Hindawi
Journal of Applied Mathematics
Volume 2023, Article ID 8975497, 10 pages
https://doi.org/10.1155/2023/8975497

Research Article

Mapping Connectivity Patterns: Degree-Based Topological Indices of Corona Product Graphs

Nasir Ali,[1] **Zaeema Kousar**,[2] **Maimoona Safdar**,[3] **Fikadu Tesgara Tolasa**,[4] **and Enoch Suleiman**[5]

[1]*Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Pakistan*
[2]*Department of Mathematics, University of Education Lahore, Vehari Campus, Pakistan*
[3]*Department of Mathematics, COMSATS University Islamabad, Vehari Campus, Pakistan*
[4]*Dambi Dollo University, Oromia, Ethiopia*
[5]*Department of Mathematics, Federal University Gashua, Yobe State, Nigeria*

Correspondence should be addressed to Fikadu Tesgara Tolasa; keebeekboonnii@gmail.com

Received 1 September 2023; Revised 22 September 2023; Accepted 30 October 2023; Published 16 November 2023

Academic Editor: Waqar A. Khan

Graph theory (GT) is a mathematical field that involves the study of graphs or diagrams that contain points and lines to represent the representation of mathematical truth in a diagrammatic format. From simple graphs, complex network architectures can be built using graph operations. Topological indices (TI) are graph invariants that correlate the physicochemical and interesting properties of different graphs. TI deal with many properties of molecular structure as well. It is important to compute the TI of complex structures. The corona product (CP) of two graphs G and H gives us a new graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the ith vertex of G to every vertex in the ith copy of H. In this paper, based on various CP graphs composed of paths, cycles, and complete graphs, the geometric index (GA) and atom bond connectivity (ABC) index are investigated. Particularly, we discussed the corona products $P_s \odot P_t$, $C_t \odot C_s$, $K_t \odot K_s$, $K_t \odot P_s$, and $P_s \odot K_t$ and GA and ABC index. Moreover, a few molecular graphs and physicochemical features may be predicted by considering relevant mathematical findings supported by proofs.

1. Introduction

Graph theory (GT) stands as a foundational mathematical discipline that explores the intricate interplay of graphs and diagrams as tools for visualizing and representing mathematical truths. Within this realm, the amalgamation of points and lines provides a canvas upon which complex relationships are elegantly portrayed in a diagrammatic format. The potency of graphs extends beyond mere visual representations, allowing for the construction of intricate network architectures through the application of various graph operations.

A central focus within graph theory is the notion of topological indices (TI), which serve as fundamental graph invariants connecting the realm of abstract graphs to the tangible world of physicochemical properties. These indices are powerful tools that encapsulate and correlate intriguing properties of graphs, reaching beyond the boundaries of pure mathematics into the realm of molecular structures. By providing insights into the structural characteristics of molecules, TI enable the deciphering of their physicochemical intricacies, offering a profound connection between mathematical abstraction and real-world phenomena.

In this context, the computation of TI, particularly within the framework of complex structures, emerges as a critical endeavor. This importance is magnified when considering the intricacies of intricate graphs generated through the corona product (CP) operation. The CP of two graphs, G and H, crafts a novel composite graph by combining a single instance of G with $|V(G)|$ copies of H. This synthesis results in each vertex of G intricately connecting to every vertex within its corresponding copy of H, yielding a versatile platform for the exploration of structural intricacies within complex graph compositions.

The present research embarks on an exploratory journey through the landscape of CP graphs, placing particular emphasis on compositions involving paths, cycles, and complete graphs. A pivotal facet of this exploration resides in the investigation of two crucial indices: the geometric index (GA) and the atom bond connectivity (ABC) index. These indices bear the responsibility of encapsulating geometric patterns and atomic bonding structures within the domain of CP graph configurations. As the inquiry unravels, an intricate tapestry of relationships emerges, shedding light on the interplay between CP operations and the nuanced behaviour of these indices.

The trajectory of this investigation is guided by a constellation of foundational references, which collectively enrich the understanding of graph theory. Among these beacons are the works of Chartrand and Lesniak [1], Carlson [2], Afzal et al. [3], Alon and Lubetzky [4], Randic [5], Cash [6], and Lamprey and Barnes [7], among others. Each of these references contributes a unique thread to the complex fabric of graph theory, stitching together the intricate narrative that spans from mathematical abstraction to real-world application. In [8], the authors have calculated degree-based topological indices of generalized subdivision double-corona product. Moreover, readers may study some more literature in [9, 10].

As this journey unfolds, not only does it deepen our comprehension of mathematical relationships, but it also opens doors to the prediction of molecular graphs and the unraveling of physicochemical attributes. The nexus of mathematical rigor and tangible application establishes the groundwork for advancing both theoretical understanding and practical prediction, exemplifying the multifaceted impact of graph theory in diverse domains.

Through a symphony of mathematical insights and tangible applications, this research seeks not only to contribute to the ongoing discourse within graph theory but also to underscore the profound symbiosis between mathematical exploration and its real-world consequences. As we delve into the intricate worlds of CP graphs and their accompanying indices, we engage in a harmonious dance between abstraction and application, deepening our understanding of both mathematical beauty and physical reality.

