Hindawi
Journal of Applied Mathematics
Volume 2023, Article ID 8300785, 14 pages
https://doi.org/10.1155/2023/8300785

Research Article

Exact Controllability for a Class of Fractional Semilinear System of Order $1 < q < 2$ with Instantaneous and Noninstantaneous Impulses

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Received 10 May 2023; Revised 21 June 2023; Accepted 24 July 2023; Published 12 August 2023

Academic Editor: Nazim I. Mahmudov

This paper is mainly concerned with the existence of mild solutions and exact controllability for a class of fractional semilinear system of order $q \in (1, 2)$ with instantaneous and noninstantaneous impulses. First, combining the Kuratowski measure of noncompactness and the Mönch fixed point theorem, we investigated the existence result for the considered system. It is remarkable that our assumptions for impulses and the nonlinear term are weaker than the Lipschitz conditions. Next, on this basis, the exact controllability for the considered system is determined. In the end, an example is provided to support the main findings.

1. Introduction

Quite a number of evolutionary processes are characterized by sudden state changes at some certain points in time. The duration of these disturbances is negligible compared to the entire evolutionary process. Thus, if we assume that these perturbations occur over relatively short periods of time, the evolutionary processes can be described in the form of pulses, even impulsive differential equations (IDEs for short). It is well known that many agricultural, biological, and medical models are designed according to impulsive influences, such as the control of infectious diseases and changes in human hormone levels under the influence of external factors. Therefore, IDEs can be seen as the accurate description of some specific problems in the real world (see [1, 2] and references therein).

On the other hand, the dynamics of some evolutionary processes, such as intravenous drugs, periodic fishing, and criteria for pest management, cannot be described by instantaneous impulsive systems. In order to solve these kinds of problems, Hernández and O'Regan [3] introduced the concept of noninstantaneous impulses, which begin at a fixed point and remain active for a finite period of time. In recent years, many scholars have made studies on these two types of IDEs in depth. For instance, Liu and O'Regan [4] investigated the functional differential equations with instantaneous impulse by applying the measure of noncompactness and the Mönch fixed point theorem. Chen et al. [5] used noncompact semigroup to deal with the semilinear evolution equations with noninstantaneous impulses.

Also, every aspect of a dynamical system cannot be covered under instantaneous impulse and noninstantaneous impulse separately. In other words, it is inevitable to consider these two types of impulse factors in a system to find out how they affect the system together. For instance, Meraj and Pandey [6] investigated a class of instantaneous and noninstantaneous impulsive systems by Sadovskii's fixed point theorem. Tian and Zhang [7] studied the existence of solutions for second-order differential equations with these two kinds of impulses by variational method.

In addition, it is widely known that many scholars have already paid considerable attention to the controllability of systems. Shukla et al. [8] studied on approximate controllability of semilinear control systems with impulses. Li et al. [9] studied the persistence of delayed cooperative models: impulsive control method. Liu et al. [10] investigated the control design for output tracking of delayed Boolean control networks. Xu et al. [11] dealt with robust set stabilization of Boolean control networks with impulses. Zhao et al. [12]

studied the controllability for a class of semilinear fractional evolution systems by resolvent operators. One of the effective ways to solve this kind of problems is transforming them into fixed point problems by some proper operators in a function space. For example, the Mönch fixed point theorem was used to deal with the controllability of differential equations by Liu and O'Regan [4]. ρ-Set contractive fixed point theorem was applied to investigate the controllability for a type of noninstantaneous impulsive systems by Meraj and Pandey [6].

Compared with the classical integer derivatives, the fractional derivatives of order $0 < q < 1$ defined by integration have the characteristics of nonlocal and memory properties. Thus, they are widely used in many fields. For example, Ge and Jhuang [13] dealt with chaos, control, and synchronization of a class of fractional system. Cheng and Yuan [14] investigated the stability for the equilibria of a kind of equation with fractional diffusion. Jia and Wang [15] studied a fast finite volume method for a type of fractional equations. Zhao [16] dealt with the controllability of a type of impulsive fractional nonlinear evolution equations. Meanwhile, many scholars have structured relevant models to study several kinds of fractional semilinear systems. For example, Shukla and Patel [17] studied controllability for fractional semilinear delay control systems. Karapinar et al. [18] got the continuity of the fractional derivative of the time-fractional semilinear pseudoparabolic systems. Kavitha Williams et al. [19] analysed the approximate controllability of the Atangana-Baleanu fractional semilinear control systems. On this basis, many scholars have found that fractional systems of order $q \in (1, 2)$ can describe more complex problems in real life and have conducted in-depth research on them. Salem and Abdullah [20] got controllability for generalized fractional differential equations. Muslim and Kumar [21] investigated the exact controllability of a control system governed by the fractional differential equation of order $\alpha \in (1, 2]$. Shukla et al. [22] dealt with the existence and approximate controllability for the fractional semilinear impulsive control system of order $r \in (1, 2)$. Niazi et al. [23] studied controllability for fuzzy fractional evolution equations. Iqbal et al. [24] investigated the existence and uniqueness of mild solutions for fractional controlled fuzzy evolution equations. The most common way to solve this kind of problem is using fixed point theorem and cosine family.

Inspired by the discussion above, we consider the exact controllability for the fractional semilinear system with instantaneous and noninstantaneous impulses as follows:

$$\begin{cases} {}^{c}D^{q}x(t) = Ax(t) + Bu(t) + f(t, x(t)), \\ t \in \cup_{i=0}^{j}(d_i, c_{i+1}) \subset T, t \neq c_i^{k_i+k}, \\ x(t) = G_i(t, x(c_i^-)), t \in \cup_{i=1}^{j}(c_i, d_i), \\ x(0) = x_0, \\ x'(0) = x_1, \\ \Delta x\left(c_i^{k_i+k}\right) = I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right), \\ \Delta x'\left(c_i^{k_i+k}\right) = \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right), \end{cases} \tag{1}$$

where $i = 0, 1, \cdots, j$, $T = [0, b]$, $b > 0$ is a constant. And $k = 1, 2, \cdots, (k_{i+1} - k_i)$, $k_0 = 0$. ${}^{c}D^{q}$ is the Caputo fractional derivative of $q \in (1, 2)$. We suppose that $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is the infinitesimal generator of a strongly continuous q-order cosine family $(C_q(t))_{t\geq 0}$, where $\mathbb{X}$ is a Banach space, and $0 = c_0 = d_0 < c_0^1 < c_0^2 < \cdots < c_0^{k_1} < c_1 < d_1 < c_1^{k_1+1} < c_1^{k_1+2} < \cdots < c_1^{k_2} < c_2 < \cdots < d_j < c_j^{k_j+1} < c_j^{k_j+2} < \cdots < c_j^{k_{j+1}} < c_{j+1} = b$. The state variable $x(\cdot) \in \mathbb{X}$. $u(\cdot) \in L^2(T, \mathbb{Y})$ is the control variable, where $\mathbb{Y}$ is another Banach space. $B : \mathbb{Y} \longrightarrow \mathbb{X}$ is a bounded linear operator. The function f: $T \times \mathbb{X} \longrightarrow \mathbb{X}$ is a function satisfying some hypotheses. $G_i : (c_i, d_i] \times \mathbb{X} \longrightarrow \mathbb{X}$ and $G_i' : (c_i, d_i] \times \mathbb{X} \longrightarrow \mathbb{X}$ are noninstantaneous impulses, and $I_i : \mathbb{X} \longrightarrow \mathbb{X}$ and $\widetilde{I_i} : \mathbb{X} \longrightarrow \mathbb{X}$ represent instantaneous impulses. The jump of the state x at time t is $\Delta x(t) = x(t^+) - x(t^-)$. In this paper, we used the Mönch fixed point theorem to get the existence of the solution without using the Lipschitz conditions.

As far as we know, no one has done research on such class of systems yet. Kumar and Abdal studied (1) in the form of classical integer derivatives in [25]. Muslim and Kumar [21] dealt with (1) without instantaneous impulses. Shukla et al. [22] investigated (1) without noninstantaneous impulses. This article has the following distinctive features. Firstly, compared with [25], (1) is in the form of fractional derivatives of order $1 < q < 2$. Secondly, the nonlinear term and the two types of impulses here are no longer required to satisfy the Lipschitz conditions which are stronger than the assumptions used in this paper. Thirdly, compared with [21, 22], we consider both types of impulses at the same time. To sum up, the research results of this paper will be able to more accurately describe and solve some complex phenomena and problems in related fields. And the results are general, which fill the gap of previous studies of fractional system of order $1 < q < 2$ with instantaneous and noninstantaneous impulses.

The structure of this article is as follows. In Section 2, we first list fundamental concepts and lemmas. In Section 3, the existence of mild solutions and exact controllability for (1) are discussed by applying the Mönch fixed point theorem and cosine family. At last, in Section 4, two reasonable examples are worked out to support the main findings.

2. Preliminaries

In this part, a set of piecewise continuous functions is presented first. Next, define a mild solution of (1). Some related definitions and lemmas are listed on the side.

Assume that $\mathbb{X}$ is a Banach space with the norm $\|\cdot\|$.

