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Research Article

A Mathematical Model of the Dynamics of Coffee Berry Disease

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Coffee berry disease (CBD) is a fungal disease caused by *Colletotrichum kahawae*. CBD is a major constraint to coffee production to Kenya and Africa at large. In this research paper, we formulate a mathematical model of the dynamics of the coffee berry disease. The model consists of coffee plant population in a plantation and *Colletotrichum kahawae* pathogen population. We derived the basic reproduction number $\mathcal{R}_{k0}$, and analyzed the dynamical behaviors of both disease-free equilibrium and endemic equilibrium by the theory of ordinary differential equations. Using the MATLAB ode45 solver, we carried out numerical simulation, and the findings are consistent with the theoretical results.

1. Introduction

Coffee berry disease (CBD) is a fungal disease caused by *Colletotrichum kahawae*. The fungus *Colletotrichum kahawae* infects all stages of the coffee crop, from flowers to mature coffee berries, causing premature fruit drop and berry rot [1].

Coffee berry disease infects coffee berries (the harvestable portion of the crop), leading to direct yield loss. Also, CBD causes the pulp to adhere to the coffee bean hence making it more difficult to process and it may lower the quality of processed coffee [2].

CBD is a major constraint to coffee production in Kenya and Africa at large. The impact of CBD in Kenya was strongly felt during the 1962/1963 and 1967/1968 crop years when losses in coffee production increased to 80% [3].

According to [4], there are around 700 thousand coffee farmers in Kenya, and it is estimated that 5 million Kenyans were hired to work in the coffee production chain. This implies that CBD threatens the livelihood of millions because direct losses of the crop reduce the income.

Many mathematical models have been created to investigate the effects of preventive and control techniques on the dynamics of plant disease spread. A study for the dynamics of the transmission of plant diseases with and without roguing mechanism was carried out by [5]. The results of the study demonstrated that roguing mechanisms help in preventing the transmission of plant diseases.

The mathematical model of induced resistance to plant disease presented by [6] divides the plant population into three compartments: susceptible plants, resistant plants, and diseased plants. The outcomes of the model showed that when the elicitor application is done on plants before the inoculation of pathogens, plants are less severely affected by the diseases.

Most of the reviews presented on coffee berry disease provide qualitative studies that describe the current status and existing strategies in managing the spread and actions of the new epidemic (see for, example, [7–10]). In this paper, we investigate the dynamics of coffee berry disease.

2. Model Formulation

The coffee plants in the plantation are divided into four groups at any time t, namely, the susceptible coffee plants $S(t)$, coffee plants exposed to *Colletotrichum kahawae* (the infected coffee plants which have not shown symptoms) $E_k(t)$, the CBD-infected coffee plants $I_k(t)$ and recovered coffee plants $R(t)$. Let $N(t)$ be the total number of coffee plants, then $N(t) = S(t) + E_k(t) + I_k(t) + R(t)$. The number of

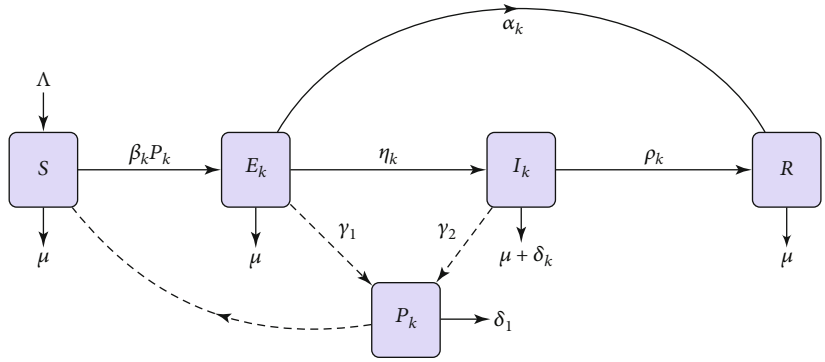

Figure 1: Flow chart of epidemic coffee plants.

Colletotrichum kahawae pathogens in the plantation at any time t is $P_k(t)$. The susceptible coffee trees are recruited at a rate of Λ. Some coffee trees will vacate all classes due to natural death at a constant rate μ. Susceptible coffee trees are exposed to the coffee berry disease through contact with *Colletotrichum kahawae* at a rate ϖ_k; thus, coffee trees in $S(t)$ class will move to $E_k(t)$ class at the rate ϖ_k. Some coffee trees in $E_k(t)$ progress to I_k at the rate η_k and others progress to $R(t)$ at the rate α_k. Also, some coffee trees in $I_k(t)$ recover and progress to $R(t)$ at the rate ρ_k. A proportion of coffee trees in $I_k(t)$ class will die from CBD-induced deaths at the rate δ_k. In addition, coffee trees $E_k(t)$ and $I_k(t)$ contribute to the increase of P_k pathogen in the environment at the rates γ_1 and γ_2, respectively. Finally, pathogens in P_k class decay at the rate δ_1.

2.1. Model Assumptions. The following are the assumptions of the model:

(i) The fungus multiplies on the coffee plant only

(ii) There is permanent immunity upon recovery

(iii) There is disease-related death of coffee plant

(iv) There is on planting once coffee plants die (dry)

2.2. Model Flow Chart and Equations. From Figure 1, we have the following equations of the model:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Lambda - (\varpi_k P_k + \mu)S,\\
\frac{dE_k}{dt} &= \varpi_k P_k S - (\alpha_k + \mu + \eta_k)E_k,\\
\frac{dI_k}{dt} &= \eta_k E_k - (\rho_k + \mu + \delta_k)I_k,\\
\frac{dP_k}{dt} &= \gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k,\\
\frac{dR}{dt} &= \alpha_k E_k + \rho_k I_k - \mu R.
\end{aligned}\right\} \tag{1}$$

3. Well-Posedness of the Model

Since the system model (1) describes coffee plants population and *Colletotrichum kahawae* pathogen population, it is essential to prove the well-posedness of the model solutions. Well-posedness of the model is proved by showing that the solutions with non-negative initial data are positive and bounded for all time $t > 0$ as follows.

