2-6m\right)x^4 + \left(3m^2-6m\right)x^3 \\
&\quad + \left(m^2+7m\right)x^2 + 9mx^1 + 6m + 4, \\
\mathrm{Gut}_{\chi^+}(J_{2,m},x) &= 4\left(\frac{1}{2}m(m-3)\right)x^4 + 2m(m-2)x^3 \\
&\quad + \left(\frac{1}{2}m(m-1) + 4(2m)\right)x^2 \\
&\quad + 2(3m)x^1 + m + 4(m+1) \\
&= \left(2m^2-6m\right)x^4 + \left(2m^2-4m\right)x^3 \\
&\quad + \left(\frac{1}{2}m^2 + \frac{15}{2}m\right)x^2 + 6mx^1 + 5m + 4, \\
H^+(J_{2.m},x,2) &= \frac{1}{2}(m(m-3))x_2^4 + m(m-2)x_{1,2}^3 \\
&\quad + \frac{1}{2}m(m-1)x_1^2 + 2mx_2^2 + 3mx_{1,2}^1, \\
S_{\chi^+}(J_{2,m},x,2) &= 2m(m-3)x_2^4 + 3m(m-2)x_{1,2}^3 \\
&\quad + m(m-1)x_1^2 + 8mx_2^2 + 9mx_{1,2}^1 + 6m + 4, \\
\mathrm{Gut}_{\chi^+}(J_{2.m},x,2) &= 2m(m-3)x_2^4 + 2m(m-2)x_{1,2}^3 \\
&\quad + \frac{1}{2}m(m-1)x_1^2 + 8\,mx_2^2 + 6\,mx_{1,2}^1 + 5m + 4.
\end{aligned}
\tag{16}
$$

Theorem 11.

$$
\begin{aligned}
S_{\chi^-}(J_{2,m},x) &= \left(m^2-3m\right)x^4 + \left(3m^2-6m\right)x^3 \\
&\quad + \left(2m^2+2m\right)x^2 + 9mx^1 + 6m + 2, \\
Gut_{\chi^-}(J_{2,m},x) &= \left(\frac{1}{2}m^2 - \frac{3}{2}m\right)x^4 + \left(2m^2-4m\right)x^3 \\
&\quad + 2m^2x^2 + 6mx^1 + 5m + 1, \\
H^-(J_{2,m},x,2) &= \frac{1}{2}(m(m-3))x_1^4 + m(m-2)x_{1,2}^3 \\
&\quad + \frac{1}{2}m(m-2)x_2^2 + 2mx_1^2 + 3mx_{1,2}^1, \\
S_{\chi^-}(J_{2,m},x,2) &= m(m-3)x_1^4 + 3m(m-2)x_{1,2}^3 \\
&\quad + 2m(m-1)x_2^2 + 4\,mx_1^2 + 9mx_{1,2}^1 \\
&\quad + 6m + 2, \\
Gut_{\chi^-}(J_{2,m},x,2) &= \frac{1}{2}m(m-3)x_1^4 + 2m(m-2)x_{1,2}^3 \\
&\quad + 2m(m-1)x_2^2 + 2mx_1^2 + 6mx_{1,2}^1 \\
&\quad + 5m + 1.
\end{aligned}
\tag{17}
$$

TABLE 1: Shows the distance sequence in Jahangir graph $J_{2,m}$.

$d(u,v)$	Color	Number of pairs
0	(c_1,c_1)	m
	(c_2,c_2)	$m+1$
1	(c_1,c_2)	3 m
2	(c_1,c_1)	$1/2m(m-1)$
	(c_2,c_2)	2 m
3	(c_1,c_2)	$m(m-2)$
4	(c_2,c_2)	$1/2(m(m-3))$

Proof. With respect to φ^-, the vertices $v_1, v_3, v_5, \cdots, v_{2m-1}$ get the color c_2, and the vertices $v_2, v_4, v_6, \cdots, v_{2m}, v_{2m+1}$ get the color c_1. We have $V_1{}^- = \{v_2, v_4, v_6, \cdots, v_{2m}, v_{2m+1}\}$, $V_2{}^- = \{v_1, v_3, v_5, \cdots, v_{2m-1}\}$. From the Table 1, we exchange each c_1 with c_2 and each c_2 with c_1, so we have

$$
\begin{aligned}
S_{\chi^-}(J_{2,m},x) &= 2\left(\frac{1}{2}m(m-3)\right)x^4 + 3m(m-2)x^3 \\
&\quad + \left(4\left(\frac{1}{2}m(m-1)\right) + 2(2m)\right)x^2 \\
&\quad + 3(3m)x^1 + 4m + 2(m+1) \\
&= \left(m^2-3m\right)x^4 + \left(3m^2-6m\right)x^3 \\
&\quad + \left(2m^2+2m\right)x^2 + 9mx^1 + 6m + 2, \\
\mathrm{Gut}_{\chi^-}(J_{2,m},x) &= 1\left(\frac{1}{2}m(m-3)\right)x^4 + 2m(m-2)x^3 \\
&\quad + \left(4\left(\frac{1}{2}m(m-1)\right) + 1(2m)\right)x^2 \\
&\quad + 2(3m)x^1 + 4m + 1(m+1) \\
&= \left(\frac{1}{2}m^2 - \frac{3}{2}m\right)x^4 + \left(2m^2-4m\right)x^3 \\
&\quad + 2m^2x^2 + 6mx^1 + 5m + 1,
\end{aligned}
$$

$$H^{-}(J_{2,m},x,2)=\frac{1}{2}(m(m-3))x_1^4+m(m-2)x_{1,2}^3+\frac{1}{2}m(m-1)x_2^2+2mx_1^2+3mx_{1,2}^1,$$
$$S_{\chi^-}(J_{2,m},x,2)=m(m-3)x_1^4+3m(m-2)x_{1,2}^3+2m(m-1)x_2^2+4mx_1^2+9mx_{1,2}^1+6m+2, \tag{18}$$
$$\mathrm{Gut}_{\chi^-}(J_{2,m},x,2)=\frac{1}{2}m(m-3))x_1^4+2m(m-2)x_{1,2}^3+2m(m-1)x_2^2+2mx_1^2+6mx_{1,2}^1+5m+1.$$