Let G and H be 2 graphs, each with a group of vertex $V(G)$ and $V(H)$ and a group of edges $E(G)$ and $E(H)$, respectively. We described the corσna product GoH as the prσduct of twσ graphs, G and H, achieved by combining each vertex of $|V(G)|$ copies of H.

$$\begin{aligned} |V(G\bigodot H)| &= |V(G)|(1+|V(H)|),\\ |E(G\bigodot H)| &= |E(G)|+|V(G)|(|V(H)|+|E(H)|. \end{aligned} \tag{1}$$

Let $G=(V,E)$ be a nontrivial, simple, or undirected graph. An independent set of vertices in an adjacent graph is known as an independent set. A graph's dominating set is a set D of vertices in which every vertex in S is not adjacent to a vertex in D. A set that is both dominant and independent in a graph is called an independent set. To determine their properties, it is crucial to know these composite molecular graphs' topological indices [3]. Topological indices of product graphs have been a fascinating area of study in recent years, and numerous articles offer formulas for various topological indices of various graph compositions [11]. Accordingly, researchers were taken by these results and were inspired to investigate the ABC index [10] and GA index [11] of the corona products of various graph architectures. The ABC index was introduced in 1998 [10]. As a result of this index, heat is used to characterize the way in which alkane production is affected by vertex degrees [10]. Detecting an independently dominating number of corona products of path, cycle, wheel, and ladder graphs will be investigated in this study. Consider a simple connected undirected graph with n vertices; then, the Randic index is defined as follows:

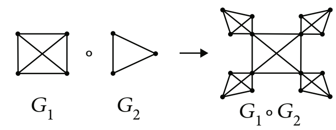

Figure 1: Corona product of two graphs.

$$X(G)=\frac{1}{\sqrt{d_u d_v}}, \tag{2}$$

where d_u is the degree of vertex u.

Consider a simple connected undirected graph $G(V,E)$ that has n nodes, and then, the ABC index is defined as follows:

$$\mathrm{ABC}(G)=\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_u d_v}}, \tag{3}$$

where d_u is the degree of vertex u.

In addition, the GA index in 2009 [12] by considering the degrees of vertices in a graph.

The GA index is defined as follows:

$$\mathrm{GA}(G)=\sum_{uv\in E(G)}\frac{2\sqrt{d_u d_v}}{(d_u+d_v)}. \tag{4}$$

The corona product of G_1 and G_2 is defined as the graph obtained by taking one copy of graph G_1 and $|V(G_1)|$ copies of G_2, where each vertex of the ith copy of G_2 relates to the ith vertex of G and is denoted by $G_1 \odot G_2$.

Figure 1 shows the corona product of two graphs. We will discuss different families of corona product graphs and calculate their topological indices.

2. Main Results

Let P_t, C_t, and K_t be a path, cycle, and complete graphs on n vertices. In this section, we discuss the ABC index and the GA index of $P_t \bigodot P_s$, $C_t \bigodot C_s$, $K_t \odot K_s$, $K_t \odot P_s$, and $P_s \odot K_t$.

Theorem 1. *The ABC index and the GA index of the corona product of two path graphs P_t and P_s are given by the following equations:*

$$\mathrm{ABC}(P_t \odot P_s) = \begin{cases} 3\sqrt{2} + \dfrac{2}{3}, & t = 2, s = 2, \\ 4\sqrt{2} + \dfrac{4(s-3)}{3} + 2(s-2)\sqrt{\dfrac{s+2}{3(s+1)}} + \dfrac{\sqrt{2s}}{s+1}, & t = 2, s > 2, \\ \dfrac{3t}{\sqrt{2}} + \sqrt{\dfrac{5}{3}} + \dfrac{(t-3)\sqrt{6}}{4}, & t > 2, s = 2, \\ 2\sqrt{2}t + \dfrac{2t(s-3)}{3} + 2(s-2)\sqrt{\dfrac{s+2}{3(s+1)}} + 2\sqrt{\dfrac{2s+1}{(s+1)(s+2)}} + \dfrac{(t-3)\sqrt{2s+2}}{s+2} + (t-2)(s-2)\sqrt{\dfrac{s+3}{3(s+2)}}, & t > 2, s > 2, \end{cases} \quad (5)$$

$$\mathrm{GA}(P_t \odot P_s) = \begin{cases} 3 + \dfrac{8\sqrt{6}}{5}, & t = 2, s = 2, \\ 1 + 2(s-3) + \dfrac{8\sqrt{6}}{5} + \dfrac{8\sqrt{2(s+1)}}{(s+3)} + \dfrac{4(s-2)\sqrt{3(s+1)}}{s+4}, & t = 2, s > 2, \\ 2t - 3 + \dfrac{4\sqrt{2}(t-2)}{3} + 8\left(\dfrac{\sqrt{3}}{7} + \dfrac{\sqrt{6}}{5}\right), & t > 2, s = 2, \\ 4\sqrt{s+1}\left(\dfrac{2\sqrt{2}}{s+3} + \dfrac{\sqrt{3}(s-2)}{s+4} + \dfrac{\sqrt{s+2}}{2s+3}\right) + 2(t-2)\sqrt{s+2}\left(\dfrac{2\sqrt{2}}{s+4} + \dfrac{\sqrt{3}(s-2)}{s+5}\right) + \dfrac{4\sqrt{6}t}{5} + t(s-3), & t > 2, s > 2. \end{cases} \quad (6)$$

Proof. Consider the corona product of two path graphs, denoted as $P_t \odot P_s$. In the case where both t and s are greater than 1, it is evident that the vertices within this composite graph can be categorized into four distinct types based on their respective degrees. Specifically,

(1) the first type encompasses vertices with a degree of 2

(2) the second type consists of vertices with a degree of 3

(3) the third type comprises vertices with a degree of $n+1$

(4) the fourth type encompasses vertices with a degree of $n+2$

Notably, within this graph, the cardinality of the vertex set $|V(P_t \odot P_s)|$ is given by $t + ts$, and the cardinality of the edge set $|E(P_t \odot P_s)|$ is equal to $ts + (t-1) + t(s-1)$, which simplifies to $2ts - 1$.