3.1. Positivity of the Solutions of the System Model (1)

Lemma 1. *Let $S_0 > 0$, $E_{k0} \geq 0$, $I_{k0} \geq 0$, $P_{k0} \geq 0$, and $R_0 \geq 0$ be the initial conditions of the system (1). Then the solutions S, E_k, I_k, P_k, and R are nonnegative $\forall t > 0$.*

Proof. From system (1), we define T as the maximum endemic time, and it is given by

$$T = \sup\{t > 0 | S(\tau) > 0, E_k(\tau) \geq 0, I_k(\tau) \geq 0, P_k(\tau) \geq 0, R(\tau) \geq 0 \forall \tau \in [0, t]\}. \tag{2}$$

Consider $S_0 > 0$, $E_{k0} \geq 0$, $I_{k0} \geq 0$, $P_{k0} \geq 0$, and $R_0 \geq 0$. Also, let us consider the first equation of system (1)

$$\frac{dS}{dt} = \Lambda - (\varpi_k P_k + \mu)S. \tag{3}$$

Equation (3) can be written as

$$\frac{dS}{dt} + (\varpi_k P_k + \mu)S = \Lambda. \tag{4}$$

upon multiplication of both sides of equation (4) by the integrating factor, we get

$$\frac{d}{dt}\left(S(t)\exp\left[\int_0^T (\varpi_k P_k + \mu)(s)ds\right]\right) = \Lambda \exp\left(\int_0^T (\varpi_k P_k + \mu)(s)ds\right). \tag{5}$$

Integrating both sides of equation (5) from 0 to T, we get

$$S(T) = \exp\left[-\int_0^T (\varpi_k P_k + \mu)(s)ds\right]\left\{S_0 + \int_0^T \Lambda \exp\left[\int_0^{\tilde{T}} (\varpi_k P_k + \mu)(\tau)d\tau\right]d\tilde{T}\right\}. \tag{6}$$

Thus, $S(t) > 0 \forall t > 0$.

For the second equation of system (1), we have

$$\begin{aligned}\frac{dE_k}{dt} &= \varpi_k P_k S - (\alpha_k + \mu + \eta_k)E_k \geq -(\alpha_k + \mu + \eta_k)E_k \Rightarrow E_k \\ &\geq E_{k0} \exp\left[-\int_0^T (\alpha_k + \mu + \eta_k)(s)ds\right] \\ &\geq E_{k0} \exp\{-(\alpha_k + \mu + \eta_k)T\} \geq 0.\end{aligned} \tag{7}$$

Hence, $E_k(t) \geq 0 \forall t > 0$.

Proving the remaining three equations in the same manner, we obtain

$$I_k(t) \geq 0, P_k(t) \geq 0, R(t) \geq 0. \tag{8}$$

Thus, all the solutions are non-negative $\forall t > 0$. □

3.2. Boundedness of the Solutions of the System Model (1). We demonstrate that every feasible solution is uniformly bounded in a proper subset $\mathscr{D}$.

Lemma 2. *Let the initial conditions of system (1) be nonnegative in* $\mathbb{R}_+^4 \times \mathbb{R}_+^1$,

$$\begin{aligned}\mathscr{D}_N &= \left\{(S, E_k, I_k, R) \in \mathbb{R}_+^4 : N(t) \leq \frac{\Lambda}{\mu}\right\}, \\ \mathscr{D}_{P_k} &= \left\{P_k \in \mathbb{R}_+^1 : P_k(t) \leq \frac{\Lambda(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)}{\mu\delta_1}\right\}.\end{aligned} \tag{9}$$

Then the set $\mathscr{D} = \mathscr{D}_N \cup \mathscr{D}_{P_k} \subset \mathbb{R}_+^4 \times \mathbb{R}_+^1$ *is positively invariant*

Proof. In this lemma, we are required to show that $\mathscr{D}_N$ and $\mathscr{D}_{P_k}$ are positively invariant. To start, we sum the first three equations and the last equation of the system (1) to get

$$\frac{dN}{dt} = \Lambda - \mu N - \delta_k I_k. \tag{10}$$

In the absence of the CBD, we have

$$\frac{dN}{dt} \leq \Lambda - \mu N. \tag{11}$$

Upon solving equation (11) for N, we get

$$N(t) \leq \frac{\Lambda}{\mu} + \left\{N_0 - \frac{\Lambda}{\mu}\right\}e^{-\mu t}. \tag{12}$$

Thus,

$$N(t) \leq \frac{\Lambda}{\mu} \text{ as } t \longrightarrow \infty. \tag{13}$$

It follows that the feasible region for the coffee plants population in the system (1) is defined by

$$\mathscr{D}_N = \left\{(S, E_k, I_k, R) \in \mathbb{R}_+^4 : N(t) \leq \frac{\Lambda}{\mu}\right\}. \tag{14}$$

Considering the fourth equation of system (1), the equation for *Colletotrichum kahawae* pathogens is

$$\frac{dP_k}{dt} = \gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k. \tag{15}$$

We rewrite it as

$$\frac{dP_k}{dt} \leq \frac{\Lambda(\gamma_1 + \gamma_2)}{\mu} - \delta_1 P_k. \tag{16}$$

Solving equation (16), we get

$$P_k(t) \leq \frac{\Lambda(\gamma_1 + \gamma_2)}{\mu\delta_1} + \left(P_{k0} - \frac{\Lambda(\gamma_1 + \gamma_2)}{\mu\delta_1}\right)e^{-\delta_1 t}. \tag{17}$$

Hence,

$$\begin{aligned}P_k(t) &\leq \frac{\Lambda(\gamma_1 + \gamma_2)}{\mu\delta_1} \text{ as } t \longrightarrow \infty. \\ \mathscr{D}_{P_k} &= \left\{P_k \in \mathbb{R}_+^1 : P_k(t) \leq \frac{\Lambda(\gamma_1 + \gamma_2)}{\mu\delta_1}\right\}.\end{aligned} \tag{18}$$

Consequently, the feasible region defined by the set $\mathscr{D} = \mathscr{D}_N \cup \mathscr{D}_{P_k} \subset \mathbb{R}_+^4 \times \mathbb{R}_+^1$ is positively invariant. □

It follows that every feasible solution of system (1) is uniformly bounded in $\mathscr{D}$; thus, the system is appropriate for the study of the dynamics of CBD infection.

