Hindawi
Journal of Applied Mathematics
Volume 2023, Article ID 5726617, 7 pages
https://doi.org/10.1155/2023/5726617

Research Article

On the Global Asymptotic Stability and 4-Period Oscillation of the Third-Order Difference Equation

M. E. Erdogan

Department of Mathematics, Selcuk University, Konya 42150, Türkiye

Correspondence should be addressed to M. E. Erdogan; m_emre448@hotmail.com

Received 23 May 2023; Revised 13 October 2023; Accepted 25 October 2023; Published 11 November 2023

Academic Editor: Sheng Zhang

The main objective of this paper is to study the global behavior and oscillation of the following third-order rational difference equation $x_{n+1} = \alpha x_n x_{n-1} x_{n-2} / \beta x_{n-1}^2 + \gamma x_{n-2}^2$, where the initial conditions x_{-2}, x_{-1}, x_0 are nonzero real numbers and α, β, γ are positive constants such that $\alpha \leq \beta + \gamma$. Visual examples supporting solutions are given at the end of the study. The figures are found with the help of MATLAB.

1. Introduction

Considering every field of biology such as physiology, genetics, development, ecology, or evolution, these fields cannot be examined without considering the time. Life occurs over time. It is not surprising that the mathematical modeling of equations created in biology is defined by temporal processes. Physiological events, such as hair growth, occur continuously over time. Processes such as population growth in populations occur more discretely over time.

Difference equations are known as mathematical expressions that are mostly used to describe a process that develops in discrete time. That is why difference equations are of great importance in applications. Since annual plants complete their life cycle in one year, they can be explained with discrete-time models. Bahar and Erdogan [1] investigated the amount of seed production required for an annual plant with seeds capable of remaining dormant underground for a maximum of 3 years. The mathematical model obtained in the study is a discrete-time 3rd-order linear difference equation:

$$P_{n+2} = \alpha\sigma\gamma P_{n+1} + \beta\sigma(1-\alpha)\sigma\gamma P_n + \theta\sigma(1-\alpha)\sigma(1-\beta)\sigma\gamma P_{(n-1)}. \tag{1}$$

Wisnoski and Shoemaker [2] conducted a study showing that competition in the seed bank alters diversity. In their study, they referred to previous studies and supported their work by presenting mathematical models.

Many scholars are interested in rational difference equations because they are more challenging to study in terms of dynamics than linear models. Actually, the fact that difference equations are present in several biological models with a wide range of applications makes them important to investigate. The Riccati difference equation is as follows:

$$x_{n+1} = \frac{a + b x_n}{c + d x_n}, \tag{2}$$

where a, b, c, d and initial condition are real numbers, describing one of the intriguing models. AlSharawi and Rhouma [3] investigated the effect of different harvesting strategies in a deterministic environment on the discrete Beverton-Holt model:

$$y_{n+1} = \frac{aK y_n}{K + (a-1) y_n} - h, \tag{3}$$

which is a special case of the Riccati difference equation.

Yang [4] investigated the global asymptotic stability of the difference equation:

$$x_{n+1} = \frac{x_{n-1}x_{n-2} + x_{n-3} + a}{x_{n-1} + x_{n-2}x_{n-3} + a}. \tag{4}$$

Kulenović et al. [5] studied the behavior of rational recursive sequence:

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}}{\gamma x_n+\delta x_{n-1}}. \tag{5}$$

Elabbasy et al. [6] investigated and study some special cases of the difference equation:

$$x_{n+1}=\frac{ax_{n-l}x_{n-k}}{bx_{n-p}-cxn-q}. \tag{6}$$

Khaliq and Elsayed [7] studied the behavior and obtained some special cases of the difference equation:

$$x_{n+1}=\frac{\alpha x_n x_{n-l}}{\beta x_{n-m}+\gamma x_{n-l}}. \tag{7}$$

See also [8–19]. Our aim is to examine the global behavior of the following third-order rational difference equation that will serve as the basis for such modelling:

$$x_{n+1}=\frac{\alpha x_n x_{n-1}x_{n-2}}{\beta x_{n-1}^2+\gamma x_{n-2}^2}, \tag{8}$$

where the initial conditions x_{-2}, x_{-1}, x_0 are nonzero real numbers and α, β, γ are positive constants such that

$$\begin{aligned}&\alpha<\beta,\\&\alpha<\gamma.\end{aligned} \tag{9}$$

Computational examples are given at the end of study and simulated solutions of some problems via MATLAB. We hope that the results of this study contribute to the development of the theory on the global stability of nonlinear rational differential equations.

