Research Article

A Mathematical Model of the Dynamics of Coffee Berry Disease

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Coffee berry disease (CBD) is a fungal disease caused by Colletotrichum kahawae. CBD is a major constraint to coffee production to Kenya and Africa at large. In this research paper, we formulate a mathematical model of the dynamics of the coffee berry disease. The model consists of coffee plant population in a plantation and Colletotrichum kahawae pathogen population. We derived the basic reproduction number $R_0$, and analyzed the dynamical behaviors of both disease-free equilibrium and endemic equilibrium by the theory of ordinary differential equations. Using the MATLAB ode45 solver, we carried out numerical simulation, and the findings are consistent with the theoretical results.

1. Introduction

Coffee berry disease (CBD) is a fungal disease caused by Colletotrichum kahawae. The fungus Colletotrichum kahawae infects all stages of the coffee crop, from flowers to mature coffee berries, causing premature fruit drop and berry rot [1].

Coffee berry disease infects coffee berries (the harvestable portion of the crop), leading to direct yield loss. Also, CBD causes the pulp to adhere to the coffee bean hence making it more difficult to process and it may lower the quality of processed coffee [2].

CBD is a major constraint to coffee production in Kenya and Africa at large. The impact of CBD in Kenya was strongly felt during the 1962/1963 and 1967/1968 crop years when losses in coffee production increased to 80% [3].

According to [4], there are around 700 thousand coffee farmers in Kenya, and it is estimated that 5 million Kenyans were hired to work in the coffee production chain. This implies that CBD threatens the livelihood of millions because direct losses of the crop reduce the income.

Many mathematical models have been created to investigate the effects of preventive and control techniques on the dynamics of plant disease spread. A study for the dynamics of the transmission of plant diseases with and without roguing mechanism was carried out by [5]. The results of the study demonstrated that roguing mechanisms help in preventing the transmission of plant diseases.

The mathematical model of induced resistance to plant disease presented by [6] divides the plant population into three compartments: susceptible plants, resistant plants, and diseased plants. The outcomes of the model showed that when the elicitor application is done on plants before the inoculation of pathogens, plants are less severely affected by the diseases.

Most of the reviews presented on coffee berry disease provide qualitative studies that describe the current status and existing strategies in managing the spread and actions of the new epidemic (see for, example, [7–10]). In this paper, we investigate the dynamics of coffee berry disease.

2. Model Formulation

The coffee plants in the plantation are divided into four groups at any time $t$, namely, the susceptible coffee plants $S(t)$, coffee plants exposed to Colletotrichum kahawae (the infected coffee plants which have not shown symptoms) $E_k(t)$, the CBD-infected coffee plants $I_k(t)$ and recovered coffee plants $R(t)$. Let $N(t)$ be the total number of coffee plants, then $N(t) = S(t) + E_k(t) + I_k(t) + R(t)$. The number of
Colletotrichum kahawae pathogens in the plantation at any time $t$ is $P_k(t)$. The susceptible coffee trees are recruited at a rate of $\Lambda$. Some coffee trees in the plantation will vacate all classes due to disease-related death at a constant rate $\mu$. Susceptible coffee trees are exposed to the disease through contact with $Colletotrichum kahawae$ at a rate $\nu_k$; thus, coffee trees in $S(t)$ class will move to $E_k(t)$ class at the rate $\nu_k$. Some coffee trees in $E_k(t)$ class will progress to $I_k(t)$ at the rate $\eta_k$ and others progress to $R_k(t)$ at the rate $\alpha_k$. Also, some coffee trees in $I_k(t)$ will progress to $R_k(t)$ at the rate $\rho_k$ and others will die from CBD-induced deaths at the rate $\delta_k$. In addition, coffee trees in $E_k(t)$ and $I_k(t)$ will contribute to the increase of $P_k$ pathogens in the environment at the rates $\gamma_1$ and $\gamma_2$, respectively.