Define $PC(T; \mathbb{X}) = \{x : T \longrightarrow \mathbb{X} | x$ is continuous at $t \neq c_i^{k_i+k}$, $t \neq c_{i+1}$, and $x(c_i^{k_i+k^-})$, $x(c_i^{k_i+k^+})$, $x(c_{i+1}^-)$, $x(c_{i+1}^+)$ exist, with $x(c_i^{k_i+k^-}) = x(c_i^{k_i+k})$ and $x(c_{i+1}^-) = x(c_{i+1})$, for $i = 0, 1, \cdots, j$, $k = 1, 2, \cdots, (k_{i+1} - k_i)\}$. Obviously, $PC(T; \mathbb{X})$ is a Banach space with the norm $\|x\|_{PC} = \sup_{t\in T}\|x(t)\|$.

Definition 1 (see [22]). If $x(t) \in C([0, b]; \mathbb{X})$, then the Riemann-Liouville integral of fractional order $q > 0$ is given by

$$J^q x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) ds, \quad (2)$$

where $C([0, b]\,; \mathbb{X})$ is the place of all continuous functions $x(t)\colon [0, b] \longrightarrow \mathbb{X}$.

Definition 2 (see [22]). The Riemann-Liouville fractional derivative of $x(t) \in C([0, b]\,; \mathbb{X})$ of order $q \in (1, 2)$ is given by

$$D_t^q x(t) = D^2 J^{2-q} x(t) = \frac{1}{\Gamma(2-r)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-q} x(s) ds. \quad (3)$$

Definition 3 (see [22]). The Caputo fractional derivative of order $q \in (1, 2)$ is given by

$${}^c D_t^q x(t) = J^{2-q} D^2 x(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} \left[\frac{d^2}{dt^2} x(t) \right] ds. \quad (4)$$

Consider fractional differential system as follows:

$${}^C D_t^q x(t) = Ax(t), x(0) = \psi, x'(0) = 0, \quad (5)$$

where $q \in (1, 2)$, $A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is a closed and densely operator defined in $\mathbb{X}$, and $D(A)$ illustrates the domain of A. By taking the Riemann-Liouville fractional integral order q on both sides of (5),

$$x(t) = \psi + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Ax(s) ds. \quad (6)$$

Definition 4 (see [22]). A family $(C_q(t))_{t \geq 0} \subset L(\mathbb{X}), q \in (1, 2]$ is called the solution operator (or a strongly continuous q-order fractional cosine family) for (5) and A is called the infinitesimal generator of $C_q(t)$ if the following conditions hold:

(i) $C_q(t)$ is strongly continuous and $C_q(0) = I$, where I denote the identity operator. And there exist constants $M_1 > 0$, $M > 0$, and $\omega \geq 0$ such that $\|C_q(t)\| \leq M_1 e^{-\omega t} \leq M$

(ii) $C_q(t) D(A) \subset D(A)$ and $AC_q(t)\eta = C_q(t) A\eta$ for all $\eta \in D(A), t \geq 0$

where $i = 1, 2, \cdots, j$.

(iii) $C_q(t)\psi$ is the solution of $x(t) = \psi + (1/\Gamma(q)) \int_0^t (t-s)^{q-1} Ax(s) ds$ for all $\psi \in D(A)$

Definition 5 (see [22]). The fractional sine family $S_q(t)\colon [0,\infty) \longrightarrow L(\mathbb{X})$ associated with $C_q(t)$ is defined by

$$S_q(t) = \int_0^t C_q(s) ds, t \geq 0. \quad (7)$$

Definition 6 (see [22]). The fractional Riemann-Liouville family $P_q(t)\colon [0,\infty) \longrightarrow L(\mathbb{X})$ associated with $C_q(t)$ is defined by

$$P_q(t) = J^{q-1} C_q(t). \quad (8)$$

Thus, for $t \in [0, b]$, according to Definition 1,

$$\begin{aligned} \|P_q(t)\| &= \|J^{q-1} C_q(t)\| \\ &= \left\| \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} C_q(s) ds \right\| \\ &\leq \frac{M}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} dt \\ &\leq \frac{M}{\Gamma(q)} \left[-(t-s)^{q-1} \right]_0^t \\ &\leq \frac{M}{\Gamma(q)} b^{q-1} =: M_P. \end{aligned} \quad (9)$$

Lemma 7 (see [21]). *The mild solution of the following fractional semilinear system of order* $1 < q < 2$ *with noninstantaneous impulses*

$$\begin{cases} {}^c D^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), t \in \cup_{i=0}^{j} (d_i, c_{i+1}), \\ x(t) = G_i(t, x(c_i^-)), t \in \cup_{i=1}^{j} (c_i, d_i], \\ x'(t) = G_i'(t, x(c_i^-)), t \in \cup_{i=1}^{j} (c_i, d_i], \\ x(0) = x_0, \\ x'(0) = x_1 \end{cases} \quad (10)$$

is be given by

$$x(t) = \begin{cases} C_q(t) x_0 + S_q(t) x_1 + \int_0^t P_q(t-s)[Bu(s) + f(s, x(s))] ds, t \in [0, c_1], \\ x(t) = G_i(t, x(c_i^-)), t \in \bigcup_{i=1}^{j} (c_i, d_i], \\ C_q(t-d_i)(G_i(d_i, x(c_i^-))) + S_q(t-d_i)\left(G_i'(d_i, x(c_i^-))\right) + \int_{d_i}^t P_q(t-s)[Bu(s) + f(s, x(s))] ds, t \in [d_i, c_{i+1}], \end{cases} \quad (11)$$

Lemma 8 (see [22]). *The mild solution of the following fractional semilinear system of order $1 < q < 2$ with instantaneous impulses*

$$\begin{cases} {}^cD^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), t \in T, \\ \Delta x(c_i) = I_i(x(c_i)), i = 0, 1, \cdots, j, \\ \Delta x'(c_i) = \widetilde{I_i}(x(c_i)), i = 0, 1, \cdots, j, \\ x(0) = x_0, \\ x'(0) = x_1 \end{cases} \tag{12}$$

is be given by

$$x(t) = C_q(t)x_0 + S_q(t)x_1 + \int_0^t P_q(t-s)[Bu(s) + f(s, x(s))]ds + \sum_{0<c_i<t} C_q(t-t_i)I_i(x(c_i)) + \sum_{0<c_i<t} S_q(t-t_i)\widetilde{I_i}(x(c_i)), t \in T. \tag{13}$$

According to the above two lemmas, similar to [25], the mild solution of (1) can be defined as follows.

Definition 9. For given $u(\cdot) \in L^2(T; \mathbb{Y})$, $x(\cdot, x_0, x_1, u)$: $T \longrightarrow \mathbb{X}$ is called a mild solution of (1), if $x \in PC(T; \mathbb{X})$ and satisfies

$$x(t) = \begin{cases} C_q(t)x_0 + S_q(t)x_1 + \int_0^t P_q(t-s)[Bu(s) + f(s, x(s))]ds + \sum_{0<c_0^k<t} C_q\left(t - c_0^k\right)I_0^k\left(x\left(c_0^{k^-}\right)\right) + \sum_{0<c_0^k<t} S_q\left(t - c_0^k\right)\widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right), t \in [0, c_1], \\ G_i(t, x(c_i^-)), t \in (c_i, d_i], \\ C_q(t-d_i)(G_i(d_i, x(c_i^-))) + S_q(t-d_i)\left(G_i'(d_i, x(c_i^-))\right) + \int_{d_i}^t P_q(t-s)[Bu(s) + f(s, x(s))]ds + \sum_{d_i<c_i^{k_i+k}<t} C_q\left(t - c_i^{k_i+k}\right)I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right) + \sum_{d_i<c_i^{k_i+k}<t} S_q\left(t - c_i^{k_i+k}\right)\widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right), t \in (d_i, c_{i+1}]. \end{cases} \tag{14}$$

Definition 10 (see [21]). System (1) is said to be exactly controllable on T if, for every $x_0, x_1 \in \mathbb{X}$, and arbitrary final state $x^b \in \mathbb{X}$, there exists a control $u \in L^2(T, \mathbb{Y})$ such that the mild solution of (1) satisfies $x(b) = x^b$.

Now, we introduce a result of the Kuratowski measure of noncompactness $\mathcal{X}$ defined on bounded subsets of the Banach space $\mathbb{X}$. For more detailed information, please see [5, 26–28] and references therein.

Lemma 11 (see [4]). *Suppose $\mathbb{X}$ is a Banach space. Let D be a countable set of strongly measurable function $x : T \longrightarrow \mathbb{X}$ such that there exists a $\mu \in L[T, R^+]$ with $\|x(t)\| \le \mu(t)$ a.e. $t \in T$ for all $x \in D$. Then, $\mathcal{X}(D(t)) \in L[J, R^+]$ and*

$$\mathcal{X}\left(\left\{\int_T x(t)dt : x \in D\right\}\right) \le 2\int_T \mathcal{X}(D(t))dt, \tag{15}$$

where $\mathcal{X}(\cdot)$ denotes the Hausdorff noncompactness measure, $T = [0, b]$.

Theorem 12 (see [4]). *Suppose $\mathbb{X}$ is a Banach space. Let D be a closed and convex subset of $\mathbb{X}$ and $u \in D$. Assume that the continuous operator $A : D \longrightarrow D$ has the following property: $C \subset D$ countable and $C \subset \bar{co}(\{u\} \cup A(C))$ imply C is relatively compact. Then, A has a fixed point in D.*

3. Existence of Solutions and Exact Controllability

In this part, we discuss the existence of mild solutions of (1) and exact controllability. To this end, we list the following assumptions in the first place.