Let us give some definitions and theorems that we need.

Definition 1 (see [18]). Let I be some interval of real numbers and let

$$f: I^3 \longrightarrow I \tag{10}$$

be a continuously differentiable function.

Then, for every set of initial conditions $x_0, x_{-1}, x_{-2}\in I$, the difference equation

$$x_{n+1}=f(x_n, x_{n-1}, x_{n-2}), \quad n=0,1,\cdots \tag{11}$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

A point $\bar{x}\in I$ is called an equilibrium point of (11) if

$$\bar{x}=f(\bar{x},\bar{x},\bar{x}); \tag{12}$$

that is,

$$x_n=\bar{x} \text{ for } n\geq 0 \tag{13}$$

is a solution of (11), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2 (see [18]). Let $\bar{x}$ be an equilibrium point of Eq. (11).

(i) The equilibrium $\bar{x}$ of Eq. (11) is called locally stable if for every $\epsilon>0$, there exists $\delta>0$ such that for all $x_0, x_{-1}, x_{-2}\in I$ with $|x_0-\bar{x}|+|x_{-1}-\bar{x}|+|x_{-2}-\bar{x}|<\delta$, we have $|x_n-\bar{x}|<\epsilon$ for all $n\geq -2$

(ii) The equilibrium $\bar{x}$ of Eq. (11) is called locally asymptotically stable if it is locally stable, and if there exists $\gamma>0$ such that for all $x_0, x_{-1}, x_{-2}\in I$ with $|x_0-\bar{x}|+|x_{-1}-\bar{x}|+|x_{-2}-\bar{x}|<\gamma$, we have $\lim_{n\longrightarrow\infty} x_n=\bar{x}$

(iii) The equilibrium $\bar{x}$ of Eq. (11) is called global attractor if for every $x_0, x_{-1}, x_{-2}\in I$, we have $\lim_{n\longrightarrow\infty} x_n=\bar{x}$

(iv) The equilibrium $\bar{x}$ of Eq. (11) is called global asymptotically stable if it is locally stable and a global attractor

(v) The equilibrium $\bar{x}$ of Eq. (11) is called unstable if it is not stable

(vi) The equilibrium $\bar{x}$ of Eq. (11) is called source or a repeller, if there exists $r>0$ such that for all x_0, $x_{-1}, x_{-2}\in I$ with $0<|x_0-\bar{x}|+|x_{-1}-\bar{x}|+|x_{-2}-\bar{x}|<r$, there exists $N\geq 1$ such that $|x_N-\bar{x}|\geq r$

The linearized equation of (11) about the equilibrium point $\bar{x}$ is

$$y_{n+1}=p_1y_n+p_2y_{n-1}+p_3y_{n-2}, n=0,1,\cdots, \tag{14}$$

where

$$\begin{aligned}p_1&=\frac{\partial f}{\partial x_n}(\bar{x},\bar{x},\bar{x}),\\p_2&=\frac{\partial f}{\partial x_{n-1}}(\bar{x},\bar{x},\bar{x}),\\p_3&=\frac{\partial f}{\partial x_{n-2}}(\bar{x},\bar{x},\bar{x}).\end{aligned} \tag{15}$$

The characteristic equation of (11) is

$$\lambda^3-p_1\lambda^2-p_2\lambda-p_3=0. \tag{16}$$

Definition 3 (see [18]). A positive semicycle of $\{x_n\}_{n=-2}^{\infty}$ of Eq. (11) consists of a "string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$, all greater than or equal to $\bar{x}$, with $l\geq -2$ and $m<\infty$ and such that either $l=-2$ or $l>-2$ and $x_{l-1}<\bar{x}$ and either $m=\infty$ or $m<\infty$ and $x_{m+1}<\%\bar{x}$.

A negative semicycle of $\{x_n\}_{n=-2}^{\infty}$ of Eq. (11) consists of a "string" of terms $\{x_l, x_{l+1}, \cdots, x_m\}$ all less than $\bar{x}$, with $l \geq -2$ and $m < \infty$ and such that either $l = -2$ or $l > -2$ and $x_{l-1} \geq \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \bar{x}$.

Theorem 4 (see [18]). *Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \cdots$. Then,*

$$\sum_{i=1}^{3} |p_i| < 1 \tag{17}$$

is a sufficient condition for the asymptotic stability of (16).

