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Research Article

Characterizations of the Generalized MPCEP Inverse of Rectangular Matrices

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In this paper, we introduce a new generalized inverse, called the G-MPCEP inverse of a complex matrix. We investigate some characterizations, representations, and properties of this new inverse. Cramer's rule for the solution of a singular equation $Ax = B$ is also presented. Moreover, the determinantal representations for the G-MPCEP inverse are studied. Finally, the G-MPCEP inverse being used in solving appropriate systems of linear equations is established.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. And the set of $m \times n$ matrices with rank r by $\mathbb{C}_r^{m\times n}$. For $A \in \mathbb{C}^{n\times n}$, the symbols A^*, rank (A), $N(A)$, and $R(A)$ stand for the conjugate transpose, the rank, the null space, and the range space of A, respectively. Moreover, I_n will refer to the $n \times n$ identity matrix. Let $A \in \mathbb{C}^{n\times n}$, the smallest positive integer k for which rank $(A^k) =$ rank (A^{k+1}) is called the index of A and is denoted by $\mathrm{Ind}(A)$. $P_{E,F}$ represents the projector on the subspace E along the subspace F. For $A \in \mathbb{C}^{n\times n}$, P_A stands for the orthogonal projection onto $R(A)$. The symbol $\mathbb{C}_n^{CM}$ represents the subset of all $n \times n$ complex matrices with index 1.

Next, let us review the definitions of some generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ of A is the unique matrix $X = A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [1]:

$$\begin{aligned} \mathrm{AXA} &= A, \\ \mathrm{XAX} &= X, \\ (\mathrm{AX})^* &= \mathrm{AX}, \\ (\mathrm{XA})^* &= \mathrm{XA}. \end{aligned} \tag{1}$$

The Moore-Penrose inverse can be used to represent orthogonal projectors $P_A = \mathrm{AA}^\dagger$ onto $R(A)$ and $Q_A = A^\dagger A$ onto $R(A^*)$, respectively. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $\mathrm{AXA} = A$ is called an inner inverse or $\{1\}$-inverse of A, and a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $\mathrm{XAX} = X$ is called an outer inverse or $\{2\}$-inverse of A.

The core-EP inverse is an outer inverse which was presented in [2] for an arbitrary square matrix. For a square matrix $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, there exists the core-EP inverse as the unique matrix $X = A^{\oplus}$ for which the next equalities hold,

$$\begin{aligned} X &= \mathrm{XAX}, \\ R\left(A^k\right) &= R(X) = R(X^*). \end{aligned} \tag{2}$$

Recall that, by [3, 4], the core-EP inverse can be expressed as $A^{\oplus} = A^D(\mathrm{AA}^D)^\dagger = A^D A^k (A^k)^\dagger$.

The (B, C)-inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(B,C)}$ [5], is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$\begin{aligned} \mathrm{XAB} &= B, \\ \mathrm{CAX} &= C, \\ N(X) &= N(C), \\ R(X) &= R(B), \end{aligned} \tag{3}$$

where $B, C \in \mathbb{C}^{n\times m}$ [5, 6].

In the paper [7], two new generalized inverses have emerged by combining the Moore-Penrose inverse and the core-EP inverse, which are the MPCEP inverse (MP-Core-EP) $A^{\dagger,\oplus}$ and the dual MPCEP inverse (*CEPMP) $A_{\oplus,\dagger}$, respectively [7]. Precisely, the MPCEP inverse of $A \in \mathbb{C}^{n\times n}$ presents a unique solution to the matrix system [7],

$$\begin{aligned} &\mathrm{XAX} = X, \\ &\mathrm{XA} = A^{\dagger}\mathrm{AA}^{\oplus}A, \\ &\mathrm{AX} = \mathrm{AA}^{\oplus}. \end{aligned} \tag{4}$$

Notice that $A^{\dagger,\oplus} = A^{\dagger}\mathrm{AA}^{\oplus}$.

The dual MPCEP inverse of $A \in \mathbb{C}^{n\times n}$ presents a unique solution to the matrix system [7],

$$\begin{aligned} &\mathrm{XA} = X_{\oplus}A, \\ &A_{\oplus,\dagger} = A^{\dagger}\mathrm{AA}_{\oplus}\mathrm{AA}^{\dagger}. \end{aligned} \tag{5}$$

And $A_{\oplus,\dagger} = A_{\oplus}\mathrm{AA}^{\dagger}$.

And in [8], representations and properties for the MPCEP inverse were introduced.

Definition 1 (see [9]). Let $A \in \mathbb{C}_r^{m\times n}$, T be a subspace of $\mathbb{C}^n$ of dimension $t \le r$, and S a subspace of $\mathbb{C}^m$ of dimension $m - t$. Then, A has a $\{2\}$ inverse X such that $R(X) = T$ and $N(X) = S$ if and only if

$$\mathrm{AT} \oplus S = \mathbb{C}^m, \tag{6}$$

in which case, X is unique and denoted by $A_{T,S}^{(2)}$.

Denote by $\mathbb{C}_{T,S}^{m\times n}$, the set of all $A \in \mathbb{C}^{m\times n}$ such that A, T, and S satisfy the abovementioned conditions, and $A_{T,S}^{(2)}$ exists.

Lemma 2 (see [9]). *Let $A \in \mathbb{C}_{T,S}^{m\times n}$. Then,*

(1) $AA_{T,S}^{(2)} = P_{AT,S}$,

(2) $A_{T,S}^{(2)}A = P_{T,(A^*S^{\perp})^{\perp}}$.

Lemma 3 (see [3]). *Let $A \in \mathbb{C}_{T,S}^{m\times n}$. Then, the system of matrix equations,*

$$\begin{aligned} &XAX = X, \\ &XA = A_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right), \\ &A, AX = AA_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right). \end{aligned} \tag{7}$$

possesses the unique solution $X := A_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{\dagger}$.

Definition 4 (see [3]). Let $A \in \mathbb{C}_{T,S}^{m\times n}$. The generalized core-EP (or g-core-EP) inverse of the given matrix A is defined as

$$A_{T,S}^{②} := A_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}. \tag{8}$$

Lemma 5 (see [3]). *Let $A \in \mathbb{C}_{T,S}^{m\times n}$. Then,*

(1) $A_{T,S}^{②}A$ *is a projector onto T along $N((AA_{T,S}^{(2)})^{*}A)$,*

(2) $AA_{T,S}^{②}$ *is the orthogonal projector onto $R(AA_{T,S}^{(2)})$;*

(3) $A_{T,S}^{②} = A^{(2)}_{T,N((AA_{T,S}^{(2)})^{*})}$.

Lemma 6 (Urquhart formula) (see [10, 11]). *If $A \in \mathbb{C}_r^{m\times n}$, $U \in \mathbb{C}_{n\times p}$, $V \in \mathbb{C}_{q\times m}$, then,*

$$X = U(VAU)^{(1)}V, \tag{9}$$

where $(VAU)^{(1)}$ is fixed, but the arbitrary element of $(VAU)\{1\}$.

