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Research Article

The Picture on the Presentation of Direct Product Group of Two Cyclic Groups

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A picture in a group presentation is a geometric configuration with an arrangement of discs and arcs within a boundary disc. The drawing of this picture does not have to follow a particular rule, only using the generator as discs and the relation as arcs. It will form a picture label pattern if drawn with a particular rule. This paper discusses the label pattern of a picture in the presentation of direct product groups. Direct product presentation is used with two cyclic groups, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ where $p, q \in \mathbb{Z}^+$ and $p, q \geq 2$. The method for forming a picture label pattern is to arrange the first generator in the initial arrangement, compile a second generator, and add a number of commutators. Furthermore, the pattern is used to calculate the length of the label on the picture. It is obtained that the picture's label is $a^{q-1}b^n ab^{q-n}$ and the length of the label is $p + 2n - q$, where n is the number of commutator discs.

1. Introduction

Let G be a group and $P = \langle x | r \rangle$ is the presentation of G, where x is a set of generators and r is a set of relations. A picture over P is an object with a particular condition. Let $\boldsymbol{Q}$ be a picture over P, and then there are various picture forms for $\boldsymbol{Q}$, but with different labels.

The theory of identity sequences over group presentations is given by [1]. The article discusses algebraic theory and an elementary and complete exposition of the theory of pictures. Moreover, it gives some examples of so-called combinatorially aspherical presentations. The author in [2] discusses combinatorial geometric techniques that determine explicit generators for the second homotopy module of the 2-complex in terms of its cell structure. The discussion focuses on the theory of pictures from a homotopy-theoretic perspective and then generalities related to the generation of second homotopy modules with some proof of its properties. The article also gives various calculations and applications for studying second homotopy modules. Furthermore, this study was expanded by [3], i.e., the extensions of a group presentation K by group presentation Q by adding reduced word on the generator of Q.

The generator's second homotopy module is calculated by [4]. The first discussion is about the relationship between the second homotopy module for two presentations. And it is then defining an isomorphic group using the Tietze transformation. A study of the picture related to the group presentation was obtained by applying the Kronecker product to the representation of the quaternion group given by [5]. Meanwhile, [6] discusses the picture of the crossed product of groups and finds the generator of its second homotopy modules.

In this paper, the author tries to study differently, i.e., to form a pattern of the picture labels of the presentation of the direct product group. It is used as the direct product of two cyclic groups, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ where $p, q \in \mathbb{Z}^+$ and $p, q \geq 2$.

The paper aims to create the pattern of the picture through the presentation of direct product groups of two cyclic groups. In this case, the pattern of the picture's label in the presentation of a cyclic group is associated with the number of arcs connected to the boundary disc.

The organization of this paper is as follows: the Preliminaries section introduces a lot of basic concepts and notations of a word, the presentation of the group, and the theory of picture, which will be used in the Result and Discussion section. The Result and Discussion section discusses the shape of the picture pattern in the presentation of the direct product by giving a formula for the picture in the presentation of a cyclic group that is associated with the number of arcs connected to the boundary disc.

Since the location of the arcs on the disc in a picture of the presentation of a cyclic group can be anywhere as much as the order of the cyclic group, it takes work to form a pattern in the picture. Thus, this paper has proven the properties of the picture in the presentation of the direct product (Theorem 12 and Corollary 13).

2. Preliminaries

We start the notion with the word, presentation groups, and picture over presentation groups.

A group presentation P is pair $\langle X\,;R\rangle$, where X is a set of generators and R is nonempty, cyclically reduced word on X (the relators).

Definition 1 (see [7]). Let $X=\{a_1,a_2,\cdots,a_n\}$ be a set of distinct elements and $X^{-1}=\{a_i^{-1}|a_i\in X\}$. Define $X^{\pm1}=X\cup X^{-1}$. A word W is a finite string with the form $a_1^{\varepsilon_1}a_2^{\varepsilon_2}\cdots a_{n-1}^{\varepsilon_{n-1}}a_n^{\varepsilon_n}$, $n\geq 0$, $a_i\in X$, $\varepsilon_i=\pm1$, and $i=1,2,\cdots,n$.