Theorem 5 (see [18]). *Let $[p, q]$ be an interval of real numbers and assume that $f : [p, q]^3 \longrightarrow [p, q]$ is a continuous function satisfying the following properties:*

(a) $f(x, y, z)$ is nondecreasing in $y, z \in [p, q]$ for each $x \in [p, q]$ and nonincreasing in $x \in [p, q]$ for each $y, z \in [p, q]$

(b) If $(m, M) \in [p, q] \times [p, q]$ is a solution of the system $M = f(m, M, M)$ and $m = f(M, m, m)$, then $m = M$

Then, Eq. (11) has a unique equilibrium $\bar{x} \in [p, q]$, and every solution of Eq. (11) converges to $\bar{x}$.

2. Dynamics of Eq. (8)

In this section, we investigate the dynamics of (8) under the assumptions that all parameters in the equation are positive and the initial conditions are nonnegative.

2.1. Local Stability of Eq. (8). Eq. (8) has a unique equilibrium point and is given by the equation

$$\bar{x} = \frac{\alpha \bar{x}^3}{\beta \bar{x}^2 + \gamma \bar{x}^2}. \tag{18}$$

So,

$$\bar{x}^3(\beta + \gamma) = \alpha \bar{x}^3. \tag{19}$$

Since $\alpha < \beta$ and $\alpha < \gamma$, then $\alpha < \beta + \gamma$, so the unique equilibrium point is $\bar{x} = 0$.

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v, t) = \frac{\alpha u v t}{\beta v^2 + \gamma t^2}. \tag{20}$$

So,

$$\begin{aligned} \frac{\partial f}{\partial u}(\bar{x}, \bar{x}, \bar{x}) &= \frac{\alpha}{\beta + \gamma}, \\ \frac{\partial f}{\partial v}(\bar{x}, \bar{x}, \bar{x}) &= \frac{\alpha(\gamma - \beta)}{(\beta + \gamma)^2}, \\ \frac{\partial f}{\partial t}(\bar{x}, \bar{x}, \bar{x}) &= \frac{\alpha(\beta - \gamma)}{(\beta + \gamma)^2}. \end{aligned} \tag{21}$$

The linearized equation of Eq. (8) is

$$y_{n+1} - \frac{\alpha}{\beta + \gamma} y_n - \frac{\alpha(\gamma - \beta)}{(\beta + \gamma)^2} y_{n-1} - \frac{\alpha(\beta - \gamma)}{(\beta + \gamma)^2} y_{n-2} = 0. \tag{22}$$

Theorem 6. *The equilibrium point of Eq. (8) is locally asymptotically stable.*

Proof. It follows by Theorem 4 that Eq. (22) is asymptotically stable if

$$\left| \frac{\alpha}{\beta + \gamma} \right| + \left| \frac{\alpha(\gamma - \beta)}{(\beta + \gamma)^2} \right| + \left| \frac{\alpha(\beta - \gamma)}{(\beta + \gamma)^2} \right| < 1. \tag{23}$$

If $\beta = \gamma$,

$$\left| \frac{\alpha}{\beta + \gamma} \right| < 1. \tag{24}$$

So

$$\alpha < \beta + \gamma. \tag{25}$$

If $\beta < \gamma$,

$$3\alpha\gamma - \alpha\beta < (\beta + \gamma)^2. \tag{26}$$

So

$$\alpha < \frac{(\beta + \gamma)^2}{3\gamma - \beta}. \tag{27}$$

From

$$\alpha < \beta + \gamma < 2\gamma < 2\gamma + \gamma - \beta = 3\gamma - \beta, \tag{28}$$

we can get

$$\alpha < \frac{(\beta + \gamma)^2}{3\gamma - \beta} < \frac{(\beta + \gamma)^2}{\beta + \gamma}, \tag{29}$$

such that

$$\alpha < \beta + \gamma \tag{30}$$

is obtained.