The main structure of this paper is as follows. In Section 2, we introduce the G-MPCEP inverse. Then, we give some representations and characterizations of the G-MPCEP inverse. In Section 3, Cramer's rule for the solution of a singular equation $Ax = B$ is presented. In Section 4, the determinantal representations of the G-MPCEP inverse are developed. In Section 5, we give the application of the G-MPCEP inverse in solving linear equations.

2. Definition, Characterizations, and Representations of the G-MPCEP Inverse

Theorem 7. *Let $A \in \mathbb{C}_r^{m\times n}$, T be a subspace of $\mathbb{C}^n$ of dimension $t \le r$, and S a subspace of $\mathbb{C}^m$ of dimension $m - t$. Suppose that $A_{T,S}^{(2)}$ exists. Then, the system of matrix equations,*

$$\begin{aligned} &XAX = X, \\ &XA = A^{\dagger}AA_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{\dagger}A, \\ &AX = P_{AA_{T,S}^{(2)}}, \end{aligned} \tag{10}$$

has the unique solution $X = A^{\dagger}AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}$.

Proof. We can easily show that (10) holds for $X = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^{\dagger}$. If different matrices X and X_1 satisfy the system of matrix Equation (10), then,

$$\begin{aligned} &\mathrm{AX}_1 = P_{\mathrm{AA}_{T,S}^{(2)}} = \mathrm{AX}, \\ &X_1A = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}A = \mathrm{XA}, \end{aligned} \tag{11}$$

which further implies

$$X = (\mathrm{XA})X = X_1(\mathrm{AX}) = X_1\mathrm{AX}_1 = X_1. \tag{12}$$

Thus, the solution X to the system (10) is unique. □

Definition 8. Let $A \in \mathbb{C}_r^{m\times n}$, T be a subspace of $\mathbb{C}^n$ of dimension $t \le r$, and S a subspace of $\mathbb{C}^m$ of dimension $m - t$. Suppose that $A_{T,S}^{(2)}$ exists. The generalized MP-Core-EP (or G-MPCEP) inverse of A is defined as

$$A_{T,S}^{\dagger,②} = A^\dagger \mathrm{AA}_{T,S}^② = A^\dagger \mathrm{AA}_{T,S}^{(2)} \left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger. \tag{13}$$

Corollary 9. *Various important special cases of the defined G-MPCEP inverse, which recover some previously studied outer inverses, are given as follows:*

(1) For $m = n$, $k = Ind(A)$ and $A_{T,S}^{(2)} = A^D$, the G-MPCEP inverse becomes the MPCEP inverse. Indeed,

$$\begin{aligned} A_{T,S}^{\dagger,②} &= A_{R(A^k),N(A^k)}^{\dagger,②} = A^\dagger AA^D\left(AA^D\right)^\dagger = A^\dagger AA^DAA^D\left(AA^D\right)^\dagger \\ &= A^\dagger AA^D P_{R\left(AA^D\right)} = A^\dagger AA^D P_{A^k} = A^\dagger AA^D A^k\left(A^k\right)^\dagger = A^\dagger AA^{\oplus}. \end{aligned} \tag{14}$$

(2) If $A_{T,S}^{(2)} = A^\dagger$, then $A_{T,S}^{\dagger,②} = A_{R(A^*),N(A^*)}^{\dagger,②} = A^\dagger$.

Let A, T, S in the following conclusions satisfy the assumption of Theorem 7.

Theorem 10. *Let $A \in \mathbb{C}^{m\times n}$. Then,*

$$A_{T,S}^{\dagger,②} = A^\dagger P_{R\left(AA_{T,S}^{(2)}\right)} = A^\dagger AA_{T,S}^{(2)} P_{R\left(AA_{T,S}^{(2)}\right)}. \tag{15}$$

Proof. Since $P_{R(\mathrm{AA}_{T,S}^{(2)})} = \mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^\dagger$, we have

$$\begin{aligned} A^\dagger P_{R\left(\mathrm{AA}_{T,S}^{(2)}\right)} &= A^\dagger \mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger = A_{T,S}^{\dagger,2}, \\ A^\dagger \mathrm{AA}_{T,S}^{(2)} P_{R\left(\mathrm{AA}_{T,S}^{(2)}\right)} &= A^\dagger \mathrm{AA}_{T,S}^{(2)} \mathrm{AA}_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^\dagger = A_{T,S}^{\dagger,2}. \end{aligned} \tag{16}$$

□

Theorem 11. *Let $A \in \mathbb{C}^{m\times n}$. Then,*

$$A_{T,S}^{\dagger,②} = P_{R(A^*)} A_{T,S}^②. \tag{17}$$

Proof. Since $A^\dagger A = P_{R(A^*)}$, $A_{T,S}^② = A_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2}{}_{T,S})^\dagger$, then the G-MPCEP inverse $A_{T,S}^{\dagger,②}$ can be expressed by

$$A_{T,S}^{\dagger,②} = A^\dagger \mathrm{AA}_{T,S}^② = P_{R(A^*)} A_{T,S}^②. \tag{18}$$

□

Theorem 12. *Let $A \in \mathbb{C}^{m\times n}$. Then,*

(1) $A_{T,S}^{\dagger,②} A$ is a projector onto $R(A^\dagger AA_{T,S}^{(2)})$ along $N((AA_{T,S}^{(2)})^* A)$,

(2) $AA_{T,S}^{\dagger,②}$ is a projector onto AT along $N((AA_{T,S}^{(2)})^*)$,

(3) $AA_{T,S}^{\dagger,②}$ is the orthogonal projector onto $R(AA_{T,S}^{(2)})$,

(4) $A_{T,S}^{\dagger,②} = A^{(2)}_{R(A^\dagger AA_{T,S}^{(2)}),N((AA_{T,S}^{(2)})^*)}$.

Proof.

