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Review Article

Mathematical Modeling and Stability Analysis of Systemic Risk in the Banking Ecosystem

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This paper investigates the dynamics of systemic risk in banking networks by analyzing equilibrium points and stability conditions. The focus is on a model that incorporates interactions among distressed and undistressed banks. The equilibrium points are determined by solving a reduced system of equations, considering both homogeneous and heterogeneous scenarios. Local and global stability analyses reveal conditions under which equilibrium points are stable or unstable. Numerical simulations further illustrate the dynamics of systemic risk, while the theoretical findings offer insights into the behavior of distressed banks under varying conditions. Overall, the model enhances our understanding of systemic financial risk and offers valuable insights for risk management and policymaking in the banking sector.

1. Introduction

Banks as well as other financial institutions play significant roles in a country's development [1]. They provide adequate liquidity and participate strategically in the selection of projects based on economic prospects. The banking system is exposed to various risks, such as credit risk, interest rate risk, market risk, and operational risk. However, credit risk remains the leading cause of banking problems around the world [2]. The severity of the global financial crisis that has been essentially remarkable in the global financial system is an example of the huge negative impact of credit risk exposures within the banking sector (see [3]). Banks have credit risk exposures to one another; the default of one bank can cause the default of the others, which can lead to the failure of the credit system of the entire banking sector. This becomes a systemic risk if such an event occurs from one bank and is transmitted to another through a systematically important channel, with the negative impact spreading from that one entity through another to the entire economy [4].

Many academics and practitioners have studied systemic risk, or the risk of failure through a systemically significant component of a financial system, in recent years. These studies provide a good understanding of how to measure systemic risk by using complex theories and network analysis [5]. Different studies have tried to understand the evolution of systemic risk over time by providing insight into the compartmental nature of the banking industry in specific countries or regions of the world, determining whether control measures are having a measurable effect based on the consequences for the financial system after a systemic event, and the propagation of default cascades through the financial network [6, 7].

In the banking system, there are various types of risks, and new risks can occur at any time. If no intervention is done to prevent the risk spread, losses become very huge.

The objective of this paper is to establish a model that captures the dynamics of the financial system and identifies the key drivers of systemic risks. Therefore, this study is aimed at developing a framework for measuring and analyzing systemic risks using a compartmental model, which can help regulators and policymakers make informed decisions and prevent financial crises. The compartmental model takes into account the interactions between various financial institutions, including banks, insurance companies, and investment firms, as well as the interdependencies between different markets, such as the stock market, bond market, and commodity market. The proposed model provides insights into the dynamics of systemic financial risk in a banking network. Our study represents an important step forward in the measurement, analysis, and control of systemic risks in the banking sector. By developing modeling tools and strategies, we aim to assist policymakers, regulators, and industry experts in their efforts to improve financial system stability and mitigate the risks of future crises. This work contributes to the ongoing dialogue on systemic risk and helps to shape the future of financial regulation and policy. Through numerical simulations, the model's behavior under different initial conditions is explored, shedding light on the potential contagion and stability of the banking system. The findings of this study contribute to a better understanding of the risks inherent in banking systems and can be utilized for risk management and policy formulation to safeguard financial stability. By using this model, the research is aimed at identifying the key drivers of systemic risks and measuring their impact on the stability of the financial system. The *UEDR* model is used to describe the interaction and relationship of the banking network.

2. Model Formulation

In the model, we divide the total number of banks (N) into four categories:

(1) $U(t)$: Risk-free banks are the number of undistressed banks which is healthy but vulnerable even though they have not yet been in distress at time t

(2) $E(t)$: The exposed banks are the number of banks that have been operating with banks at risk and have started to show weak performances in their operations, but the expected loss has not yet been remarkable at time t

(3) $D(t)$: Risk contagious banks, which is the number of banks that are in distress of credit risk at time t and experiencing potential loss

(4) $R(t)$: Recovered is the number of banks that are out of credit risk at time t

(5) $L(t)$: Number of banks that have been distressed and then liquidated at time t

2.1. Assumptions

(1) We assume that every bank in the system under consideration is vulnerable to suffering from credit risk

(2) Every bank is equally likely to be contaminated by the contagious bank(s) in the case of interaction with a risky bank and then become exposed

(3) The risk-exposed banks may recover without being risky and become undistressed, or they may become distressed and move into the class (D).

(4) When the bank is contaminated with the risk, there is no risk management failure; the bank recovers through the banking management system or is liquidated, and the recovered banks move to the recovered compartment

(5) The recovered can lose immunity and become undistressed again

Figure 1 is the schematic presentation of the risk of contagion in the banking network.

The model is expressed as a system of ordinary differential equations as in

$$
\begin{aligned}
U &= -\beta UD + \alpha E + \theta R, \\
\dot{E} &= \beta UD - (\alpha + \sigma)E, \\
\dot{D} &= \sigma E - (\gamma_1 + \gamma_2)D, \\
\dot{R} &= \gamma_1 D - \theta R, \\
\dot{L} &= \gamma_2 D,
\end{aligned}
\tag{1}
$$

where the parameters are described in Table 1.