Similarly, if $\gamma < \beta$,

$$3\alpha\beta - \alpha\gamma < (\beta + \gamma)^2. \tag{31}$$

So

$$\alpha < \frac{(\beta + \gamma)^2}{3\beta - \gamma}. \tag{32}$$

From

$$\begin{aligned} &\alpha < \beta + \gamma < 2\beta < 2\beta + \beta - \gamma = 3\beta - \gamma, \\ &\alpha < \frac{(\beta+\gamma)^2}{3\beta - \gamma} < \frac{(\beta+\gamma)^2}{\beta+\gamma}, \end{aligned} \tag{33}$$

such that

$$\alpha < \beta + \gamma \tag{34}$$

is regained. This completes the proof. □

2.2. Global Asymptotic Stability of $\bar{x}$ of Eq. (8)

Theorem 7. *The equilibrium point of $\bar{x}$ of Eq. (8) is globally asymptotically stable.*

Proof. Let p, q be real numbers and assume that $f : [p, q]^3 \longrightarrow [p, q]$ is a function defined by $f(u, v, t) = \alpha uvt/(\beta v^2 + \gamma t^2)$. Then, we can easily see that the function is increasing in u and decreasing in v, t. Suppose that (m, M) is a solution of the system

$$\begin{aligned} M &= f(m, M, M), \\ m &= f(M, m, m). \end{aligned} \tag{35}$$

Then, from Eq. (8),

$$\begin{aligned} M &= \frac{\alpha m M^2}{\beta M^2 + \gamma M^2}, \\ m &= \frac{\alpha M m^2}{\beta m^2 + \gamma m^2}, \\ (\alpha + \beta &+ \gamma)(M - m) = 0. \end{aligned} \tag{36}$$

Thus,

$$M = m. \tag{37}$$

By Theorem 5, $\bar{x}$ is a global attractor of Eq. (8). From Theorem 6 and Definition 1, $\bar{x}$ is globally asymptotically stable of Eq. (8) and the proof is complete. □

2.3. Boundedness of Solutions of Eq. (8)

Theorem 8. *Every solution of Eq. (8) is bounded.*

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (8). Let $M = \max\{x_{n-1}, x_{n-2}\}$. From Eq. (8),

$$x_{n+1} = \frac{\alpha x_n x_{n-1} x_{n-2}}{\beta x_{n-1}^2 + \gamma x_{n-2}^2} \le \frac{\alpha x_n M M}{\beta M^2 + \gamma M^2} < \frac{(\beta + \gamma) x_n M^2}{\beta M^2 + \gamma M^2}, \tag{38}$$

which implies that $x_{n+1} < x_n$ for $n \ge 0$. Then,

$$\lim_{n \longrightarrow \infty} x_n = \bar{x}. \tag{39}$$

Then, the proof is complete. □

2.4. Oscillation of Eq. (8).

Theorem 9. *Assume that $\alpha = \beta + \gamma$; then, Eq. (8) possesses the prime period 2 solutions:*

$$\cdots, \phi, \psi, \phi, \psi, \cdots. \tag{40}$$

Furthermore, every solution of Eq. (8) converges to a period 2 solution (40) with $\phi \ge 0$.

Proof. Let

$$\cdots, \phi, \psi, \phi, \psi, \cdots \tag{41}$$

be a period two solution of Eq. (8). Then,

$$\begin{aligned} \psi &= \frac{(\beta + \gamma)\phi^2 \psi}{\beta \psi^2 + \gamma \phi^2}, \\ \phi &= \frac{(\beta + \gamma)\psi^2 \phi}{\beta \phi^2 + \gamma \psi^2}. \end{aligned} \tag{42}$$

So

$$\phi^2 = \psi^2. \tag{43}$$

Then, this implies either $\phi = \psi$ or $\phi = -\psi$. However, from $\phi \ne \psi$ which contradicts to $\phi = \psi$, the solution becomes $\phi = -\psi$.

This completes the proof. □

Theorem 10. *Assume that $\alpha = \beta + \gamma$ and $\beta = \gamma$; then, there are four periodic solutions of Eq. (8) as*

$$\cdots, \phi, \phi, \psi, \psi, \cdots. \tag{44}$$

Proof. Let ϕ and ψ be real numbers such that $\phi \ne \psi$.

Let

$$\cdots, \phi, \phi, \psi, \psi, \cdots \tag{45}$$

be a periodic solution of Eq. (8) with prime period four. Then, we have four cases:

(i) $\psi = ((\beta + \gamma)\psi\phi^2)/(\beta\phi^2 + \gamma\phi^2)$; hence, $\beta\phi^2\psi + \gamma\phi^2\psi = (\beta + \gamma)\psi\phi^2$. So every ϕ and ψ real numbers provide the equation

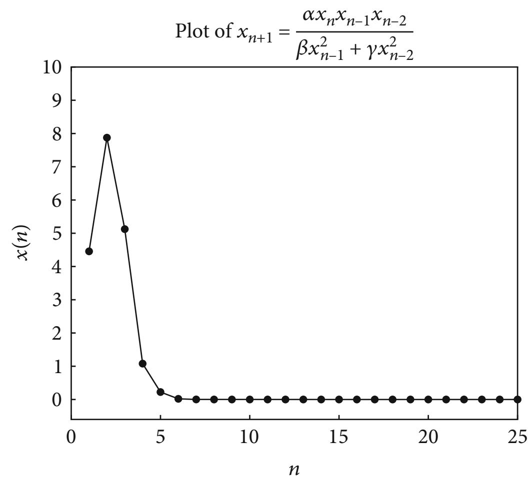

Figure 1: Stability of the solutions of Eq. (8) under the conditions $\alpha < \beta + \gamma$ and $\beta < \gamma$.