□

Theorem 12. *For m is even, we have*

$$S_{\chi^+}(J_{3,m},x)=\left(10m^2-25m\right)x^4+\left(10m^2-10m\right)x^3+\left(\frac{5m^2}{2}+\frac{39m}{2}\right)x^2+\frac{37m}{2}x^1+15m+2,$$
$$Gut_{\chi^+}(J_{3,m},x)=\left(\frac{25m^2}{2}-31m\right)x^4+\left(\frac{25m^2}{2}-13m\right)x^3+\left(\frac{25m^2}{8}+\frac{85m}{4}\right)x^2+\frac{41m}{2}x^1+\frac{39m}{2}+1,$$
$$H^{+}(J_{3,m},x,3)=\frac{m}{2}(m-2)x_2^4+m(m-3)x_{2,3}^4+\frac{m}{2}(m-2)x_3^4+\frac{m}{2}(m-2)x_2^3+m^2x_{2,3}^3+\frac{m}{2}(m-2)x_3^3+mx_{1,2}^2+mx_{1,3}^2+\left(\frac{m^2}{8}+\frac{5m}{4}\right)x_2^2+\frac{m^2}{4}x_{2,3}^2+\left(\frac{m^2}{8}+\frac{5m}{4}\right)x_3^2+\frac{m}{2}x_{1,2}^1+\frac{m}{2}x_{1,3}^1+3mx_{2,3}^1,$$
$$S_{\chi^+}(J_{3,m},x,3)=2m(m-2)x_2^4+5m(m-3)x_{2,3}^4+3m(m-2)x_3^4+2m(m-2)x_2^3+5m^2x_{2,3}^3+3m(m-2)x_3^3+3mx_{1,2}^2+4mx_{1,3}^2+\left(\frac{m^2}{2}+5m\right)x_2^2+\frac{5m^2}{4}x_{2,3}^2+3\left(\frac{m^2}{4}+\frac{5m}{2}\right)x_3^2+\frac{3m}{2}x_{1,2}^1+2mx_{1,3}^1+15mx_{2,3}^1+15m+2,$$
$$Gut_{\chi^+}(J_{3,m},x,3)=2m(m-2)x_2^4+6m(m-3)x_{2,3}^4+\frac{9m}{2}(m-2)x_3^4+2m(m-2)x_2^3+6m^2x_{2,3}^3+\frac{9m}{2}(m-2)x_3^3+2mx_{1,2}^2+3mx_{1,3}^2+\left(\frac{m^2}{2}+5m\right)x_2^2+\frac{3m^2}{2}x_{2,3}^2+\left(\frac{9m^2}{8}+\frac{45m}{4}\right)x_3^2+mx_{1,2}^1+\frac{3m}{2}x_{1,3}^1+18mx_{2,3}^1+\frac{39m}{2}+1,$$
$$S_{\chi^-}(J_{3,m},x)=\left(6m^2-15m\right)x^4+\left(6m^2-6m\right)x^3+\left(\frac{3m^2}{2}+\frac{33m}{2}\right)x^2+\frac{27m}{2}x^1+9m+6,$$
$$Gut_{\chi^-}(J_{3,m},x)=\left(\frac{9m^2}{2}-11m\right)x^4+\left(\frac{9m^2}{2}-5m\right)x^3+\left(\frac{9m^2}{8}+\frac{61m}{4}\right)x^2+\frac{21m}{2}x^1+\frac{15m}{2}+9,$$
$$H^{-}(J_{3,m},x,3)=\frac{m}{2}(m-2)x_1^4+m(m-3)x_{1,2}^4+\frac{m}{2}(m-2)x_2^4+\frac{m}{2}(m-2)x_1^3+m^2x_{1,2}^3+\frac{m}{2}(m-2)x_2^3+\left(\frac{m^2}{8}+\frac{5m}{4}\right)x_1^2+\frac{m^2}{4}x_{1,2}^2+mx_{1,3}^2+\left(\frac{m^2}{8}+\frac{5m}{4}\right)x_2^2+mx_{2,3}^2+3mx_{1,2}^1+\frac{m}{2}x_{1,3}^1+\frac{m}{2}x_{2,3}^1,$$
$$S_{\chi^-}(J_{3,m},x,3)=m(m-2)x_1^4+3m(m-3)x_{1,2}^4+2m(m-2)x_2^4+m(m-2)x_1^3+3m^2x_{1,2}^3+2m(m-2)x_2^3+\left(\frac{m^2}{4}+\frac{5m}{2}\right)x_1^2+\frac{3m^2}{4}x_{1,2}^2+4mx_{1,3}^2+\left(\frac{m^2}{2}+5m\right)x_2^2+5mx_{2,3}^2+9mx_{1,2}^1+2mx_{1,3}^1+\frac{5m}{2}x_{2,3}^1+9m+6,$$
$$Gut_{\chi^-}(J_{3,m},x,3)=\frac{m}{2}(m-2)x_1^4+3m(m-3)x_{1,2}^4+2m(m-2)x_2^4+\frac{m}{2}(m-2)x_1^3+2m^2x_{1,2}^3+2m(m-2)x_2^3+\left(\frac{m^2}{8}+\frac{5m}{4}\right)x_1^2+\frac{m^2}{2}x_{1,2}^2+3mx_{1,3}^2+\left(\frac{m^2}{2}+5m\right)x_2^2+6mx_{2,3}^2+6mx_{1,2}^1+\frac{3m}{2}x_{1,3}^1+3mx_{2,3}^1+\frac{15m}{2}+9. \tag{19}$$

Proof. Let number $V=\{v_1,v_2,v_3,\cdots,v_{3m},v_{3m+1}\}$ be the vertices of $J_{3,m}$. It is obvious that $\chi(J_{3,m})=3$ because C_{3m}

contains an even number of vertices, therefore $\chi(C_{2m})=2$ and vertex v_{3m+1} taking a different color of color C_{3m}. Let c_1, c_2, c_3 be the three colors we use for coloring $J_{3,m}$.