3. Mathematical Analysis of the Model

3.1. Positivity and Boundedness of the Solution

Theorem 1. *From system (1), let the initial conditions be $U(0) \geq 0$, $E(0) \geq 0$, $D(0) \geq 0$, $L(0) \geq 0$, and $R(0) \geq 0$. Then, the components of the solution $U(t)$, $E(t)$, $D(t)$, $L(t)$, and $R(t)$ are positive and bounded for all $t \geq 0$.*

Proof. Since the UEDRL model is used here to model systemic financial risk in a banking population, it is reasonable to assume that the parameters and variables in all classes are nonnegative; that is, $t \geq 0$. We provide proof that all variables of the model are nonnegative for all given nonnegative initial conditions.

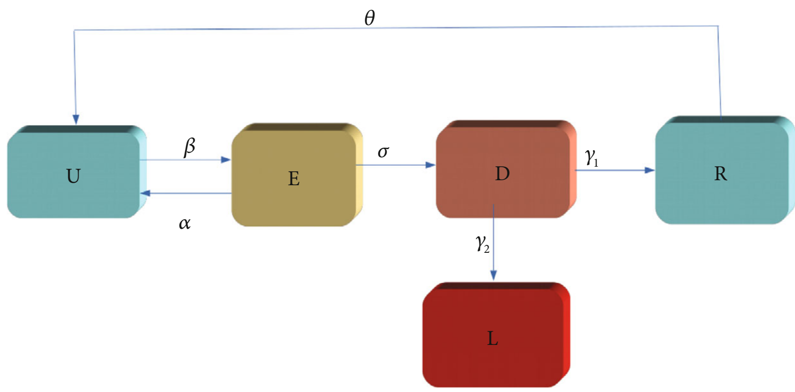

Figure 1: Flow diagram of risk contagion in the banking network.

Table 1: Parameters and their meaning.

Parameter	Description
β	The new rate of contagion risk caused by interactions between undistressed or risk-free banks and banks who are distressed
α	The rate at which the risk-exposed banks turn back to the undistressed banks
σ	The rate at which the risk-exposed banks move to the risk-distressed class
γ_1	The rate at which the distressed banks become recovered and move to the recovered class
γ_2	The rate at which the distressed banks become liquidated
θ	The rate at which the recovered banks loose immunity and turn back to the vulnerable class

From system (1), we have the following:

$$\begin{aligned}
\left.\frac{dU}{dt}\right|_{U=0} &= \alpha E + \theta R \geq 0, \text{since } \alpha \geq 0, \theta \geq 0, E \geq 0, R \geq 0,\\
\left.\frac{dE}{dt}\right|_{E=0} &= \beta U D \geq 0, \text{since } \beta \geq 0, U \geq 0, D \geq 0,\\
\left.\frac{dD}{dt}\right|_{D=0} &= \sigma E \geq 0, \text{since } \sigma \geq 0, E \geq 0,\\
\left.\frac{dR}{dt}\right|_{R=0} &= \gamma_1 D \geq 0, \text{since } \gamma_1 \geq 0, D \geq 0,\\
\left.\frac{dL}{dt}\right|_{L=0} &= \gamma_2 D \geq 0, \text{since } \gamma_2 \geq 0, D \geq 0.
\end{aligned} \tag{2}$$

This shows that the solution of system (1) is always positive $\forall t \geq 0$.

For the boundedness of the solution, consider the total number of banks to be as follows:

$$N(t) = U(t) + E(t) + D(t) + R(t) + L(t). \tag{3}$$

The derivatives of both sides give us

$$\frac{dN(t)}{dt} = \frac{dU(t)}{dt} + \frac{dE(t)}{dt} + \frac{dD(t)}{dt} + \frac{dR(t)}{dt} + \frac{dL(t)}{dt}. \tag{4}$$

From system (1), $dN(t)/dt = 0$ which implies that $N(t) = N$. This shows that each of the components of the solution $U(t)$, $E(t)$, $D(t)$, $R(t)$, and $L(t)$ is bounded between zero and the total initial number of banks N.

□

3.2. Feasible Solution. All solutions of the model in system (1) are bounded. The feasible region for the banking population is $\Omega = \{(U(t), E(t), D(t), R(t), L(t)) \in \mathscr{R}^5 | U(t) + E(t) + D(t) + R(t) + L(t) \leq N\}$. The region Ω is a positively invariant region with respect to the model in system (1). Hence, the model is mathematically and systemically financially well posed in Ω.

3.3. Risk-Free Equilibrium (RFE) Point for Systemic Risk. In order to obtain the equilibrium points of the system, we equate the system of equations to zeros, i.e., $\dot{U} = \dot{E} = \dot{D} = \dot{R} = \dot{L} = 0$. Since the last equation of system (1) is independent of the others, we have the following reduced system:

$$-\beta U D + \alpha E + \theta(N - U - E - D) = 0, \tag{5}$$

$$\beta U D - (\alpha + \sigma)E = 0, \tag{6}$$

$$\sigma E - (\gamma_1 + \gamma_2)D = 0, \tag{7}$$

$$\gamma_1 D - \theta R = 0. \tag{8}$$

Equilibrium points for risk-free are conditions where there is no systemic risk, that is $E = D = 0$. From equation (5), we have $\theta(N - U) = 0 \Rightarrow U = N$, and then, the equilibrium point of the risk-free for credit risk is $P_0 = (N, 0, 0, 0)$.