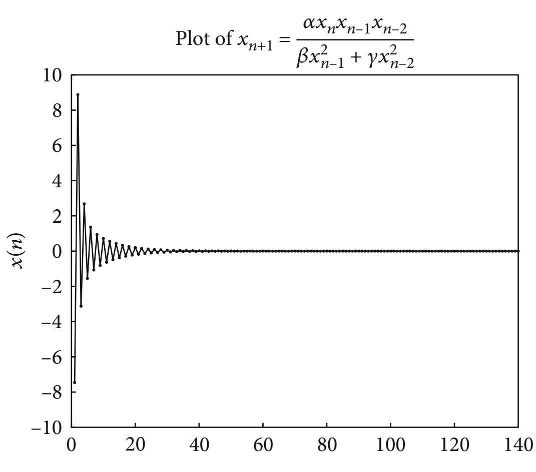

Figure 2: Behavior of Eq. (8) under the conditions $\alpha < \beta + \gamma$ and $\gamma < \beta$.

(ii) $\phi = ((\beta + \gamma)\psi^2\phi)/(\beta\psi^2 + \gamma\phi^2)$; hence, $\beta\phi\psi^2 + \gamma\phi^3 = (\beta + \gamma)\psi^2\phi$, that is, $\phi^2 = \psi^2$. So it becomes $\phi = -\psi$ like in Theorem 9

(iii) $\phi = ((\beta + \gamma)\psi^2\phi)/(\beta\psi^2 + \gamma\psi^2)$; hence, $\beta\phi\psi^2 + \gamma\phi\psi^2 = (\beta + \gamma)\phi\psi^2$. So every ϕ and ψ real numbers provide the equation

(iv) $\psi = ((\beta + \gamma)\psi\phi^2)/(\beta\phi^2 + \gamma\psi^2)$; hence, $\beta\phi^2\psi + \gamma\psi^3 = (\beta + \gamma)\phi^2\psi$, that is, $\phi^2 = \psi^2$. So it becomes $\phi = -\psi$ like in Theorem 9

In these cases, the proof is complete. □

3. Computational Examples

In this section, I perform computational examples to illustrate the validity of the main results. In order to better express the numerical samples, a graph of the solutions

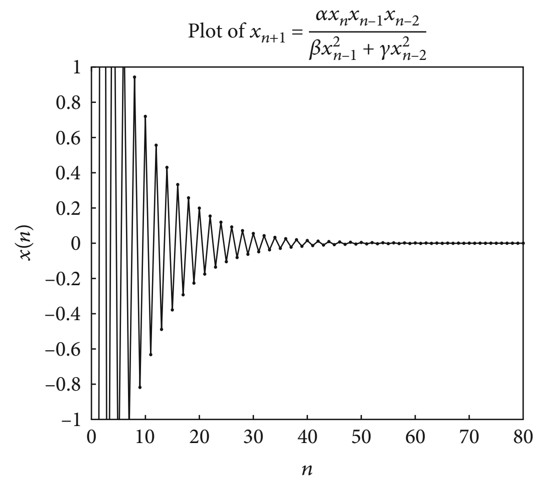

Figure 3: The zoomed version of Figure 2.

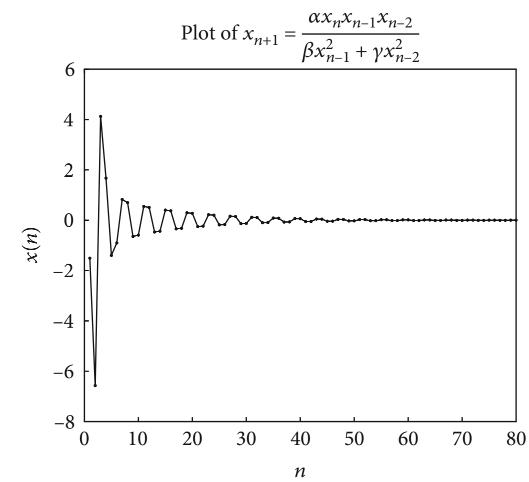

Figure 4: Behavior of Eq. (8) under the conditions $\alpha < \beta + \gamma$ and $\beta = \gamma$.

was obtained by using MATLAB. These graphs are drawn with different parameters and different starting conditions.