By carefully considering the degrees of these distinct vertex types, we discern a total of ten distinct types of edge partitions. These partitions, each characterized by specific vertex degree combinations, lend to a comprehensive understanding of the connectivity patterns within the composite graph. The specifics of these edge partitions can be observed in Table 1, encapsulating the diverse ways in which vertices of different degrees interact and contribute to the graph's structure.

Now, substitute the value in Table 1 for each case. □

Case 1. $t = s = 2$.

$$\begin{aligned} \mathrm{ABC}(P_t \odot P_s) = {} & 4\sqrt{\frac{(s+1)+2-2}{2(s+1)}} + 2(s-2)\sqrt{\frac{(s+1)+3-2}{3(s+1)}} \\ & + \sqrt{\frac{2(s+1)-2}{(s+1)(s+1)}} + 2(t-2)\sqrt{\frac{(s+2)+2-2}{2(s+2)}} \\ & + t\sqrt{\frac{2+2-2}{2^2}} = 3\sqrt{2} + \frac{2}{3}. \end{aligned} \quad (7)$$

Table 1: Number of the edges in each partition of $P_t \odot P_s$ based on the degree of the end vertices of each edge.

(d_u, d_v)	Number of edges
$(s+1, 2)$	4
$(s+1, 3)$	$2(s-2)$
$(s+1, s+1)$	$1\,;t=2$ $0\,;t>2$
$(s+2, s+2)$	$0\,;t=2$ $t-3, t>2$
$(s+1, s+2)$	$0\,;t=2$ $2\,;t>2$
$(s+2, 2)$	$0\,;t=2$ $2(t-2)\,(s-2), t>2$
$(s+2, 3)$	$0\,;t=2$ $(t-2)(s-2), t>2$
$(2, 2)$	$t\,;s=2$ $0\,;s>2$
$(2, 3)$	$0\,;s=2$ $2t\,;s>2$
$(3, 3)$	$0\,;s=2$ $t(s-3)\,;s>2$

Case 2. $t=2, s>2$.

$$\begin{aligned}\mathrm{ABC}(P_t \odot P_s) &= 4\sqrt{\frac{(s+1)+2-2}{2(s+1)}} + 2(s-2)\sqrt{\frac{(s+1)+3-2}{3(s+1)}} \\ &\quad + \sqrt{\frac{2(s+1)-2}{(s+1)(s+1)}} + 2(t-2)\sqrt{\frac{(s+2)+2-2}{2(s+2)}} \\ &\quad + 2t\sqrt{\frac{2+3-2}{6}} + t(s-3)\sqrt{\frac{3+3-2}{9}} \\ &= 4\sqrt{2} + \frac{4(s-3)}{3} + 2(s-2)\sqrt{\frac{s+2}{3(s+1)}} + \frac{\sqrt{2s}}{(s+1)}.\end{aligned} \tag{8}$$

Case 3. $t>2, s=2$.

$$\begin{aligned}\mathrm{ABC}(P_t \odot P_s) &= 4\sqrt{\frac{(s+1)+2-2}{2(s+1)}} + 2(s-2)\sqrt{\frac{(s+1)+3-2}{3(s+1)}} \\ &\quad + 2\sqrt{\frac{(s+1)+(s+2)-2}{(s+1)(s+2)}} + (t-3)\sqrt{\frac{2(s+2)-2}{(s+2)(s+2)}} \\ &\quad + 2(t-2)\sqrt{\frac{(s+2)+2-2}{2(s+2)}} \\ &\quad + (t-2)(s-2)\sqrt{\frac{(s+2)+3-2}{3(s+2)}} + t\sqrt{\frac{2+2-2}{2^2}} \\ &= \frac{3t}{\sqrt{2}} + \sqrt{\frac{5}{3}} + \frac{(t-3)\sqrt{6}}{4}.\end{aligned} \tag{9}$$

Case 4. $t>2, s>2$.

$$\begin{aligned}\mathrm{ABC}(P_t \odot P_s) &= 4\sqrt{\frac{(s+1)+2-2}{2(s+1)}} + 2(s-2)\sqrt{\frac{(s+1)+3-2}{3(s+1)}} \\ &\quad + 2\sqrt{\frac{(s+1)+(s+2)-2}{(s+1)(s+2)}}(t-3)\sqrt{\frac{2(s+2)-2}{(s+2)(s+2)}} \\ &\quad + 2(t-2)\sqrt{\frac{(s+2)+2-2}{2(s+2)}} \\ &\quad + (t-2)(s-2)\sqrt{\frac{(t+2)+3-2}{3(s+2)}} \\ &\quad + 2t\sqrt{\frac{2+3-2}{6}} + t(s-3)\sqrt{\frac{3+3-2}{3^2}} \\ &= 2\sqrt{2t} + \frac{2t(s-3)}{3} + 2(s-2)\sqrt{\frac{s+2}{3(s+1)}} \\ &\quad + 2\sqrt{\frac{2s+1}{(s+1)(s+2)}} + \frac{(t-3)\sqrt{2s+2}}{s+2} \\ &\quad + (t-2)(s-2)\sqrt{\frac{s+3}{3(s+2)}}.\end{aligned} \tag{10}$$

Similarly, using Equation (6) and the values in Table 1, we obtain the required result for the $G(P_t \bigodot P_s)$, which completes the proof.