H1. $f : \mathcal{T} \times \mathbb{X} \longrightarrow \mathbb{X}$ is continuous. There exist $a(t) \in L^2[T, R^+]$ and $b(t) \in L^1[T, R^+]$ such that

$$\|f(t, x)\| \le a(t)\|x\| + b(t), \forall x \in \mathbb{X}, t \in T. \tag{16}$$

And there exists $\xi(t) \in L^1[\mathcal{T}, R^+]$ such that

$$\mathcal{X}(f(t, D)) \le \xi(t)\mathcal{X}(D), \tag{17}$$

for arbitrary bounded set $D \subset \mathbb{X}$.

H2. $G_i : T_i \times \mathbb{X} \longrightarrow \mathbb{X}$ are derivable and $G_i' : T_i \times \mathbb{X} \longrightarrow \mathbb{X}$ are continuous, $T_i = (c_i, d_i]$, $i = 1, 2, \cdots, j$. There exist $g_i(t)$, $h_i(t)$, $\bar{g}_i(t)$, $\bar{h}_i(t)$, $\mathscr{H}_i(t)$, and $\widetilde{\mathscr{H}_i(t)} \in L^1[T, R^+]$ such that

$$\begin{aligned} \|G_i(t, x)\| &\le g_i(t)\|x\| + h_i(t), \\ \left\|G_i'(t, x)\right\| &\le \bar{g}_i(t)\|x\| + \bar{h}_i(t), \end{aligned} \tag{18}$$

for $\forall t \in T_i$ and $\forall x \in \mathbb{X}$,

$$\mathcal{X}(G_i(t, D)) \leq \mathcal{H}_i(t)\mathcal{X}(D), \quad \mathcal{X}\left(G_i'(t, D)\right) \leq \widetilde{\mathcal{H}_i(t)}\mathcal{X}(D), \tag{19}$$

for any bounded $D \subset \mathbb{X}$ and $\forall t \in T_i$.

H3. $I_i^{k_i+k}, \widetilde{I_i^{k_i+k}} : \mathbb{X} \longrightarrow \mathbb{X}$ are continuous for $i = 0, 1, \cdots, j$, $k = 1, 2, \cdots, (k_{i+1} - k_i)$, and there exist positive constants $\mathcal{I}_i^{k_i+k}$, $\widetilde{\mathcal{I}_i^{k_i+k}}$, $\mathcal{K}_i^{k_i+k}$, $\widetilde{\mathcal{K}_i^{k_i+k}}$, m_i, and $\widetilde{m_i}$ such that

$$\left\|I_i^{k_i+k}(x)\right\| \leq \mathcal{I}_i^{k_i+k}\|x\| + m_i, \quad \left\|\widetilde{I_i^{k_i+k}}(x)\right\| \leq \widetilde{\mathcal{I}_i^{k_i+k}}\|x\| + \widetilde{m_i}, \tag{20}$$

for all $x \in \mathbb{X}$,

$$\mathcal{X}\left(I_i^{k_i+k}(D)\right) \leq \mathcal{K}_i^{k_i+k}\mathcal{X}(D), \quad \mathcal{X}\left(\widetilde{I_i^{k_i+k}}(D)\right) \leq \widetilde{\mathcal{K}_i^{k_i+k}}\mathcal{X}(D), \tag{21}$$

for any bounded $D \subset \mathbb{X}$.

H4. The linear operator $W_{d_i}^{c_{i+1}} : L^2(T, \mathbb{Y}) \longrightarrow \mathbb{X}$ defined by

$$W_{d_i}^{c_{i+1}}u = \int_{d_i}^{c_{i+1}} P_q(c_{i+1} - s)Bu(s)ds, \; i = 0, 1, 2, \cdots, j, \tag{22}$$

has a bounded invertible operator $(W_{d_i}^{c_{i+1}})^{-1}$. It takes values in $L^2(T, \mathbb{Y})/\mathrm{Ker}_{d_i}^{c_{i+1}}$. In addition, there exists a positive constant K such that

$$\left\|\left(W_{d_i}^{c_{i+1}}\right)^{-1}\right\| \leq K, \tag{23}$$

and $\eta(t) \in L^1[T, R^+]$ such that

$$\mathcal{X}\left(\left(W_{d_i}^{c_{i+1}}\right)^{-1}(D)(t)\right) \leq \eta(t)\mathcal{X}(D), \; t \in T, \tag{24}$$

for any bounded set $D \subset \mathbb{X}$.

For convenience, denote

$$\begin{aligned}
\mathcal{F}_1 &= \|a(\cdot)\|_{L^2[[0,c_1],R^+]}, \\
\mathcal{F}_2 &= \|a(\cdot)\|_{L^2[[d_i,c_{i+1}],R^+]}, \\
\mathcal{F}_3 &= \|b(\cdot)\|_{L^1[[0,c_1],R^+]}, \\
\mathcal{F}_4 &= \|b(\cdot)\|_{L^1[[d_i,c_{i+1}],R^+]}, \\
g &= \sup_{t\in(c_i,d_i],i=1,2,\cdots,j} g_i(t), \\
\bar{g} &= \sup_{t\in(c_i,d_i],i=1,2,\cdots,j} \bar{g}_i(t), \\
h &= \sup_{t\in(c_i,d_i],i=1,2,\cdots,j} h_i(t), \\
\bar{h} &= \sup_{t\in(c_i,d_i],i=1,2,\cdots,j} \bar{h}_i(t).
\end{aligned} \tag{25}$$

We define the control as follows:

$$u(t, x^{c_1}, x^{c_{i+1}}, x) = \begin{cases} \left(W_0^{c_1}\right)^{-1}\left[x^{c_1} - C_q(b)x_0 - S_q(x^{c_1})x_1 - \int_0^{c_1} P_q(x^{c_1} - s)f(s, x(s))ds - \sum_{0<c_0^k<c_1} C_q\left(x^{c_1} - c_0^k\right)I_0^k\left(x\left(c_0^{k^-}\right)\right) - \sum_{0<c_0^k<c_1} S_q\left(x^{c_1} - c_0^k\right)\widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right)\right](t), \; t \in [0, c_1], \\ \left(W_{d_i}^{c_{i+1}}\right)^{-1}\left[x^{c_{i+1}} - C_q(x^{c_{i+1}} - d_i)(G_i(d_i, x(c_i^-))) - S_q(x^{c_{i+1}} - d_i)\left(G_i'(d_i, x(c_i^-))\right) - \int_{d_i}^{c_{i+1}} P_q(b - s)f(s, x(s))ds - \sum_{d_i<c_i^{k_i+k}<c_{i+1}} C_q\left(x^{c_{i+1}} - c_i^{k_i+k}\right)I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right) - \sum_{d_i<c_i^{k_i+k}<c_{i+1}} S_q\left(x^{c_{i+1}} - c_i^{k_i+k}\right)\widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right)\right](t), \; t \in (d_i, c_{i+1}], \end{cases} \tag{26}$$

where $i = 1, \cdots, j$, for $x^{c_1}, x^{c_{i+1}} \in \mathbb{X}$, $t \in T$, and $x \in PC(T, \mathbb{X})$.

Now, we show that the control operator $u(t, x^{c_1}, x^{c_{i+1}}, \cdot)$ is bounded. For convenience, denote $u(t, x^{c_1}, x^{c_{i+1}}, x)$ as $u(t, x)$.

Lemma 13. *Assume that H1-H4 hold. Then, for $x^{c_1}, x^{c_{i+1}} \in \mathbb{X}$, the set $\{u(t, x): x \in B_\delta\}$ is bounded on T, where $B_\delta = \{x \in PC(T; \mathbb{X}): \|x\| \leq \delta\}$.*

Proof. Notice that by H1-H4,

$$
\begin{aligned}
\|u(t,x)\| &= \Big\| \left(W_0^{c_1}\right)^{-1}\Big[x^{c_1} - C_q(c_1)x_0 - S_q(c_1)x_1 \\
&\quad - \int_0^{c_1} P_q(c_1-s)f(s,x(s))ds - \sum_{0<c_0^k<c_1} C_q\left(c_1-c_0^k\right)I_0^k\left(x\left(c_0^{k^-}\right)\right) \\
&\quad - \sum_{0<c_0^k<c_1} S_q\left(c_1-c_0^k\right)\widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right)\Big](t)\Big\| \\
&\le K\Big[\|x^{c_1}\| + M\|x_0\| + Mc_1\|x_1\| + M_p\mathscr{F}_1\delta(c_1)^{1/2} \\
&\quad + \mathscr{F}_3 + \sum_{0<c_0^k<c_1} M\left(\mathscr{I}_0^k\delta + m_0\right) \\
&\quad + \sum_{0<c_0^k<c_1} Mc_1\left(\widetilde{\mathscr{I}_0^k}\delta + \widetilde{m_0}\right)\Big] =: u_1,
\end{aligned} \tag{27}
$$

for $t \in (0, c_1]$.