3. Well-Posedness of the Model

Since the system model (1) describes coffee plants population and $Colletotrichum kahawae$ pathogen population, it is essential to prove the well-posedness of the model solutions. Well-posedness of the model is proved by showing that the solutions with non-negative initial data are positive and bounded for all time $t > 0$ as follows.

3.1. Positivity of the Solutions of the System Model (1)

Lemma 1. Let $S_0 > 0$, $E_{k0} \geq 0$, $I_{k0} \geq 0$, $P_{k0} \geq 0$, and $R_0 \geq 0$ be the initial conditions of the system (1). Then the solutions $S$, $E_k$, $I_k$, $P_k$, and $R$ are nonnegative for all time $t > 0$.

Proof. From system (1), we define $T$ as the maximum endemic time, and it is given by

$$ T = \sup \{ t > 0 | S(t) > 0, E_k(t) \geq 0, I_k(t) \geq 0, P_k(t) \geq 0, R_k(t) \geq 0, 0 \leq r \leq t \}. $$

3.2. Model Assumptions. The following are the assumptions of the model:

(i) The fungus multiplies on the coffee plant only

(ii) There is permanent immunity upon recovery

(iii) There is disease-related death of coffee plant

(iv) There is on planting once coffee plants die (dry)

2.2. Model Flow Chart and Equations. From Figure 1, we have the following equations of the model:

$$
\begin{align*}
\frac{dS}{dt} &= \Lambda - (\alpha_k P_k + \mu)S, \\
\frac{dE_k}{dt} &= \alpha_k P_k S - (\alpha_k + \mu + \eta_k)E_k, \\
\frac{dI_k}{dt} &= \eta_k E_k - (\rho_k + \mu + \delta_k)I_k, \\
\frac{dP_k}{dt} &= \gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k, \\
\frac{dR}{dt} &= \alpha_k E_k + \rho_k I_k - \mu R.
\end{align*}
$$

(1)

Upon multiplication of both sides of equation (4) by the integrating factor, we get

$$
\frac{d}{dt} \left( S(t) \exp \left[ \int_0^t (\alpha_k P_k + \mu) (s) ds \right] \right) = \Lambda \exp \left[ \int_0^t (\alpha_k P_k + \mu) (s) ds \right].
$$

(5)
Integrating both sides of equation (5) from 0 to T, we get

$$S(T) = \exp \left[ \int_0^T (\alpha_t P_k + \mu)(\tau) d\tau \right] \left\{ S_0 + \int_0^T (\alpha_t P_k + \mu)(\tau) d\tau \right\}.$$  

(6)

Thus, $S(t) > 0 \forall t > 0$.

For the second equation of system (1), we have

$$\frac{dE_k}{dt} = \phi_k P_k S - (\alpha_k + \mu + \eta_k) E_k \geq -(\alpha_k + \mu + \eta_k) E_k \Rightarrow E_k \geq E_{k_0} \exp \left[ - \int_0^T (\alpha_k + \mu + \eta_k)(\tau) d\tau \right] \geq E_{k_0} \exp \left\{ -(\alpha_k + \mu + \eta_k) T \right\} \geq 0. $$

(7)

Hence, $E_k(t) \geq 0 \forall t > 0$.

Proving the remaining three equations in the same manner, we obtain

$$I_k(t) \geq 0, P_k(t) \geq 0, R(t) \geq 0. $$

Thus, all the solutions are non-negative $\forall t > 0$. □

3.2. Boundedness of the Solutions of the System Model (1).

We demonstrate that every feasible solution is uniformly bounded in a proper subset $\mathcal{D}$.