Theorem 2. *The ABC index and the GA index and the corona product of the two cycles C_t and C_s are given by the following equations:*

$$\begin{aligned}ABC(C_t \bigodot C_s) &= \frac{2ts}{3} + \frac{t\sqrt{2s+2}}{s+2} + ts\sqrt{\frac{s+3}{3(s+2)}}, \\ GA(C_t \bigodot C_s) &= t(s+1) + \frac{2ts\sqrt{3(s+2)}}{s+5}.\end{aligned} \tag{11}$$

Proof. The theorem's verification is straightforward: for $t$and s both exceeding 2, it becomes evident that the cardinality of the vertex set in the corona product of two cycle graphs, denoted as $|V(C_t \bigodot C_s)|$, equals $n+nm$. Additionally, the edge set's cardinality, represented by $|E(C_t \bigodot C_s)|$, amounts to $t+2ts$.

Furthermore, within this composite graph, a classification of vertices into two distinct types based on their degrees emerges. The first type encompasses vertices with a degree of 3, while the second type comprises vertices with a degree of $s+2$. These differing degrees illuminate the diverse connectivity patterns within the graph, offering insights into the way vertices of distinct degrees interact and contribute to the overall structural makeup.

To concretize this insight, a comprehensive depiction of the edge partitions, discerned through a careful consideration of each vertex's degree, is presented in Table 2. This table succinctly captures the distinct arrangements of edges

Table 2: Number of edges in each the partition of $C_t \odot C_s$ based on the degree of end vertices of each edge.

(d_u, d_v)	Number of edges
$(s+2, s+2)$	t
$(s+2, 3)$	ts
$(3, 3)$	ts

based on the degrees of the respective vertices, shedding light on the intricate relationships and connectivity dynamics within the composite graph.

By substituting values in Table 2 in Equation (2) and simplifying the formula, we obtain

$$\begin{aligned} \mathrm{ABC}(C_t \odot C_s) &= t\sqrt{\frac{(s+2)+(s+2)-2}{2(s+2)}} \\ &\quad + ts\sqrt{\frac{(s+2)+3-2}{3(s+2)}} + ts\sqrt{\frac{3+3-2}{9}} \\ &= \frac{2ts}{3} + \frac{t\sqrt{2s+2}}{s+2} + ts\sqrt{\frac{s+3}{3(s+2)}}. \end{aligned} \tag{12}$$

Similarly, using Equation (3) and the values in Table 2, we obtain the required result for $G(C_t \odot C_s)$. □

Theorem 3. *For the corona product, of two complete graphs K_t and K_s, ABC index and GA index are equal to the following equations, respectively:*

$$ABC(K_t \odot K_s) = t\sqrt{\frac{(s-1)^3}{2}} + \frac{s}{\sqrt{t+s-1}} \cdot \left(\sqrt{s(t+2s-3)} + (t-1)\sqrt{\frac{t+s-2}{2(t+s-1)}}\right), \tag{13}$$

$$GA(K_t \odot K_s) = \frac{t((s-1)+t-1)}{2} + \frac{2ts\sqrt{s(t+s-1)}}{t+2s-1}. \tag{14}$$

Proof. By invoking the corona product definition, it becomes evident that when both t and s surpass 1, the cardinality of the vertex set in the corona product of two complete graphs, represented as $|V(K_t \odot K_s)|$, is $t+ts$. Additionally, the cardinality of the edge set, denoted as $|E(K_t \odot K_s)|$, equates to $ts + C(t,2) + tC(s,2)$, where $C(n,k)$ represents the binomial coefficient.

It is worth highlighting that, within this composite graph, a classification of vertices unfolds based on their degrees. Specifically, one classification pertains to vertices possessing a degree of s, while the other involves vertices with a degree of $t+s-1$. This duality of vertex degrees underscores the diverse interactions and contributions of vertices to the graph's overall structure.

Consequently, this classification engenders the existence of three distinctive types of edge partitions, as eloquently displayed in Table 3. Each of these partitions corresponds to different configurations of edges, contingent upon the degrees of the participating vertices. This delineation illuminates the intricate interplay between vertex degrees and edge connections within the composite graph.

By substituting the values in Table 3 in Equation (13) and simplifying the formula, we obtain

$$\mathrm{ABC}(K_t \odot K_s) = t\sqrt{\frac{(s-1)^3}{2}} + \frac{t}{\sqrt{t+s-1}} \cdot \left(\sqrt{s(t+2s-3)} + (t-1)\sqrt{\frac{t+s-2}{2(t+s-1)}}\right). \tag{15}$$

Similarly, by substituting the values in Table 3 to Equation (14) and simplifying the formula, we have

$$\mathrm{GA}(K_t \odot K_s) = \frac{t((s-1)+t-1)}{2} + \frac{2ts\sqrt{s(t+s-1)}}{t+2s-1}. \tag{16}$$