3.4. The Basic Reproduction Number of the UEDR Model for Systemic Risk

Definition 2. We define the basic reproduction number S_0 as the average number of secondary distressed banks that occur when one distressed bank is interacting with a completely undistressed sample.

Lemma 3. *The basic reproduction number of system (1) is given by*

$$S_0 = \frac{\sigma\beta U_0}{(\alpha+\sigma)(\gamma_1+\gamma_2)}, \tag{9}$$

where U_0 is the number of undistressed bank at the risk-free equilibrium point.

Proof. The basic reproduction number is determined using the matrix generation method, based on equation (1). More details on the matrix generation method can be found in [8]. Let F and V represent the transitional inflows and the transitional outflows, respectively,

$$F = \begin{pmatrix} 0 & \beta U \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\alpha+\sigma) & 0 \\ -\sigma & \gamma_1+\gamma_2 \end{pmatrix}. \tag{10}$$

This implies

$$V^{-1} = \frac{1}{(\alpha+\sigma)(\gamma_1+\gamma_2)} \begin{pmatrix} (\gamma_1+\gamma_2) & 0 \\ \sigma & (\alpha+\sigma) \end{pmatrix}. \tag{11}$$

Then

$$FV^{-1} = \begin{pmatrix} 0 & \beta U \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{(\alpha+\sigma)} & 0 \\ \frac{\sigma}{(\alpha+\sigma)\gamma} & \frac{1}{\gamma} \end{pmatrix} = \begin{pmatrix} \frac{\sigma\beta U}{(\alpha+\sigma)(\gamma_1+\gamma_2)} & \frac{\beta U}{\gamma_1+\gamma_2} \\ 0 & 0 \end{pmatrix}. \tag{12}$$

Then by the matrix generation method, we find that

$$S_0 = \frac{\sigma\beta U_0}{(\alpha+\sigma)(\gamma_1+\gamma_2)}. \tag{13}$$

□

3.5. Risk Persistence Equilibrium (RPE) Point. In order to indicate the possibility of credit risk spreading, we determine the risk persistence equilibrium point. Since in persistence conditions the risk spreads, the number of banks is $U \neq 0$, $E \neq 0$, $D \neq 0$, and $R \neq 0$. From equations (5)–(8), we obtain the risk persistence equilibrium point for systemic risk as

$$\begin{aligned} U^* &= \frac{U_0}{S_0}, \\ E^* &= \frac{\theta(\gamma_1+\gamma_2)U_0(S_0-1)}{(\sigma+\theta)(\gamma_1+\gamma_2)+\sigma\theta}, \\ D^* &= \frac{\theta\sigma U_0(S_0-1)}{(\sigma+\theta)(\gamma_1+\gamma_2)+\sigma\theta}, \\ R^* &= \frac{\gamma_1\sigma U_0(S_0-1)}{(\sigma+\theta)(\gamma_1+\gamma_2)+\sigma\theta}. \end{aligned} \tag{14}$$

3.6. Stability Analysis of the Model

3.6.1. Local Stability of the Risk-Free Equilibrium Point

Theorem 4. *The RFE of system (1) exhibits local asymptotic stability when the basic reproduction number $S_0 < 1$. Conversely, it becomes unstable when $S_0 \geq 1$.*

Proof. Based on equations (5)–(8), we find the Jacobian matrix as

$$J = \begin{pmatrix} -\beta D - \theta & \alpha-\theta & -\beta U - \theta & 0 \\ \beta D & -(\alpha+\sigma) & \beta U & 0 \\ 0 & \sigma & -(\gamma_1+\gamma_2) & 0 \\ 0 & 0 & \gamma_1 & -\theta \end{pmatrix}. \tag{15}$$

Finding the eigenvalues of the matrix in equation (15), thus, obtain

$$\det(\lambda I - J) = \begin{vmatrix} \lambda+\beta D+\theta & \alpha-\theta & -\beta U-\theta & 0 \\ \beta D & \lambda+(\alpha+\sigma) & \beta U & 0 \\ 0 & \sigma & \lambda+(\gamma_1+\gamma_2) & 0 \\ 0 & 0 & \gamma_1 & \lambda+\theta \end{vmatrix} = 0. \tag{16}$$

Replacing the values $U = N$ and $D = 0$, we find then

$$\begin{vmatrix} \lambda+\theta & \alpha-\theta & -\beta N-\theta & 0 \\ 0 & \lambda+(\alpha+\sigma) & \beta N & 0 \\ 0 & \sigma & \lambda+\gamma_1+\gamma_2 & 0 \\ 0 & 0 & \gamma_1 & \lambda+\theta \end{vmatrix} = 0. \quad (17)$$

Then, the associated characteristic equation is given by

$$(\lambda+\theta)^2[(\lambda+\alpha+\sigma)(\lambda+\gamma_1+\gamma_2)-\sigma\beta N] = 0. \quad (18)$$

That is

$$(\lambda+\theta)^2\left(\lambda^2+a_1\lambda+a_2\right) = 0, \quad (19)$$

where $a_1 = \alpha+\sigma+\gamma_1+\gamma_2$ and $a_2 = (\alpha+\sigma)(\gamma_1+\gamma_2)-\sigma\beta N$. According to the Routh-Hurwitz criteria [9], the system is stable at P_0 if $a_1 > 0$ and $a_2 > 0$. Then, $a_2 > 0$ if $(\alpha+\sigma)(\gamma_1+\gamma_2)-\sigma\beta N > 0$. Therefore, the risk-free equilibrium point is locally asymptotically stable if $S_0 < 1$. □