With respect to φ^+, the vertices v_{3m+1} get the color c_1, the vertices $v_2, v_4, v_6, \cdots . v_{3m}$ get the color c_2, and the vertices $v_1, v_3, v_5, \cdots . v_{3m-1}$ get the color c_3. We have ${V_1}^+ = \{v_{3m+1}\}$, ${V_2}^+ = \{v_2, v_4, v_6, \cdots, v_{3m}\}$, ${V_3}^+ = \{v_1, v_3, v_5, \cdots, v_{3m-1}\}$. As we can see in Figure 3, the black vertices take the color which is c_3, the white vertices take the color which is c_2, and the black and white vertices take the color c_1. □

From the Table 2, we have

$$S_{\chi^+}(J_{3,m}, x) = \left(4\frac{m}{2}(m-2) + 5m(m-3) + 6\frac{m}{2}(m-2)\right)x^4 + \left(4\frac{m}{2}(m-2) + 5m^2 + 6\frac{m}{2}(m-2)\right)x^3 + \left(3m + 4m + 4\left(\frac{m^2}{8} + \frac{5m}{4}\right) + 5\frac{m^2}{4} + 6\left(\frac{m^2}{8} + \frac{5m}{4}\right)\right)x^2 + \left(3\frac{m}{2} + 4\frac{m}{2} + 5(3m)\right)x^1 + 2(1) + 4\frac{3m}{2} + 6\frac{3m}{2}$$
$$= \left(10m^2 - 25m\right)x^4 + \left(10m^2 - 10m\right)x^3 + \left(\frac{5m^2}{2} + \frac{39m}{2}\right)x^2 + \frac{37m}{2}x^1 + 15m + 2,$$

$$\mathrm{Gut}_{\chi^+}(J_{3,m}, x) = \left(4\frac{m}{2}(m-2) + 6m(m-3) + 9\frac{m}{2}(m-2)\right)x^4 + \left(4\frac{m}{2}(m-2) + 6m^2 + 9\frac{m}{2}(m-2)\right)x^3 + \left(2m + 3m + 4\left(\frac{m^2}{8} + \frac{5m}{4}\right) + 6\frac{m^2}{4} + 9\left(\frac{m^2}{8} + \frac{5m}{4}\right)\right)x^2 + \left(2\frac{m}{2} + 3\frac{m}{2} + 6(3m)\right)x^1 + 1(1) + 4\frac{3m}{2} + 9\frac{3m}{2}$$
$$= \left(\frac{25m^2}{2} - 31m\right)x^4 + \left(\frac{25m^2}{2} - 13m\right)x^3 + \left(\frac{25m^2}{8} + \frac{85m}{4}\right)x^2 + \frac{41m}{2}x^1 + \frac{39m}{2} + 1,$$

$$H^+(J_{3,m}, x, 3) = \frac{m}{2}(m-2)x_2^4 + m(m-3)x_{2,3}^4 + \frac{m}{2}(m-2)x_3^4 + \frac{m}{2}(m-2)x_2^3 + m^2x_{2,3}^3 + \frac{m}{2}(m-2)x_3^3 + mx_{1,2}^2 + mx_{1,3}^2 + \left(\frac{m^2}{8} + \frac{5m}{4}\right)x_2^2 + \frac{m^2}{4}x_{2,3}^2 + \left(\frac{m^2}{8} + \frac{5m}{4}\right)x_3^2 + \frac{m}{2}x_{1,2}^1 + \frac{m}{2}x_{1,3}^1 + 3mx_{2,3}^1,$$

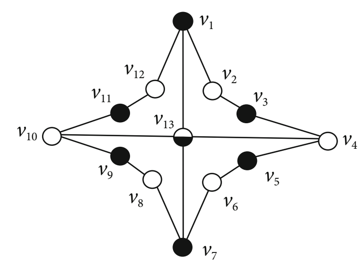

FIGURE 3: Jahangir graph $J_{3,4}$.

TABLE 2: Shows the distance sequence in Jahangir graph $J_{3,m}$ for m is even.

$d(u,v)$	Color	Number of pairs
0	(c_1, c_1)	1
	(c_2, c_2)	$3m/2$
	(c_3, c_3)	$3m/2$
1	(c_1, c_2)	$m/2$
	(c_1, c_3)	$m/2$
	(c_2, c_3)	$3m$
2	(c_1, c_2)	m
	(c_1, c_3)	m
	(c_2, c_2)	$(m^2/8) + (5m/4)$
	(c_2, c_3)	$m^2/4$
	(c_3, c_3)	$(m^2/8) + (5m/4)$
3	(c_2, c_2)	$m/2(m-2)$
	(c_2, c_3)	m^2
	(c_3, c_3)	$m/2(m-2)$
4	(c_2, c_2)	$m/2(m-2)$
	(c_2, c_3)	$m(m-3)$
	(c_3, c_3)	$m/2(m-2)$

$$S_{\chi^+}(J_{3,m}, x, 3) = 2m(m-2)x_2^4 + 5m(m-3)x_{2,3}^4 + 3m(m-2)x_3^4 + 2m(m-2)x_2^3 + 5m^2x_{2,3}^3 + 3m(m-2)x_3^3 + 3mx_{1,2}^2 + 4mx_{1,3}^2 + \left(\frac{m^2}{2} + 5m\right)x_2^2 + \frac{5m^2}{4}x_{2,3}^2 + 3\left(\frac{m^2}{4} + \frac{5m}{2}\right)x_3^2 + \frac{3m}{2}x_{1,2}^1 + 2mx_{1,3}^1 + 15mx_{2,3}^1 + 15m + 2,$$

$$Gut_{\chi^+}(J_{3,m}, x, 3) = 2m(m-2)x_2^4 + 6m(m-3)x_{2,3}^4 + \frac{9m}{2}(m-2)x_3^4 + 2m(m-2)x_2^3 + 6m^2x_{2,3}^3 + \frac{9m}{2}(m-2)x_3^3 + 2mx_{1,2}^2$$

$$
\begin{aligned}
&+3mx_{1,3}^{2}+\left(\frac{m^{2}}{2}+5m\right)x_{2}^{2}+\frac{3m^{2}}{2}x_{2,3}^{2}\\
&+\left(\frac{9m^{2}}{8}+\frac{45m}{4}\right)x_{3}^{2}+mx_{1,2}^{1}+\frac{3m}{2}x_{1,3}^{1}\\
&+18mx_{2,3}^{1}+\frac{39m}{2}+1.
\end{aligned}\tag{20}
$$