Similarly,

$$
\begin{aligned}
\|u(t,x)\| &= \Big\| \left(W_{d_i}^{c_{i+1}}\right)^{-1}\Big[x^{c_{i+1}} - C_q(c_{i+1}-d_i)(G_i(d_{i-1},x(c_i^-))) \\
&\quad - S_q(c_{i+1}-d_i)\left(G_i'(d_i,x(c_i^-))\right) - \int_{d_i}^{c_{i+1}} P_q(c_{i+1}-s)f(s,x(s))ds \\
&\quad - \sum_{d_i<c_i^{k_i+k}<c_{i+1}} C_q\left(c_{i+1}-c_i^{k_i+k}\right)I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right) \\
&\quad - \sum_{d_i<c_i^{k_i+k}<c_{i+1}} S_q\left(c_{i+1}-c_i^{k_i+k}\right)\widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right)\Big](t)\Big\| \\
&\le K\Big[\|x^{c_{i+1}}\| + M(g\delta+h) + M(c_{i+1}-d_i)(\bar{g}\delta+\bar{h}) \\
&\quad + M_p\mathscr{F}_2\delta(c_{i+1}-d_i)^{1/2} + \mathscr{F}_4 + \sum_{d_i<c_i^{k_i+k}<c_{i+1}} M\left(\mathscr{I}_i^{k_i+k}\delta + m_i\right) \\
&\quad + \sum_{d_i<c_i^{k_i+k}<c_{i+1}} M(d_{i+1}-c_i)\left(\widetilde{\mathscr{I}_i^{k_i+k}}\delta + \widetilde{m_i}\right)\Big] =: u_2,
\end{aligned} \tag{28}
$$

for $t \in (d_i, c_{i+1}]$, where $i = 1, \cdots, j$.

Now, we try to prove the existence of mild solutions and exact controllability for (1).

For convenience, we denote

$$
\begin{aligned}
M^* &= 4M_p^2\|B\|\left(\int_0^{c_1}\int_0^{c_1}\eta(t)\xi(s)dtds\right) + 2M_p\|B\|M \\
&\quad \cdot \left(\int_0^{c_1}\eta(t)dt\right)\sum_{k=1}^{k_1}\mathscr{K}_0^k + M\sum_{k=1}^{k_1}\mathscr{K}_0^k + Mc_1\sum_{k=1}^{k_1}\widetilde{\mathscr{K}_0^k} \\
&\quad + 2M_p\|B\|M\left(\int_0^{c_1}\eta(t)dt\right)\sum_{k=1}^{k_1}\widetilde{\mathscr{K}_0^k} + 2M_p\left(\int_0^{c_1}\xi(t)dt\right),
\end{aligned}
$$

$$
\begin{aligned}
M^{**} &= \sup_{t\in(d_i,c_{i+1}]}\Big[M\mathscr{H}_i(t) + M(c_{i+1}-d_i)\widetilde{\mathscr{H}_i}(t) + 2M_p\|B\|M \\
&\quad \cdot \left(\int_{d_i^{c_{i+1}}}\eta(t)\mathscr{H}_i(t)dt\right) + 2M_p\|B\|M(c_{i+1}-d_i) \\
&\quad \cdot \left(\int_{d_i^{c_{i+1}}}\eta(t)\widetilde{\mathscr{H}_i}(t)dt\right) + 4M_p^2\|B\|M \\
&\quad \cdot \left(\int_{d_i}^{c_{i+1}}\int_{d_i}^{c_{i+1}}\eta(t)\xi(s)dtds\right) + 2M_p\|B\|M \\
&\quad \cdot \left(\int_{d_i}^{c_{i+1}}\eta(t)dt\right)\sum_{i=1}^{k_{i+1}-k_i}\mathscr{K}_i^{k_i+k} + 2M_p\|B\|M \\
&\quad \cdot \left(\int_{d_i}^{c_{i+1}}\eta(t)dt\right)\sum_{i=1}^{k_{i+1}-k_i}(c_{i+1}-d_i)\widetilde{\mathscr{K}_i^{k_i+k}} + 2M_p \\
&\quad \cdot \left(\int_{d_i}^{c_{i+1}}\xi(t)dt\right) + M\sum_{i=1}^{k_{i+1}-k_i}\mathscr{K}_i^{k_i+k} \\
&\quad + M\sum_{i=1}^{k_{i+1}-k_i}(c_{i+1}-d_i)\widetilde{\mathscr{K}_i^{k_i+k}}\Big],
\end{aligned}
$$

$$
H^* = \sup_{t\in(c_i,d_i]}\mathscr{H}_i(t),
$$

$$
P_0 =: \left[M_p\mathscr{F}_1(c_1)^{1/2} + \sum_{0<c_0^k<c_1} M\mathscr{I}_0^k + \sum_{0<c_0^k<c_1} M\widetilde{\mathscr{I}_0^k}c_1\right](1+K),
$$

$$
\begin{aligned}
P_i =: &\Big[Mg + M(c_{i+1}-d_i)\bar{g} + M_p\mathscr{F}_2(c_{i+1}-d_i)^{1/2} \\
&+ \sum_{d_i<c_i^{k_i+k}<c_{i+1}} M\mathscr{I}_i^{k_i+k} + \sum_{d_i<c_i^{k_i+k}<c_{i+1}} M\widetilde{\mathscr{I}_i^{k_i+k}}(d_{i+1}-c_i)\Big](1+K).
\end{aligned} \tag{29}
$$

□

Theorem 14. *Assume that H1-H4 hold. Then, (1) is exactly controllable on T provided that*

$$
\begin{gathered}
M^* + M^{**} + H^* < 1, \\
P_0 \in (0,1), \\
P_i \in (0,1), \\
g \in (0,1),
\end{gathered} \tag{30}
$$

where $i = 1, 2, \cdots, j$.

Proof. In order to get the existence, define operator Y on $PC(T, \mathbb{X})$ as follows:

$$
(Yx)(t) = \begin{cases} C_q(t)x_0 + S_q(t)x_1 + \int_0^t P_q(t-s)[Bu(s)+f(s,x(s))]ds + \sum_{0<c_0^k<t} C_q\left(t-c_0^k\right)I_0^k\left(x\left(c_0^{k^-}\right)\right) + \sum_{0<c_0^k<t} S_q\left(t-c_0^k\right)\widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right), t\in[0,c_1], G_i(t,x(c_i^-)), t\in(c_i,d_i], \\ C_q(t-d_i)(G_i(d_i,x(c_i^-))) + S_q(t-d_i)\left(G_i'(d_i,x(c_i^-))\right) + \int_{d_i}^t P_q(t-s)[Bu(s)+f(s,x(s))]ds + \sum_{d_i<c_i^{k_i+k}<t} C_q\left(t-c_i^{k_i+k}\right)I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right) + \sum_{d_i<c_i^{k_i+k}<t} S_q\left(t-c_i^{k_i+k}\right)\widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right), t\in(d_i,c_{i+1}], \end{cases} \tag{31}
$$

where $i = 1, 2, \cdots, j$ and $u(t)$ is defined as (26).

Obviously, the existence of fixed points of Y is equivalent to the existence of mild solutions of (1).

The proof will be divided into the following three steps:

Step 1: show that there exists a constant $\delta > 0$ such that $Y(B_\delta) \subset B_\delta$.

Choose δ satisfying

$$\delta \geq \max_{i=1,2,\cdots j} \left[\frac{Q_0}{1 - P_0}, \frac{Q_i}{1 - P_i}, \frac{h}{1 - g} \right], \tag{32}$$

where

$$\begin{aligned} Q_0 &= M\|x_0\| + Mc_1\|x_1\| + c_1 M_p \|B\| K \\ &\quad \cdot \left[\|x^{c_1}\| + M\|x_0\| + Mc_1\|x_1\| + \mathscr{F}_3 + Mm_0 + Mc_1\widetilde{m_0} \right] \\ &\quad + \mathscr{F}_3 + Mm_0 + Mc_1\widetilde{m_0}, \\ Q_i &= Mh + M(c_{i+1} - d_i)\bar{h} + (c_{i+1} - d_i)M_p\|B\|K \\ &\quad \cdot \left[\|x^{c_{i+1}}\| + Mh + M(c_{i+1} - d_i)\bar{h} + \mathscr{F}_4 + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} Mm_i + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} M(d_{i+1} - c_i)\widetilde{m_i} \right] \\ &\quad + \mathscr{F}_4 + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} M(d_{i+1} - c_i)\widetilde{m_i} + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} Mm_i. \end{aligned} \tag{33}$$

It is time to claim that $Y(B_\delta) \subset B_\delta$.

By H1-H4 and (32), one can get that for $x \in B_\delta$,

$$\begin{aligned} \|(Yx)(t)\| &\leq M\|x_0\| + Mc_1\|x_1\| + c_1 M_p \|B\| u_1 \\ &\quad + M_p \mathscr{F}_1 \delta (c_1)^{1/2} + \mathscr{F}_3 + \sum_{0 < c_0^k < c_1} M\left(\mathscr{I}_0^k \delta + m_0 \right) \\ &\quad + \sum_{0 < c_0^k < c_1} Mc_1 \left(\widetilde{\mathscr{I}_0^k} \delta + \widetilde{m_0} \right) \leq Q_0 + P_0 \delta \leq \delta, \end{aligned} \tag{34}$$

for $t \in [0, c_1]$.

So, $\|(Yx)(t)\| \leq Q_0 + P_0\delta \leq \delta$ for $t \in [0, c_1]$.

Similarly,

$$\|(Yx)(t)\| \leq g\delta + h \leq \delta, \tag{35}$$

for $t \in (c_i, d_i]$, where $i = 1, 2, \cdots, j$.

Therefore, $\|(Yx)(t)\| \leq \delta$ for $t \in [c_i, d_i]$.