There are four operations on words on X, that is

(i) If W contains a subword $a_i^{\varepsilon_i}a_i^{-\varepsilon_i}$, $x_i\in X$, and $\varepsilon_i=\pm1$, then delete it

(ii) Insert a word $a_i^{\varepsilon_i}a_i^{-\varepsilon_i}$, $x_i\in X$, and $\varepsilon_i=\pm1$ at any position on W

(iii) If W contains a subword A^{ε}, $A\in X$, and $\varepsilon=\pm1$, then delete it

(iv) Insert A^{ε}, $A\in X$, and $\varepsilon=\pm1$ at any position in W

The set of words will be denoted by $(X^{\pm1})^*$. Two words, V and W, are equivalent (relative to P) if one of the words can be obtained from the other by finite operations on words. If V and W are equivalent, then they are symbolized by $V\sim_P W$. Relation "$\sim_P$" is an equivalence relation in $(X^{\pm1})^*$.

Let $[W]_P$ be the equivalence class containing W, then a set of equivalence classes $\{[W]_P\,;W\in(X^{\pm1})^*\}$ with binary operation $[V]_P[W]_P=[VW]_P$ for every $V,W\in(X^{\pm1})^*$ form a group. The group defined by P denoted by $G(P)$ with identity in $G(P)$ is $[1]_P$ and $[W]_P^{-1}=[W^{-1}]_P$. If P is understood to be a group presentation for $G(P)$, then we refer to simply G.

Two words, V and W, are equivalent (relative to X) if one of the words can be obtained from the other by finite operations on words. If V and W are equivalent, then they are symbolized by $V\sim_X W$. The relation $\sim_X$ is an equivalence relation on $\{[W]_X\,;W\in(X^{\pm1})^*\}$, and $[W]_X$ is the equivalence class containing W. Set $\{[W]_X\,;W\in(X^{\pm1})^*\}$ with binary operation $[V]_X[W]_X=[VW]_X$ form a group, i.e., the free group containing X as a basis, denoted by $F(X)$. We usually dispense with equivalence class notation and write W for $[W]_X$. If $[V]_X=[W]_X$, then we write $V\approx W$ and say V and W are freely equivalent.

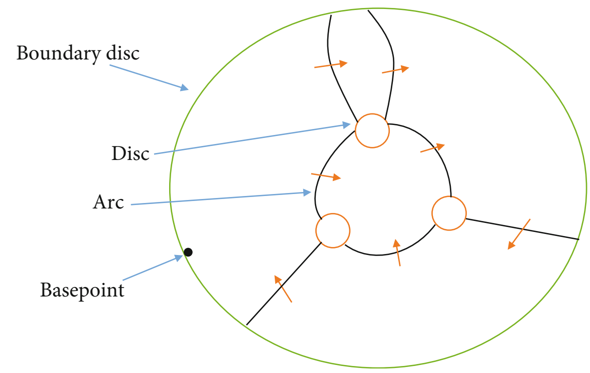

Figure 1: Picture $\mathbf{Q}$ over P.

We have the following properties:

Theorem 2 (see [8]). *If X is any set, there is a free group F_X having X as a free basis.*

Lemma 3 (see [9]). *If F is free on X, then X generates F.*

Theorem 4 (see [10] Characterization of freeness). *Let G be a group and X be a subset of G. Then G is free with basis X if and only if the following both hold:*

(1) X generates G, and

(2) If W is a word on X and $W=_G 1$, then W is not freely reduced, that is, W must contain an inverse pair

Definition 5 (see [2]). Let $P=\langle X\,;R\rangle$ be a group presentation. A picture $\mathbf{Q}$ over P is a geometric configuration consisting of the following:

(i) A boundary disc with a basepoint

(ii) Discs with labels read an element of R (clockwise or anticlockwise)

(iii) Disjoints arcs with label elements of X

The picture can be illustrated in Figure 1.