With respect to φ^{-}, the vertices v_{3m+1} get the color c_3, the vertices $v_2, v_4, v_6, \cdots . v_{3m}$ get the color c_2, and the vertices $v_1, v_3, v_5, \cdots . v_{3m-1}$ get the color c_1. We have $V_1^{-} = \{v_1, v_3, v_5, \cdots, v_{3m-1}\}$, $V_2^{-} = \{v_2, v_4, v_6, \cdots, v_{3m}\}$, $V_3^{-} = \{v_{3m+1}\}$. From the Table 2, we exchange each c_1 with c_3 and each c_3 with c_1, so we have

$$
\begin{aligned}
S_{\chi^-}(J_{3,m},x)&=\left(2\frac{m}{2}(m-2)+3m(m-3)+4\frac{m}{2}(m-2)\right)x^{4}\\
&\quad+\left(2\frac{m}{2}(m-2)+3m^{2}+4\frac{m}{2}(m-2)\right)x^{3}\\
&\quad+\left(2\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)+3\frac{m^{2}}{4}+4m\right.\\
&\quad\left.+4\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)+5m\right)x^{2}\\
&\quad+\left(3(3m)+4\frac{m}{2}+5\frac{m}{2}\right)x^{1}+2\frac{3m}{2}\\
&\quad+4\frac{3m}{2}+6(1)\\
&=\left(6m^{2}-15m\right)x^{4}+\left(6m^{2}-6m\right)x^{3}\\
&\quad+\left(\frac{3m^{2}}{2}+\frac{33m}{2}\right)x^{2}+\frac{27m}{2}x^{1}+9m+6,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Gut}_{\chi^-}(J_{3,m},x)&=\left(1\frac{m}{2}(m-2)+2m(m-3)+4\frac{m}{2}(m-2)\right)x^{4}\\
&\quad+\left(1\frac{m}{2}(m-2)+4\frac{m}{2}(m-2)+2m^{2}\right)x^{3}\\
&\quad+\left(1\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)+2\frac{m^{2}}{4}+3m+6m\right.\\
&\quad\left.+4\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)\right)x^{2}\\
&\quad+\left(2(3m)+3\frac{m}{2}+6\frac{m}{2}\right)x^{1}\\
&\quad+1\frac{3m}{2}+4\frac{3m}{2}+9(1)\\
&=\left(\frac{9m^{2}}{2}-11m\right)x^{4}+\left(\frac{9m^{2}}{2}-5m\right)x^{3}\\
&\quad+\left(\frac{9m^{2}}{8}+\frac{61m}{4}\right)x^{2}+\frac{21m}{2}x^{1}+\frac{15m}{2}+9,
\end{aligned}
$$

$$
\begin{aligned}
H^{-}(J_{3,m},x,3)&=\frac{m}{2}(m-2)x_{1}^{4}+m(m-3)x_{1,2}^{4}+\frac{m}{2}(m-2)x_{2}^{4}\\
&\quad+\frac{m}{2}(m-2)x_{1}^{3}+m^{2}x_{1,2}^{3}+\frac{m}{2}(m-2)x_{2}^{3}\\
&\quad+\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)x_{1}^{2}+\frac{m^{2}}{4}x_{1,2}^{2}+mx_{1,3}^{2}\\
&\quad+\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)x_{2}^{2}+mx_{2,3}^{2}+3mx_{1,2}^{1}\\
&\quad+\frac{m}{2}x_{1,3}^{1}+\frac{m}{2}x_{2,3}^{1},
\end{aligned}
$$

$$
\begin{aligned}
S_{\chi^-}(J_{3,m},x,3)&=m(m-2)x_{1}^{4}+3m(m-3)x_{1,2}^{4}+2m(m-2)x_{2}^{4}\\
&\quad+m(m-2)x_{1}^{3}+3m^{2}x_{1,2}^{3}+2m(m-2)x_{2}^{3}\\
&\quad+\left(\frac{m^{2}}{4}+\frac{5m}{2}\right)x_{1}^{2}+\frac{3m^{2}}{4}x_{1,2}^{2}+4mx_{1,3}^{2}\\
&\quad+\left(\frac{m^{2}}{2}+5m\right)x_{2}^{2}+5mx_{2,3}^{2}+9mx_{1,2}^{1}\\
&\quad+2mx_{1,3}^{1}+\frac{5m}{2}x_{2,3}^{1}+9m+6,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Gut}_{\chi^-}(J_{3,m},x,3)&=\frac{m}{2}(m-2)x_{1}^{4}+3m(m-3)x_{1,2}^{4}\\
&\quad+2m(m-2)x_{2}^{4}+\frac{m}{2}(m-2)x_{1}^{3}+2m^{2}x_{1,2}^{3}\\
&\quad+2m(m-2)x_{2}^{3}+\left(\frac{m^{2}}{8}+\frac{5m}{4}\right)x_{1}^{2}\\
&\quad+\frac{m^{2}}{2}x_{1,2}^{2}+3mx_{1,3}^{2}+\left(\frac{m^{2}}{2}+5m\right)x_{2}^{2}\\
&\quad+6mx_{2,3}^{2}+6mx_{1,2}^{1}+\frac{3m}{2}x_{1,3}^{1}+3mx_{2,3}^{1}\\
&\quad+\frac{15m}{2}+9.
\end{aligned}\tag{21}
$$