3.6.2. Local Stability of the Risk Persistence Equilibrium Point

Theorem 5. *The RPE of system (1) is globally asymptotically stable if the basic reproduction number $S_0 \geq 1$.*

Proof. In equation (16), we replace U by U^* and D by D^*; then, the matrix now becomes

$$\det(\lambda I - J) = \begin{vmatrix} \lambda+\beta D^*+\theta & \alpha-\theta & -\beta U^*-\theta & 0 \\ \beta D^* & \lambda+(\alpha+\sigma) & \beta U^* & 0 \\ 0 & \sigma & \lambda+(\gamma_1+\gamma_2) & 0 \\ 0 & 0 & \gamma_1 & \lambda+\theta \end{vmatrix} = 0. \quad (20)$$

Let subdivide determinant (20) in block determinants. Then, we have

$$\det(\lambda I - J) = \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \quad (21)$$

where

$$A = \begin{pmatrix} \lambda+\beta D^*+\theta & \alpha-\theta \\ \beta D^* & \lambda+\alpha+\sigma \end{pmatrix}, \quad B = \begin{pmatrix} -\beta U^*-\theta & 0 \\ \beta U^* & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda+\gamma_1+\gamma_2 & 0 \\ \gamma_1 & \lambda+\theta \end{pmatrix}. \quad (22)$$

Then, assuming that matrix A is a nonsingular matrix:

$$\det(\lambda I - J) = \det(A)\det\left(D - CA^{-1}B\right) = 0, \quad (23)$$

where $D - CA^{-1}B$ is the Schur complement (see [10]). Thus, by equation (23), we have

$$\begin{aligned} &\det(A)\det\left(D-CA^{-1}B\right) = 0 \\ &\Rightarrow (\lambda+\theta)\left[\lambda^2+(\alpha+\sigma+\theta+\beta D^*)\lambda+(\alpha+\sigma)\theta+(\sigma+\theta)\beta D^*\right] \\ &\quad \cdot \left[\lambda^3+(\alpha+\sigma+\theta+\gamma_1+\gamma_2+\beta D^*)\lambda^2+(\beta(\sigma+\theta+\gamma_1+\gamma_2)D^*\right. \\ &\quad +(\alpha+\sigma)\theta)+(\alpha+\sigma+\theta)(\gamma_1+\gamma_2)+\sigma\beta U^*)\lambda+(\beta(\sigma+\theta)D^* \\ &\quad +(\alpha+\sigma)\theta)(\gamma_1+\gamma_2)] = 0, \end{aligned} \quad (24)$$

which implies that $\lambda_1 = -\theta < 0$.

$$\lambda^2+(\alpha+\sigma+\theta+\beta D^*)\lambda+(\alpha+\sigma)\theta+(\sigma+\theta)\beta D^* = 0, \quad (25)$$

$$\begin{aligned} &\lambda^3+(\alpha+\sigma+\theta+\gamma_1+\gamma_2+\beta D^*)\lambda^2 \\ &\quad +(\beta(\sigma+\theta+\gamma_1+\gamma_2)D^*+(\alpha+\sigma)\theta) \\ &\quad +(\alpha+\sigma+\theta)(\gamma_1+\gamma_2)+\sigma\beta U^*)\lambda \\ &\quad +(\beta(\sigma+\theta)D^*+(\alpha+\sigma)\theta)(\gamma_1+\gamma_2) = 0. \end{aligned} \quad (26)$$

From equation (25), we have $\lambda^2+b_1\lambda+b_2 = 0$ with $b_1 = \alpha+\sigma+\theta+\beta D^*$ and $b_2 = (\alpha+\sigma)\theta+(\sigma+\theta)\beta D^*$. Thus, by Routh-Hurwitz criteria, $b_1, b_2 > 0$ if and only if $S_0 \geq 1$ and $\lambda_2, \lambda_3 < 0$.

From equation (26), we have $\lambda^3+c_1\lambda^2+c_2\lambda+c_3 = 0$ with

$$\begin{aligned} c_1 &= \alpha+\sigma+\theta+\gamma_1+\gamma_2+\beta D^*, \\ c_2 &= \beta(\sigma+\theta+\gamma_1+\gamma_2)D^*+(\alpha+\sigma)\theta) \\ &\quad +(\alpha+\sigma+\theta)(\gamma_1+\gamma_2)+\sigma\beta U^*, \\ c_3 &= (\beta(\sigma+\theta)D^*+(\alpha+\sigma)\theta)(\gamma_1+\gamma_2). \end{aligned} \quad (27)$$

Since $c_1 > 0, c_3 > 0$, so if $c_1c_2 > c_3$ with $S_0 \geq 1$, then by Routh-Hurwitz criteria $\lambda_i < 0, \forall i \in \{2,3,4,5,6\}$. Therefore,

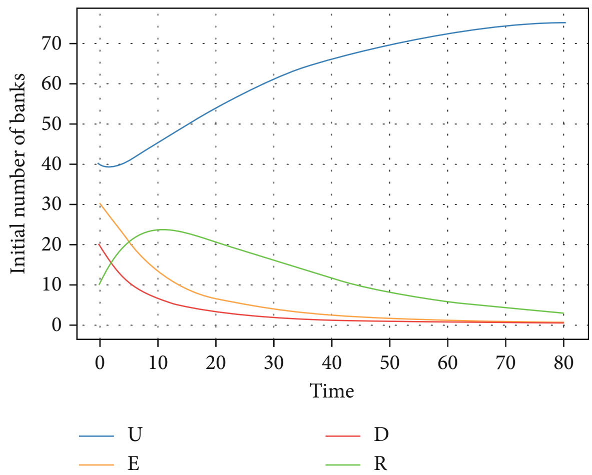

FIGURE 2: Simulation of the model for $S_0 = 0.444 < 1$.