In addition,

$$\begin{aligned} \|(Yx)(t)\| &\leq M(g\delta + h) + M(c_{i+1} - d_i)\left(\bar{g}\delta + \bar{h} \right) \\ &\quad + (c_{i+1} - d_i) M_p \|B\| u_2 + M_p \mathscr{F}_2 \delta (c_{i+1} - d_i)^{1/2} \\ &\quad + \mathscr{F}_4 + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} M\left(\mathscr{I}_i^{k_i+k} \delta + m_i \right) \\ &\quad + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} M(d_{i+1} - c_i) \left(\widetilde{\mathscr{I}_i^{k_i+k}} \delta + \widetilde{m_i} \right) \\ &\leq Q_i + P_i \delta \leq \delta, \end{aligned} \tag{36}$$

for $t \in (d_i, c_{i+1}]$, $i = 1, 2, \cdots, j$.

Thus, $\|(Yx)(t)\| \leq Q_i + P_i\delta \leq \delta$ for $t \in (d_i, c_{i+1}]$.

Combining (34)-(36), one can obtain that $\|(Yx)(t)\| \leq \delta$ for $t \in \mathscr{T}$. That is, $Y(B_\delta) \subset B_\delta$.

Step 2: claim that Y is continuous on B_δ.

We first show that the control $u(t, x)$ is continuous with respect to x on B_δ. Assume that $\{x_n\}_{n=1}^{\infty}$ is a sequence satisfying $x_n \longrightarrow x$ as $n \longrightarrow \infty$ on B_δ. Then,

$$\begin{aligned} \|u(t, x_n) - u(t, x)\| &\leq KM_p \int_0^{c_1} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &\quad + \sum_{0 < c_0^k < c_1} M \left\| I_0^k\left(x_n\left(c_0^{k^-} \right) \right) - I_0^k\left(x\left(c_0^{k^-} \right) \right) \right\| \\ &\quad + \sum_{0 < c_0^k < c_1} Mc_1 \left\| \widetilde{I_0^k}\left(x_n\left(c_0^{k^-} \right) \right) - \widetilde{I_0^k}\left(x\left(c_0^{k^-} \right) \right) \right\|, \end{aligned} \tag{37}$$

for $t \in [0, c_1]$.

$$\begin{aligned} \|u(t, x_n) - u(t, x)\| &\leq M\|G_i(d_i, x_n(c_i^-)) - G_i(d_i, x(c_i^-))\| \\ &\quad + M(c_{i+1} - d_i)\left\| G_i'(d_i, x_n(c_i^-)) - G_i'(d_i, x(c_i^-)) \right\| \\ &\quad + M_p \int_{d_i}^{c_{i+1}} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &\quad + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} M \left\| I_i^{k_i+k}\left(x_n\left(c_i^{k_i+k^-} \right) \right) - I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-} \right) \right) \right\| \\ &\quad + \sum_{d_i < c_i^{k_i+k} < c_{i+1}} M(c_{i+1} - d_i) \left\| \widetilde{I_i^{k_i+k}}\left(x_n\left(c_i^{k_i+k^-} \right) \right) - \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-} \right) \right) \right\|, \end{aligned} \tag{38}$$

for $t \in [d_i, c_{i+1}]$, where $i = 1, 2, \cdots, j$.

(37) and (38) together with H1-H3 imply

$$\|u(t, x_n) - u(t, x)\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \tag{39}$$

for arbitrary $t \in T$.

So, control function $u(t, x)$ is continuous with respect to x on B_δ.

Next, we prove that Y is continuous on B_δ. Assume that $\{x_n\}_{n=1}^{\infty}$ is a sequence on B_δ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

Notice that for $t \in [0, c_1]$,

$$\begin{aligned} \|(Yx_n)(t) - (Yx)(t)\| &\leq M_p\|B\| \int_0^{c_1} \|u(s, x_n(s)) - u(s, x(s))\| ds \\ &\quad + M_p \int_0^{c_1} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &\quad + \sum_{0 < c_0^k < t} M \left\| I_0^k\left(x_n\left(c_0^{k^-} \right) \right) - I_0^k\left(x\left(c_0^{k^-} \right) \right) \right\| \\ &\quad + \sum_{0 < c_0^k < t} Mc_1 \left\| \widetilde{I_0^k}\left(x_n\left(c_0^{k^-} \right) \right) - \widetilde{I_0^k}\left(x\left(c_0^{k^-} \right) \right) \right\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \tag{40}$$

For $t \in (c_i, d_i]$,

$$\|(Yx_n)(t) - (Yx)(t)\| = \|G_i(t, x_n(c_i^-)) - G_i(t, x(c_i^-))\| \longrightarrow 0, \text{as } n \longrightarrow \infty. \tag{41}$$

For $t \in (d_i, c_{i+1}]$,

$$\begin{aligned}\|(Yx_n)(t) - (Yx)(t)\| &\le M\|G_i(d_i, x(c_i^-)) - G_i(d_i, x_n(c_i^-))\| \\ &+ M(c_{i+1} - d_i)\left\|\widetilde{G_i}(d_i, x_n(c_i^-)) - \widetilde{G_i}(d_i, x(c_i^-))\right\| \\ &+ M_p\|B\|\int_{d_i}^{c_{i+1}} \|u(s, x_n(s)) - u(s, x(s))\| ds \\ &+ M_p\int_{d_i}^{c_{i+1}} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &+ \sum_{d_i < c_i^{k_i+k} < t} M\left\|I_i^{k_i+k}\left(x_n\left(c_i^{k_i+k^-}\right)\right) - I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right)\right\| \\ &+ \sum_{d_i < c_i^{k_i+k} < t} M(c_{i+1} - d_i)\left\|\widetilde{I_i^{k_i+k}}\left(x_n\left(c_i^{k_i+k^-}\right)\right) - \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right)\right\|.\end{aligned} \tag{42}$$

Combining (40)-(42), we get that Y is continuous on B_δ.

Step 3: show that D is relatively compact if

$$D \subset \bar{co}(\{u_0\} \cup Y(D)), \tag{43}$$

and $D \subset B^\delta$ is countable, where $u_0 \in B^\delta$.

Without losing generality, let $D = \{\bar{x}_n\}_{n=1}^{\infty}$. We prove that $\{Y\bar{x}_n\}_{n=1}^{\infty}$ is equicontinuous on $[0, c_1]$, $(c_i, d_i]$, and $(d_i, c_{i+1}]$, $i = 1, \cdots, j$. In this case, assuming it is true, $\bar{co}(\{u_0\} \cup Y(D))$ would also be equicontinuous on the above intervals.

We notice that for each $x \in D$ and $t_1, t_2 \in [0, c_1]$,

$$\begin{aligned}\|(Yx)(t_1) - (Yx)(t_2)\| &\le \|C_q(t_1) - C_q(t_2)\|\|x_0\| + \|S_q(t_1) - S_q(t_2)\|\|x_1\| \\ &+ \left\|\int_0^{t_1} P_q(t_1 - s)[Bu(s) + f(s, x(s))]ds - \int_0^{t_2} P_q(t_2 - s)[Bu(s) + f(s, x(s))]ds\right\| \\ &+ \left\|\sum_{0 < c_0^k < t_1} C_q\left(t_1 - c_0^k\right) I_0^k\left(x\left(c_0^{k^-}\right)\right) - \sum_{0 < c_0^k < t_2} C_q\left(t_2 - c_0^k\right) I_0^k\left(x\left(c_0^{k^-}\right)\right)\right\| \\ &+ \left\|\sum_{0 < c_0^k < t_1} S_q\left(t_1 - c_0^k\right) \widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right) - \sum_{0 < c_0^k < t_2} S_q\left(t_2 - c_0^k\right) \widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right)\right\| \\ &=: L_1 + L_2 + L_3 + L_4 + L_5.\end{aligned} \tag{44}$$

Obviously, $L_1 \longrightarrow 0$ and $L_2 \longrightarrow 0$ as $(t_2 - t_1) \longrightarrow 0$. And

$$\begin{aligned}L_3 &\le \left\|\int_{t_1}^{t_2} P_q(t_1 - s)[Bu(s) + f(s, x(s))]ds\right\| \\ &+ \left\|\int_0^{t_2} \left[P_q(t_1 - s) - P_q(t_2 - s)\right][Bu(s) + f(s, x(s))]ds\right\|,\end{aligned}$$

$$\begin{aligned}L_4 &\le \left\|\sum_{t_2 < c_0^k < t_1} C_q\left(t_1 - c_0^k\right) I_0^k\left(x\left(c_0^{k^-}\right)\right)\right\| \\ &+ \left\|\sum_{0 < c_0^k < t_2} \left[C_q\left(t_1 - c_0^k\right) - C_q\left(t_2 - c_0^k\right)\right] I_0^k\left(x\left(c_0^{k^-}\right)\right)\right\|,\end{aligned}$$

$$\begin{aligned}L_5 &\le \left\|\sum_{t_2 < c_0^k < t_1} S_q\left(t_1 - c_0^k\right) \widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right)\right\| \\ &+ \left\|\sum_{0 < c_0^k < t_2} \left[S_q\left(t_1 - c_0^k\right) - S_q\left(t_2 - c_0^k\right)\right] \widetilde{I_0^k}\left(x\left(c_0^{k^-}\right)\right)\right\|.\end{aligned} \tag{45}$$

Through calculation, we conclude $L_i \longrightarrow 0$ for $i = 3, 4, 5$, as $(t_2 - t_1) \longrightarrow 0$. Thus, $\|(Yx)(t_1) - (Yx)(t_2)\| \longrightarrow 0$ as $(t_2 - t_1) \longrightarrow 0$.