The picture $\mathbf{Q}$ over P is spherical if it has at least one disc, and no arc of $\mathbf{Q}$ meets the boundary disc.

A picture $\mathbf{Q}$ over P becomes a based picture over P when it is equipped with a basepoint as follows:

(i) Each disc has one base point, a selected point in the interior of a basic corner

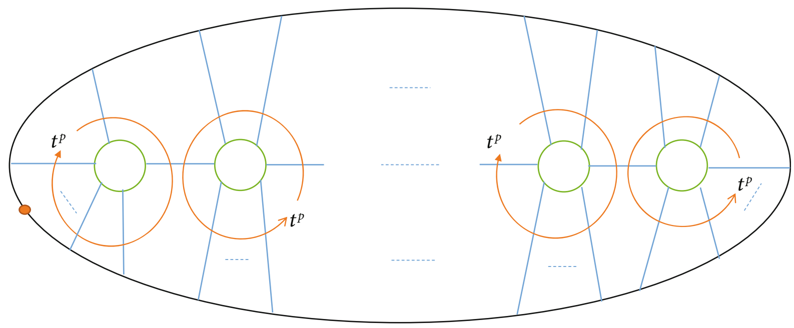

Figure 2: Picture of a presentation of a cyclic group with one lane.

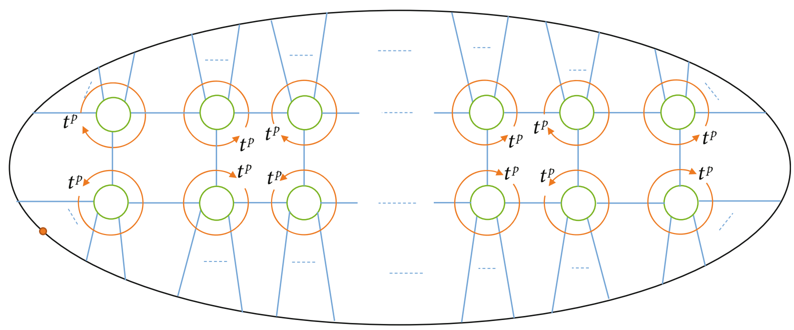

Figure 3: Picture of a presentation of a cyclic group with two lanes.

(ii) Picture $\mathbf{Q}$ has a global base point, a selected point in the boundary disc that does not lie on any arc of $\mathbf{Q}$

Two pictures will be equivalent if one can be transformed to the other by a finite number of delete/insert floating circle, delete/insert canceling pairs, bridge move, and replace(X), where X be a set of based spherical pictures over P.

Let $\mathbf{Q}$ be a picture over P labeled by $W(\mathbf{Q})$, then the length of $W(\mathbf{Q})$, symbolized by $|W(\mathbf{Q})|$, defined as the arcs connected to the boundary disc that is read clockwise.

Definition 6 (see [7]). Let G be a cyclic group with the generator t, $G = \langle t \rangle = \{t^n | n \in \mathbb{Z}\}$. Then, the presentation of a cyclic group order n is defined as $\langle t\,;t^n \rangle$.

Proposition 7 (see [7]). *A presentation of a cyclic group $\langle t\,; t^n \rangle$ is isomorphic to $\mathbb{Z}_n$, ($\langle t\,;t^n\rangle \cong \mathbb{Z}_n$).*

Definition 8 (see [7]). Let G_1 and G_2 be groups. The direct product $G_1 \times G_2$ of G_1 and G_2 is the set of all ordered pairs $\{(g,h) | g \in G_1, h \in G_2\}$ with the operation $(g_1, h_1)\bullet(g_2, h_2) = (g_1 g_2, h_1 h_2)$.