This completes the proof. □

Theorem 4. *For the corona product of the complete graph and path graph K_t and P_s, ABC index and GA index are equal to the following equations, respectively:*

$$\mathrm{ABC}(K_t \odot P_s) = \begin{cases} \dfrac{3t}{\sqrt{2}} + \dfrac{t(t-1)\sqrt{2t}}{2(t+1)}, & t \geq 2, s > 2, \\ \dfrac{t}{\sqrt{t+s-1}}\left((s-2)\sqrt{\dfrac{t+s}{3}} + \dfrac{t-1}{2}\sqrt{\dfrac{2t+2s-4}{t+s-1}}\right) + \dfrac{4t}{\sqrt{2}} + \dfrac{2t(s-3)}{3}, & t \geq 2, s > 2, \end{cases} \tag{17}$$

$$\mathrm{GA}(K_t \odot P_s) = \begin{cases} \dfrac{t(t+1)}{2} + \dfrac{4t\sqrt{2(t+1)}}{t+3}, & t \geq 2, s = 2, \\ \dfrac{t(t-1)}{2} + \dfrac{4t\sqrt{2(t+s-1)}}{t+s-1} + t(s-3) + \dfrac{2t(s-2)\sqrt{3(t+s-1)}}{t+s+2} + \dfrac{4t\sqrt{6}}{5}, & t \geq 2, m > 2. \end{cases} \tag{18}$$

TABLE 3: Number of edges in each partition of $K_t \odot K_s$ based on the degree of the end vertices of each edge.

(d_u, d_v)	Number of edges
(s, s)	$t\,C_2^s$
$(s, t+s-1)$	ts
$(t+s-1, t+s-1)$	C_2^t

TABLE 4: Number of edges in each partition of $K_t \odot P_s$ based on the degree of end vertices of each edge.

(d_u, d_v)	Number of edges
$(t+s-1, t+s-1)$	C_2^t
$(t+s-1, 2)$	$2t$
$(t+s-1, 3)$	$t(s-2)$
$(2, 2)$	$t\,; s=2$ $0\,; m>2$
$(2, 3)$	$0\,; s=2$ $2t\,; s>2$
$(3, 3)$	$0\,; s=2$ $t(s-3)\,; s>2$

Proof. Considering the corona product of $K_t \odot P_s$, where both t and s exceed 1, a distinctive classification of vertices emerges based on their degrees. This categorization yields three primary vertex types:

(1) Vertices with a degree of 2 constitute the first type

(2) The second type encompasses vertices possessing a degree of 3

(3) The third type comprises vertices with a degree of $n+m-1$

In the context of this composite graph, the cardinality of the vertex set, $|V(K_t \odot P_s)|$, equates to $n+nm$. Correspondingly, the cardinality of the edge set, $|E(K_t \odot P_s)|$, is characterized by $ts + t(s-1) + C(t,2)$.

An insightful observation emerges upon evaluating the degrees of the vertices: the graph exhibits six distinct types of edge partitions, as eloquently depicted in Table 4. Each partition encapsulates a unique configuration of edges, shaped by the degrees of the connected vertices. This revelation accentuates the intricate interplay between vertex degrees and edge connections within the composite graph.

Now, substitute the values in Table 4 in Equation (17) for both cases. □

Case 1. $n \geq 2, m = 2$.

$$\mathrm{ABC}(K_t \odot P_s) = {}^tC_2\sqrt{\frac{2(t+s-1)-2}{(t+s-1)^2}} + 2t\sqrt{\frac{2+(t+s-1)-2}{2(t+s-1)}} + t(s-2)\sqrt{\frac{3+(t+s-1)-2}{3(t+s-1)}} + t\sqrt{\frac{2+2-2}{2^2}}. \tag{19}$$

By simplifying the formula, we obtain

$$\mathrm{ABC}(K_t \odot p_s) = \frac{3t}{\sqrt{2}} + \frac{t(t-1)\sqrt{2t}}{2(t+1)}. \tag{20}$$

Case 2. $n \geq 2, m > 2$.

$$\mathrm{ABC}(K_t \odot P_s) = {}^tC_2\sqrt{\frac{2(t+s-1)-2}{(t+s-1)^2}} + 2t\sqrt{\frac{2+(t+s-1)-2}{2(t+s-1)}} + t(s-2)\sqrt{\frac{3+(t+s-1)-2}{3(t+s-1)}} + 2t\sqrt{\frac{2+3-2}{2(3)}} + t(s-3)\sqrt{\frac{3+3-2}{3^2}}. \tag{21}$$

Through the process of simplifying the formula, we achieve the desired outcome. Similarly, by plugging in the values from Table 4 into equation (18) and then simplifying the formula, we attain the necessary results for the GA index of the corona product graph $(K_t \odot P_s)$.

Theorem 5. *The ABC index and the GA index of the corona product of the path graph and complete graph P_s and K_t are given by the following equations:*

$$\mathrm{ABC}(P_s \odot K_t) = \begin{cases} \dfrac{\sqrt{2t}}{t+1} + (t-1)\sqrt{2(t-1)} + 2t\sqrt{\dfrac{2t-1}{t(t+1)}}, & t \geq 2, s = 2, \\ \dfrac{s(t-1)\sqrt{2t-2}}{2} + 2t\sqrt{\dfrac{2t-1}{t(t+1)}} + 2\sqrt{\dfrac{2t+1}{(t+1)(t+2)}} + \dfrac{(s-3)\sqrt{2(t+1)}}{t+2} + t(s-2)\sqrt{\dfrac{2}{t+2}}, & t \geq 2, s > 2, \end{cases} \tag{22}$$

$$\mathrm{GA}(P_s \odot K_t) = \begin{cases} 1 + t(t-1) + \dfrac{4t\sqrt{t(t+1)}}{2t+1}, & t \geq 2, s = 2, \\ \dfrac{ts(t-1)}{2} + \dfrac{4t\sqrt{t(t+1)}}{2t+1} + s - 3 + \dfrac{4\sqrt{(t+1)(t+2)}}{2t+3} + \dfrac{t(s-2)\sqrt{t(t+2)}}{t+1}, & t \geq 2, s > 2. \end{cases} \tag{23}$$

Table 5: Number of edges in each partition of $P_s \bigodot K_t$ based on the degree of end vertices of each edge.