**Lemma 2.** Let the initial conditions of system (1) be nonnegative in $\mathbb{R}_+^4 \times \mathbb{R}_+^1$,

$$\mathcal{D}_N = \left\{ (S, E_k, I_k, R) \in \mathbb{R}_+^4 : N(t) \leq \frac{\Lambda}{\mu} \right\},$$

$$\mathcal{D}_{P_k} = \left\{ P_k \in \mathbb{R}_+^4 : P_k(t) \leq \frac{\Lambda(Y_1 + Y_2 + Y_3 + Y_4)}{\mu \delta_1} \right\}. $$

Then the set $\mathcal{D} = \mathcal{D}_N \cup \mathcal{D}_{P_k} \subset \mathbb{R}_+^4 \times \mathbb{R}_+^1$ is positively invariant.

**Proof.** In this lemma, we are required to show that $\mathcal{D}_N$ and $\mathcal{D}_{P_k}$ are positively invariant. To start, we sum the first three equations and the last equation of the system (1) to get

$$\frac{dN}{dt} = A - \mu N - \delta_1 I_k.$$  

(10)

In the absence of the CBD, we have

$$\frac{dN}{dt} \leq A - \mu N.$$  

(11)

Upon solving equation (11) for $N$, we get

$$N(t) \leq \frac{A}{\mu} + \left\{ N_0 - \frac{A}{\mu} \right\} e^{-\mu t}.$$  

(12)

Thus,

$$N(t) \leq \frac{A}{\mu} \text{ as } t \rightarrow \infty.$$  

(13)

It follows that the feasible region for the coffee plants population in the system (1) is defined by

$$\mathcal{D}_N = \left\{ (S, E_k, I_k, R) \in \mathbb{R}_+^4 : N(t) \leq \frac{\Lambda}{\mu} \right\}. $$

(14)

Considering the fourth equation of system (1), the equation for Colletotrichum kahawae pathogens is

$$\frac{dP_k}{dt} = \gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k.$$  

(15)

We rewrite it as

$$\frac{dP_k}{dt} \leq \frac{\Lambda(y_1 + y_2)}{\mu \delta_1} - \delta_1 P_k.$$  

(16)

Solving equation (16), we get

$$P_k(t) \leq \frac{\Lambda(y_1 + y_2)}{\mu \delta_1} + \left( P_{k_0} - \frac{\Lambda(y_1 + y_2)}{\mu \delta_1} \right) e^{-\delta_1 t}.$$  

(17)

Hence,

$$P_k(t) \leq \frac{\Lambda(y_1 + y_2)}{\mu \delta_1} \text{ as } t \rightarrow \infty.$$  

(18)

Consequently, the feasible region defined by the set $\mathcal{D} = \mathcal{D}_N \cup \mathcal{D}_{P_k} \subset \mathbb{R}_+^4 \times \mathbb{R}_+^1$ is positively invariant. □

It follows that every feasible solution of system (1) is uniformly bounded in $\mathcal{D}$; thus, the system is appropriate for the study of the dynamics of CBD infection.

3.3. CBD Disease-Free Equilibrium Point (DFE).

The DFE for CBD is a situation in which there is no CBD infection in the plant population. Therefore, DFE for CBD model (1) is given by

$$\mathcal{S}_{k_0}^0 = (S^0, E_k^0, I_k^0, P_k^0, R^0) = \left( \frac{\Lambda}{\mu}, 0, 0, 0, 0 \right).$$  

(19)

3.4. Reproduction Number ($R_{k_0}$). According to [11], $R_{k_0}$ is the average number of secondary infections produced by a “typical" infected plant in a completely susceptible plant population. To compute $R_{k_0}$, the next-generation method [12] is applied. Using this method, $R_{k_0}$ is given by $\rho(FV^{-1})$ (the spectral radius of $FV^{-1}$) where $F$ is the Jacobian of $\mathcal{F}_i$ at $\mathcal{S}_{k_0}^0$ and $\mathcal{F}_i$ is the rate at which new infections appear in compartment $i$, and $V$ is the Jacobian of $\mathcal{V}_i$ at $\mathcal{S}_{k_0}^0$ and $\mathcal{V}_i$ is the rate of progression of plants into and out of compartment $i$. In view of the
system model (1), the infected compartments are given by the following system:

\[
\begin{align*}
\frac{dE_k}{dt} &= \omega_k P_k S - (\alpha_k + \mu + \eta_k)E_k, \\
\frac{dI_k}{dt} &= \eta_k E_k - (\rho_k + \mu + \delta_k)I_k, \\
\frac{dP_k}{dt} &= \gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k.
\end{align*}
\]  