3.3. CBD Disease-Free Equilibrium Point (DFE). The DFE for CBD is a situation in which there is no CBD infection in the plant population. Therefore, DFE for CBD model (1) is given by

$$\mathscr{E}_k^0 = \left(S^0, E_k^0, I_k^0, P_k^0, R^0\right) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0\right). \tag{19}$$

3.4. Reproduction Number ($\mathscr{R}_{k0}$). According to [11], $\mathscr{R}_{k0}$ is the average number of secondary infections produced by a "typical"infected plant in a completely susceptible plant population. To compute $\mathscr{R}_{k0}$, the next-generation method [12] is applied. Using this method, $\mathscr{R}_{k0}$ is given by $\rho(FV^{-1})$ (the spectral radius of FV^{-1}) where F is the Jacobian of $\mathscr{F}_i$ at $\mathscr{E}_k^0$ and $\mathscr{F}_i$ is the rate at which new infections appear in compartment i, and V is the Jacobian of $\mathscr{V}_i$ at $\mathscr{E}_k^0$ and $\mathscr{V}_i$ is the rate of progression of plants into and out of compartment i. In view of the

system model (1), the infected compartments are given by the following system:

$$\left.\begin{aligned}\frac{dE_k}{dt} &= \varpi_k P_k S - (\alpha_k + \mu + \eta_k)E_k,\\ \frac{dI_k}{dt} &= \eta_k E_k - (\rho_k + \mu + \delta_k)I_k,\\ \frac{dP_k}{dt} &= \gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k.\end{aligned}\right\} \tag{20}$$

From the system (20), we derive

$$\begin{aligned}\mathscr{F}_i &= \begin{bmatrix}\varpi_k P_k S\\ 0\\ 0\end{bmatrix},\\ \mathscr{V}_i &= \begin{bmatrix}(\alpha_k + \mu + \eta_k)E_k\\ -\eta_k E_k + (\rho_k + \mu + \delta_k)I_k\\ -\gamma_1 E_k - \gamma_2 I_k + \delta_1 P_k\end{bmatrix}.\end{aligned} \tag{21}$$

And it follows that

$$\begin{aligned}F &= \begin{bmatrix}0 & 0 & \dfrac{\varpi_k \Lambda}{\mu}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\\ V &= \begin{bmatrix}(\alpha_k + \mu + \eta_k) & 0 & 0\\ -\eta_k & (\rho_k + \mu + \delta_k) & 0\\ -\gamma_1 & -\gamma_2 & \delta_1\end{bmatrix}.\end{aligned} \tag{22}$$

The inverse of V is given by

$$V^{-1} = \begin{bmatrix}\dfrac{1}{(\alpha_k + \mu + \eta_k)} & 0 & 0\\ \dfrac{\eta_k}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)} & \dfrac{1}{(\rho_k + \mu + \delta_k)} & 0\\ \dfrac{(\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\delta_1} & \dfrac{\gamma_2}{(\rho_k + \mu + \delta_k)\delta_1} & \dfrac{1}{\delta_1}\end{bmatrix}. \tag{23}$$

Computing the product of F and V^{-1}, it obtains

$$FV^{-1} = \begin{bmatrix}\dfrac{\varpi_k \Lambda((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\mu\delta_1} & \dfrac{\varpi_k \Lambda \gamma_2}{(\rho_k + \mu + \delta_k)\mu\delta_1} & \dfrac{\varpi_k \Lambda}{\mu\delta_1}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{24}$$

Clearly, the dominant eigenvalue of FV^{-1} is $\varpi_k \Lambda((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)/(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\mu\delta_1$. Hence,

$$\mathscr{R}_{k0} = \frac{\varpi_k \Lambda((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\mu\delta_1}. \tag{25}$$

3.5. Local Stability of the DFE

Theorem 3. *The DFE of coffee berry disease, $\mathscr{E}_k^0$, is locally asymptotically stable if $\mathscr{R}_{k0} < 1$ and unstable if $\mathscr{R}_{k0} > 1$.*

Proof. If the Jacobian matrix's eigenvalues at $\mathscr{E}_k^0$ have negative real parts, $\mathscr{E}_k^0$ is considered to be locally asymptotically stable. Evaluating the Jacobian matrix of system (1) at $\mathscr{E}_k^0$, we get

$$J(\mathscr{E}_k^0) = \begin{bmatrix}-\mu & 0 & 0 & \dfrac{-\varpi_k \Lambda}{\mu} & 0\\ 0 & -(\alpha_k + \mu + \eta_k) & 0 & \dfrac{\varpi_k \Lambda}{\mu} & 0\\ 0 & \eta_k & -(\rho_k + \mu + \delta_k) & 0 & 0\\ 0 & \gamma_1 & \gamma_2 & -\delta_1 & 0\\ 0 & \alpha_k & \rho_k & 0 & -\mu\end{bmatrix}. \tag{26}$$