(1) Since $A_{T,S}^{\dagger,②}$ is an outer inverse of A, we have $A_{T,S}^{\dagger,②}A$ which is a projector. Notice that

$$\begin{aligned} R\left(A_{T,S}^{\dagger,②}A\right) &\subseteq R\left(A^\dagger \mathrm{AA}_{T,S}^{(2)}\right) = R\left(A^\dagger \mathrm{AA}_{T,S}^{(2)} \mathrm{AA}_{T,S}^{(2)}\right) \\ &= R\left(A^\dagger \mathrm{AA}_{T,S}^{(2)} \mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger \mathrm{AA}_{T,S}^{(2)}\right) \subseteq R\left(A_{T,S}^{\dagger,②}A\right). \end{aligned} \tag{19}$$

Also, we obtain

$$N\left(A_{T,S}^{\dagger,②}A\right) = N\left(A_{T,S}^②A\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger A\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^* A\right). \tag{20}$$

(2) Since

$$\mathrm{AA}_{T,S}^{\dagger,②} = \mathrm{AA}^\dagger \mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger = \mathrm{AA}_{T,S}^② = \mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger, \tag{21}$$

we get

$$\begin{aligned} R\left(\mathrm{AA}_{T,S}^{\dagger,②}\right) &= R\left(\mathrm{AA}_{T,S}^②\right) \subseteq R\left(\mathrm{AA}_{T,S}^{(2)}\right) = R\left(\mathrm{AA}_{T,S}^{(2)}\mathrm{AA}_{T,S}^{(2)}\right) \\ &= R\left(\mathrm{AA}_{T,S}^{(2)}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger \mathrm{AA}_{T,S}^{(2)}\right) \subseteq R\left(\mathrm{AA}_{T,S}^{\dagger,②}\right). \end{aligned} \tag{22}$$

By Lemma 2, we have $R(\mathrm{AA}_{T,S}^{(2)}) = \mathrm{AT}$.
On the other hand,

$$N\left(\mathrm{AA}_{T,S}^{\dagger,②}\right) = N\left(\mathrm{AA}_{T,S}^②\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^\dagger\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^*\right). \tag{23}$$

(3) By (2) and Lemma 5, we can obtain $\mathrm{AA}_{T,S}^{\dagger,②}$ which is the orthogonal projector onto $R(\mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^\dagger) = R(\mathrm{AA}_{T,S}^{(2)})$.

(4) This part is clear by

$$R\left(A_{T,S}^{\dagger,②}\right) = R\left(A_{T,S}^{\dagger,②}A\right) = R\left(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\right), N\left(A_{T,S}^{\dagger,②}\right) = N\left(\mathrm{AA}_{T,S}^{\dagger,②}\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^{*}\right). \tag{24}$$

□

Theorem 13. *Let $A \in \mathbb{C}^{m\times n}$. Then, $A_{T,S}^{\dagger,②}$ is the $(A^{\dagger}AA_{T,S}^{(2)}, (AA_{T,S}^{(2)})^{\dagger})$-inverse of A.*

Proof. From the definition of the (B, C)-inverse, we can get

$$A_{T,S}^{\dagger,②}\mathrm{AA}^{\dagger}\mathrm{AA}_{T,S}^{(2)} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{\dagger}\mathrm{AA}^{\dagger}\mathrm{AA}_{T,S}^{(2)} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}, \tag{25}$$

and

$$\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}\mathrm{AA}_{T,S}^{\dagger,②} = \left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}\mathrm{AA}^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger} = \left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}. \tag{26}$$

On the other hand, from Theorem 12, we have

$$R\left(A_{T,S}^{\dagger,②}\right) = R\left(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\right), N\left(A_{T,S}^{\dagger,②}\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^{*}\right) = N\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}\right). \tag{27}$$

□

Theorem 14. *Let $A \in \mathbb{C}^{m\times n}$. Then,*

$$A_{T,S}^{\dagger,②} = A^{\dagger}AA_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{*}. \tag{28}$$

Proof. Using $A_{T,S}^{\dagger,②} = A_{R(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}),N((\mathrm{AA}_{T,S}^{(2)})^{*})}$, we obtain, on the basis of the Urquhart formula [10, 11] (see Lemma 6),

$$A_{T,S}^{\dagger,②} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\left(\mathrm{AA}_{T,S}^{(2)}\right)^{*}\mathrm{AA}^{\dagger}\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{*} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{*}. \tag{29}$$

□

Theorem 15. *Let $A \in \mathbb{C}^{m\times n}$. Then $X = A_{T,S}^{\dagger,②}$ is the unique solution to the constrained matrix equation*

$$R(X) \subseteq R\left(A^{\dagger}AA_{T,S}^{(2)}\right), \quad AX = P_{R(AT)}. \tag{30}$$

Proof. By Theorem 12, notice that (30) holds for $A_{T,S}^{\dagger,②}$. Let (30) be satisfied for different $X, X_1 \in \mathbb{C}_{T,S}^{m\times n}$. Because

$$A(X - X_1) = P_{R(\mathrm{AT})} - P_{R(\mathrm{AT})} = 0, \tag{31}$$

we see $A^{\dagger}\mathrm{AA}_{T,S}^{(2)}A(X - X_1) = 0$ and $R(X - X_1) \subseteq N(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}A)$. Then, the inclusions $R(X) \subseteq R(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}) = R(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}A)$ and $R(X_1) \subseteq R(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}A)$ yield $R(X - X_1) \subseteq N(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}A) \cap R(A^{\dagger}\mathrm{AA}_{T,S}^{(2)}A) = \{0\}$. So, $X = X_1$, i.e., $A_{T,S}^{\dagger,②}$ is the unique solution to the constrained system (30). □

Corollary 16. *If $A \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$, the MPCEP inverse of A is the unique matrix satisfying*

$$R(X) \subseteq R\left(A^{k}\right) \text{ and } AX = P_{R\left(A^{k}\right)}. \tag{32}$$

Theorem 17. *Let $A \in \mathbb{C}^{m\times n}$. The following statements are mutually equivalent for $X \in \mathbb{C}^{m\times n}$.*

(1) $X = A_{T,S}^{\dagger,②}$;

(2) $AXA = AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}A, XAX = X, AX = AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}, XA = A^{\dagger}AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}A$;

(3) $AX = AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}, X = A^{\dagger}AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}AX$;

(4) $AX = AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}, A^{\dagger}AA_{T,S}^{(2)}AX = X, (AA_{T,S}^{(2)})AX = AA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger}$;

(5) $XAA_{T,S}^{(2)} = A^{\dagger}AA_{T,S}^{(2)}, A_{T,S}^{(2)}AXAA_{T,S}^{(2)} = A_{T,S}^{(2)}, XAA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger} = X$;

(6) $A^{\dagger}AA_{T,S}^{(2)}AXAA_{T,S}^{(2)}(AA_{T,S}^{(2)})^{\dagger} = X, AXAA_{T,S}^{(2)} = AA_{T,S}^{(2)}$.

Proof.

(1)⇒(2): On the basis of $X = A_{T,S}^{\dagger,②}$, one can verify $\mathrm{AXA} = \mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^{\dagger}A$. The rest of the proof is evident by Theorem 7.