Theorem 13. *For m is odd, we have*

$$
\begin{aligned}
S_{\chi^+}(J_{3,m},x)&=\left(10m^{2}-29m+10\right)x^{4}\\
&\quad+\left(10m^{2}-11m-2\right)x^{3}\\
&\quad+\left(\frac{5m^{2}}{2}+20m-\frac{11}{2}\right)x^{2}+\left(\frac{37m}{2}-\frac{5}{2}\right)x^{1}\\
&\quad+15m-1,
\end{aligned}
$$

$$
\begin{aligned}
Gut_{\chi^+}(J_{3,m},x)&=\left(\frac{25m^{2}}{2}-41m+\frac{47}{2}\right)x^{4}\\
&\quad+\left(\frac{25m^{2}}{2}-\frac{31m}{2}-3\right)x^{3}\\
&\quad+\left(\frac{25m^{2}}{8}+\frac{45m}{2}-\frac{93}{8}\right)x^{2}\\
&\quad+\left(\frac{41m}{2}-\frac{13}{2}\right)x^{1}+\frac{39m}{2}-\frac{9}{2},
\end{aligned}
$$

$$
\begin{aligned}
H^{+}(J_{3,m},x,3)&=(m-3)x_{1,2}^{4}+(m-2)x_{1,3}^{4}+\frac{m^{2}-2m-1}{2}x_{2}^{4}\\
&\quad+\left(m^{2}-4m+4\right)x_{2,3}^{4}+\frac{m^{2}-4m+3}{2}x_{3}^{4}\\
&\quad+\frac{m+1}{2}x_{1,2}^{3}+\frac{m-1}{2}x_{1,3}^{3}+\frac{m^{2}-3m+4}{2}x_{2}^{3}
\end{aligned}
$$

$$+\frac{2m^2-m-5}{2}x_{2,3}^3+\frac{m^2-2m+1}{2}x_3^3+x_1^2+(m+1)x_{1,2}^2+mx_{1,3}^2+\frac{m^2+8m-9}{8}x_2^2+\frac{m^2+3}{4}x_{2,3}^2+\frac{m^2+12m-13}{8}x_3^2+\frac{m+1}{2}x_{1,2}^1+\frac{m+3}{2}x_{1,3}^1+(3m-2)x_{2,3}^1,$$

$$S_{\chi^+}(J_{3,m},x,3)=3(m-3)x_{1,2}^4+4(m-2)x_{1,3}^4+2\left(m^2-2m-1\right)x_2^4+5\left(m^2-4m+4\right)x_{2,3}^4+3\left(m^2-4m+3\right)x_3^4+\frac{3m+3}{2}x_{1,2}^3+2(m-1)x_{1,3}^3+2\left(m^2-3m+4\right)x_2^3+\frac{10m^2-5m-25}{2}x_{2,3}^3+3\left(m^2-2m+1\right)x_3^3+2x_1^2+3(m+1)x_{1,2}^2+4mx_{1,3}^2+\frac{m^2+8m-9}{2}x_2^2+\frac{5m^2+15}{4}x_{2,3}^2+\frac{3m^2+36m-39}{4}x_3^2+\frac{3m+3}{2}x_{1,2}^1+2(m+3)x_{1,3}^1+5(3m-2)x_{2,3}^1+15m-1,$$

$$Gut_{\chi^+}(J_{3,m},x,3)=2(m-3)x_{1,2}^4+3(m-2)x_{1,3}^4+2\left(m^2-2m-1\right)x_2^4+6\left(m^2-4m+4\right)x_{2,3}^4+\frac{9m^2-36m+27}{2}x_3^4+(m+1)x_{1,2}^3+\frac{3m-3}{2}x_{1,3}^3+2\left(m^2-3m+4\right)x_2^3+3\left(2m^2-m-5\right)x_{2,3}^3+\frac{9m^2-18m+9}{2}x_3^3+x_1^2+2(m+1)x_{1,2}^2+3mx_{1,3}^2+\frac{m^2+8m-9}{2}x_2^2+\frac{3m^2+9}{2}x_{2,3}^2+\frac{9m^2+108m-117}{8}x_3^2+(m+1)x_{1,2}^1+\frac{3m+9}{2}x_{1,3}^1+6(3m-2)x_{2,3}^1+\frac{39m}{2}-\frac{9}{2},$$

$$S_{\chi^-}(J_{3,m},x)=\left(6m^2-11m-10\right)x^4+\left(6m^2-5m+2\right)x^3+\left(\frac{3m^2}{2}+16m+\frac{1}{2}\right)x^2+\left(\frac{27m}{2}+\frac{5}{2}\right)x^1+9m+9.$$

$$Gut_{\chi^-}(J_{3,m},x)=\left(\frac{9m^2}{2}-5m-\frac{33}{2}\right)x^4+\left(\frac{9m^2}{2}-\frac{7m}{2}+5\right)x^3+\left(\frac{9m^2}{2}+\frac{29m}{2}+\frac{83}{8}\right)x^2+\left(\frac{21m}{2}+\frac{7}{2}\right)x^1+\frac{15m}{2}+\frac{31}{2},$$

$$H^-(J_{3,m},x,3)=\frac{m^2-4m+3}{2}x_1^4+\left(m^2-4m+4\right)x_{1,2}^4+(m-2)x_{1,3}^4+\frac{m^2-2m-1}{2}x_2^4+(m-3)x_{2,3}^4+\frac{m^2-2m+1}{2}x_1^3+\frac{2m^2-m-5}{2}x_{1,2}^3+\frac{m-1}{2}x_{1,3}^3+\frac{m^2-3m+4}{2}x_2^3+\frac{m+1}{2}x_{2,3}^3+\frac{m^2+12m-13}{8}x_1^2+\frac{m^2+3}{4}x_{1,2}^2+mx_{1,3}^2+\frac{m^2+8m-9}{8}x_2^2+(m+1)x_{2,3}^2+x_{3,3}^2+(3m-2)x_{1,2}^1+\frac{m+3}{2}x_{1,3}^1+\frac{m+1}{2}x_{2,3}^1,$$

$$S_{\chi^-}(J_{3,m},x,3)=\left(m^2-4m+3\right)x_1^4+3\left(m^2-4m+4\right)x_{1,2}^4+4(m-2)x_{1,3}^4+2\left(m^2-2m-1\right)x_2^4+5(m-3)x_{2,3}^4+\left(m^2-2m+1\right)x_1^3+\frac{6m^2-3m-15}{2}x_{1,2}^3+2(m-1)x_{1,3}^3+2\left(m^2-3m+4\right)x_2^3+\frac{5m+5}{2}x_{2,3}^3+\frac{m^2+12m-13}{4}x_1^2+\frac{3m^2+9}{4}x_{1,2}^2+4mx_{1,3}^2+\frac{m^2+8m-9}{2}x_2^2+5(m+1)x_{2,3}^2+6x_{3,3}^2+3(3m-2)x_{1,2}^1+2(m+3)x_{1,3}^1+\frac{5m+5}{2}x_{2,3}^1+9m+9,$$