It is clear that $\lambda_1 = -\mu$ and $\lambda_2 = -\mu$ are the eigenvalues of matrix (26). Thus, we reduce the matrix to get

$$J_1(\mathscr{E}_k^0) = \begin{bmatrix}-(\alpha_k + \mu + \eta_k) & 0 & \dfrac{\varpi_k \Lambda}{\mu}\\ \eta_k & -(\rho_k + \mu + \delta_k) & 0\\ \gamma_1 & \gamma_2 & -\delta_1\end{bmatrix}. \tag{27}$$

To determine the eigenvalues of the matrix (27), we express it as follows

$$\begin{vmatrix}-(\alpha_k + \mu + \eta_k) - \lambda & 0 & \dfrac{\varpi_k \Lambda}{\mu}\\ \eta_k & -(\rho_k + \mu + \delta_k) - \lambda & 0\\ \gamma_1 & \gamma_2 & -\delta_1 - \lambda\end{vmatrix} = 0. \tag{28}$$

From equation (28), we have the following characteristic equation

$$\begin{aligned}&\big[\lambda^3 + ((\alpha_k + \mu + \eta_k) + (\rho_k + \mu + \delta_k) + \delta_1)\lambda^2 + (\delta_1(\rho_k + \mu + \delta_k)\\ &\quad + (\alpha_k + \mu + \eta_k)((\rho_k + \mu + \delta_k) + \delta_1))\lambda + (\alpha_k + \mu + \eta_k)\\ &\quad \cdot (\rho_k + \mu + \delta_k)\delta_1] - \frac{\varpi_k \Lambda \gamma_1}{\mu}\lambda - \frac{\varpi_k \Lambda((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{\mu} = 0.\end{aligned} \tag{29}$$

Upon simplification of equation (29), we obtain

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0, \tag{30}$$

where

$$p_1 = (\alpha_k + \mu + \eta_k) + (\rho_k + \mu + \delta_k) + \delta_1,$$
$$p_2 = \delta_1(\rho_k + \mu + \delta_k) + (\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k) + (\alpha_k + \mu + \eta_k)\delta_1 - \frac{\varpi_k \Lambda \gamma_1}{\mu}$$
$$p_3 = (\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\delta_1 - \frac{\varpi_k \Lambda((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{\mu}. \tag{31}$$

According to Routh-Hurwitz criterion, equation (30) has roots with negative real parts if

$$p_1, p_2, p_3 > 0, \quad p_1 p_2 > p_3. \tag{32}$$

Considering the coefficients p_1, p_2, and p_3, it is clear that $p_1 > 0$. In order to show that $p_2, p_3 > 0$, we first express p_2 in terms of $\mathscr{R}_{k0}$. Thus, we rewrite the equation (25) as

$$\mathscr{R}_{k0}(\alpha_k + \mu + \eta_k)\delta_1 - \frac{\varpi_k \Lambda \eta_k \gamma_2}{\mu(\rho_k + \mu + \delta_k)} = \frac{\varpi_k \Lambda \gamma_1}{\mu}. \tag{33}$$

Substituting the equation (33) in p_2, we get

$$p_2 = \delta_1(\rho_k + \mu + \delta_k) + (\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k) + \frac{\varpi_k \Lambda \eta_k \gamma_2}{\mu(\rho_k + \mu + \delta_k)} + (\alpha_k + \mu + \eta_k)\delta_1(1 - \mathscr{R}_{k0}). \tag{34}$$

Therefore, $p_2, p_3 > 0$ when $\mathscr{R}_{k0} < 1$. Also it is clear that $p_2, p_3 < 0$ when $\mathscr{R}_{k0} > 1$. Hence, $\mathscr{E}_k^0$ is locally asymptotically stable if $\mathscr{R}_{k0} < 1$ and unstable if $\mathscr{R}_{k0} > 1$. □

3.6. Global Stability of Disease-Free Equilibrium

Theorem 4. *$\mathscr{E}_k^0$ is globally asymptotically stable if $\mathscr{R}_{k0} < 1$ and unstable if $\mathscr{R}_{k0} > 1$.*

Proof. Consider the Lyapunov function,

$$\mathscr{L} = \frac{(\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)} E_k + \gamma_2 I_k + (\rho_k + \mu + \delta_k) P_k. \tag{35}$$

Taking derivative of $\mathscr{L}$, we get

$$\frac{d\mathscr{L}}{dt} = \frac{(\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)} \frac{dE_k}{dt} + \gamma_2 \frac{dI_k}{dt} + (\rho_k + \mu + \delta_k) \frac{dP_k}{dt}, \tag{36}$$

substituting the values of dE_k/dt, dI_k/dt, and dP_k/dt in equation (36), we get

$$\frac{d\mathscr{L}}{dt} = \frac{(\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2 d)}{(\alpha_k + \mu + \eta_k)} [\varpi_k P_k S - (\alpha_k + \mu + \eta_k) E_k] + \gamma_2 [\eta_k E_k - (\rho_k + \mu + \delta_k) I_k] + (\rho_k + \mu + \delta_k) \cdot [\gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k]. \tag{37}$$

Upon simplifying equation (37), we obtain

$$\frac{d\mathscr{L}}{dt} = \left(\frac{\varpi_k(\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)} S - (\rho_k + \mu + \delta_k)\delta_1 \right) P_k. \tag{38}$$

Since $S \le S^0 = \Lambda/\mu$, equation (38) can be rewritten as

$$\frac{d\mathscr{L}}{dt} \le \left(\frac{\varpi_k \Lambda(\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)\mu} - (\rho_k + \mu + \delta_k)\delta_1 \right) P_k. \tag{39}$$