(2)⇒(3): Using $\mathrm{XA} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^{\dagger}A$ and $\mathrm{XAX} = X$, we obtain

$$A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{\dagger}\mathrm{AX} = \mathrm{XAX} = X. \tag{33}$$

(3)⇒(4): By $\mathrm{AX} = \mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^{\dagger}$, we get

$$\left(\mathrm{AA}_{T,S}^{(2)}\right)\mathrm{AX} = \mathrm{AA}_{T,S}^{(2)}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger} = \mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}. \tag{34}$$

The assumption $X = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}(\mathrm{AA}_{T,S}^{(2)})^{\dagger}\mathrm{AX}$ implies

$$A^{\dagger}\mathrm{AA}_{T,S}^{(2)}AX = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger} = A^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}\mathrm{AX} = X. \tag{35}$$

(4)⇒(5): Since $AX = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$, $A^{\dagger}AA^{(2)}_{T,S}AX = X$, we have

$$\begin{aligned} XAA^{(2)}_{T,S} &= A^{\dagger}AA^{(2)}_{T,S}AXAA^{(2)}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S} \\ &= A^{\dagger}AA^{(2)}_{T,S}AA^{(2)}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}, \\ A^{(2)}_{T,S}AXAA^{(2)}_{T,S} &= A^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S} = A^{(2)}_{T,S}. \end{aligned} \tag{36}$$

Because $XAA^{(2)}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}$, we see that

$$\begin{aligned} XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} &= A^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{\dagger}AA^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} \\ &= A^{\dagger}AA^{(2)}_{T,S}AX = X. \end{aligned} \tag{37}$$

(5)⇒(6): For the assumptions, we have

$$\begin{aligned} A^{\dagger}AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} &= A^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} \\ &= XAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = X \\ AXAA^{(2)}_{T,S} &= AA^{\dagger}AA^{(2)}_{T,S} = AA^{(2)}_{T,S}. \end{aligned} \tag{38}$$

(6)⇒(1): Note that

$$\begin{aligned} X &= A^{\dagger}AA^{(2)}_{T,S}AXAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{\dagger}AA^{(2)}_{T,S}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} \\ &= A^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}. \end{aligned} \tag{39}$$

□

Theorem 18. *Let $A \in \mathbb{C}^{m\times n}$ and let $U, V \in \mathbb{C}^{n\times m}$ such that $V \in A\{1\}$. Then the following statements are equivalent.*

(1) $A^{\dagger,②}_{T,S} = VAU$;

(2) $AU = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$, $A^{\dagger}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} = VAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$;

(3) $N(AU) = N((AA^{(2)}_{T,S})^{*})$, $R(VAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}) \subseteq R(A^{*})$ $A^{\dagger}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} = VAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$, $AUA = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}A$;

(4) $U = A^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} + (I_n - A^{\dagger}A)Y$, $V = A^{\dagger} + Z(I_m - AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger})$, *for arbitrary* $Y, Z \in \mathbb{C}^{n\times m}$.

Proof. (1)⇒(2): The equalities $AVA = A$, and $A^{\dagger,②}_{T,S} = A^{\dagger}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$ imply

$$\begin{aligned} AU = A(VAU) = AA^{\dagger,②}_{T,S} &= AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}, \\ A^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{\dagger,②}_{T,S} &= V(AU) = VAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}. \end{aligned} \tag{40}$$

(2)⇒(3): The assumption $AU = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$ yields $AUA = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}A$ and $N(AU) = N(AA^{②}_{T,S}) = N((AA^{(2)}_{T,S})^{*})$. Applying the condition $A^{\dagger}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} = VAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$, we get

$$\begin{aligned} R\left(VAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}\right) &= R\left(A^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}\right) \\ &\subseteq R\left(A^{\dagger}AA^{(2)}_{T,S}\right) \subseteq R\left(A^{\dagger}\right) = R(A^{*}). \end{aligned} \tag{41}$$

(3)⇒(1): Since $R(I_m - (AA^{(2)}_{T,S})^{*}) = N((AA^{(2)}_{T,S})^{*}) = N(AU)$, and $AUA = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}A$, we obtain

$$\begin{aligned} AU &= AUAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} \\ &= AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}. \end{aligned} \tag{42}$$

By $A^{\dagger}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} = VAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$, we have

$$VAU = VAA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{\dagger}AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} = A^{\dagger,②}_{T,S}. \tag{43}$$

(1)⇒(4): Recall that all solutions of the equation $AU = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$ are obtained as a sum of a particular solution of $AU = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$ and the general solution of the homogeneous equation $AU = 0$. According to [9], the general solution of $AU = AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$ is given by

$$U = A^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger} + \left(I_n - A^{\dagger}A\right)Y \tag{44}$$

for arbitrary $Y \in \mathbb{C}^{n\times m}$. In the same way, the general solution of $A^{\dagger}AA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger} = VAA^{(2)}_{T,S}(AA^{(2)}_{T,S})^{\dagger}$ is given by

$$V = A^{\dagger} + Z\left(I_m - AA^{(2)}_{T,S}\left(AA^{(2)}_{T,S}\right)^{\dagger}\right), \tag{45}$$

for arbitrary $Z \in \mathbb{C}^{n\times m}$.

(4)⇒(1): If $U = A^{(2)}_{T,S}(\mathrm{AA}^{(2)}_{T,S})^{\dagger} + (I_n - A^{\dagger}A)Y$ and $V = A^{\dagger} + Z(I_m - \mathrm{AA}^{(2)}_{T,S}(\mathrm{AA}^{(2)}_{T,S})^{\dagger})$, for arbitrary $Y, Z \in \mathbb{C}^{n\times m}$, then

$$VAU = A^{\dagger}\mathrm{AA}^{(2)}_{T,S}\left(\mathrm{AA}^{(2)}_{T,S}\right)^{\dagger} = A^{\dagger,②}_{T,S}. \tag{46}$$

□

Using the full-rank decomposition of A, we develop the next expression for the G-MPCEP inverse $A^{\dagger,②}_{T,S}$.

Theorem 19. *Let $A \in \mathbb{C}^{m\times n}$. If A has the full-rank decomposition $A = PQ$ and $l \geq Ind(A)$, then*

$$A^{\dagger,②}_{T,S} = Q^{*}(P^{*}AQ^{*})^{-1}P^{*}P_{R\left(AA^{(2)}_{T,S}\right)}. \tag{47}$$

Proof. The proof follows from $A^{\dagger} = Q^{*}(P^{*}\mathrm{AQ}^{*})^{-1}P^{*}$ in SHU conjunction with Theorem 10. □

Theorem 20. *Let $A \in \mathbb{C}^{m\times n}$. Suppose that $G \in \mathbb{C}^{n\times m}_{s}$ satisfies $R(G) = T$ and $N(G) = S$. Assume that $G = PQ$ is full rank factorization of G and QAP is invertible. Then,*

$$A^{\dagger,②}_{T,S} = P_{R(A^{*})}P(QAP)^{-1}Q\left(AP(QAP)^{-1}Q\right)^{\dagger}. \tag{48}$$

Proof. By [12], we observe that $A^{(2)}_{T,S} = P(\mathrm{QAP})^{-1}Q$. An application of Theorem 10 leads to

$$A^{\dagger,②}_{T,S} = P_{R(A^{*})}P(\mathrm{QAP})^{-1}Q\left(\mathrm{AP}(\mathrm{QAP})^{-1}Q\right)^{\dagger}, \tag{49}$$

which was our goal. □

Corollary 21. *Let A, T, S satisfy the assumption of Theorem 7. Suppose A has the full-rank decomposition $A = PQ$. If $Ind(A) = k$ and $G = P_1Q_1$ is the full-rank decomposition of G, then*

$$A^{\dagger,②}_{T,S} = Q^{*}(P^{*}AQ^{*})^{-1}P^{*}AP_1(Q_1AP_1)^{-1}Q\left(AP_1(Q_1AP_1)^{-1}Q_1\right)^{\dagger}. \tag{50}$$

The G-MPCEP inverse $A^{\dagger}\mathrm{AA}^{(2)}_{T,S}(\mathrm{AA}^{(2)}_{T,S})^{\dagger}$ can be further characterized under the assumptions $R(U) = E$ and $N(V) = F$ for some $U \in \mathbb{C}^{n\times k}$ and $V \in \mathbb{C}^{l\times m}$. The notation $\Omega_{U,V} = A^{(2)}_{R(U),N(V)}$ will be used in order to simplify presentations.