Theorem 9 (see [9]). *Let G_1 and G_2 be groups defined by presentations $P_1 = \langle X_1\,; R_1 \rangle$ and $P_2 = \langle X_2\,; R_2 \rangle$, respectively. Then, the presentation $P_1 \times P_2 = \langle X_1, X_2\,; R_1, R_2, x_1 x_2 = x_2 x_1$ $(x_1 \in X_1, x_2 \in X_2)\rangle$ defines the presentation of the direct product group $G_1 \times G_2$.*

Based on Definition 8, we have a presentation of $\mathbb{Z}_p \times \mathbb{Z}_q$: let $P_1 = \langle a\,; a^p \rangle$ and $P_2 = \langle b\,; b^q \rangle$ be the presentation for $\mathbb{Z}_p$ and $\mathbb{Z}_q$, respectively, then $P = P_1 \times P_2$ and $P = \langle a, b\,; a^p, b^q, aba^{-1}b^{-1} \rangle$ or $P = \langle a, b\,; a^p, b^q, [a,b] \rangle$

3. Results and Discussion

To form the picture pattern of $P = \langle a, b\,; a^p, b^q, [a,b] \rangle$, the following rules are used:

(1) Establish the desired picture pattern by setting the first disc's location as disc a^p (and continuing another disc on the right side as needed)

(2) Each number of disc a^p and disc b^q is one

(3) Picture pattern based on the number of discs $[a, b]$

The picture is introduced in the presentation of a cyclic group with two forms, i.e., a picture with one lane and two lanes.

The following theorem discusses the number of arcs connected to the boundary disc for the picture of the presentation of a cyclic group.

Theorem 10. *Let $\mathbf{Q}$ be a picture over $P = \langle t\,; t^p \rangle$ and $p \geq 4$ with the number of disjoint discs ($= n$) and $n \geq 3$, and every disjoint disc is connected with one arc (see Figures 2 and 3). Then, the number of arcs connected to the boundary disc is as follows:*

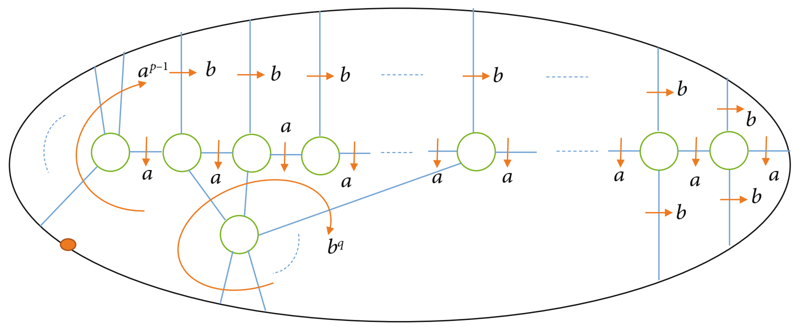

FIGURE 4: Picture over $P = \langle a, b\,; a^p, b^q, [a, b]\rangle$.

(a) $(p-2)n+2$, $n \in \mathbb{Z}^+$, $n \geq 3$, *and* $\mathbf{Q}$ *is a picture with a one lane*

(b) $4(p-2)+(p-3)(n-4)$, n *is even, and* $\mathbf{Q}$ *is a picture with two lanes*

Remark 11.

(i) For $p = 2$, we have two arcs connected to the boundary disc for a picture with one lane, and there is no picture with two lanes

(ii) For $p = 3$, we have $n+2$ arcs connected to the boundary disc, where $n \in \mathbb{Z}^+$ for a picture with one lane and the number of arcs connected to the boundary disc is 4, where n is even for a picture with two lanes, and there is no picture with two lanes, where n is odd

Proof of Theorem 10.

(a) The proof is divided into two cases, for n is odd and even

(i) n is odd

Consider that $\triangle_1$, $\triangle_2$, and $\triangle_3$ are disjoint discs in picture $\mathbf{Q}$ and $\theta(\triangle_i)$, $i = 1, 2, 3$ is a symbol for the number of arcs from $\triangle_i$. It is found that $\theta(\triangle_1)$ and $\theta(\triangle_3)$ are $(p-1)$, and $\theta(\triangle_2) = p-2$. Thus, $\sum_{i=1}^{3}\theta(\triangle_i) = 3(p-2)+2$, so it is true for $n = 3$. Suppose it is true for $n = k$, k is odd, we have $\sum_{i=1}^{k+2}\theta(\triangle_i) = (k+2)(p-2)+2$. So, it is true for $n = k+2$.