Clearly $d\mathscr{L}/dt \le 0$ when $\mathscr{R}_{k0} < 1$ and $d\mathscr{L}/dt = 0$ when $P_k = 0$. Therefore, if $P_k \longrightarrow 0$ as $t \longrightarrow \infty$, then $(S, E_k, I_k, P_k, R) \longrightarrow (S^0, 0, 0, 0, 0) = (\Lambda/\mu, 0, 0, 0, 0)$ as $t \longrightarrow \infty$. Hence, $\mathscr{E}_k^0$ is the largest invariant set of $\{\mathscr{D} = \mathscr{D}_N \cup \mathscr{D}_{P_k} \subset \mathbb{R}_+^4 \times \mathbb{R}_+^1 : d\mathscr{L}/dt = 0\}$ According to LaSalle's invariance principle [13], $\mathscr{E}_k^0$ is globally asymptotically stable in $\mathscr{D}$ provided that $\mathscr{R}_{k0} < 1$. □

3.7. Existence of Endemic Equilibrium ($\mathscr{E}_k^*$) of Coffee Berry Disease. Equating the right hand side of system (1) to zero and substituting $S = S^*$, $E_k = E_k^*$, $I_k = I_k^*$, $P_k = P_k^*$, and $R = R^*$, we obtain

$$\begin{aligned} 0 &= \Lambda - (\varpi_k P_k^* + \mu) S^*, \\ 0 &= \varpi_k P_k^* S^* - (\alpha_k + \mu + \eta_k) E_k^*, \\ 0 &= \eta_k E_k^* - (\rho_k + \mu + \delta_k) I_k^*, \\ 0 &= \gamma_1 E_k^* + \gamma_2 I_k^* - \delta_1 P_k^*, \\ 0 &= \alpha_k E_k^* + \rho_k I_k^* - \mu R^*. \end{aligned} \tag{40}$$

From system (40), we solve for S^*, E_k^*, I_k^*, P_k^*, and R^* to get

$$\begin{aligned} S^* &= \frac{\Lambda}{\mu \mathscr{R}_{k0}}, \\ E_k^* &= \frac{\mu \delta_1 (\rho_k + \mu + \delta_k)(\mathscr{R}_{k0} - 1)}{\varpi_k(\gamma_1(\rho_k + \mu + \delta_k) + \gamma_2 \eta_k)}, \\ I_k^* &= \frac{\eta_k}{(\rho_k + \mu + \delta_k)} E_k^*, \\ P_k^* &= \frac{\mu(\mathscr{R}_{k0} - 1)}{\varpi_k}, \\ R^* &= \frac{\alpha_k(\rho_k + \mu + \delta_k) + \rho_k \eta_k}{(\rho_k + \mu + \delta_k)\mu} E_k^*. \end{aligned} \tag{41}$$

Thus, the following theorem hold.

Table 1: Parameter values of the model.

Parameter symbol	Value	Range	Source
Λ	0.00133/day	—	[16]
μ	0.00056/day	—	[16]
ϖ_k	0.0007954551/day	0-1.0	Assumed
α_k	0.001/day	0-1.0	Assumed
η_k	0.01/day	0-1.0	Assumed
ρ_k	0.005/day	0-1.0	Assumed
δ_k	0.0001/day	0-1.0	Assumed
γ_1	0.0587364/day	0-1.0	Assumed
γ_2	0.0487364/day	0-1.0	Assumed
δ_1	0.00900982/day	0-1.0	Assumed

Theorem 5. *There exist a unique positive* $\mathscr{E}_k^* = (S^*, E_k^*, I_k^*, P_k^*, R^*)$ *if* $\mathscr{R}_{k0} > 1$.

3.8. Local Stability of Endemic Equilibrium

Theorem 6. *The endemic equilibrium point of coffee berry disease* $\mathscr{E}_k^*$ *is locally asymptotically stable if* $\mathscr{R}_{k0} > 1$.

Proof. The Jacobian of system (1) at $\mathscr{E}_k^* = (S^*, E_k^*, I_k^*, P_k^*, R^*))$ is given by

$$J(\mathscr{E}_k^*) = \begin{bmatrix} -(\mu + \varpi_k P^*) & 0 & 0 & -\varpi_k S^* & 0 \\ \varpi_k P^* & -(\alpha_k + \mu + \eta_k) & 0 & \varpi_k S^* & 0 \\ 0 & \eta_k & -(\rho_k + \mu + \delta_k) & 0 & 0 \\ 0 & \gamma_1 & \gamma_2 & -\delta_1 & 0 \\ 0 & \alpha_k & \rho_k & 0 & -\mu \end{bmatrix}. \tag{42}$$

Clearly, from matrix (42), $\lambda_1 = -\mu$ is one of the eigenvalues. Thus, we consider the reduced matrix

$$J_1(\mathscr{E}_k^*) = \begin{bmatrix} -(\mu + \varpi_k P^*) & 0 & 0 & -\varpi_k S^* \\ \varpi_k P^* & -(\alpha_k + \mu + \eta_k) & 0 & \varpi_k S^* \\ 0 & \eta_k & -(\rho_k + \mu + \delta_k) & 0 \\ 0 & \gamma_1 & \gamma_2 & -\delta_1 \end{bmatrix}. \tag{43}$$

The trace of matrix (43) is given by

$$tr(J_1(\mathscr{E}_k^*)) = -\{(\mu + \varpi_k P^*) + (\alpha_k + \mu + \eta_k) + (\rho_k + \mu + \delta_k) + \delta_1\} < 0, \tag{44}$$

and the determinant is given by

$$\det\,(J_1(\mathscr{E}_k^*)) = \mu(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\delta_1(\mathscr{R}_{k0} - 1). \tag{45}$$

In view of equation (45), $\det\,(J_1(\mathscr{E}_k^*)) > 0$ when $\mathscr{R}_{k0} > 1$. Thus, by Routh-Hurwitz criteria, since matrix (43) has positive determinant when $\mathscr{R}_{k0} > 1$ and negative trace, it follows that the all the eigenvalues of matrix (43) have negative real parts. Therefore, $\mathscr{E}_k^*$ is locally asymptotically stable if $\mathscr{R}_{k0} > 1$ □

3.9. Global Stability of the Endemic Equilibrium Point

Theorem 7. *The endemic equilibrium point* $\mathscr{E}_k^*$ *of the system (1) is globally asymptotically stable if* $\mathscr{R}_{k0} > 1$.