TABLE 2: Simulation values for parameters.

Parameter	Simulation for $S_0 < 1$	Simulation for $S_0 > 1$
α	0.2	0.03
β	0.01	0.04
θ	0.05	0.05
σ	0.1	0.1
γ_1	0.2	0.2
γ_2	0.1	0.1

it follows that the system is locally asymptotically stable at the risk persistence equilibrium point. □

3.6.3. Global Stability of the Risk-Free Equilibrium Point

Theorem 6. *The RFE of system (1) is globally asymptotically stable if the basic reproduction number $S_0 < 1$.*

Proof. Let the Lyapunov function $G : \Omega \longrightarrow \mathbb{R}$ be defined as follows:

$$G = aE + bD. \tag{28}$$

The derivative of G is given by

$$\begin{aligned} \dot{G} &= a\dot{E} + b\dot{D} = a(\beta UD - (\alpha+\sigma)E) + b(\sigma E - (\gamma_1+\gamma_2)D) \\ &\le a\beta U_0 D - a(\alpha+\sigma)E + b\sigma E - b(\gamma_1+\gamma_2)D. \end{aligned} \tag{29}$$

Let the coefficient of E correspond to zero, and the values of a and b are given by

$$b = \frac{a(\alpha+\sigma)}{\sigma}, \forall a \in \mathbb{R}_+. \tag{30}$$

Then, combining equations (29) and (30), $\dot{G}$ can be written as

$$\begin{aligned} \dot{G} &\le \beta U_0 D - \frac{(\alpha+\sigma)(\gamma_1+\gamma_2)D}{\sigma} \\ &= \frac{\sigma\beta U_0 D - (\alpha+\sigma)(\gamma_1+\gamma_2)D}{\sigma} \\ &= \frac{(\alpha+\sigma)(\gamma_1+\gamma_2)(S_0-1)D}{\sigma}, \end{aligned} \tag{31}$$

□

Then, this implies that $\dot{G} \le 0$ if $S_0 \le 1$. Hence, it follows from Lasalle's invariance principle that the system is globally asymptotically stable at P_0.

3.6.4. Global Stability of the Risk Persistence Equilibrium Point

Theorem 7. *The RPE of system (1) is globally asymptotically stable.*

Proof. Let a Lyapunov function be defined as

$$\begin{aligned} V = &\left(U - U^* - U^* \ln \frac{U}{U*}\right) + \left(E - E^* - E^* \ln \frac{E}{E^*}\right) \\ &+ \left(D - D^* - D^* \ln \frac{D}{D^*}\right), \end{aligned} \tag{32}$$

$$\begin{aligned} \dot{V} &= \left(1 - \frac{U^*}{U}\right)\dot{U} + \left(1 - \frac{E^*}{E}\right)\dot{E} + \left(1 - \frac{D^*}{D}\right)\dot{D} \\ &= \left(1 - \frac{U^*}{U}\right)(-\beta UD + \alpha E + \theta R) + \left(1 - \frac{E^*}{E}\right) \\ &\quad - (\beta UD - (\alpha+\sigma)E) + \left(1 - \frac{D^*}{D}\right)(\sigma E - (\gamma_1+\gamma_2)D) \\ &= (-\beta UD + \alpha E + \theta R) - \frac{U^*}{U}(-\beta UD + \alpha E + \theta R) + \beta UD \\ &\quad - (\alpha+\sigma)E - \frac{E^*}{E}(\beta UD - (\alpha+\sigma)E) + (\sigma E - (\gamma_1+\gamma_2)D) \\ &\quad - \frac{D^*}{D}(\sigma E - (\gamma_1+\gamma_2)D). \end{aligned} \tag{33}$$

Summing the term with D but without D^* or E^*, we have

$$\beta U^* D - (\gamma_1+\gamma_2)D = 0 \Rightarrow \beta U^* = (\gamma_1+\gamma_2). \tag{34}$$