(20)

From the system (20), we derive

\[
\begin{align*}
\mathcal{F}_i &= \begin{bmatrix} \omega_k P_k S \\ 0 \\ 0 \end{bmatrix}, \\
\mathcal{V}_i &= \begin{bmatrix} (\alpha_k + \mu + \eta_k)E_k \\ -\eta_k E_k + (\rho_k + \mu + \delta_k)I_k \\ -\gamma_1 E_k - \gamma_2 I_k + \delta_1 P_k \end{bmatrix}.
\end{align*}
\]  

(21)

And it follows that

\[
F = \begin{bmatrix} 0 & 0 & \frac{-\omega_k A}{\mu} \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix},
\]

\[
V = \begin{bmatrix} (\alpha_k + \mu + \eta_k) & 0 & 0 \\
-\eta_k & (\rho_k + \mu + \delta_k) & 0 \\
-\gamma_1 & -\gamma_2 & \delta_1 \end{bmatrix}.
\]

The inverse of \(V\) is given by

\[
V^{-1} = \begin{bmatrix} \frac{1}{(\alpha_k + \mu + \eta_k)} & 0 & 0 \\
\frac{\eta_k}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)} & 1 & 0 \\
\frac{(\eta_k - \gamma_1)(\rho_k + \mu + \delta_k)\gamma_2}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\delta_1} & \frac{1}{\delta_1} & 0 \end{bmatrix}.
\]  

(23)

Computing the product of \(F\) and \(V^{-1}\), it obtains

\[
FV^{-1} = \begin{bmatrix} \frac{\omega_k A}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\delta_1} & \frac{\omega_k A}{(\rho_k + \mu + \delta_k)\mu \delta_1} & \frac{\omega_k A}{\mu \delta_1} \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}.
\]  

(24)

Clearly, the dominant eigenvalue of \(FV^{-1}\) is \(\omega_k A((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)/(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\mu \delta_1\). Hence,

\[
\mathcal{R}_{ko} = \frac{\omega_k A((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{(\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\mu \delta_1}.
\]  

(25)

3.5. Local Stability of the DFE

**Theorem 3.** The DFE of coffee berry disease, \(\mathcal{E}_0\), is locally asymptotically stable if \(\mathcal{R}_{ko} < 1\) and unstable if \(\mathcal{R}_{ko} > 1\).

**Proof.** If the Jacobian matrix’s eigenvalues at \(\mathcal{E}_0\) have negative real parts, \(\mathcal{E}_0\) is considered to be locally asymptotically stable. Evaluating the Jacobian matrix of system (1) at \(\mathcal{E}_0\), we get

\[
J(\mathcal{E}_0) = \begin{bmatrix} -\mu & 0 & 0 \\
0 & -\mu & 0 \\
0 & 0 & 0 \end{bmatrix}.
\]

(26)

It is clear that \(\lambda_1 = -\mu\) and \(\lambda_2 = -\mu\) are the eigenvalues of matrix (26). Thus, we reduce the matrix to get

\[
J_1(\mathcal{E}_0) = \begin{bmatrix} -\mu & 0 & 0 \\
0 & -\mu & 0 \\
0 & 0 & 0 \end{bmatrix}.
\]

(27)

To determine the eigenvalues of the matrix (27), we express it as follows

\[
-\mu \begin{bmatrix} 0 & 0 & \omega_k A \\
\eta_k & -\mu & 0 \\
\gamma_1 & \gamma_2 & -\delta_1 \end{bmatrix} = 0.
\]