(i) In Figure 1, Eq. (8) is shown to be globally asymptotically stable under the initial conditions $x_{-2} = 4.456$, $x_{-1} = 7.875$, and $x_0 = 5.124$ and the parameters $\alpha = 3$, $\beta = 5.845$, and $\gamma = 6.931$ that meet the conditions $\alpha < \beta + \gamma$ and $\beta < \gamma$

(ii) In Figures 2 and 3, Eq. (8) is shown to be globally asymptotically stable under the initial conditions $x_{-2} = -7.456$, $x_{-1} = 8.875$, and $x_0 = -3.124$ and the parameters $\alpha = 13$, $\beta = 7.845$, and $\gamma = 6.931$ that meet the conditions $\alpha < \beta + \gamma$ and $\gamma < \beta$

(iii) In Figure 4, Eq. (8) is shown to be globally asymptotically stable under the initial conditions

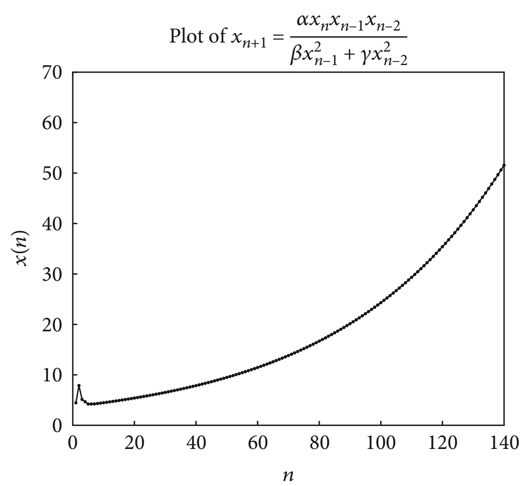

Figure 5: Unboundness solutions of Eq. (8) under the conditions $\beta + \gamma < \alpha$ and $\beta < \gamma$.

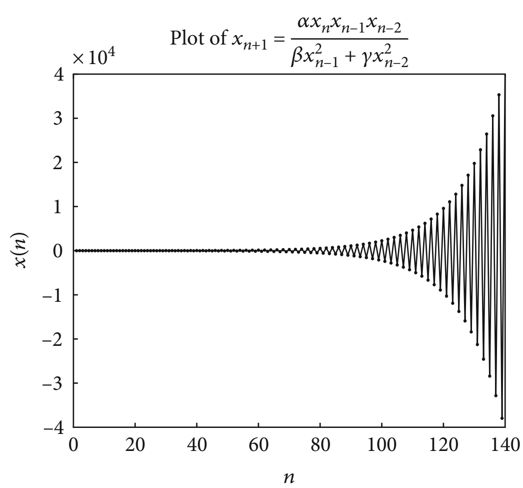

Figure 6: Unboundness solutions of Eq. (8) under the conditions $\beta + \gamma < \alpha$ and $\gamma < \beta$.

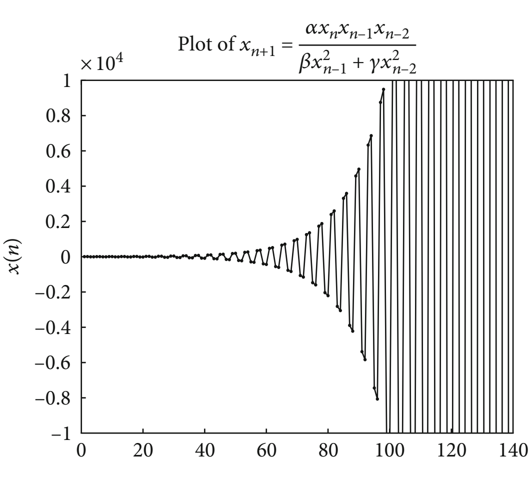

Figure 7: Unboundness solutions of Eq. (8) under the conditions $\beta + \gamma < \alpha$ and $\beta = \gamma$.