(ii) n is even

Consider that $\triangle_1$, $\triangle_2$, $\triangle_3$, and $\triangle_4$ are disjoint discs in picture $\mathbf{Q}$. It is found that $\theta(\triangle_1)$ and $\theta(\triangle_4)$ are $(p-1)$ and $\theta(\triangle_2) = p-2$. Thus, $\sum_{i=1}^{4}\theta(\triangle_i) = 4(p-2)+2$, so it is true for $n = 4$. Suppose it is true for $n = k$, k is even, we have $\sum_{i=1}^{k+2}\theta(\triangle_i) = (k+2)(p-2)+2$.

(b) Since $\mathbf{Q}$ is a picture with two lanes, the number of discs in $\mathbf{Q}$ is even. Consider that $\triangle_1$, $\triangle_2$, $\triangle_3$, and $\triangle_4$ are disjoint discs in picture $\mathbf{Q}$. It is found that $\theta(\triangle_i) = p-2$ for $i = 1, 2, 3, 4$. Thus, $\sum_{i=1}^{4}\theta(\triangle_i) = 4(p-2) = 4(p-2)+(p-3)(n-4)$. So, it is true for $n = 4$. Suppose it is true for $n = k$, k is even, we have $\sum_{i=1}^{k+2}\theta(\triangle_i) = (k+2)(p-2)+(p-3)((k+2)-4) = 4(p-2)+(p-3)(n-4)$. So, it is true for $n = k+2$

Since $\mathbf{Q}$ is a picture with two lanes, the number of discs in $\mathbf{Q}$ is even. □

Theorem 12. *Let* $\mathbf{Q}$ *be a picture over* $P = \langle a, b\,; a^p, b^q, [a, b]\rangle$, *where* $p, q \geq 2$. *Then, picture* $\mathbf{Q}$ *has the label* $a^{p-1}b^n ab^{q-n}$, *where* n *is the number disc of the commutator and* $n \in \mathbb{N}$.

Proof. The proof is divided into two cases: for $n = 1, 2, \cdots, q$, and for $n > q$.

Case $n = 1, 2, \cdots, q$. We have to consider the following picture.

Let $\mathbf{Q}$ be a picture over $P = \langle a, b\,; a^p, b^q, aba^{-1}b^{-1}\rangle$. Picture $\mathbf{Q}$ has three types of discs, i.e., discs with p arcs $(\triangle_{a^p})$, discs with q arcs $(\triangle_{b^q})$, and discs with $aba^{-1}b^{-1}$ arcs $(\triangle_{[a,b]})$.

For $n = 1, 2, \cdots, q$, by taking the basepoint as in Figure 4.

The number of arcs connected to the boundary disc is obtained.

(i) The number arcs a on $\triangle_{a^p}$ are $p-1$

(ii) There is one arc b on each disc $\triangle_{[a,b]}$ for $n = 1, 2, \cdots, q$ (n is the number of commutators)

(iii) The number of arc b on $\triangle_{b^q}$ for $n = 1, \cdots, q$ is $q-n$

□

For more details, it can be seen in Table 1.

Note that the label for picture $\mathbf{Q}$ can be written in the form $a^{p-1}b^n ab^{q-n}$ for $n = 1, 2, \cdots, q$. Furthermore, with the operation on the picture and a transformation on the picture, any picture with any determination of base points is equivalent to the picture labeled $a^{p-1}b^n ab^{q-n}$ (see [4]).

Case $n > q$

For $n > q$, consider the picture in Figure 5.