(28)

From equation (28), we have the following characteristic equation

\[
[\lambda^2 + ((\alpha_k + \mu + \eta_k) + (\rho_k + \mu + \delta_k) + \delta_1)\lambda^2 + (\delta_1)(\rho_k + \mu + \delta_k) \\
+ (\alpha_k + \mu + \eta_k)((\rho_k + \mu + \delta_k) + \delta_1))\lambda + (\alpha_k + \mu + \eta_k) - \frac{\omega_k A \gamma_1}{\mu} - \frac{\omega_k A((\rho_k + \mu + \delta_k)\gamma_1 + \eta_k \gamma_2)}{\mu} = 0.
\]

(29)

Upon simplification of equation (29), we obtain

\[
\lambda^2 + \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3 = 0,
\]

(30)
where
\[ p_1 = (\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k) + \delta_1, \]
\[ p_2 = \delta_1(\rho_k + \mu + \delta_k) + (\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k) + (\alpha_k + \mu + \eta_k)\delta_1 - \frac{\omega_k \Lambda Y_1}{\mu}, \]
\[ p_3 = (\alpha_k + \mu + \eta_k)(\rho_k + \mu + \delta_k)\delta_1 - \frac{\omega_k \Lambda ((\rho_k + \mu + \delta_k)Y_1 + \eta_k Y_2)}{\mu}. \]

Substituting the equation (33) in \( R \) when
\[ R > 0 \]
we obtain
\[ \frac{dR}{dt} = \frac{\omega_k \Lambda (\rho_k + \mu + \delta_k)Y_1 + \eta_k Y_2}{(\alpha_k + \mu + \eta_k)} - (\rho_k + \mu + \delta_k)\delta_1 P_k. \]

Upon simplifying equation (37), we obtain
\[ \frac{d\mathcal{E}}{dt} = \frac{\omega_k \Lambda (\rho_k + \mu + \delta_k)Y_1 + \eta_k Y_2}{(\alpha_k + \mu + \eta_k)} \left[ \frac{(\rho_k + \mu + \delta_k)Y_1 + \eta_k Y_2}{(\alpha_k + \mu + \eta_k)} \right] \frac{[\gamma_1 E_k + \gamma_2 I_k - \delta_1 P_k]}{1}. \]

Since \( S \leq S^0 = \Lambda/\mu \), equation (38) can be rewritten as
\[ \frac{d\mathcal{E}}{dt} = \frac{\omega_k \Lambda (\rho_k + \mu + \delta_k)Y_1 + \eta_k Y_2}{(\alpha_k + \mu + \eta_k)} - (\rho_k + \mu + \delta_k)\delta_1 P_k. \]

Clearly \( d\mathcal{E}/dt \leq 0 \) when \( R < 1 \) and \( d\mathcal{E}/dt = 0 \) when \( P_k = 0 \). Therefore, if \( P_k \rightarrow 0 \) as \( t \rightarrow \infty \), then \( (S, E_k, I_k, P_k, R) \rightarrow (S^0, 0, 0, 0, 0) \) as \( t \rightarrow \infty \). Hence, \( \mathcal{E}^0 \) is the largest invariant set of \( \{ \mathcal{D} = \mathbb{D} \cup \mathbb{D}_P \subset \mathbb{R}^4 \times \mathbb{R}^1 : d\mathcal{E}/dt = 0 \} \) according to LaSalle’s invariance principle [13]. \( \mathcal{E}^0 \) is globally asymptotically stable in \( \mathbb{D} \) provided that \( R < 1 \).