Next, for each $x \in D$ and $t_1, t_2 \in (c_i, d_i]$,

$$\|(Yx)(t_1) - (Yx)(t_2)\| = \|G_i(t_1, x(c_i^-) - G_i(t_2, x(c_i^-)\| \longrightarrow 0, \text{as } (t_2 - t_1) \longrightarrow 0. \tag{46}$$

That is, $Y(D)$ is equicontinuous on $(c_i, d_i]$.

In the end, for each $x \in D$ and $t_1, t_2 \in (d_i, c_{i+1}]$,

$$\begin{aligned}\|(Yx)(t_1) - (Yx)(t_2)\| &\le \|C_q(t_1 - d_i) - C_q(t_2 - d_i)\| \\ &\cdot \|G_i(d_i, x(c_i^-))\| + \|S_q(t_1 - d_i) - S_q(t_2 - d_i)\|\|G_i'(d_i, x(c_i^-))\| \\ &+ \left\|\int_{d_i}^{t_1} P_q(t_1 - s)[Bu(s) + f(s, x(s))]ds\right. \\ &\left. - \int_{d_i}^{t_2} P_q(t_2 - s)[Bu(s) + f(s, x(s))]ds\right\| \\ &+ \left\|\sum_{d_i < c_i^{k_i+k} < t_1} C_q\left(t_1 - c_i^{k_i+k}\right) I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right)\right. \\ &\left. - \sum_{d_i < c_i^{k_i+k} < t_2} C_q\left(t_2 - c_i^{k_i+k}\right) I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right)\right\| \\ &+ \left\|\sum_{d_i < c_i^{k_i+k} < t_1} S_q\left(t_1 - c_i^{k_i+k}\right) \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right)\right. \\ &\left. - \sum_{d_i < c_i^{k_i+k} < t_2} S_q\left(t_2 - c_i^{k_i+k}\right) \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right)\right\| =: l_1 + l_2 + l_3 + l_4 + l_5.\end{aligned} \tag{47}$$

Similarly, $l_1 \longrightarrow 0$ and $l_2 \longrightarrow 0$ as $(t_2 - t_1) \longrightarrow 0$.

$$l_3 \le \left\| \int_{t_1}^{t_2} P_q(t_1 - s)[Bu(s) + f(s, x(s))]ds \right\| + \left\| \int_{d_i}^{t_2} [P_q(t_1 - s) - P_q(t_2 - s)][Bu(s) + f(s, x(s))]ds \right\|,$$

$$l_4 \le \left\| \sum_{t_2 < c_i^{k_i+k} < t_1} C_q\left(t_1 - c_i^{k_i+k}\right) I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right) \right\| + \left\| \sum_{d_i < c_i^{k_i+k} < t_2} \left[C_q\left(t_1 - c_i^{k_i+k}\right) - C_q\left(t_2 - c_i^{k_i+k}\right)\right] I_i^{k_i+k}\left(x\left(c_i^{k_i+k^-}\right)\right) \right\|,$$

$$l_5 \le \left\| \sum_{t_2 < c_i^{k_i+k} < t_1} S_q\left(t_1 - c_i^{k_i+k}\right) \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right) \right\| + \left\| \sum_{d_i < c_i^{k_i+k} < t_2} \left[S_q\left(t_1 - c_i^{k_i+k}\right) - S_q\left(t_2 - c_i^{k_i+k}\right)\right] \widetilde{I_i^{k_i+k}}\left(x\left(c_i^{k_i+k^-}\right)\right) \right\|. \tag{48}$$

By calculation, one can get that $l_i \longrightarrow 0$ for $i = 3, 4, 5$, as $(t_2 - t_1) \longrightarrow 0$. That is, $\|(Yx)(t_1) - (Yx)(t_2)\| \longrightarrow 0$ as $(t_2 - t_1) \longrightarrow 0$.

Combining (44)-(48), $\{Yx_n\}_{n=1}^{\infty}$ is equicontinuous on $[0, c_1]$, $(c_i, d_i]$, and $(d_i, c_{i+1}]$, respectively, where $i = 1, \cdots, j$. Therefore,

$$\mathscr{X}_{PC}\left(\{Yx_n\}_{n=1}^{\infty}\right) = \sup_{1 \le i \le j} \sup_{t \in [0, c_1], (c_i, d_i] \text{ and } (d_i, c_{i+1}]} \mathscr{X}\left(\{Yx_n(t)\}_{n=1}^{\infty}\right). \tag{49}$$

Notice that

$$\|x_n(t) - x_m(t)\| \le \|x_n - x_m\|_{PC}, \forall t \in T, \tag{50}$$

implies

$$\mathscr{X}\left(\{x_n(t)\}_{n=1}^{\infty}\right) \le \mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right), \forall t \in T, \tag{51}$$

where $\mathscr{X}_{PC}$ denotes the Kuratowski measure of noncompactness of a bounded set in $PC(T, \mathbb{X})$.

According to Lemma 11, for $t \in [0, c_1]$,

$$\begin{aligned}
\mathscr{X}\left(\{(Yx_n)(t)\}_{n=1}^{\infty}\right) &\le 2M_p\|B\|\int_0^{c_1}\mathscr{X}(u(t))dt \\
&+ 2M_p\int_0^{c_1}\mathscr{X}\left(\{f(t, x_n(t))\}_{n=1}^{\infty}\right)dt + \sum_{k=1}^{k_1} M\mathscr{X}\left(\left\{I_0^k(x_n(t))\right\}_{n=1}^{\infty}\right) \\
&+ \sum_{k=1}^{k_1} Mc_1\mathscr{X}\left(\left\{\widetilde{I_0^k}(x_n(t))\right\}_{n=1}^{\infty}\right) \le 4M_p^2\|B\| \\
&\cdot \left(\int_0^{c_1}\int_0^{c_1}\eta(t)\xi(s)dtds\right)\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) \\
&+ 2M_p\|B\|M\left(\int_0^{c_1}\eta(t)dt\right)\sum_{k=1}^{k_1}\mathscr{K}_0^k\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) \\
&+ 2M_p\|B\|M\left(\int_0^{c_1}\eta(t)dt\right)\sum_{k=1}^{k_1}\widetilde{\mathscr{K}_0^k}\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) \\
&+ 2M_p\left(\int_0^{c_1}\xi(t)dt\right)\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) + M\sum_{k=1}^{k_1}\mathscr{K}_0^k\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) \\
&+ Mc_1\sum_{k=1}^{k_1}\widetilde{\mathscr{K}_0^k}\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) = \left[4M_p^2\|B\|\left(\int_0^{c_1}\int_0^{c_1}\eta(t)\xi(s)dtds\right)\right. \\
&+ 2M_p\|B\|M\left(\int_0^{c_1}\eta(t)dt\right)\sum_{k=1}^{k_1}\mathscr{K}_0^k + M\sum_{k=1}^{k_1}\mathscr{K}_0^k + Mc_1\sum_{k=1}^{k_1}\widetilde{\mathscr{K}_0^k} \\
&+ 2M_p\|B\|M\left(\int_0^{c_1}\eta(t)dt\right)\sum_{k=1}^{k_1}\widetilde{\mathscr{K}_0^k} \\
&+ 2M_p\left(\int_0^{c_1}\xi(t)dt\right)\Big]\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right) = M^*\mathscr{X}_{PC}\left(\{x_n\}_{n=1}^{\infty}\right).
\end{aligned} \tag{52}$$

For $t \in (c_i, d_i]$,

$$\begin{aligned}
\mathscr{X}\left(\{(Yx_n)(t)\}_{n=1}^{\infty}\right) &\le \mathscr{X}\left(\{G_i(t, x_n(t))\}_{n=1}^{\infty}\right) \\
&\le \mathscr{H}_i(t)\mathscr{X}\left(\{x_n(t)\}_{n=1}^{\infty}\right) \\
&\le H^*\mathscr{X}\left(\{x_n(t)\}_{n=1}^{\infty}\right).
\end{aligned} \tag{53}$$

For $t \in (d_i, c_{i+1}]$,

$$\begin{aligned}
\mathscr{X}\left(\{(Y\bar{x}_n)(t)\}_{n=1}^{\infty}\right) &\le 2M_p\|B\|\int_{d_i}^{c_{i+1}}\mathscr{X}(u(t))dt \\
&+ 2M_p\int_{d_i}^{c_{i+1}}\mathscr{X}\left(\{f(t, \bar{x}_n(t))\}_{n=1}^{\infty}\right)dt \\
&+ \sum_{i=1}^{k_{i+1}-k_i} M\mathscr{X}\left(\left\{I_i^{k_i+k}(\bar{x}_n(t))\right\}_{n=1}^{\infty}\right) \\
&+ \sum_{i=1}^{k_{i+1}-k_i} M(c_{i+1} - d_i)\mathscr{X}\left(\left\{\widetilde{I_i^{k_i+k}}(\bar{x}_n(t))\right\}_{n=1}^{\infty}\right) \\
&+ M\mathscr{X}\left(\{G_i(t, \bar{x}_n(t))\}_{n=1}^{\infty}\right) \\
&+ M(c_{i+1} - d_i)\mathscr{X}\left(G_i'(t, \bar{x}_n(t))\right\}_{n=1}^{\infty}\Big) \\
&\le M\mathscr{H}_i(t)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right) \\
&+ M(c_{i+1} - d_i)\widetilde{\mathscr{H}}_i(t)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)
\end{aligned}$$