$$Gut_{\chi^-}(J_{3,m},x,3)=\frac{m^2-4m+3}{2}x_1^4+2\left(m^2-4m+4\right)x_{1,2}^4+3(m-2)x_{1,3}^4+2\left(m^2-2m-1\right)x_2^4+6(m-3)x_{2,3}^4+\frac{m^2-2m+1}{2}x_1^3+\left(2m^2-m-5\right)x_{1,2}^3+\frac{3m-3}{2}x_{1,3}^3+2\left(m^2-3m+4\right)x_2^3+3(m+1)x_{2,3}^3+\frac{m^2+12m-13}{8}x_1^2+\frac{m^2+3}{2}x_{1,2}^2$$

$$+ 3mx_{1,3}^2 + \frac{m^2+8m-9}{2}x_2^2 + 6(m+1)x_{2,3}^2 + 9x_{3,3}^2 + 2(3m-2)x_{1,2}^1 + \frac{3m+9}{2}x_{1,3}^1 + 3(m+1)x_{2,3}^1 + \frac{15m}{2} + \frac{31}{2}. \quad (22)$$

Proof. Let number $V = \{v_1, v_2, v_3, \cdots, v_{3m}, v_{3m+1}\}$ be the vertices of $J_{3,m}$. It is obvious that $\chi(J_{3,m}) = 3$ because C_{3m} contains an odd number of vertices, therefore $\chi(C_{3m}) = 3$ and vertex v_{3m+1} are taking the same color of v_{3m}. Let c_1, c_2, c_3 be the three color we use for coloring $J_{3,m}$.

With respect to φ^+, the vertices v_{3m+1}, v_{3m} get the color c_1, the vertices $v_2, v_4, v_6, \cdots . v_{3m-1}$ get the color c_2, and the vertices $v_1, v_3, v_5, \cdots . v_{3m-2}$ get the color c_3. We have $V_1{}^+ = \{v_{3m+1}, v_{3m}\}$, $V_2{}^+ = \{v_2, v_4, \cdots, v_{3m-1}\}$, $V_3{}^+ = \{v_1, v_3, \cdots, v_{3m-2}\}$. As we can see in Figure 4, the black vertices take the color which is c_3, the white vertices take the color which is c_2, and the black and white vertices take the color c_1. □

From the Table 3, we have

$$\begin{aligned} S_{\chi^+}(J_{3,m}, x) = & \Bigg(3(m-3) + 4(m-2) + 4\frac{m^2-2m-1}{2} + 5(m^2-4m+4) + 6\frac{m^2-4m+3}{2}\Bigg)x^4 \\ & + \Bigg(3\frac{m+1}{2} + 4\frac{m-1}{2} + 4\frac{m^2-3m+4}{2} + 5\frac{2m^2-m-5}{2} + 6\frac{m^2-2m+1}{2}\Bigg)x^3 \\ & + \Bigg(2(1) + 3(m+1) + 4m + 4\frac{m^2+8m-9}{8} + 5\frac{m^2+3}{4} + 6\frac{m^2+12m-13}{8}\Bigg)x^2 \\ & + \Bigg(3\frac{m+1}{2} + 4\frac{m+3}{2} + 5(3m-2)\Bigg)x^1 + 2(2) + 4\frac{3m-1}{2} + 6\frac{3m-1}{2} \\ = & \left(10m^2 - 29m + 10\right)x^4 + \left(10m^2 - 11m - 2\right)x^3 + \left(\frac{5m^2}{2} + 20m - \frac{11}{2}\right)x^2 + \left(\frac{37m}{2} - \frac{5}{2}\right)x^1 + 15m - 1, \end{aligned}$$

$$\begin{aligned} \mathrm{Gut}_{\chi^+}(J_{3,m}, x) = & \Bigg(2(m-3) + 3(m-2) + 4\frac{m^2-2m-1}{2} + 6(m^2-4m+4) + 9\frac{m^2-4m+3}{2}\Bigg)x^4 \\ & + \Bigg(2\frac{m+1}{2} + 3\frac{m-1}{2} + 4\frac{m^2-3m+4}{2} + 6\frac{2m^2-m-5}{2} + 9\frac{m^2-2m+1}{2}\Bigg)x^3 \\ & + \Bigg(1(1) + 2(m+1) + 3m + 4\frac{m^2+8m-9}{8} + 6\frac{m^2+3}{4} + 9\frac{m^2+12m-13}{8}\Bigg)x^2 \\ & + \Bigg(2\frac{m+1}{2} + 3\frac{m+3}{2} + 6(3m-2)\Bigg)x^1 \end{aligned}$$

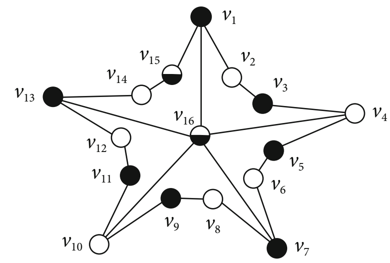

FIGURE 4: Jahangir graph $J_{3,5}$.

TABLE 3: Shows the distance sequence in Jahangir graph $J_{3,m}$ for m is odd.