Theorem 22. *Let $U \in \mathbb{C}^{n\times k}$, $V \in \mathbb{C}^{l\times m}$, and $A \in \mathbb{C}^{m\times n}_{R(U),N(V)}$. Then the next assertions are equivalent.*

(1) $A^{\dagger,②}_{T,S}$ just coincides with $X \in \mathbb{C}^{n\times m}$, defined by

$$X = A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = A^{\dagger}AU(VAU)^{(1)}V\left(AU(VAU)^{(1)}V\right)^{\dagger}, \tag{51}$$

(2) $VAX = V(A\Omega_{U,V})^{\dagger}$, $A^{\dagger}A\Omega_{U,V}AX = X$,

(3) $VAXA\Omega_{U,V} = V$, $A^{\dagger}A\Omega_{U,V}AXA\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = X$.

Proof. By [9], rank (AB) = rank $(B) \Leftrightarrow B(\mathrm{AB})^{(1)}\mathrm{AB} = B$. The equality $R(\Omega_{U,V}) = R(U)$ implies, for $U^{(1)} \in U\{1\}$,

$$\Omega_{U,V} = \mathrm{UU}^{(1)}\Omega_{U,V} \text{ and } \Omega_{U,V}\mathrm{AU} = U. \tag{52}$$

By [9], rank (AB) = rank $(A) \Leftrightarrow \mathrm{AB}(\mathrm{AB})^{(1)}A = A$. By the equality $R(\Omega_{U,V}) = N(V)$, we get, for $V^{(1)} \in V\{1\}$,

$$\Omega_{U,V} = \Omega_{U,V}V^{(1)}V \text{ and } \mathrm{VA}\Omega_{U,V} = V. \tag{53}$$

(1)⇒(2): Applying $X = A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}$, we conclude

$$\begin{aligned}\mathrm{VAX} &= \mathrm{VAA}^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = (VA\Omega_{U,V})(A\Omega_{U,V})^{\dagger}\\ &= V(A\Omega_{U,V})^{\dagger},\end{aligned}$$

$$\begin{aligned}A^{\dagger}A\Omega_{U,V}\mathrm{AX} &= A^{\dagger}A\Omega_{U,V}\mathrm{AA}^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}\\ &= A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = X.\end{aligned} \tag{54}$$

(2)⇒(1): Notice that $A^{\dagger}\Omega_{U,V}\mathrm{AX} = \Omega_{U,V}(A\Omega_{U,V})^{\dagger}$ and $\mathrm{VAX} = V(A\Omega_{U,V})^{\dagger}$ give

$$\begin{aligned}X &= A^{\dagger}A\Omega_{U,V}\mathrm{AX} = A^{\dagger}A\Omega_{U,V}V^{(1)}(\mathrm{VAX})\\ &= A^{\dagger}A\left(\Omega_{U,V}V^{(1)}V\right)(A\Omega_{U,V})^{\dagger} = A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}.\end{aligned} \tag{55}$$

(1)⇒(3): Because $X = A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}$, it follows that

$$\begin{aligned}\mathrm{VAXA}\Omega_{U,V} &= \mathrm{VAA}^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}A\Omega_{U,V}\\ &= \mathrm{VA}\Omega_{U,V}(A\Omega_{U,V})^{\dagger}A\Omega_{U,V} = \mathrm{VA}\Omega_{U,V} = V,\end{aligned}$$

$$\begin{aligned}&A^{\dagger}A\Omega_{U,V}\mathrm{AXA}\Omega_{U,V}(A\Omega_{U,V})^{\dagger}\\ &\quad= A^{\dagger}A\Omega_{U,V}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}\\ &\quad= A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = X.\end{aligned} \tag{56}$$

(3)⇒(1): From the assumptions $\mathrm{VAXA}\Omega_{U,V} = V$ and $A^{\dagger}A\Omega_{U,V}\mathrm{AXA}\Omega_{U,V}(A\Omega_{U,V})^{\dagger} = X$, we get

$$\begin{aligned}X &= A^{\dagger}A\Omega_{U,V}\mathrm{AXA}\Omega_{U,V}(A\Omega_{U,V})^{\dagger}\\ &= A^{\dagger}A\Omega_{U,V}V^{(1)}(\mathrm{VAXA}\Omega_{U,V})(A\Omega_{U,V})^{\dagger}\\ &= A^{\dagger}A\Omega_{U,V}V^{(1)}V(A\Omega_{U,V})^{\dagger} = A^{\dagger}A\Omega_{U,V}(A\Omega_{U,V})^{\dagger}.\end{aligned} \tag{57}$$

□

3. Cramer's Rule for the Solution of a Singular Equation $Ax = B$

Theorem 23. *Let $A \in \mathbb{C}^{m\times n}$ with $Ind(A) = k$, $\text{rank}(A^{\dagger}AA^{(2)}_{T,S}) = \text{rank}((AA^{(2)}_{T,S})^{*}) = r$. Suppose that $U \in C^{m\times(m-r)}$ and $V^{*} \in C^{n\times(n-r)}$ are full-column rank matrices such that*

$$R\left(A^{\dagger}AA^{(2)}_{T,S}\right) = N(V), N\left(\left(AA^{(2)}_{T,S}\right)^{*}\right) = R(U). \tag{58}$$

Then the bordered matrix

$$X = \begin{pmatrix} A & U \\ V & 0 \end{pmatrix} \tag{59}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{\dagger,②}_{T,S} & \left(I_n - A^{\dagger,②}_{T,S}A\right)V^{\dagger} \\ U^{\dagger}\left(I_m - AA^{\dagger,②}_{T,S}\right) & -U^{\dagger}\left(A - AA^{\dagger,②}_{T,S}A\right)V^{\dagger} \end{pmatrix}. \tag{60}$$