$$+2M_p\|B\|M\left(\int_{d_i^{c_{i+1}}}\eta(t)\mathscr{H}_i(t)dt\right)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+2M_p\|B\|M(c_{i+1}-d_i)\left(\int_{d_i^{c_{i+1}}}\eta(t)\widetilde{\mathscr{H}_i}(t)dt\right)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+4M_p^2\|B\|M\left(\int_{d_i}^{c_{i+1}}\int_{d_i}^{c_{i+1}}\eta(t)\xi(s)dtds\right)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+2M_p\|B\|M\left(\int_{d_i}^{c_{i+1}}\eta(t)dt\right)\sum_{i=1}^{k_{i+1}-k_i}\mathscr{K}_i^{k_i+k}\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+2M_p\|B\|M\left(\int_{d_i}^{c_{i+1}}\eta(t)dt\right)\sum_{i=1}^{k_{i+1}-k_i}(c_{i+1}-d_i)\widetilde{\mathscr{K}_i^{k_i+k}}\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+2M_p\left(\int_{d_i}^{c_{i+1}}\xi(t)dt\right)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+M\sum_{i=1}^{k_{i+1}-k_i}\mathscr{K}_i^{k_i+k}\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$+M\sum_{i=1}^{k_{i+1}-k_i}(c_{i+1}-d_i)\widetilde{\mathscr{K}_i^{k_i+k}}\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$=\Big[M\mathscr{H}_i(t)+M(c_{i+1}-d_i)\widetilde{\mathscr{H}_i}(t)$$
$$+2M_p\|B\|M\left(\int_{d_i}^{c_{i+1}}\eta(t)\mathscr{H}_i(t)dt\right)$$
$$+2M_p\|B\|M(c_{i+1}-d_i)\left(\int_{d_i}^{c_{i+1}}\eta(t)\widetilde{\mathscr{H}_i}(t)dt\right)$$
$$+4M_p^2\|B\|M\left(\int_{d_i}^{c_{i+1}}\int_{d_i}^{c_{i+1}}\eta(t)\xi(s)dtds\right)$$
$$+2M_p\|B\|M\left(\int_{d_i}^{c_{i+1}}\eta(t)dt\right)\sum_{i=1}^{k_{i+1}-k_i}\mathscr{K}_i^{k_i+k}$$
$$+2M_p\|B\|M\left(\int_{d_i}^{c_{i+1}}\eta(t)dt\right)\sum_{i=1}^{k_{i+1}-k_i}(c_{i+1}-d_i)\widetilde{\mathscr{K}_i^{k_i+k}}$$
$$+2M_p\left(\int_{d_i}^{c_{i+1}}\xi(t)dt\right)+M\sum_{i=1}^{k_{i+1}-k_i}\mathscr{K}_i^{k_i+k}$$
$$+M\sum_{i=1}^{k_{i+1}-k_i}(c_{i+1}-d_i)\widetilde{\mathscr{K}_i^{k_i+k}}\Big]\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)$$
$$\leq M^{**}\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right). \tag{54}$$

Combining (52), (53), and (54), one can get that

$$\mathscr{X}\left(\{(Y\bar{x}_n)(t)\}_{n=1}^{\infty}\right)\leq(M^*+M^{**}+H^*)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right). \tag{55}$$

Furthermore,

$$\begin{aligned}\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right)&\leq\mathscr{X}_{PC}\left(\{Y\bar{x}_n\}_{n=1}^{\infty}\right)\\&\leq(M^*+M^{**}+H^*)\mathscr{X}_{PC}\left(\{\bar{x}_n\}_{n=1}^{\infty}\right).\end{aligned} \tag{56}$$

According to the assumption of Theorem 14, we conclude that $\mathscr{X}_{PC}(\{\bar{x}_n\}_{n=1}^{\infty})=0$. Therefore, $D=\{\bar{x}_n\}_{n=1}^{\infty}$ is relatively compact. By Theorem 12, it yields that, in B^{δ}, Y has at least one fixed point.

To sum up, (1) is exactly controllable on T. □

4. Examples

To demonstrate the effectiveness of the obtained results, two examples are presented in this section.

Example 1.

$$\begin{cases}{}^cD^{3/2}x(t,\eta)=\dfrac{\partial^2}{\partial\eta^2}x(t,\eta)+u(t,\eta)+\varepsilon_1\dfrac{e^{-t}x(t,\eta)}{e^{-t}+e^t}+\varepsilon_2\displaystyle\int_0^t e^{-(s-t)}x(s,\eta)ds,\\ \eta\in(0,1),\ t\in\left(0,\dfrac{1}{5}\right]\cup\left(\dfrac{4}{5},1\right],\\ x(t,\eta)=\varepsilon_3e^{-\left(t-\frac{1}{5}\right)}\dfrac{x(-(1/5),\eta)}{1+x(-(1/5),\eta)},\ t\in\left(\dfrac{1}{5},\dfrac{4}{5}\right],\eta\in(0,1),\\ \dfrac{\partial x(t,\eta)}{\partial t}=-\varepsilon_3e^{-\left(t-\frac{1}{5}\right)}\dfrac{x(-(1/5),\eta)}{1+x(-(1/5),\eta)},\ t\in\left(\dfrac{1}{5},\dfrac{4}{5}\right],\eta\in(0,1),\\ x(t,0)=x(t,1)=0,\\ x(0,\eta)=x_0(\eta),\\ \dfrac{\partial x(0,\eta)}{\partial t}=x_1(\eta),\\ \Delta x(t)(\eta)|_{t=1/7}=\varepsilon_4\displaystyle\int_0^{1/7}\cos\left(\dfrac{1}{7}-s\right)x(s,\eta)ds,\eta\in(0,1),\\ \Delta\dfrac{\partial x(t)(\eta)}{\partial t}\Big|_{t=1/7}=\varepsilon_4x\left(\dfrac{1}{7},\eta\right),\eta\in(0,1),\end{cases} \tag{57}$$

where $t\in T=[0,1]$.

Conclusion of Example 1. (57) is exactly controllable on T.

Proof. (57) can be seen as a system in the form (1), where

$$\begin{aligned}q&=\frac{3}{2},\\ b&=c_2=1,\\ c_0&=d_0=0,\\ c_1&=\frac{1}{5},\\ d_1&=\frac{4}{5},\\ c_0^1&=\varepsilon_4,\\ f(t,x(t))&=\varepsilon_1\frac{e^{-t}x(t,\eta)}{e^{-t}+e^t}+\varepsilon_2\int_0^t e^{-(s-t)}x(s,\eta)ds,\\ G_1(t,x)&=\varepsilon_3e^{-(t-1/5)}\frac{x(-(1/5),\eta)}{1+x(-(1/5),\eta)},\\ G_1'(t,x)&=-\varepsilon_3e^{-(t-1/5)}\frac{x(-(1/5),\eta)}{1+x(-(1/5),\eta)},\\ I_0^1\left(x\left(\frac{1^-}{7}\right)\right)&=\varepsilon_4\int_0^{1/7}\cos\left(\frac{1}{7}-s\right)x(s,\eta)ds,\\ \widetilde{I_0^1}\left(x\left(\frac{1^-}{7}\right)\right)&=\varepsilon_4x\left(\frac{1}{7},\eta\right).\end{aligned} \tag{58}$$

Let $\mathbb{X} = L^2([0,1])$ be equipped with the norm defined by

$$\|x\| = \left(\int_0^1 |x(t)|^2 dt \right)^{1/2}, x \in \mathbb{X}. \quad (59)$$

Define $Ax = x''$, and

$$D(A) = \left\{ x \in \mathbb{X} : x, x' \text{ are absolutely continuous and } x'' \in \mathbb{X}, x(0) = x(1) = 0 \right\}. \quad (60)$$

Thus,

$$Ax = \sum_{n=1}^{\infty} -n^2 \langle x, e_n \rangle e_n, x \in D(A), \quad (61)$$

where $e_n(\eta) = \sqrt{2/\pi} \sin(n\eta), 0 \leq \eta \leq 1, n = 1, 2, \cdots$.

We now introduce the cosine function

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, e_n \rangle e_n, t \in \mathbb{R}. \quad (62)$$

According to the subordinate principle [29], it follows that A is the infinitesimal generator of a strongly continuous exponentially bounded fractional cosine family $C_q(t)$ such that $C_q(0) = I$ and

$$C_q(t) = \int_0^{\infty} \varphi_{t,q/2}(s) ds, t > 0, \quad (63)$$

where $\varphi_{t,q/2}(s) = t^{-q/2} \phi_{q/2}(st^{-q/2})$, and

$$\phi_\gamma(\tau) = \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n! \Gamma(-\gamma n + 1 - \gamma)}, 0 < \gamma < 1. \quad (64)$$

The fractional Riemann-Liouville family $P_q(t)$: $[0,\infty) \longrightarrow L(\mathbb{X})$ associated with $C_q(t)$ is defined by

$$P_q(t) = J^{q-1} C_q(t). \quad (65)$$

Obviously, $W_{d_i}^{c_{i+1}} : L^2(T, \mathbb{Y}) \longrightarrow \mathbb{X}$ defined by

$$W_{d_i}^{c_{i+1}} u = \int_{d_i}^{c_{i+1}} P_q(c_{i+1} - s) Bu(s) ds, i = 0, 1, \quad (66)$$

has a bounded invertible operator $(W_{d_i}^{c_{i+1}})^{-1}$ taking values in $L^2(T, \mathbb{Y}) / \mathrm{Ker}_{d_i}^{c_{i+1}}$. In addition, there exists a positive constant K such that

$$\left\| \left(W_{d_i}^{c_{i+1}} \right)^{-1} \right\| \leq K. \quad (67)$$

Therefore, H4 holds.