$d(u,v)$	Color	Number of pairs
0	(c_1, c_1)	2
	(c_2, c_2)	$(3m-1)/2$
	(c_3, c_3)	$(3m-1)/2$
1	(c_1, c_2)	$(m+1)/2$
	(c_1, c_3)	$(m+3)/2$
	(c_2, c_3)	$3m-2$
2	(c_1, c_1)	1
	(c_1, c_2)	$m+1$
	(c_1, c_3)	m
	(c_2, c_2)	$(m^2+8m-9)/8$
	(c_2, c_3)	$(m^2+3)/4$
	(c_3, c_3)	$(m^2+12m-13)/8$
3	(c_1, c_2)	$(m+1)/2$
	(c_1, c_3)	$(m-1)/2$
	(c_2, c_2)	$(m^2-3m+4)/2$
	(c_2, c_3)	$(2m^2-m-5)/2$
	(c_3, c_3)	$(m^2-2m+1)/2$
4	(c_1, c_2)	$m-3$
	(c_1, c_3)	$m-2$
	(c_2, c_2)	$(m^2-2m-1)/2$
	(c_2, c_3)	m^2-4m+4
	(c_3, c_3)	$(m^2-4m+3)/2$

$$+1(2)+4\frac{3m-1}{2}+9\frac{3m-1}{2}$$
$$=\left(\frac{25m^2}{2}-41m+\frac{47}{2}\right)x^4$$
$$+\left(\frac{25m^2}{2}-\frac{31m}{2}-3\right)x^3$$
$$+\left(\frac{25m^2}{8}+\frac{45m}{2}-\frac{93}{8}\right)x^2$$
$$+\left(\frac{41m}{2}-\frac{13}{2}\right)x^1+\frac{39m}{2}-\frac{9}{2},$$

$$H^+(J_{3,m},x,3)=(m-3)x_{1,2}^4+(m-2)x_{1,3}^4$$
$$+\frac{m^2-2m-1}{2}x_2^4+\left(m^2-4m+4\right)x_{2,3}^4$$
$$+\frac{m^2-4m+3}{2}x_3^4+\frac{m+1}{2}x_{1,2}^3$$
$$+\frac{m-1}{2}x_{1,3}^3+\frac{m^2-3m+4}{2}x_2^3$$
$$+\frac{2m^2-m-5}{2}x_{2,3}^3+\frac{m^2-2m+1}{2}x_3^3+x_1^2$$
$$+(m+1)x_{1,2}^2+mx_{1,3}^2+\frac{m^2+8m-9}{8}x_2^2$$
$$+\frac{m^2+3}{4}x_{2,3}^2+\frac{m^2+12m-13}{8}x_3^2$$
$$+\frac{m+1}{2}x_{1,2}^1+\frac{m+3}{2}x_{1,3}^1+(3m-2)x_{2,3}^1,$$

$$S_{\chi^+}(J_{3,m},x,3)=3(m-3)x_{1,2}^4+4(m-2)x_{1,3}^4$$
$$+2\left(m^2-2m-1\right)x_2^4$$
$$+5\left(m^2-4m+4\right)x_{2,3}^4$$
$$+3\left(m^2-4m+3\right)x_3^4+\frac{3m+3}{2}x_{1,2}^3$$
$$+2(m-1)x_{1,3}^3+2\left(m^2-3m+4\right)x_2^3$$
$$+\frac{10m^2-5m-25}{2}x_{2,3}^3$$
$$+3\left(m^2-2m+1\right)x_3^3+2x_1^2+3(m+1)x_{1,2}^2$$
$$+4mx_{1,3}^2+\frac{m^2+8m-9}{2}x_2^2$$
$$+\frac{5m^2+15}{4}x_{2,3}^2+\frac{3m^2+36m-39}{4}x_3^2$$
$$+\frac{3m+3}{2}x_{1,2}^1+2(m+3)x_{1,3}^1$$
$$+5(3m-2)x_{2,3}^1+15m-1,$$

$$\mathrm{Gut}_{\chi^+}(J_{3,m},x,3)=2(m-3)x_{1,2}^4+3(m-2)x_{1,3}^4$$
$$+2\left(m^2-2m-1\right)x_2^4$$
$$+6\left(m^2-4m+4\right)x_{2,3}^4$$
$$+\frac{9m^2-36m+27}{2}x_3^4+(\mathrm{m}+1)x_{1,2}^3$$
$$+\frac{3m-3}{2}x_{1,3}^3+2\left(m^2-3m+4\right)x_2^3+3\left(2m^2-m-5\right)x_{2,3}^3$$
$$+\frac{9m^2-18m+9}{2}x_3^3+x_1^2+2(m+1)x_{1,2}^2+3mx_{1,3}^2$$
$$+\frac{m^2+8m-9}{2}x_2^2+\frac{3m^2+9}{2}x_{2,3}^2+\frac{9m^2+108m-117}{8}x_3^2$$
$$+(m+1)x_{1,2}^1+\frac{3m+9}{2}x_{1,3}^1+6(3m-2)x_{2,3}^1+\frac{39m}{2}-\frac{9}{2}. \tag{23}$$

With respect to φ^-, the vertices v_{3m+1}, v_{3m} get the color c_3, the vertices $v_2, v_4, v_6, \cdots .v_{3m-1}$ get the color c_2, and the vertices $v_1, v_3, v_5, \cdots .v_{3m-2}$ get the color c_1. We have $V_1{}^- = \{v_1, v_3, \cdots, v_{3m-2}\}$, $V_2{}^- = \{v_2, v_4, \cdots, v_{3m-1}\}$, $V_3{}^- = \{v_{3m}, v_{3m+1}\}$. From the Table 3, we exchange each c_1 with c_3 and each c_3 with c_1, so we have

$$S_{\chi^-}(J_{3,m},x)=\left(2\frac{m^2-4m+3}{2}+3\left(m^2-4m+4\right)\right.$$
$$\left.+4\frac{m^2-2m-1}{2}+4(m-2)+5(m-3)\right)x^4$$
$$+\left(2\frac{m^2-2m+1}{2}+3\frac{2m^2-m-5}{2}+4\frac{m-1}{2}\right.$$
$$\left.+4\frac{m^2-3m+4}{2}+5\frac{m+1}{2}\right)x^3$$
$$+\left(2\frac{m^2+12m-13}{8}+3\frac{m^2+3}{4}+4m\right.$$
$$\left.+4\frac{m^2+8m-9}{8}+5(m+1)+1(1)\right)x^2$$
$$+\left(3(3m-2)+4\frac{m+3}{2}+5\frac{m+1}{2}\right)x^1$$
$$+2\frac{3m-1}{2}+4\frac{3m-1}{2}+6(2)$$
$$=\left(6m^2-11m-10\right)x^4$$
$$+\left(6m^2-5m+2\right)x^3+\left(\frac{3m^2}{2}+16m+\frac{1}{2}\right)x^2$$
$$+\left(\frac{27m}{2}+\frac{5}{2}\right)x^1+9m+9,$$