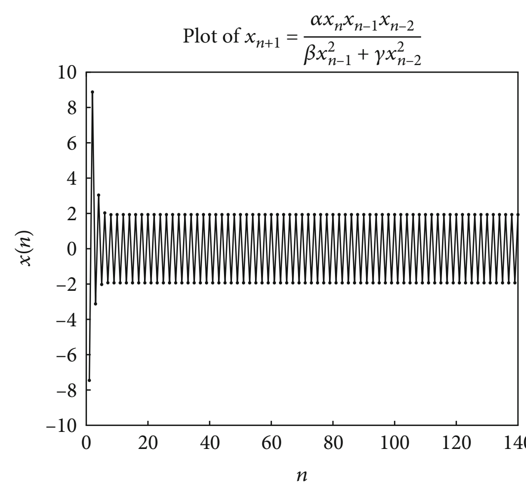

Figure 8: Prime period two solutions of Eq. (8) under the conditions $\alpha = \beta + \gamma$ and $\gamma < \beta$.

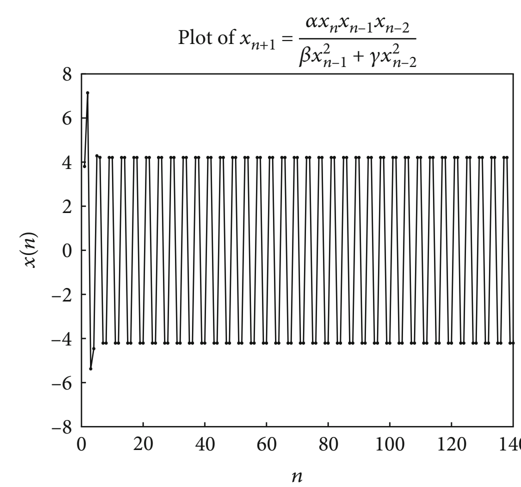

Figure 9: Periodic solution of Eq. (8) with prime period four under the conditions $\alpha = \beta + \gamma$ and $\gamma = \beta$.

$x_{-2} = -1.508$, $x_{-1} = -6.57$, and $x_0 = 4.124$ and the parameters $\alpha = 13$, $\beta = 7$, and $\gamma = 7$ that meet the conditions $\alpha < \beta + \gamma$ and $\beta = \gamma$

(iv) In Figure 5, Eq. (8) is shown to be not globally asymptotically stable under the initial conditions $x_{-2} = 4.456$, $x_{-1} = 7.875$, and $x_0 = 5.124$ and the parameters $\alpha = 13$, $\beta = 5.845$, and $\gamma = 6.931$ that meet the conditions $\beta + \gamma < \alpha$ and $\beta < \gamma$

(v) In Figure 6, Eq. (8) is shown to be not globally asymptotically stable under the initial conditions $x_{-2} = -7.456$, $x_{-1} = 8.875$, and $x_0 = -3.124$ and the parameters $\alpha = 16$, $\beta = 7.845$, and $\gamma = 6.931$ that meet the conditions $\beta + \gamma < \alpha$ and $\gamma < \beta$

(vi) In Figure 7, Eq. (8) is shown to be not globally asymptotically stable under the initial conditions $x_{-2} = 3.802$, $x_{-1} = 7.141$, and $x_0 = -5.375$ and the parameters $\alpha = 12.124$, $\beta = 5.572$, and $\gamma = 5.572$ that meet the conditions $\beta + \gamma < \alpha$ and $\beta = \gamma$

(vii) In Figure 8, Eq. (8) is shown to be globally asymptotically stable with prime period two under the initial conditions $x_{-2} = -7.456$, $x_{-1} = 8.875$, and $x_0 = -3.124$ and the parameters $\alpha = 13$, $\beta = 7$, and $\gamma = 6$ that meet the conditions $\alpha = \beta + \gamma$ and $\gamma < \beta$

(viii) In Figure 9, Eq. (8) is shown to be globally asymptotically stable with prime period four under the initial conditions $x_{-2} = -7.456$, $x_{-1} = 8.875$, and $x_0 = -3.124$ and the parameters $\alpha = 13$, $\beta = 7$, and $\gamma = 6$ that meet the conditions $\alpha = \beta + \gamma$ and $\gamma = \beta$

4. Conclusion

It is very interesting for researchers to examine the dynamics of rational difference equations, especially high-period dynamics. In this article, the solutions of a third-order rational difference equation with four-period oscillations are examined. First of all, the local stability of eq. (8) and then the global asymptotic stability are examined. Afterwards, the oscillation of equation (8) was examined, and the obtained theoretical results are supported by numerical examples and graphics of solutions.