(d_u, d_v)	Number of edges
$(t+1, t+1)$	$1\,; s = 2$ $0\,; s > 2$
(t, t)	$s\,C_2^t$
$(t, t+1)$	$2t$
$(t+1, t+2)$	$0\,; s = 2$ $2\,; s > 2$
$(t+2, t+2)$	$0\,; s = 2$ $s-3\,; s > 2$
$(t+2, t)$	$0\,; s = 2$ $t(s-2)\,; s > 2$

Proof. Let us delve into the corona product of the path graph and the complete graph, denoted as $P_s \odot K_t$, where both t and s are greater than 1. Within this context, a classification of vertices emerges based on their respective degrees, leading to the identification of three distinct vertex types:

(1) The first type encompasses vertices with a degree of t

(2) The second type consists of vertices possessing a degree of $t+1$

(3) The third type comprises vertices with a degree of $t+2$

In relation to this composite graph, the cardinality of the vertex set, denoted as $|V(P_s \odot K_t)|$, corresponds to $s + ts$. Simultaneously, the edge set's cardinality, represented by $|E(P_s \odot K_t)|$, can be expressed as $ts + (s-1) + sC(t, 2)$.

A significant observation materializes as we assess the vertex degrees: the graph is characterized by six distinct types of edge partitions, as vividly portrayed in Table 5. Each partition embodies a unique arrangement of edges, intricately shaped by the degrees of the vertices they connect. This insight highlights the intricate interplay between vertex degrees and the network of edge connections within the composite graph.

Now, substitute the values in Table 5 in Equation (22) for both cases. □

Case 1. $n \geq 2, m = 2$.

$$\mathrm{ABC}(P_s \odot K_t) = \sqrt{\frac{2(t+1)-2}{(t+1)^2}} + s^t C_2 \sqrt{\frac{2t-2}{t^2}} + 2t\sqrt{\frac{t+(t+1)-2}{t(t+1)}}. \tag{24}$$

By simplifying the formula, we obtain

$$\mathrm{ABC}(P_2 \odot K_t) = \frac{\sqrt{2t}}{t+1} + (t-1)\sqrt{2(t-1)} + 2t\sqrt{\frac{2t-1}{t(t+1)}}. \tag{25}$$

Case 2. $n \geq 2, m > 2$.

$$\begin{aligned}\mathrm{ABC}(P_s \odot K_t) = {} & s^t C_2 \sqrt{\frac{2t-2}{t^2}} + 2t\sqrt{\frac{t+(t+1)-2}{t(t+1)}} \\ & + 2\sqrt{\frac{(t+1)+(t+2)-2}{(t+1)(t+2)}} \\ & + (s-3)\sqrt{\frac{2(t+2)-2}{(t+2)^2}} \\ & + t(s-2)\sqrt{\frac{t+(t+2)-2}{t(t+2)}}.\end{aligned} \tag{26}$$

Through the process of simplifying the formula, we achieve the desired outcome. In a similar vein, by inserting the values from Table 5 into Equation (23) and subsequently simplifying the formula, we acquire the necessary results for the GA index of the corona product graph $(P_s \odot K_t)$.

Corollary 6. *The Randic index, ABC index, and GA index of the corona product of graphs W_n and P_m are defined by the following equations (for particular cases, general result still can be worked out for such products):*

(i) Randic index $(W_n \bigodot P_m) = (n = 3, m = 6) = 12.898$

(ii) ABC index $(W_n \bigodot P_m) = (n = 3, m = 6) = 31.718$

(iii) GA index $(W_n \bigodot P_m) = (n = 3, m = 6) = 45.865$

Proof. Let us examine the corona product of two graphs, $W_n \bigodot P_m$, where n and m are both greater than 2. By analyzing the degrees of vertices, we can categorize them into four distinct types.

The first type consists of vertices with a degree of 3, while the second type comprises vertices with a degree of 4. The third type encompasses vertices with a degree of $n+3$, and the fourth type includes vertices with a degree of $m+2$. By delving into the degrees of these vertices, we can identify a total of six different partition types, as illustrated in Table 6.

Now, substitute the value in Table 6 for each case. □

Case 1. $n = 3, p = 6$.

$$\begin{gathered} X(G) = \frac{4}{\sqrt{3X2}} + \frac{4}{\sqrt{2X3}} + \frac{12}{\sqrt{3X3}} + \frac{8}{\sqrt{2X9}} + \frac{16}{\sqrt{3X9}} + \frac{6}{\sqrt{9X9}}, \\ X(G) = 12.898. \end{gathered} \tag{27}$$

Case 2. $n = 3, p = 6$.

$$\begin{aligned} \mathrm{ABC}(G) &= \sqrt{\frac{12}{6}} + \sqrt{\frac{12}{6}} + \sqrt{\frac{48}{9}} + \sqrt{\frac{72}{18}} + \sqrt{\frac{160}{27}} + \sqrt{\frac{96}{81}}, \\ \mathrm{ABC}(G) &= 31.718. \end{aligned} \tag{28}$$

Table 6: Number of edges in each partition of $W_n \odot P_m$ based on the degree of end vertices of each edge.