Proof. Consider the following Lyapunov function

$$\begin{aligned} \mathscr{L}(S, E_k, I_k, P_k, R) = {} & \left(S - S^* \ln \frac{S}{S^*}\right) + \left(E_k - E_k^* \ln \frac{E_k}{E_k^*}\right) \\ & + \left(I_k - I_k^* \ln \frac{I_k}{I_k^*}\right) + \left(P_k - P_k^* \ln \frac{P_k}{P_k^*}\right) \\ & + \left(R - R^* \ln \frac{R}{R^*}\right). \end{aligned} \tag{46}$$

Differentiating L with respect to t, we get

$$\begin{aligned} \frac{d\mathscr{L}}{dt} = {} & \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{E_k^*}{E_k}\right)\frac{dE_k}{dt} + \left(1 - \frac{I_k^*}{I_k}\right)\frac{dI_k}{dt} \\ & + \left(1 - \frac{P_k^*}{P_k}\right)\frac{dP_k}{dt} + \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt}. \end{aligned} \tag{47}$$

Using system (1), we express equation (47) as

$$\begin{aligned} \frac{d\mathscr{L}}{dt} = {} & \left(1 - \frac{S^*}{S}\right)(\Lambda - (\varpi_k P_k + \mu)S) + \left(1 - \frac{E_k^*}{E_k}\right) \\ & \cdot (\varpi_k P_k S - (\alpha_k + \mu + \eta_k)E_k) + \left(1 - \frac{I_k^*}{I_k}\right) \\ & \cdot (\eta_k E_k - (\rho_k + \mu + \delta_k)I_k) + \left(1 - \frac{P_k^*}{P_k}\right) \\ & \cdot (\gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k) + \left(1 - \frac{R^*}{R}\right)(\alpha_k E_k + \rho_k I_k - \mu R). \end{aligned} \tag{48}$$

Rearranging system (40), we obtain

$$\begin{aligned} \Lambda &= (\varpi_k P_k^* + \mu)S^* \\ (\alpha_k + \mu + \eta_k) &= \frac{\varpi_k P_k^* S^*}{E_k^*}, \\ (\rho_k + \mu + \delta_k) &= \frac{\eta_k E_k^*}{I_k^*}, \\ \delta_1 &= \frac{\gamma_1 E_k^* + \gamma_2 I_k^*}{P_k^*}, \\ \mu &= \frac{\alpha_k E_k^* + \rho_k I_k^*}{R^*}. \end{aligned} \tag{49}$$