According to Definition 4, there exists a constant $M > 0$ such that $\|C_q(t)\| \leq M$. Put $x(t) = x(t, \eta)$, that is, $x(t)(\eta) = x(t, \eta), t \in [0,1], \eta \in [0,1]$. $B : \mathbb{Y} \longrightarrow \mathbb{X}$ which is defined as $Bu(t) = z(t, \eta)$ is a bounded linear operator.

Clearly,

$$\begin{aligned} \|f(t, x(t))\| &\leq \varepsilon_1 \left\| \frac{e^{-t} x(t, \eta)}{e^{-t} + e^t} \right\| + \varepsilon_2 \left\| \int_0^t e^{-(s-t)} x(s, \eta) ds \right\| \\ &\leq \varepsilon_1 \|x\| + \varepsilon_2 e^t \|x\| = \left(\varepsilon_1 + \varepsilon_2 e^t \right) \|x\|, \end{aligned} \quad (68)$$

$$\|G_1(t, x)\| \leq \varepsilon_3 \|x\|, \quad (69)$$

$$\left\|G_1'(t, x)\right\| \leq \varepsilon_3 \|x\|, \quad (70)$$

$$\left\| I_0^1(x(\varepsilon_4^-)) \right\| \leq \varepsilon_4 \|x\|, \quad (71)$$

$$\left\| \widetilde{I_0^1}(x(\varepsilon_4^-)) \right\| \leq \varepsilon_4 \|x\|. \quad (72)$$

Combining (68)-(72), the assumptions H1-H3 hold with

$$\begin{aligned} a(t) &= \varepsilon_1 + \varepsilon_2 e^t, \\ b(t) &= 0, \\ g_1(t) &= \bar{g}_1(t) = \varepsilon_3, \\ h_1(t) &= \bar{h}_1(t) = 0, \\ \mathcal{I}_0^1 &= \varepsilon \sin \varepsilon_4, \\ \widetilde{\mathcal{I}_0^1} &= \varepsilon, \\ m_0 &= \widetilde{m_0} = 0, \\ M_p &= \frac{M}{\Gamma(3/2)} = \frac{2M}{\sqrt{\pi}}, \\ \mathcal{F}_1 &= \frac{\varepsilon_1^2}{5} + \left(\varepsilon_1 \varepsilon_2 + \frac{\varepsilon_2^2}{2} \right) \left(e^{2/5} - 1 \right), \\ \mathcal{F}_2 &= \frac{\varepsilon_1^2}{5} + \left(e^2 - e^{8/5} \right) \varepsilon_1 \varepsilon_2 + \left(e^2 - e^{8/5} \right) \frac{\varepsilon_2^2}{2}, \\ \mathcal{F}_3 &= \mathcal{F}_4 = 0, \\ g &= \bar{g} = \varepsilon_3, \\ h &= \bar{h} = 0. \end{aligned} \quad (73)$$

Therefore, when ε_1, ε_2, ε_3, and ε_4 are sufficiently small, the conditions $P_0 \in (0,1)$ and $P_1 \in (0,1)$ in Theorem 14 are guaranteed. That is, (57) is exactly controllable on $[0,1]$. □

Example 2. The design of digital filters has attracted a lot of attention of a wide range of researchers since the last century. A digital filter is a system that can manipulate the sampled digital signal, usually to enhance or reduce some properties of the signal being processed. Digital filters are often used in industry because of their significant advantages of stable input and output, phase linearity, and low coefficient sensitivity. We introduced filter mode for system (1), as shown in the attached picture.

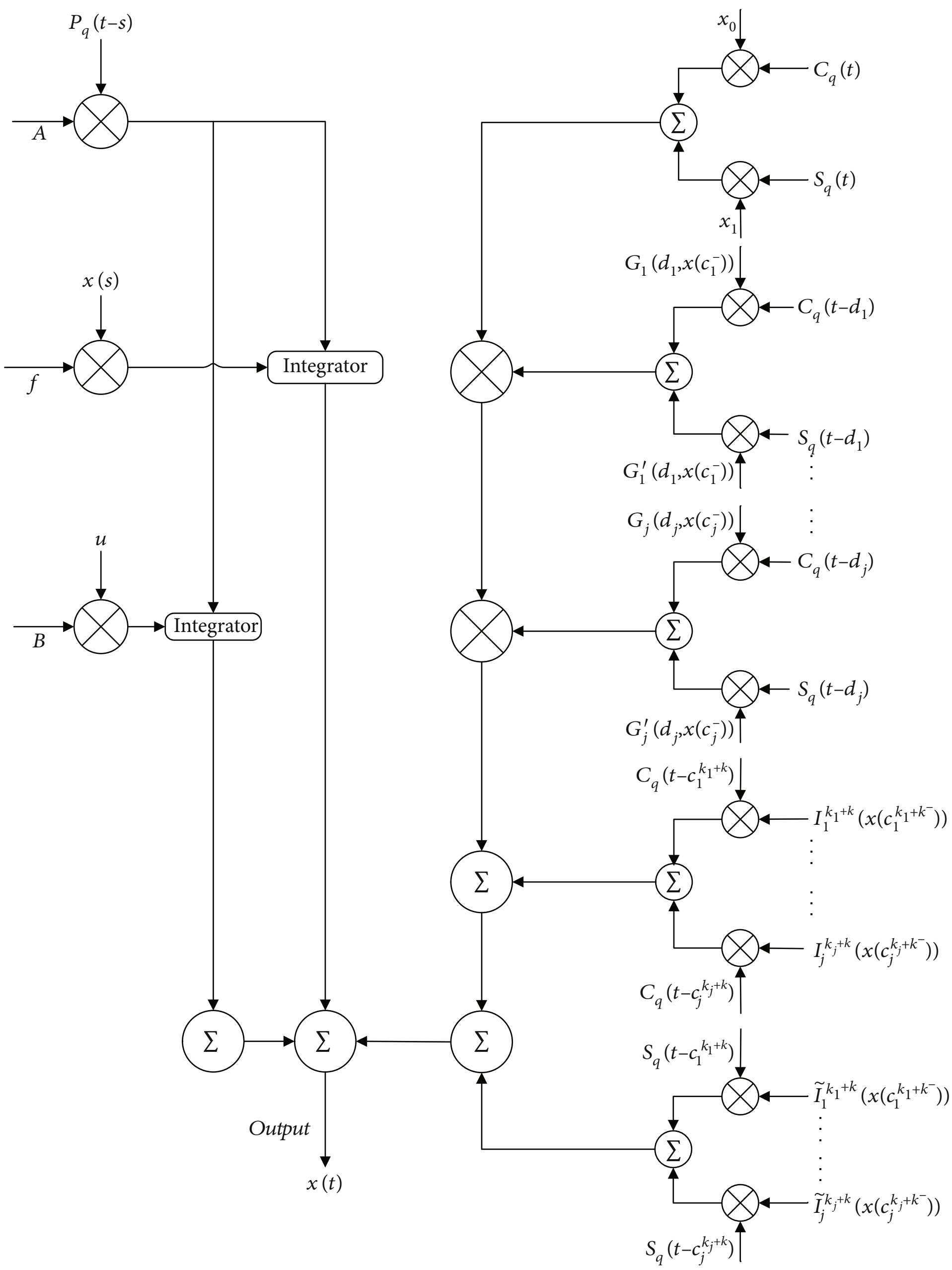

Figure 1: Digital filter model.

We present our filter system in Figure 1, which shows the rough pattern of the digital filter model based on the filter systems defined in [30–35].

(1) Product modulator 1 gets the input A and $P_q(t-s)$ turns out the output as $AP_q(t-s)$

(2) Product modulator 2 gets the input f and $x(s)$ turns out the output as $f(s, x(s))$

(3) Product modulator 3 gets the input B and u turns out the output as Bu

(4) Here, integrators performed the integral of $P_q(t-s)$ $[f(s, x(s)) + Bu(s)]$, over the period t

In addition,

(i) inputs $P_q(t-s)$ and $f(s, x(s))$ are combined and multiplied with an output of integrator over $(0, t)$

(ii) inputs $P_q(t-s)$ and $Bu(s)$ are combined and multiplied with an output of integrator over $(0, t)$

In the end, move all the outputs from the integrators to the summer network. Therefore, the output of $x(t)$ is made, which is bounded and exactly controllable.

5. Conclusions

The paper obtained the existence of mild solutions and exact controllability for a type of fractional semilinear system of order $q \in (1, 2)$ with instantaneous and noninstantaneous impulses. The existence result is investigated by the Kuratowski measure of noncompactness and Mönch fixed point theorem. It is remarkable that we do not use the Lipschitz conditions, because they are stronger than our assumptions for nonlinear term and the impulses. On this basis, the exact controllability for the considered system is obtained. The conclusions of this paper are important and general, which fill the gap of previous studies of fractional systems of $q \in (1, 2)$ with instantaneous and noninstantaneous impulses. In the future, we will study the systems with time delay and nonlocal conditions and try to investigate the systems in the form of other fractional derivatives such as the Hilfer fractional derivatives. They can make the systems describe more complex phenomena.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the NNSF of China (62073202) and Natural Science Foundation of Shandong Province (ZR2020MA007).

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