Assume that the base point as in Figure 5, the number of arcs connected to the boundary disc is

(i) The number arc a on $\triangle_{a^p}$ is $p-1$

Table 1: Explanation of Figure 4.

n: the number of commutators	The number of arcs a connected to the boundary disc	The number of arcs b connected to the boundary disc	Label of picture $\mathbf{Q}$
1	p	q	$a^{p-1}b^n ab^{q-1}$
2	p	q	$a^{p-1}b^n ab^{q-2}$
3	p	q	$a^{p-1}b^n ab^{q-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n = q$	p	q	$a^{p-1}b^n ab^{q-n}$

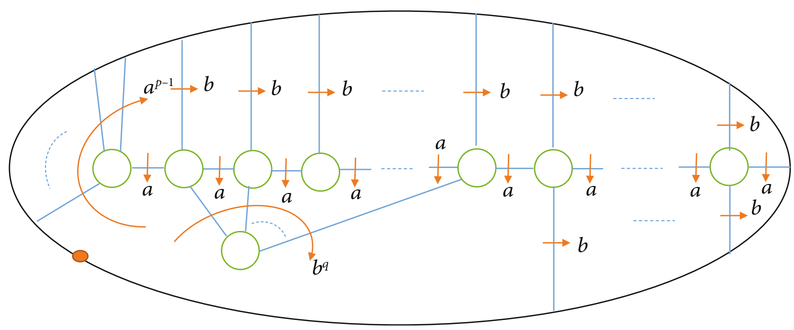

Figure 5: Picture over $P = \langle a, b\,;a^p, b^q, aba^{-1}b^{-1}\rangle$.

Table 2: Explanation of Figure 5.

n: the number of commutators	The number of arcs a connected to the boundary disc	The number of arcs b connected to the boundary disc	Label of picture $\mathbf{Q}$
$q+1$	p	$2n-q$	$a^{p-1}b^n ab^{q-(q+1)}$
$q+2$	p	$2n-q$	$a^{p-1}b^n ab^{q-(q+2)}$
$q+3$	p	$2n-q$	$a^{p-1}b^n ab^{q-(q+3)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	p	$2n-q$	$a^{p-1}b^n ab^{q-n}$

(ii) In the opposite direction, there is one arc b on $\triangle_{[a,b]}$, respectively, for $n = 1, 2, \cdots, q$, and two arcs n, respectively, for $n > q$

(iii) No arc b connected to the boundary disc on $\triangle_{b^q}$ for $n = 1, \cdots, q$

For more details, it can be seen in Table 2.

Thus, the label of picture $\mathbf{Q}$ is $a^{p-1}b^n ab^{q-n}$ for $n > q$. Furthermore, with the operation on the picture, any picture over $P = \langle a, b\,; a^p, b^q, [a, b]\rangle$ is equivalent to the picture labeled by $a^{p-1}b^n ab^{q-n}$.

Corollary 13. *Let $\mathbf{Q}$ be a picture over $P = \langle a, b\,; a^p, b^q, [a, b]\rangle$, where $p, q \geq 2$. Then*

$$|W(Q)| = \begin{cases} p+q, \mathit{for}\ n = 1, 2, \cdots, q \\ 2n + p - q\ \mathit{for}\ n > q \end{cases}. \quad (1)$$

Proof. Let $\mathbf{Q}$ be a picture over $P = \langle a, b\,; a^p, b^q, [a, b]\rangle$. Based on Theorem 12, any picture over P has the label $a^{p-1}b^n a b^{q-n}$. So $|a^{p-1}b^n ab^{q-n}| = p+q$ for $n = 1, 2, \cdots, q$ and $|a^{p-1}b^n a b^{q-n}| = p + 2n - q$ for $n > q$. □

4. Conclusions

This article provides a picture pattern of the presentation of the direct product of two cyclic groups by setting a particular disc position at the beginning; in this case, the disc a^p. The new picture pattern with this condition is given. Based on this pattern is one way to determine the label of the picture of the presentation.

Data Availability

The data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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