Proof. Since $R(A^{\dagger,②}_{T,S}) = R(A^{\dagger}\text{AA}^{(2)}_{T,S}) = N(V)$, we obtain $\text{V}\,\text{A}^{\dagger,②}_{T,S} = 0$. By

$$\begin{aligned} R\left(I_m - \text{AA}^{\dagger,②}_{T,S}\right) &= N\left(\text{AA}^{\dagger,②}_{T,S}\right) = N\left(A^{\dagger,②}_{T,S}\right) = N\left(\left(\text{AA}^{(2)}_{T,S}\right)^{*}\right) \\ &= R(U) = R\left(\text{UU}^{\dagger}\right), \end{aligned} \tag{61}$$

we can obtain

$$\text{UU}^{\dagger}\left(I_m - \text{AA}^{\dagger,②}_{T,S}\right) = I_m - \text{AA}^{\dagger,②}_{T,S}. \tag{62}$$

Let

$$Y = \begin{pmatrix} A^{\dagger,②}_{T,S} & \left(I_n - A^{\dagger,②}_{T,S}A\right)V^{\dagger} \\ U^{\dagger}\left(I_m - \text{AA}^{\dagger,②}_{T,S}\right) & -U^{\dagger}\left(A - \text{AA}^{\dagger,②}_{T,S}A\right)V^{\dagger} \end{pmatrix}, \tag{63}$$

we have

$$\begin{aligned} \text{XY} &= \begin{pmatrix} \text{AA}^{\dagger,②}_{T,S} + \text{UU}^{\dagger}\left(I_m - \text{AA}^{\dagger,②}_{T,S}\right) & A\left(I_n - A^{\dagger,②}_{T,S}A\right)V^{\dagger} - \text{UU}^{\dagger}\left(A - \text{AA}^{\dagger,②}_{T,S}A\right)V^{\dagger} \\ \text{VA}^{\dagger,②}_{T,S} & V\left(I_n - A^{\dagger,②}_{T,S}A\right)V^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} \text{AA}^{\dagger,②}_{T,S} + \left(I_m - \text{AA}^{\dagger,②}_{T,S}\right) & A\left(I_n - A^{\dagger,②}_{T,S}A\right)V^{\dagger} - \text{UU}^{\dagger}\left(I_m - \text{AA}^{\dagger,②}_{T,S}\right)\text{AV}^{\dagger} \\ \text{VA}^{\dagger,②}_{T,S} & \text{VV}^{\dagger} - \text{VA}^{\dagger,②}_{T,S}\text{AV}^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} I_m & A\left(I_n - A^{\dagger,②}_{T,S}A\right)V^{\dagger} - \left(I_m - \text{AA}^{\dagger,②}_{T,S}\right)AV^{\dagger} \\ \text{VA}^{\dagger,②}_{T,S} & \text{VV}^{\dagger} \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-r} \end{pmatrix} = I_{m+n-r}. \end{aligned} \tag{64}$$

Thus, X is nonsingular and $X^{-1} = Y$. □

Using the relationship between the G-MPCEP inverse and a nonsingular bordered matrix, we give Cramer's rule for solving a singular linear equation, $Ax = B$. $A(\text{ij} \longrightarrow b_j)$ denotes the matrix obtained by replacing ith column of A with b_j, where b_j is the jth column of B.

Theorem 24. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times n}_{T,S}$. If $R(B) \subseteq AT$, then the restricted matrix equation*

$$AX = B, R(X) \subseteq R\left(A^{\dagger}AA^{(2)}_{T,S}\right) \tag{65}$$

is consistent and it has the unique solution $X = A^{\dagger,②}_{T,S}B$.

Proof. Since $R(B) \subseteq \text{AT}$, then $\text{AA}^{\dagger,②}_{T,S}B = P_{\text{AT}}B = B$. Clearly, $X = A^{\dagger,②}_{T,S}B$ is a solution of $\text{AX} = B$. $X = A^{\dagger,②}_{T,S}B$ also satisfies the restricted condition because $R(X) \subseteq R(A^{\dagger,②}_{T,S}) = R(A^{\dagger}\text{A}\,\text{A}^{(2)}_{T,S})$. Finally, we show the uniqueness of X. If X_1 also satisfies (65), we can get $R(X_1) \subseteq R(A^{\dagger,②}_{T,S})$, then

$$X = A^{\dagger,②}_{T,S}B = A^{\dagger,②}_{T,S}\text{AX}_1 = P_{\text{AT}}X_1 = X_1. \tag{66}$$

□

Theorem 25. *Let $A, B \in \mathbb{C}^{m\times n}$. Suppose $U \in \mathbb{C}^{m\times(m-n+r)}$ and $V^* \in \mathbb{C}^{n\times r}$ having full column rank such that*

$$R\left(A_{T,S}^{\dagger,②}\right) = R\left(A^{\dagger}AA_{T,S}^{(2)}\right) = N(V) \text{ and } R(U) = N\left(A_{T,S}^{\dagger,②}\right). \tag{67}$$

If $R(B) \subseteq AT$, then the unique solution $X = A_{T,S}^{\dagger,②}B$ of the singular linear equation (65) is given by

$$x_{ij} = \frac{\det\begin{pmatrix} A(i \longrightarrow b_j) & U \\ V(i \longrightarrow 0) & 0 \end{pmatrix}}{\det\begin{pmatrix} A & U \\ V & 0 \end{pmatrix}}, i = 1, 2, \cdots m, j = 1, 2, \cdots n. \tag{68}$$

Proof. Since $X = A_{T,S}^{\dagger,②}B \in \mathrm{AT} = N(V)$ and $B \in R(\mathrm{AA}_{T,S}^{\dagger,②}) = A(\mathrm{AT})$, we have

$$\mathrm{VX} = 0, \left(I_m - \mathrm{AA}_{T,S}^{\dagger,②}\right)B = 0. \tag{69}$$

It follows from (69) that the solution of $\mathrm{AX} = B$ satisfies

$$\begin{pmatrix} A & U \\ V & 0 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{70}$$

By Theorem 23, the coefficient matrix of (70) is nonsingular. Using (60) and (69), we can obtain

$$\begin{aligned}\begin{pmatrix} X \\ 0 \end{pmatrix} &= \begin{pmatrix} A_{T,S}^{\dagger,②} & \left(I_n - A_{T,S}^{\dagger,②}A\right)V^{\dagger} \\ U^{\dagger}\left(I_m - AA_{T,S}^{\dagger,②}\right) & -U^{\dagger}\left(A - \mathrm{AA}_{T,S}^{\dagger,②}A\right)V^{\dagger} \end{pmatrix}\begin{pmatrix} B \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} A_{T,S}^{\dagger,②}B \\ 0 \end{pmatrix}.\end{aligned} \tag{71}$$

Therefore, $X = A_{T,S}^{\dagger,②}B$ and (68) follows from the classical Cramer's rule [1]. □

4. The Determinantal Representation of the G-MPCEP Inverse

The following notations are used in order to express determinantal representations of the generalized inverses.