3.7. Existence of Endemic Equilibrium \((\mathcal{E}^*_k)\) of Coffee Berry Disease. Equating the right hand side of system (1) to zero and substituting \( S = S^*, E_k = E^*_k, I_k = I^*_k, P_k = P^*_k \), and \( R = R^* \), we obtain
\[ 0 = \Lambda - (\omega_k P_k^* + \mu) S^*, \]
\[ 0 = \omega_k P_k^* S^* - (\alpha_k + \mu + \eta_k) E^*_k, \]
\[ 0 = \eta_k E^*_k - (\rho_k + \mu + \delta_k) I^*_k, \]
\[ 0 = \gamma_1 E^*_k + \gamma_2 I^*_k - \delta_1 P^*_k, \]
\[ 0 = \alpha_k E^*_k + \rho_k P^*_k - \mu R^*. \]

From system (40), we solve for \( S^*, E^*_k, I^*_k, P^*_k \), and \( R^* \) to get
\[ S^* = \frac{\Lambda}{\mu R^0}, \]
\[ E^*_k = \frac{\mu \delta_1 (\rho_k + \mu + \delta_k)(R^0 - 1)}{\omega_k (\gamma_1 (\rho_k + \mu + \delta_k) + \gamma_2 \eta_k)}, \]
\[ I^*_k = \frac{\eta_k (\rho_k + \mu + \delta_k) E^*_k}{(\rho_k + \mu + \delta_k)}, \]
\[ P^*_k = \frac{\mu (R^0 - 1)}{\omega_k}, \]
\[ R^* = \frac{\alpha_k (\rho_k + \mu + \delta_k) + \rho_k \eta_k E^*_k}{(\rho_k + \mu + \delta_k) \mu}. \]

Thus, the following theorem hold.
Table 1: Parameter values of the model.

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<tr>
<th>Parameter symbol</th>
<th>Value</th>
<th>Range</th>
<th>Source</th>
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<td>—</td>
<td>[16]</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.00056/day</td>
<td>—</td>
<td>[16]</td>
</tr>
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<tr>
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<td>Assumed</td>
</tr>
</tbody>
</table>

3.8. Local Stability of Endemic Equilibrium

Theorem 5. There exist a unique positive $S^*_k = (S^*_k, E^*_k, I^*_k, P^*_k, R^*_k)$ if $R_{k0} > 1$.

Proof. The Jacobian of system (1) at $S^*_k = (S^*_k, E^*_k, I^*_k, P^*_k, R^*_k)$ is given by

$$J(S^*_k) = \begin{bmatrix} -\mu - \omega_k P^* & 0 & 0 & -\omega_k S^* & 0 \\ \omega_k P^* & -\gamma_k + \mu + \eta_k & 0 & \omega_k S^* & 0 \\ 0 & \eta_k & -\rho_k + \mu + \delta_k & 0 & 0 \\ 0 & \gamma_1 & \gamma_2 & -\delta_k & 0 \\ 0 & \alpha_k & \rho_k & 0 & -\mu \end{bmatrix}$$

Clearly, from matrix (42), $\lambda_1 = -\mu$ is one of the eigenvalues. Thus, we consider the reduced matrix

$$J_1(S^*_k) = \begin{bmatrix} -\mu & 0 & 0 & -\omega_k S^* & 0 \\ \omega_k P^* & -\gamma_k + \mu + \eta_k & 0 & \omega_k S^* & 0 \\ 0 & \eta_k & -\rho_k + \mu + \delta_k & 0 & 0 \\ 0 & \gamma_1 & \gamma_2 & -\delta_k & 0 \\ 0 & \alpha_k & \rho_k & 0 & -\mu \end{bmatrix}$$

The trace of matrix (43) is given by

$$tr(J_1(S^*_k)) = -\{(\mu + \omega_k P^*) + (\alpha_k + \mu + \eta_k) + (\rho_k + \mu + \delta_k) + \delta_1\} < 0.$$  

and the determinant is given by

$$\det(J_1(S^*_k)) = \mu(\alpha_k + \gamma_k)(\rho_k + \mu + \delta_k)\delta_1(R_{k0} - 1).$$  

In view of equation (45), $\det(J_1(S^*_k)) > 0$ when $R_{k0} > 1$. Thus, by Routh-Hurwitz criteria, since matrix (43) has positive determinant when $R_{k0} > 1$ and negative trace, it follows that the all the eigenvalues of matrix (43) have negative real parts. Therefore, $S^*_k$ is locally asymptotically stable if $R_{k0} > 1$.