Edge point	Number of pair
(3,9)	16
(9,9)	6
(2,9)	8
(2,3)	4
(3,3)	12
(3,2)	4

Case 3. $n = 3, p = 6$.

$$\mathrm{GA}(G) = \frac{18\sqrt{6}}{5} + \frac{12\sqrt{9}}{3} + \frac{16\sqrt{18}}{11} + \frac{8\sqrt{27}}{3} + \frac{2\sqrt{81}}{3},$$
$$GA(G) = 45.865. \tag{29}$$

Corollary 7. *The Randic index, ABC index, and GA index of the corona product of graphs k_n and L_m are defined by the following equations:*

(i) *Randic index $(k_n \odot L_m) = (n = 4, m = 4) = 17.028$*

(ii) *ABC index $(K_n \odot L_m) = (n = 4, m = 4) = 46.243$*

(iii) *GA index $(K_n \odot L_m) = (n = 4, m = 4) = 73.117$*

Proof. Let us explore the corona product of two graphs, $K_n \odot L_m$, where both n and m exceed 3. By examining the vertex degrees, we can distinguish four distinct types of vertices. The initial type comprises vertices with a degree of 4, while the second type encompasses vertices with a degree of 5. The third type consists of vertices with a degree of n, and the fourth type encompasses vertices with a degree of $n + m - 1$. By analyzing the vertex degrees in this manner, we can identify a total of seven distinct partition types, as illustrated in Table 7.

Now, substitute the value in Table 7 for each case. □

Case 1. $n = 4, m = 4$.

$$X(G) = \frac{8}{\sqrt{3X3}} + \frac{8}{\sqrt{4X3}} + \frac{8}{\sqrt{3X4}} + \frac{16}{\sqrt{4X4}} + \frac{16}{\sqrt{3X11}} + \frac{16}{\sqrt{4X11}} + \frac{6}{\sqrt{11X11}},$$
$$X(G) = 17.028. \tag{30}$$

Case 2. $n = 4, m = 4$.

$$\mathrm{ABC}(G) = \sqrt{\frac{32}{9}} + \sqrt{\frac{40}{12}} + \sqrt{\frac{40}{12}} + \sqrt{\frac{96}{16}} + \sqrt{\frac{192}{33}} + \sqrt{\frac{208}{44}} + \sqrt{\frac{120}{121}},$$
$$\mathrm{ABC}(G) = 46.243. \tag{31}$$

Table 7: Number of edges in each partition of $K_n \odot L_m$ on the basis of the degree of end vertices of each edge.

Edge point	Number of pair
(3,11)	16
(4,11)	16
(11,11)	6
(3,4)	8
(4,4)	16
(3,3)	8
(4,3)	8

Case 3. $n = 4, m = 4$.

$$\mathrm{GA}(G) = \frac{16\sqrt{9}}{6} + \frac{32\sqrt{12}}{7} + \frac{16\sqrt{33}}{7} + \frac{32\sqrt{44}}{15} + \frac{6\sqrt{121}}{11},$$
$$\mathrm{GA}(G) = 73.117. \tag{32}$$

Corollary 8. *The Randic index, ABC index, and GA index of the corona product of graphs L_n and K_m are defined by the following equations:*

(i) *Randic index $(L_n \odot K_m) = (n = 4, m = 6) = 34.872$*

(ii) *ABC index $(L_n \odot K_m) = (n = 4, m = 6) = 90.728$*

(iii) *GA index $(L_n \odot K_m) = (n = 4, m = 6) = 117.195$*

Proof. Let us examine the corona product of two graphs, $L_n \odot K_m$, where both n and m are greater than 4. By analyzing the degrees of vertices, we can classify them into four specific types.

The initial type encompasses vertices with a degree of 5, while the second type comprises vertices with a degree of 6. The third type consists of vertices with a degree of $n + 1$, and the fourth type includes vertices with a degree of $m + 2$. Through this analysis of vertex degrees, we can identify a total of nine distinct partition types, as depicted in Table 8.

Now, substitute the value in Table 8 for each case. □

Case 1. $n = 4, m = 6$.

$$X(G) = \frac{8}{\sqrt{5X6}} + \frac{112}{\sqrt{6X6}} + \frac{2}{\sqrt{5X8}} + \frac{22}{\sqrt{6X8}} + \frac{2}{\sqrt{8X8}} + \frac{2}{\sqrt{9X8}} + \frac{24}{\sqrt{6X9}} + \frac{2}{\sqrt{8X9}} + \frac{4}{\sqrt{9X9}},$$
$$(G) = 34.872. \tag{33}$$

Table 8: Number of edges in each partition of $L_n \odot K_m$ on the basis of the degree of end vertices of each edge.