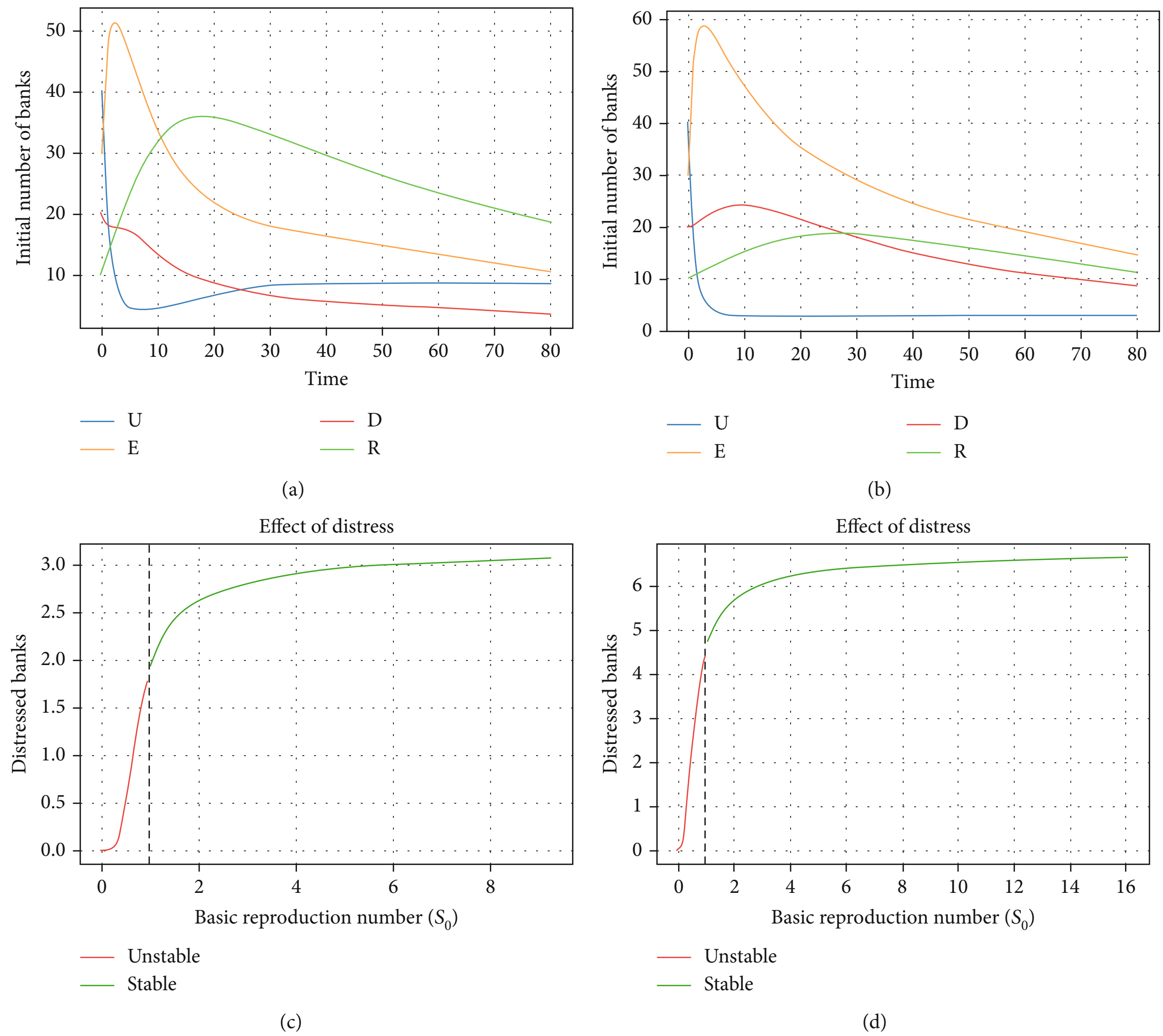

FIGURE 3: (a) Simulation of the model for $S_0 = 4 > 1$, heterogeneous case. (b) Simulation of the model for $S_0 = 10 > 1$, homogeneous case. (c) Asymptotic behavior of distress in terms of S_0 for the heterogeneous case. (d) Asymptotic behavior of distress in terms of S_0 for the homogeneous case.

Inserting (34) in (33) yields

$$\begin{aligned}
\dot{V} &= \theta R - \frac{U^*}{U}(\alpha E + \theta R) - \frac{E^*}{E}(\beta UD - (\alpha+\sigma)E) \\
&\quad - \frac{D^*}{D}(\sigma E - (\gamma_1+\gamma_2)D) \\
&= \theta R - \frac{U^*}{U}\left(\left(\frac{\beta U^* D^* - \theta R^*}{E^*}\right)E + \theta R\right) \\
&\quad - \beta UD\frac{E^*}{E} + (\alpha+\sigma)E^* - \frac{\sigma D^* E}{D} + (\gamma_1+\gamma_2)D^* \\
&= \theta R - \frac{\beta U^{*2} D^* E}{UE^*} + \theta\frac{U^* R^* E}{UE^*} - \theta\frac{U^* R}{U} - \frac{\beta UDE^*}{E} \\
&\quad + \beta U^* D^* - \frac{\sigma D^* E}{D} + \beta U^* D^* \\
&= \beta U^* D^*\left(2 - \frac{U^* E}{UE^*} - \frac{UDE^*}{UD^* E}\right) + \theta R \\
&\quad + \theta\frac{U^* R^* E}{UE^*} - \frac{\theta U^* R}{U} - \frac{\sigma D^* E}{D} \\
&= \beta U^* D^*\left(2 - \frac{U^* E}{UE^*} - \frac{UDE^*}{U^* D^* E}\right) + \theta R\left(\frac{U-U^*}{U}\right) \\
&\quad + \frac{\sigma E^* U^* E}{UE^*} - \frac{\sigma D^* E}{D} \text{ since } \theta R^* \\
&= \sigma E^* \beta U^* D^*\left(2 - \frac{U^* E}{UE^*} - \frac{UDE^*}{U^* D^* E}\right) \\
&\quad + \theta R\left(\frac{U-U^*}{U}\right) + \sigma E\left(\frac{U^* D - UD^*}{UD}\right).
\end{aligned} \tag{35}$$