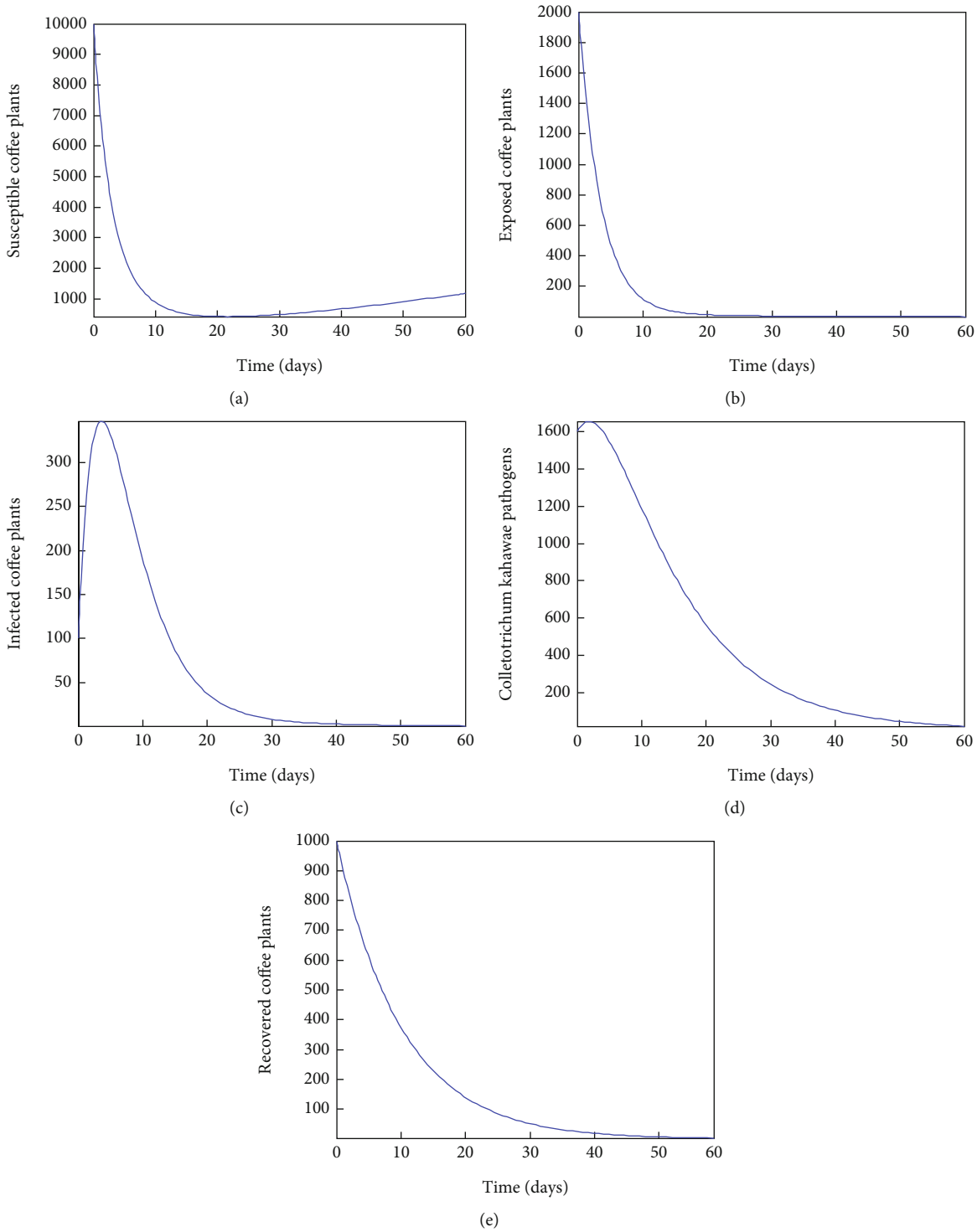

FIGURE 2: Graphs showing the dynamics of model system (1) when $\mathscr{R}_{k0} = 0.00095 < 1$ for susceptible coffee plants (a), exposed coffee plants (b), the CBD-infected coffee plants (c), *Colletotrichum kahawae* pathogen (d), and recovered coffee plants (e) based on parameter values: $\Lambda = 0.00133$, $\mu = 0.00056$, $\varpi_k = 0.0007954551$, $\alpha_k = 0.001$, $\eta_k = 0.01$, $\rho_k = 0.005$, $\delta_k = 0.0001$, $\gamma_1 = 0.0587364$, $\gamma_2 = 0.0487364$, and $\delta_1 = 0.00900982$.

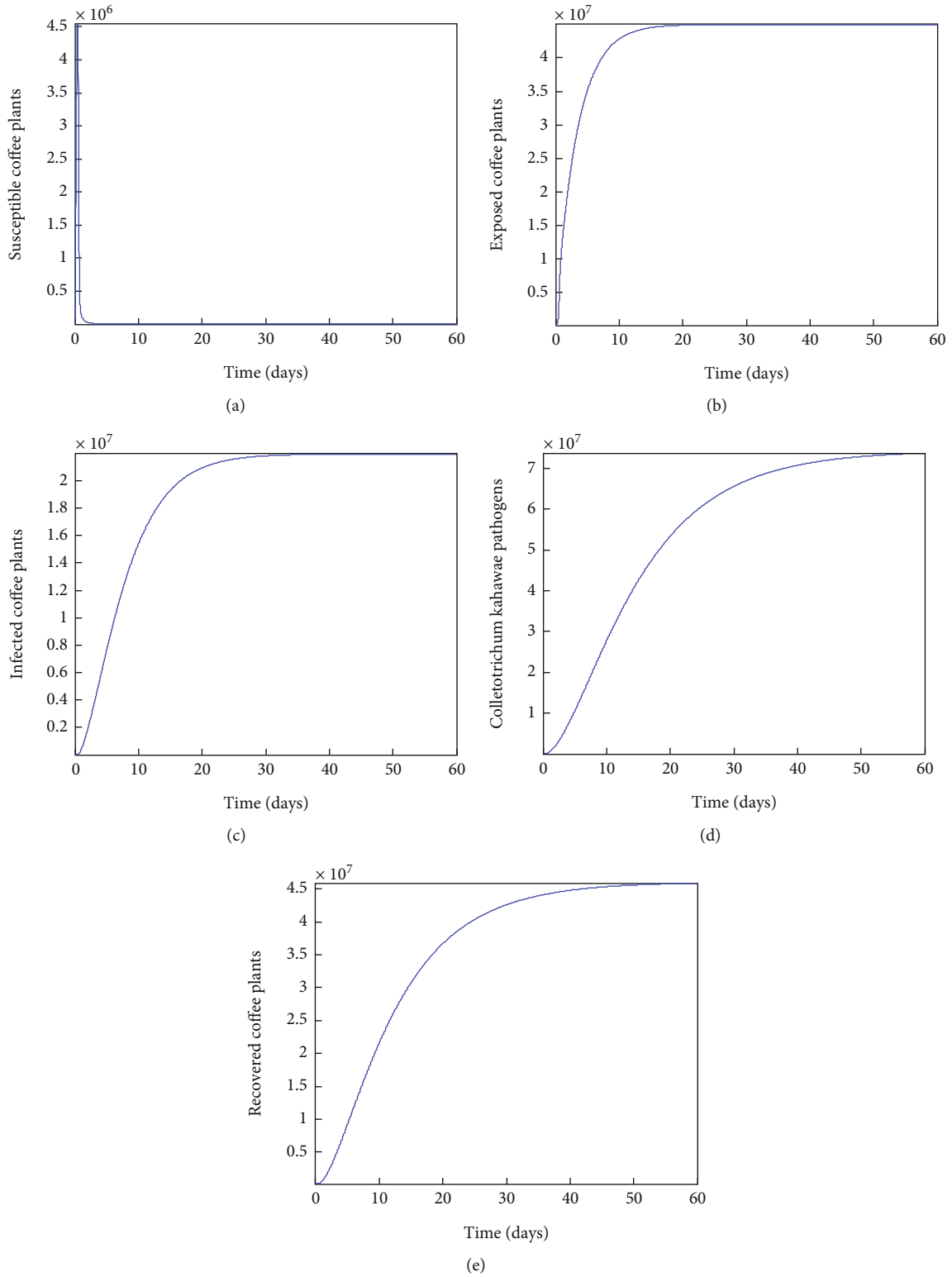

FIGURE 3: Graphs showing the dynamics of model system (1) when $\mathscr{R}_{k0} = 1.1391 > 1$ for susceptible coffee plants (a), exposed coffee plants (b), the CBD-infected coffee plants (c), *Colletotrichum kahawae* pathogen (d), and recovered coffee plants (e) based on parameter values: $\Lambda = 0.00133$, $\mu = 0.056$, $\varpi_k = 0.0007954551$, $\alpha_k = 0.1$, $\eta_k = 0.01$, $\rho_k = 0.005$, $\delta_k = 0.1$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$, and $\delta_1 = 0.0900982$.