$$\mathrm{Gut}_{\chi^-}(J_{3,m},x)=\left(1\frac{m^2-4m+3}{2}+2\left(m^2-4m+4\right)+3(m-2)\right.$$
$$\left.+4\frac{m^2-2m-1}{2}+6(m-3)\right)x^4$$
$$+\left(1\frac{m^2-2m+1}{2}+2\frac{2m^2-m-5}{2}+3\frac{m-1}{2}\right.$$
$$\left.+4\frac{m^2-3m+4}{2}+6\frac{m+1}{2}\right)x^3$$
$$+\left(1\frac{m^2+12m-13}{8}+2\frac{m^2+3}{4}+3m\right.$$
$$\left.+4\frac{m^2+8m-9}{8}+6(m+1)+9(1)\right)x^2$$

$$
\begin{aligned}
&+\left(2(3m-2)+3\frac{m+3}{2}+6\frac{m+1}{2}\right)x^1\\
&+1\frac{3m-1}{2}+4\frac{3m-1}{2}+9(2)\\
&=\left(\frac{9m^2}{2}-5m-\frac{33}{2}\right)x^4\\
&+\left(\frac{9m^2}{2}-\frac{7m}{2}+5\right)x^3\\
&+\left(\frac{9m^2}{2}+\frac{29m}{2}+\frac{83}{8}\right)x^2\\
&+\left(\frac{21m}{2}+\frac{7}{2}\right)x^1+\frac{15m}{2}+\frac{31}{2},
\end{aligned}
$$

$$
\begin{aligned}
H^-(J_{3,m},x,3)={}&\frac{m^2-4m+3}{2}x_1^4+\left(m^2-4m+4\right)x_{1,2}^4\\
&+(m-2)x_{1,3}^4+\frac{m^2-2m-1}{2}x_2^4\\
&+(m-3)x_{2,3}^4+\frac{m^2-2m+1}{2}x_1^3\\
&+\frac{2m^2-m-5}{2}x_{1,2}^3+\frac{m-1}{2}x_{1,3}^3\\
&+\frac{m^2-3m+4}{2}x_2^3+\frac{m+1}{2}x_{2,3}^3\\
&+\frac{m^2+12m-13}{8}x_1^2+\frac{m^2+3}{4}x_{1,2}^2+mx_{1,3}^2\\
&+\frac{m^2+8m-9}{8}x_2^2+(m+1)x_{2,3}^2+x_{3,3}^2\\
&+(3m-2)x_{1,2}^1+\frac{m+3}{2}x_{1,3}^1+\frac{m+1}{2}x_{2,3}^1,
\end{aligned}
$$

$$
\begin{aligned}
S_{\chi^-}(J_{3,m},x,3)={}&\left(m^2-4m+3\right)x_1^4+3\left(m^2-4m+4\right)x_{1,2}^4\\
&+4(m-2)x_{1,3}^4+2\left(m^2-2m-1\right)x_2^4\\
&+5(m-3)x_{2,3}^4+\left(m^2-2m+1\right)x_1^3\\
&+\frac{6m^2-3m-15}{2}x_{1,2}^3+2(m-1)x_{1,3}^3\\
&+2\left(m^2-3m+4\right)x_2^3+\frac{5m+5}{2}x_{2,3}^3\\
&+\frac{m^2+12m-13}{4}x_1^2+\frac{3m^2+9}{4}x_{1,2}^2\\
&+4mx_{1,3}^2+\frac{m^2+8m-9}{2}x_2^2+5(m+1)x_{2,3}^2\\
&+6x_{3,3}^2+3(3m-2)x_{1,2}^1+2(m+3)x_{1,3}^1\\
&+\frac{5m+5}{2}x_{2,3}^1+9m+9,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Gut}_{\chi^-}(J_{3,m},x,3)={}&\frac{m^2-4m+3}{2}x_1^4+2\left(m^2-4m+4\right)x_{1,2}^4\\
&+3(m-2)x_{1,3}^4+2\left(m^2-2m-1\right)x_2^4\\
&+6(m-3)x_{2,3}^4+\frac{m^2-2m+1}{2}x_1^3\\
&+\left(2m^2-m-5\right)x_{1,2}^3+\frac{3m-3}{2}x_{1,3}^3\\
&+2\left(m^2-3m+4\right)x_2^3+3(m+1)x_{2,3}^3\\
&+\frac{m^2+12m-13}{8}x_1^2+\frac{m^2+3}{2}x_{1,2}^2+3mx_{1,3}^2\\
&+\frac{m^2+8m-9}{2}x_2^2+6(m+1)x_{2,3}^2+9x_{3,3}^2+2(3m-2)x_{1,2}^1\\
&+\frac{3m+9}{2}x_{1,3}^1+3(m+1)x_{2,3}^1+\frac{15m}{2}+\frac{31}{2}.
\end{aligned}
\tag{24}
$$

3. Conclusion

In this paper, we computed the chromatic Schultz and Gutman polynomials of Jahangir graph $J_{n,m}$ for $n = 2, 3$. Also, we suggested three new definitions; the first definition is the expanded Hosoya polynomial, where there are expanded H^- − Hosoya polynomial and expanded H^+ − Hosoya polynomial. The second definition is the expanded chromatic Schultz polynomial, where there are expanded χ^+ − chromatic Schultz polynomial and expanded χ^- − chromatic Schultz polynomial. The third definition is the expanded chromatic Gutman polynomial where there are expanded χ^+ − chromatic Gutman polynomial and expanded χ^- − chromatic Gutman polynomial. We computed these suggested definitions of Jahangir graph $J_{n,m}$ for $n = 2, 3$. The importance of these suggested definitions is that they give the colors of vertices in the beginning and the end of every path in the graph.

Data Availability

The data used to support the findings of the study are available upon the author's reasonable request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

The authors contributed equally in the analysis and write-up of the manuscript.

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