Edge point	Number of pair
(6,8)	22
(6,9)	24
(6,6)	112
(5,6)	8
(8,8)	2
(9,9)	4
(8,9)	2
(9,8)	2

Case 2. $n = 4, m = 6$.

$$\begin{aligned} \mathrm{ABC}(G) &= \sqrt{\frac{72}{30}} + \sqrt{\frac{1120}{36}} + \sqrt{\frac{22}{40}} + \sqrt{\frac{264}{48}} + \sqrt{\frac{28}{64}} \\ &\quad + \sqrt{\frac{30}{72}} + \sqrt{\frac{312}{54}} + \sqrt{\frac{30}{72}} + \sqrt{\frac{64}{81}}, \\ \mathrm{ABC}(G) &= 90.728. \end{aligned} \tag{34}$$

Case 3. $n = 4, m = 6$.

$$\begin{aligned} \mathrm{GA}(G) &= \frac{16\sqrt{30}}{11} + \frac{56\sqrt{36}}{3} + \frac{4\sqrt{40}}{13} + \frac{22\sqrt{48}}{7} + \frac{\sqrt{64}}{4} \\ &\quad + \frac{4\sqrt{72}}{17} + \frac{48\sqrt{54}}{15} + \frac{4\sqrt{72}}{17} + \frac{4\sqrt{81}}{9}, \\ \mathrm{GA}(G) &= 177.195. \end{aligned} \tag{35}$$

3. Material and Methods

The authors studied the concepts of different operations for the graphs and found the corona product operation very interesting and useful. It has a waste application in chemistry, physics, and computer fields for constructing complex networks; some applications may be studied in [7, 13, 14]. Some recent developments and benefits of studying the degree-based topological indices for such graphs were greatly beneficial. Using the idea and developing special corona product graphs, we used the techniques to calculate some topological indices of these products by considering some special cases. Moreover, a combination of studying different graphs under different degrees, distances, or edge-based invariants can be seen in [4, 8–10, 13–17].

4. Conclusion

In this study, we generalized the concept of topological indices of different families of corona product graphs and described the corona graphs of the P_t, P_s, K_t, K_s, and C_t, C_s . In the future, you will be able to find the topological indices based on the order and distance of these graph operations. The results are convenient for constructing and investigating topological indexes of complex network structures. In this study, we created corona products' ABC and GA indexes when synthesizing paths, cycles, and complete graphs. In addition, a generalized corona product graph calculation example is given in different cases.

Open problems: find the GA and ABC index for the following families of corona products $W_n \odot P_m$, $K_n \odot L_m$, and $L_n \odot K_m$.

Data Availability

The data is provided on the request to the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors equally contributed towards this work and agree to publish this paper under academic ethics.

References

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd edition3rd edition, , 2000.

[2] K. Carlson, "Generalized books and cm-snakes are prime graphs," *Ars Combinatoria*, vol. 80, pp. 215–221, 2006.

[3] F. Afzal, M. Farahani, M. Cancan, F. Arshad, D. Afzal, and F. Chaudhry, "Topological indices of the system of generalized prisms via m-polynomial approach," *Eurasian Chemical Communications*, vol. 3, no. 5, pp. 296–300, 2021.

[4] N. Alon and E. Lubetzky, "Independent sets in tensor graph powers," *Journal of Graph Theory*, vol. 54, no. 1, pp. 73–87, 2007.

[5] M. Randic, "On history of the Randić index and emerging hostility toward chemical graph theory," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 59, pp. 5–124, 2008.

[6] G. G. Cash, "Relationship between the Hosoya polynomial and the hyper-Wiener index," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 893–895, 2002.

[7] R. H. Lamprey and B. H. Barnes, "Products of graph and applications," *Modelling and Simulation*, vol. 5, pp. 1119–1123, 1974.

[8] Y. Wang, M. Waheed, M. K. Jamil, S. M. Shah, A. Aslam, and H. Aram, "Degree-based topological indices of generalized subdivision double-corona product," *Journal of Chemistry*, vol. 2022, Article ID 3119739, 14 pages, 2022.

[9] N. De, "Application of corona product of graphs in computing topological indices of some special chemical graphs," in *Handbook of Research on Applied Cybernetics and Systems Science*, pp. 82–101, IGI Global, 2017.

[10] R. Tabish, M. K. Jamil, and M. Waheed, "Further results on topological indices of two corona variants of graph products," 2021, http://arxiv.org/abs/2111.11006.

[11] D. Vukičević and B. Furtula, "Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges," *Journal of Mathematical Chemistry*, vol. 46, no. 4, pp. 1369–1376, 2009.

[12] F. Harry and R. Melter, "On the metric dimension of a graph," *An rs Combinatoria*, vol. 2, p. 1911, 1976.

[13] M. Elias and B. Taheri, "Four new sums of graphs and their Wiener indices," *Discrete Applied Mathematics*, vol. 157, no. 4, pp. 794–803, 2009.

[14] M. Bilal, M. K. Jamil, M. Waheed, and A. Alameri, "Three topological indices of two new variants of graph products," *Mathematical Problems in Engineering*, vol. 2021, Article ID 7724177, 9 pages, 2021.

[15] S. W. Saputro, "On local metric dimension of (n−3)-regular graph," *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 98, pp. 43–54, 2016.

[16] B. Bresar, W. Imrich, S. Klavzar, and B. Zmazek, "Hypercubes as direct products," *SIAM Journal on Discrete Mathematics*, vol. 18, no. 4, pp. 778–786, 2005.

[17] A. Graovac and T. Pisanski, "On the Wiener index of a graph," *Journal of Mathematical Chemistry*, vol. 8, no. 1, pp. 53–62, 1991.