Substituting equation (49) in (48), we get

$$\begin{aligned}\frac{d\mathscr{L}}{dt} &= \left(1-\frac{S^*}{S}\right)((\varpi_k P_k^* + \mu)S^* - (\varpi_k P_k + \mu)S) \\ &+ \left(1-\frac{E_k^*}{E_k}\right)\left(\varpi_k P_k S - \frac{\varpi_k P_k^* S^*}{E_k^*}E_k\right) + \left(1-\frac{I_k^*}{I_k}\right) \\ &\cdot \left(\eta_k E_k - \frac{\eta_k E_k^*}{I_k^*}I_k\right) + \left(1-\frac{P_k^*}{P_k}\right) \\ &\cdot \left(\gamma_1 E_k + \gamma_2 I_k - \frac{\gamma_1 E_k^* + \gamma_2 I_k^*}{P_k^*}P_k\right) \\ &+ \left(1-\frac{R^*}{R}\right)\left(\alpha_k E_k + \rho_k I_k - \frac{\alpha_k E_k^* + \rho_k I_k^*}{R^*}R\right).\end{aligned} \tag{50}$$

Equation (50) can be written as

$$\begin{aligned}\frac{d\mathscr{L}}{dt} &= -\mu\frac{(S-S^*)^2}{S} + \varpi_k P_k^* S^*\left(1-\frac{1}{w}\right) - \varpi_k P_k^* S^*(xw - x) \\ &+ \varpi_k P_k^* S^*\left(wz - x - \frac{wz}{x} + 1\right) + \eta_k E_k^*\left(x - y - \frac{x}{y} + 1\right) \\ &+ \gamma_1 E_k^*\left(x - z - \frac{x}{z} + 1\right) + \gamma_2 I_k^*\left(y - z - \frac{y}{z} + 1\right) \\ &+ \alpha_k E_k^*\left(x - q - \frac{x}{q} + 1\right) + \rho_k I_k^*\left(y - q - \frac{y}{q} + 1\right).\end{aligned} \tag{51}$$

where

$$\begin{aligned} w &= \frac{S}{S^*}, \\ x &= \frac{E_k}{E_k^*}, \\ y &= \frac{I_k}{I_k^*}, \\ z &= \frac{P_k}{P_k^*}, \\ q &= \frac{R}{R^*}. \end{aligned} \tag{52}$$

Upon simplifying equation (51), we get

$$\frac{d\mathscr{L}}{dt} = -\mu\frac{(S-S^*)^2}{S} + f(q,w,x,y,z), \tag{53}$$

where

$$\begin{aligned}f(q,w,x,y,z) &= \varpi_k P_k^* S^*\left(2 - \frac{1}{w} - wx + wz - \frac{wz}{x}\right) \\ &+ \eta_k E_k^*\left(x - y - \frac{x}{y} + 1\right) + \gamma_1 E_k^*\left(x - z - \frac{x}{z} + 1\right) \\ &+ \gamma_2 I_k^*\left(y - z - \frac{y}{z} + 1\right) + \alpha_k E_k^*\left(x - q - \frac{x}{q} + 1\right) \\ &+ \rho_k I_k^*\left(y - q - \frac{y}{q} + 1\right).\end{aligned} \tag{54}$$

Using geometric mean inequality [14], we obtain

$$\begin{aligned} &2 - \frac{1}{w} - wx + wz - \frac{wz}{x} \le 0, \\ &x - y - \frac{x}{y} + 1 \le 0, \\ &x - z - \frac{x}{z} + 1 \le 0, \\ &y - z - \frac{y}{z} + 1 \le 0, \\ &x - q - \frac{x}{q} + 1 \le 0, \\ &y - q - \frac{y}{q} + 1 \le 0. \end{aligned} \tag{55}$$

Therefore, $f(q,w,x,y,z) \le 0$, and it follows that $d\mathscr{L}/dt \le 0$ in $\mathscr{D}$. The equality $d\mathscr{L}/dt = 0$ if $q = w = x = y = z = 1$ and $S = S^*$. Hence, according to LaSalle's invariance principle [13], $\mathscr{E}_k^0$ is globally asymptotically stable in $\mathscr{D}$. □

4. Numerical Simulation

We carry out numerical simulations of the model (1), using MATLAB ode45 solver as carried out in [15]. The parameter values used are presented in Table 1, and the initial populations are taken to be $S(0) = 10000$, $E_k(0) = 2000$, $I_k(0) = 100$, $P_k(0) = 1600$, and $R(0) = 1000$.

From Figure 2, it can be seen that when $\mathscr{R}_{k0} < 1$, the number of exposed coffee plants, infected coffee plants, recovered coffee plants, and *Colletotrichum kahawae* pathogens converges to zero. However, the susceptible coffee plants tend to a constant Λ/μ. This demonstrates that only susceptible coffee plants remain after CBD infection dies out. The given CBD model system tends to the DFE which is consistent with Theorem 3. Also, Figure 3 demonstrates that the plants in various classes of CBD model converge to the endemic equilibrium point when $\mathscr{R}_{k0} > 1$, thus the CBD endemic would persist.

5. Conclusion

In this paper, we have formulated a mathematical model of the dynamics of CBD. We have calculated the equilibrium points of the system model, derived the basic reproduction number $\mathscr{R}_{k0}$, and demonstrated that the CBD dies out when $\mathscr{R}_{k0} < 1$. We have also demonstrated that CBD persists in the coffee plant population if $\mathscr{R}_{k0} > 1$. The findings of the numerical simulation are consistent with theoretical results in stability analysis.

Data Availability

All the data is within the text.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

This work was carried out in collaboration with all authors. Author DMM designed the study; author HON wrote the first draft of the manuscript and performed mathematical analysis of the study; author GWG managed literature searches; and author WNM performed numerical simulation. All authors read and approved the final manuscript.

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