Data Availability

The authors confirm that the data supporting the findings of this study are available within the article.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

[1] Z. Bahar and M. E. Erdogan, "Propagation of annual plants," in *Selcuk 8th International Conference on Applied Sciences*, pp. 214–222, Konya, Turkey, 2023.

[2] N. I. Wisnoski and L. G. Shoemaker, "Seed banks alter metacommunity diversity: the interactive effects of competition, dispersal and dormancy," *Ecology Letters*, vol. 25, no. 4, pp. 740–753, 2022.

[3] Z. AlSharawi and M. B. H. Rhouma, "The Beverton–Holt model with periodic and conditional harvesting," *Journal of Biological Dynamics*, vol. 3, no. 5, pp. 463–478, 2009.

[4] X. Yang, "On the global asymptotic stability of the difference equation $x_n = x_{n-1}x_{n-2} + x_{n-3} + a/x_{n-1} + x_{n-2}x_{n-3} + a$," *Applied Mathematics and Computation*, vol. 171, pp. 857–861, 2005.

[5] M. R. S. Kulenović, G. Ladas, and W. S. Sizer, "On the recursive sequence $x_{n+1} = \alpha x_n + \beta x_{n-1}/\gamma x_n + \delta x_{n-1}$," *Mathematical Sciences Research Hot-Line*, vol. 2, pp. 1–16, 1998.

[6] E. M. Elabbasy, H. A. El-Metwally, and E. M. Elsayed, "Global behavior of the solutions of some difference equations," *Advances in Difference Equations*, vol. 2011, no. 1, 2011.

[7] A. Khaliq and E. M. Elsayed, "Qualitative study of a higher order rational difference equation," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, pp. 1128–1143, 2018.

[8] M. E. Erdogan, C. Cinar, and I. Yalcinkaya, "On the dynamics of the recursive sequence $x_{n+1} = x_{n-1}/\beta + \gamma x_{n-2}^2 x_{n-4} + \gamma x_{n-2} x_{n-4}^2$," *Computers \& Mathematics with Applications*, vol. 61, pp. 533–537, 2011.

[9] M. E. Erdogan, C. Cinar, and I. Yalcinkaya, "On the dynamics of the recursive sequence $x_{n+1} = \alpha x_{n-1}/\beta + \gamma \sum_{k=1}^{t} x_{n-2k} \prod_{k=1}^{t} x_{n-2k}$," *Mathematical and Computer Modelling*, vol. 54, pp. 1481–1485, 2011.

[10] M. E. Erdogan and C. Cinar, "On the dynamics of the recursive sequence $x_{n+1} = \alpha x_{n-1}/\beta + \gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}$," *Fasciculi Mathematici*, vol. 50, pp. 59–66, 2013.

[11] R. Abo-Zeid and M. A. Al-Shabi, "Global behavior of a third order difference equation," *Tamkang Journal of Mathematics*, vol. 43, no. 3, pp. 375–384, 2012.

[12] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Chapman & Hall/CRC Pure and Applied Mathematics, 2000.

[13] G. L. Karakostas, "Convergence of a difference equation via the full limiting sequences method," *Differential Equations and Dynamical Systems*, vol. 1, pp. 289–294, 1993.

[14] F. Belhannache, N. Touafek, and R. Abo-Zeid, "Dynamics of a third-order rational difference equation," *Bulletin Mathematique de La Societe Des Sciences Math. de Roumanie*, vol. 59, pp. 13–22, 2016.

[15] A. E. Hamza, A. M. Ahmed, and A. M. Youssef, "On the recursive sequence $x_{n+1} = a + bx_n/A + Bx_{n-1}^k$," *Arab Journal of Mathematical Sciences*, vol. 17, pp. 31–44, 2011.

[16] A. E. Hamza and M. A. El-Sayed, "Stability problem of some nonlinear difference equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 21, no. 2, p. 340, 1998.

[17] E. A. Grove, G. Ladas, M. Predescu, and M. Radin, "On the global character of the difference equation $x_{n+1} = \alpha + \gamma x_{n-(2k+1)} + \delta x_{n-2l}/A + x_{n-2l}$," *Journal of Difference Eq. and Applications*, vol. 9, pp. 171–199, 2003.

[18] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures*, Chapman and Hall/CRC, 2001.

[19] M. E. Erdogan, "Stability of the third order rational difference equation," *Manas Journal of Engineering*, vol. 8, no. 1, pp. 68–76, 2020.