Let us denote $x_1 = U^* E/UE^*$ and $x_2 = UDE^*/U^* D^* E$. If $D = D^* \Rightarrow x_1 \cdot x_2 = 1$, using the relation

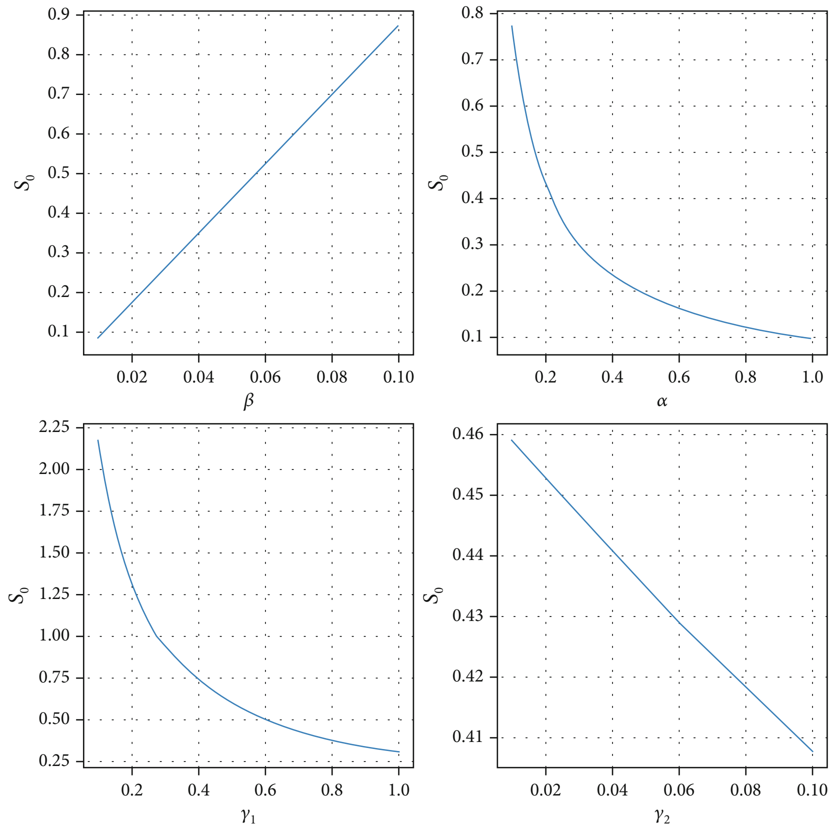

Figure 4: The basic reproduction number S_0 with respect to the parameters.

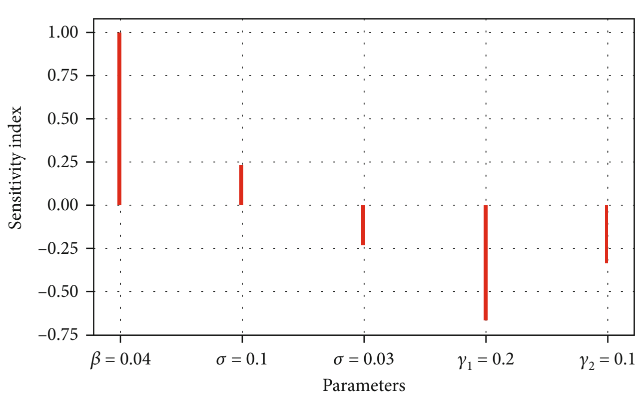

Figure 5: The sensitivity indices for S_0.

between harmonic and geometric means (see [11]), we have

$$\frac{x_1 + x_2}{2} \geq \sqrt{x_1 \cdot x_2}, x_1, x_2 \geq 0, \tag{36}$$

where this implies that $x_1 + x_2 \geq 2$ with the equality attained if $x_1 = x_2 = 1$. Hence, we obtain $dV/dt \leq 0$ for $U = U^*$, with $dV/dt = 0$ on the set $\{(U, E, D); U = U^*, D = D^*, E = E^*\}$. Therefore, it follows from Lasalle's invariance principle that the system is globally asymptotically stable at the risk persistence equilibrium point. □

4. Forward Sensitivity Analysis of the Basic Reproduction Number

According to [12], the forward normalized sensitivity index is utilized to determine how variable S_0 responds to changes in parameter p. This sensitivity is expressed in terms of partial derivatives as follows:

$$\mathcal{F}_p^{S_0} = \frac{\partial S_0}{\partial p} \times \frac{p}{S_0}. \tag{37}$$

Consequently from equation (13), we obtain the following relationship:

$$\begin{aligned}
&\frac{\partial S_0}{\partial \beta} \times \frac{\beta}{S_0} = 1,\\
&\frac{\partial S_0}{\partial \sigma} \times \frac{\sigma}{S_0} = 1,\\
&\frac{\partial S_0}{\partial \alpha} \times \frac{\alpha}{S_0} = -\frac{\alpha}{\alpha + \sigma},\\
&\frac{\partial S_0}{\partial \gamma_1} \times \frac{\gamma_1}{S_0} = -\frac{\gamma_1}{\gamma_1 + \gamma_2},\\
&\frac{\partial S_0}{\partial \gamma_2} \times \frac{\gamma_2}{S_0} = -\frac{\gamma_2}{\gamma_1 + \gamma_2}.
\end{aligned} \tag{38}$$