3.9. Global Stability of the Endemic Equilibrium Point

Theorem 7. The endemic equilibrium point $S^*_k$ of the system (1) is globally asymptotically stable if $R_{k0} > 1$.

Proof. Consider the following Lyapunov function

$$L(S, E_k, I_k, P_k, R_k) = \left(1 - \frac{S}{S^*}\ln \frac{S}{S^*}\right) + \left(1 - \frac{E_k}{E_k^*}\ln \frac{E_k}{E_k^*}\right) + \left(1 - \frac{I_k}{I_k^*}\ln \frac{I_k}{I_k^*}\right) + \left(1 - \frac{P_k}{P_k^*}\ln \frac{P_k}{P_k^*}\right) + \left(1 - \frac{R_k}{R_k^*}\ln \frac{R_k}{R_k^*}\right).$$

Differentiating $L$ with respect to $t$, we get

$$\frac{dL}{dt} = \left(1 - \frac{S}{S^*}\right) \frac{dS}{dt} + \left(1 - \frac{E_k}{E_k^*}\right) \frac{dE_k}{dt} + \left(1 - \frac{I_k}{I_k^*}\right) \frac{dI_k}{dt} + \left(1 - \frac{P_k}{P_k^*}\right) \frac{dP_k}{dt} + \left(1 - \frac{R_k}{R_k^*}\right) \frac{dR_k}{dt}.$$

Using system (1), we express equation (47) as

$$\frac{dL}{dt} = \left(1 - \frac{S}{S^*}\right) \left(\Lambda - (\omega_k P_k + \mu)S\right) \left(1 - \frac{E_k}{E_k^*}\right) \left(1 - \frac{I_k}{I_k^*}\right) \left(1 - \frac{P_k}{P_k^*}\right) \left(1 - \frac{R_k}{R_k^*}\right) \left(\alpha_k E_k + \rho_k I_k - \mu R\right).$$