Let $\beta = \beta_1, \cdots, \beta_k \subseteq \{1, 2, \cdots, n\}$ and $\alpha = \alpha_1, \cdots, \alpha_k \subseteq \{1, 2, \cdots, m\}$ be subsets with $1 \le k \le \min\{m, n\}$. Suppose A_{β}^{α} stands for a submatrix of $A \in \mathbb{C}^{m\times n}$ with rows and columns indexed by α and β. Then, A_{α}^{α} and $|A_{\alpha}^{\alpha}|$ denote a principal minor. Assume that

$$L_{k,n} = \{\alpha : \alpha = (\alpha_1, \cdots, \alpha_k), 1 \le \alpha_1 < \cdots < \alpha_k \le n\} \tag{72}$$

represents strictly increasing sequences $1 \le k \le n$ of integers chosen from $\{1,\ldots,n\}$. For fixed $i \in \alpha$ and $j \in \beta$, put

$$I_{r,m}\{i\} = \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}, J_{r,n}\{j\} = \{\beta : \beta \in L_{r,n}, j \in \beta\}. \tag{73}$$

Denote by $a_{\cdot j}$ and $a_{i\cdot}$ the jth column and the ith row of A, respectively. The notations $A_{i\cdot}(b)$ and $A_{\cdot j}(c)$ stand for matrices arising from A stating the row vector b instead of its ith row and the column vector c instead of its jth column.

Lemma 26. *[13] Let $A \in \mathbb{C}_r^{m\times n}$. Then the Moore-Penrose inverse $A^{\dagger} \in \mathbb{C}^{n\times m}$ possesses the determinantal representations*

$$A^{\dagger} = \left(\frac{\sum_{\beta\in J_{r,n}\{i\}}\left|(A^*A)_{\cdot i}\left(a_{\cdot j}^*\right)_{\beta}^{\beta}\right|}{\sum_{\beta\in J_{r,n}}\left|(A^*A)_{\beta}^{\beta}\right|}\right)_{n\times m}, \tag{74}$$

$$= \left(\frac{\sum_{\alpha\in I_{r,m}\{j\}}\left|(A^*A)_{j\cdot}\left(a_{i\cdot}^*\right)_{\alpha}^{\alpha}\right|}{\sum_{\beta\in I_{r,m}}\left|(A^*A)_{\alpha}^{\alpha}\right|}\right)_{n\times m}. \tag{75}$$

Lemma 27. *[14] Let $A \in \mathbb{C}^{m\times n}$ be of rank r, T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose that $G \in \mathbb{C}_s^{n\times m}$ satisfies $R(G) = T$ and $N(G) = S$. If $A_{T,S}^{(2)}$ exists, then*

$$Ind(AG) = Ind(GA) = 1. \tag{76}$$

Furthermore,

$$A_{T,S}^{(2)} = G(AG)_g = (GA)_g G. \tag{77}$$

Lemma 28. *[15] Let A, T, S and G be the same as in Lemma 27. Write $G = (g_{ij})$. Suppose that the generalized inverse $A_{T,S}^{(2)}$ of A exists. Then, $A_{T,S}^{(2)}$ can be represented as follows:*

$$A_{T,S}^{(2)} = \left(\frac{\sum_{\beta\in Q_{t,n}\{i\}}\left|\left((GA)_{\cdot i}\left(g_{\cdot j}\right)\right)_{\beta}^{\beta}\right|}{\sum_{\beta\in Q_{t,n}}\left|(GA)_{\beta}^{\beta}\right|}\right)_{n\times m}, \tag{78}$$

$$= \left(\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((AG)_{j\cdot}\left(g_{i\cdot}\right)\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(AG)_{\alpha}^{\alpha}\right|}\right)_{n\times m}. \tag{79}$$

Theorem 29. *Let $A \in \mathbb{C}_r^{m\times n}$, $Ind(A) = k$ and A, T, S and G be the same as in Lemma 27. Write $G = (g_{ij})$. Then its*

G-MPCEP inverse $A_{T,S}^{\dagger,②}$ has the following determinantal representations

$$A_{T,S}^{\dagger,②}=\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|(AG)_{j.}\left(v_{i.}^{(1)}\right)_{\alpha}^{\alpha}\right|}{\sum_{\beta\in J_{r,n}}\left|(A^{*}A)_{\beta}^{\beta}\right|\sum_{\alpha\in Q_{t,m}}\left|(AG)_{\alpha}^{\alpha}\right|}\left(A\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((AG)_{j.}(g_{i.})\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(AG)_{\alpha}^{\alpha}\right|}\right)^{\dagger},\tag{80}$$

$$=\frac{\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}\left(v_{.j}^{(2)}\right)_{\beta}^{\beta}\right|}{\sum_{\beta\in J_{r,n}}\left|(A^{*}A)_{\beta}^{\beta}\right|\sum_{\alpha\in Q_{t,m}}\left|(AG)_{\alpha}^{\alpha}\right|}\left(A\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((AG)_{j.}(g_{i.})\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(AG)_{\alpha}^{\alpha}\right|}\right)^{\dagger},\tag{81}$$

where

$$v_{i.}^{(1)}=\left[\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}\left(\widetilde{A}_{.f}\right)_{\beta}^{\beta}\right|\right]\in\mathbb{C}^{1\times n},f=1,\cdots,n,$$

$$v_{.j}^{(2)}=\left[\sum_{\alpha\in Q_{t,m}\{j\}}\left|(AG)_{j.}\left(\widetilde{A}_{l.}\right)_{\alpha}^{\alpha}\right|\right]\in\mathbb{C}^{m\times 1},l=1,\cdots,n.\tag{82}$$

Proof. According to Definition 8 for $A_{T,S}^{\dagger,②}$, we obtain

$$A_{T,S}^{\dagger,②}=\sum_{l=1}^{m}\sum_{f=1}^{n}A_{\mathrm{il}}^{\dagger}A_{lf}A_{T,S(fj)}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}.\tag{83}$$