5. Numerical Simulation of the Model

Numerical simulation of the model is performed through the use of Python with the initial conditions assumed to be $U(0) = 40$, $E(0) = 30$, $D(0) = 20$, and $R(0) = 10$, and the parameter values are chosen with the assumption that we consider the existence of systemic risk, that is, when $S_0 > 1$, and the case when there is no systemic risk in the banking system (that is, when $S_0 < 1$) as follows:

Figure 2 shows the simulation for the parameter's values in the second column in Table 2. The x-axis shows the time in quarters of the year, as it is when the financial statements are often published. The curves show the behavior of the undistressed, exposed, distressed, and recovered banks at the risk-free equilibrium point, which is discussed in the following section.

Figure 3(a) shows the behavior of the undistressed, exposed, distressed, and recovered banks at the risk persistence equilibrium point for the heterogeneous case, when the parameter's values are given in column 3 of Table 2.

Figure 3(b) above shows the behavior of the undistressed, exposed, distressed, and recovered banks at the risk persistence equilibrium point for the homogeneous case, that is, when all the parameters α, β, γ_1, γ_2, and θ are equal to 0.05.

Figure 3(c) shows the heterogeneous simulation of the behavior of the distressed banks in terms of the basic reproduction number and the stability at the risk persistence equilibrium point, while Figure 3(d) shows the homogeneous simulation of the behavior of the distressed banks in terms of the basic reproduction number and the stability at the risk persistence equilibrium point.

Figure 4 shows the behavior of the basic reproduction number S_0 with respect to the parameters, while Figure 5 shows the most sensitive parameters in the model.

The numerical simulations presented in this study play a pivotal role in enhancing the understanding of systemic risk within the banking sector. The primary objectives of the simulations were to investigate the behavior of the mathematical model under different scenarios and assess its implications for systemic risk management and policy formulation. In the homogeneous case, where key parameter values are equal, the simulations show a rapid increase in systemic risk, resulting in a significant number of banks experiencing distress over a long-term period. This scenario suggests the need for regulatory intervention aimed at minimizing risk promptly by reducing interactions between undistressed and distressed banks.

6. Discussion and Conclusion

From the numerical results obtained, it can be observed that in the scenario of homogeneity, where the rates $\alpha, \beta, \sigma, \gamma_1, \gamma_2$, and θ are equal, the basic reproduction number exhibits a higher value, the peak of distressed banks is reached rapidly, and a significant number of banks experience distress during a long-term period (over 20 years).

In the case of heterogeneity (based on the parameter values provided in Table 2), Figure 3(a) illustrates that the peak of distressed banks occurs at a lower level compared to the homogeneous scenario. However, the risk continues to persist over a long-term period.

This contrast implies that regulatory intervention would vary in these two specific scenarios. In the homogenous case, regulators would likely intervene by reducing the interaction rate between undistressed and distressed banks, aiming to minimize risk as swiftly as possible. On the other hand, in the heterogeneous scenario, the intervention strategy would involve increasing the recovery rate for distressed banks, with the goal of gradually eliminating them from the system.

From Figure 3(c), we can see that the risk persistence equilibrium point for the heterogeneous is stable at a lower number of banks (5) compared to the homogeneous case, where its stability starts at almost $D = 5$. This implies that as long as the decision-maker is in a situation where all the parameters are equal or not, the decision-maker will pay attention to which parameter to intervene with priority.

Observing Figure 4, we notice that an elevation in the risk transmission parameter, β, and the exposure risk, σ, leads to an increase in systemic risk, as represented by the basic reproduction number, S_0. Conversely, an increase in the parameters α, γ_1, and γ_2 results in a reduction of risk transmission. Figure 5 illustrates that the transmission rates, followed by the recovery risk, are the most sensitive parameters when it comes to systemic risk.

Conversely, in the heterogeneous case, the simulations indicate a lower peak for distressed banks compared to the homogeneous scenario. The systemic risk persists over the long term, suggesting that intervention should focus on gradually eliminating distressed banks from the system by increasing recovery rates. Furthermore, sensitivity analysis highlights the critical role of specific parameters, particularly the transmission rate and recovery risk, in influencing systemic risk. These findings provide regulators and policymakers with valuable guidance for prioritizing interventions and crafting effective risk management strategies.

In conclusion, this paper makes a valuable contribution to the understanding of systemic risk within the banking sector through its comprehensive mathematical model and stability analysis. The analysis presented here sheds light on several key aspects, ranging from the model's formulation to its implications for systemic risk management and policy formulation. The paper's use of numerical simulations enhances the robustness of its findings by illustrating the model's behavior under various scenarios. The insights gained from the model's formulation, stability analyses, and numerical simulations provide valuable tools for policymakers and regulators. The identification of the S_0 threshold as a pivotal point for systemic stability underscores the importance of effective risk management strategies to prevent the escalation of systemic risk. Overall, this study provides a solid foundation for further research and practical applications in risk management and policy formulation within the banking sector.

Data Availability

Simulated data was used for this study.

Conflicts of Interest

The authors have no conflicts of interest to disclose in the publication of this paper.

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