By substituting (74) and (79) for the determinantal representations of $A^{\dagger}$ and $A_{T,S}^{(2)}$ in (83), we get

$$\begin{aligned}A_{T,S}^{\dagger,②}&=\sum_{l=1}^{m}\sum_{f=1}^{n}\frac{\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}\left(a_{l}^{*}\right)_{\beta}^{\beta}\right|}{\sum_{\beta\in J_{r,n}}\left|(A^{*}A)_{\beta}^{\beta}\right|}A_{\mathrm{lf}}\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((\mathrm{AG})_{j.}\left(g_{f.}\right)\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(\mathrm{AG})_{\alpha}^{\alpha}\right|}\\&\quad\cdot\left(A\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((\mathrm{AG})_{j.}(g_{i.})\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(\mathrm{AG})_{\alpha}^{\alpha}\right|}\right)^{\dagger}\\&=\sum_{l=1}^{m}\sum_{f=1}^{n}\frac{\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}(e_{.l})_{\beta}^{\beta}\right|}{\sum_{\beta\in J_{r,n}}\left|(A^{*}A)_{\beta}^{\beta}\right|}\widetilde{A}_{lf}\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((\mathrm{AG})_{j.}\left(e_{f.}\right)\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(\mathrm{AG})_{\alpha}^{\alpha}\right|}\\&\quad\cdot\left(A\frac{\sum_{\alpha\in Q_{t,m}\{j\}}\left|\left((\mathrm{AG})_{j.}(g_{i.})\right)_{\alpha}^{\alpha}\right|}{\sum_{\alpha\in Q_{t,m}}\left|(\mathrm{AG})_{\alpha}^{\alpha}\right|}\right)^{\dagger},\end{aligned}\tag{84}$$

where $e_{.l}$ is the unit lth column vector, $e_{f.}$ is the unit fth row vector, and $\widetilde{\mathrm{A}}_{\mathrm{lf}}$ is the (lf)th element of the matrix $\tilde{A}=A^{*}(\mathrm{AG})$. If we put

$$v_{\mathrm{if}}^{(1)}:=\sum_{l=1}^{m}\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}(e_{.l})_{\beta}^{\beta}\right|\widetilde{A}_{\mathrm{lf}}=\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}\left(\widetilde{A}_{.f}\right)_{\beta}^{\beta}\right|\tag{85}$$

as the fth component of the row vector $v_{i.}^{(1)}=[v_{i1}^{(1)},\cdots,v_{\mathrm{in}}^{(1)}]$, then from

$$\sum_{f=1}^{n}v_{\mathrm{if}}^{(1)}\sum_{\alpha\in Q_{t,m}\{j\}}\left|(\mathrm{AG})_{j.}\left(e_{f.}\right)_{\alpha}^{\alpha}\right|=\sum_{\alpha\in Q_{t,m}\{j\}}\left|(\mathrm{AG})_{j.}\left(v_{i.}^{(1)}\right)_{\alpha}^{\alpha}\right|,\tag{86}$$

get (80). If we initially obtain

$$v_{\mathrm{lj}}^{(2)}:=\sum_{f=1}^{n}\widetilde{A}_{\mathrm{lf}}\sum_{\alpha\in Q_{t,m}\{j\}}\left|(\mathrm{AG})_{j.}\left(e_{f.}\right)_{\alpha}^{\alpha}\right|=\sum_{\alpha\in Q_{t,m}\{j\}}\left|(\mathrm{AG})_{j.}\left(\widetilde{A}_{l.}\right)_{\alpha}^{\alpha}\right|,\tag{87}$$

for the lth component of the column vector $v_{.j}^{(2)}=[v_{1j}^{(2)},\cdots,v_{\mathrm{in}}^{(1)}]$, then by

$$\sum_{l=1}^{m}\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}(e_{.l})_{\beta}^{\beta}\right|v_{lj}^{(2)}=\sum_{\beta\in J_{r,n}\{i\}}\left|(A^{*}A)_{.i}\left(v_{.j}^{(2)}\right)_{\beta}^{\beta}\right|,\tag{88}$$

get (81). □

5. Application

In this section, we will give an application of the G-MPCEP inverse in solving linear equations.

Theorem 30. *Let $A\in\mathbb{C}^{m\times n}$ and $b\in\mathbb{C}^{m}$. Then, the equation*

$$Ax=AA_{T,S}^{(2)}\left(AA_{T,S}^{(2)}\right)^{\dagger}b\tag{89}$$

is consistent and its general solution is

$$x=A_{T,S}^{\dagger,②}b+\left(I_{n}-A^{\dagger}A\right)y,\tag{90}$$

for arbitrary $y\in\mathbb{C}^{n}$.

Proof. For x given by (90), we get

$$Ax=\mathrm{AA}_{T,S}^{\dagger,②}b=\mathrm{AA}^{\dagger}\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}b=\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}b,\tag{91}$$

and so x is a solution to (89).

If x is a solution to 5.1, we have

$$A_{T,S}^{\dagger,②}b=A^{\dagger}\left(\mathrm{AA}_{T,S}^{(2)}\left(\mathrm{AA}_{T,S}^{(2)}\right)^{\dagger}b\right)=A^{\dagger}\mathrm{AX}.\tag{92}$$

Thus,

$$x=A_{T,S}^{\dagger,②}b+x-A^{\dagger}\mathrm{AX}=A_{T,S}^{\dagger,②}b+\left(I_{n}-A^{\dagger}A\right)x,\tag{93}$$

i.e., the solution x is of the form (90). □

Theorem 31. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then, the general solution of*

$$A^\dagger Ax = A^{\dagger,②}_{T,S} b \tag{94}$$

is given by

$$x = A^{\dagger,②}_{T,S} b + \left(I_n - A^\dagger A\right) y, \tag{95}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Notice that x of the form (95) is a solution to (94):

$$A^\dagger Ax = A^\dagger \mathrm{A} A^{\dagger,②}_{T,S} b = A^{\dagger,②}_{T,S} b. \tag{96}$$

Let x be a solution to (94). Then, by $A^\dagger Ax = A^{\dagger,②}_{T,S} b$, we deduce that x has the form (95):

$$x = A^{\dagger,②}_{T,S} b + x - A^\dagger Ax = A^{\dagger,②}_{T,S} b + \left(I_n - A^\dagger A\right) x. \tag{97}$$

□

Theorem 32. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then, the general solution to*

$$A^\dagger Ax = A^\dagger b, b \in R\left(A^\dagger A A^{(2)}_{T,S}\right), \tag{98}$$

is given by

$$x = A^{\dagger,②}_{T,S} b + \left(I_n - A^\dagger A\right) y, = A^\dagger b + \left(I_n - A^\dagger A\right) y \tag{99}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. If x is represented by (99), then

$$A^\dagger Ax = A^\dagger \mathrm{A} A^{\dagger,②}_{T,S} b = A^\dagger P_{R\left(A^\dagger \mathrm{A} A^{(2)}_{T,S}\right)} b = A^\dagger b. \tag{100}$$

Hence, x is a solution of (98).

On the other hand, assume that x is a solution to (98). Using

$$A^{\dagger,②}_{T,S} b = A^\dagger P_{R\left(A^\dagger \mathrm{A} A^{(2)}_{T,S}\right)} b = A^\dagger b = A^\dagger Ax, \tag{101}$$

one can conclude that

$$x = A^{\dagger,②}_{T,S} b + x - A^\dagger Ax = A^{\dagger,②}_{T,S} b + \left(I_n - A^\dagger A\right) x. \tag{102}$$

Thus, the solution x to (98) possesses the form (99). Since $b \in R(A^\dagger \mathrm{A} A^{(2)}_{T,S})$, we observe the identities $A^{\dagger,②}_{T,S} b = A^\dagger P_{R(A)} b = A^\dagger b = A^\dagger Ax$, which confirm the second identity in (99). □

Data Availability

There is no data in this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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