Rearranging system (40), we obtain

$$\Lambda = (\omega_k P_k^* + \mu)S^*$$

$$\left(\alpha_k + \gamma_k\right) = \frac{\omega_k P_k^* S^*}{E_k^*},$$

$$\left(\rho_k + \delta_k\right) = \frac{\eta_k E_k^*}{I_k^*},$$

$$\delta_1 = \frac{\gamma_1 E_k^* + \gamma_2 I_k^*}{P_k^*},$$

$$\mu = \frac{\alpha_k E_k^* + \rho_k I_k^*}{R_k^*}.$$
Figure 2: Graphs showing the dynamics of model system (1) when $R_{00} = 0.00095 < 1$ for susceptible coffee plants (a), exposed coffee plants (b), the CBD-infected coffee plants (c), Colletotrichum kahawae pathogen (d), and recovered coffee plants (e) based on parameter values: $\Lambda = 0.00133$, $\mu = 0.00056$, $\nu_k = 0.0007954551$, $\sigma_k = 0.001$, $\eta_k = 0.01$, $\rho_k = 0.005$, $\delta_k = 0.0001$, $\gamma_1 = 0.0587364$, $\gamma_2 = 0.0487364$, and $\delta_1 = 0.00900982$. 
Figure 3: Graphs showing the dynamics of model system (1) when $R_0 = 1.1391 > 1$ for susceptible coffee plants (a), exposed coffee plants (b), the CBD-infected coffee plants (c), Colletotrichum kahawae pathogen (d), and recovered coffee plants (e) based on parameter values: $\Lambda = 0.00133, \mu = 0.056, \omega_k = 0.0007954551, \alpha_k = 0.1, \eta_k = 0.01, \rho_k = 0.005, \delta_k = 0.1, \gamma_1 = 0.1, \gamma_2 = 0.1$, and $\delta_1 = 0.0900982$. 
Substituting equation (49) in (48), we get
\[
\frac{dS}{dt} = \left(1 - \frac{S^*}{S}\right)\left(\frac{\alpha x P + \mu S^* - (\alpha + \mu)S}{S}\right) + \left(1 - \frac{E^*_k}{E^*_k}\right)\left(\frac{\alpha x P + \mu S^* - (\alpha + \mu)S}{E^*_k}\right) + \left(1 - \frac{I^*_k}{I^*_k}\right)\left(\frac{\alpha x P + \mu S^* - (\alpha + \mu)S}{I^*_k}\right)
\]
\[
+ \left(\frac{\eta E^*_k - \eta E^*_k I^*_k}{I^*_k}\right) + \left(1 - \frac{P^*_k}{P^*_k}\right)\left(\frac{\alpha x P + \mu S^* - (\alpha + \mu)S}{P^*_k}\right) + \left(1 - \frac{R^*_k}{R^*_k}\right)\left(\frac{\alpha x P + \mu S^* - (\alpha + \mu)S}{R^*_k}\right).
\]

Equation (50) can be written as
\[
\frac{dS}{dt} = -\mu\left(\frac{S - S^*}{S}\right)^2 + \alpha x P^*_k S^* \left(1 - \frac{1}{w}\right) - \alpha x P^*_k S^* \left(\frac{wz - x + \frac{wz}{x} + 1}{w} + \eta E^*_k \left(\frac{x - y - 1}{y}\right) + y^*_k \left(\frac{x - z - x}{z} + 1\right) + \eta^*_k \left(\frac{y - z - y}{z} + 1\right) + \alpha E^*_k \left(\frac{x - q - x}{q} + 1\right) + \rho^*_k \left(\frac{y - q - y}{q} + 1\right).
\]

Using geometric mean inequality [14], we obtain
\[
2 - \frac{1}{w} - w + wz = \frac{wz}{x} \leq 0,
\]
\[
x - y - \frac{x}{y} + 1 \leq 0,
\]
\[
x - z - \frac{x}{z} + 1 \leq 0,
\]
\[
y - z - \frac{y}{z} + 1 \leq 0,
\]
\[
x - q - \frac{x}{q} + 1 \leq 0,
\]
\[
y - q - \frac{y}{q} + 1 \leq 0.
\]

Therefore, \( f(q, w, x, y, z) \leq 0 \), and it follows that \( dS/dt \leq 0 \) in \( \mathcal{D} \). The equality \( dS/dt = 0 \) if \( q = w = x = y = z = 1 \) and \( S = S^* \). Hence, according to LaSalle’s invariance principle [13], \( \mathcal{D}^* \) is globally asymptotically stable in \( \mathcal{D} \).

4. Numerical Simulation

We carry out numerical simulations of the model (1), using MATLAB ode45 solver as carried out in [15]. The parameter values used are presented in Table 1, and the initial populations are taken to be \( S(0) = 10000, E^*_k(0) = 2000, I^*_k(0) = 100, P^*_k(0) = 1600, \) and \( R(0) = 1000 \).

From Figure 2, it can be seen that when \( R_{\infty} < 1 \), the number of exposed coffee plants, infected coffee plants, recovered coffee plants, and Colletotrichum kahawae pathogens converges to zero. However, the susceptible coffee plants tends to a constant \( \Lambda/\mu \). This demonstrates that only susceptible coffee plants remain after CBD infection dies out. The given CBD model system tends to the DFE which is consistent with Theorem 3. Also, Figure 3 demonstrates that the plants in various classes of CBD model converge to the endemic equilibrium point when \( R_{\infty} > 1 \), thus the CBD endemic would persist.

5. Conclusion

In this paper, we have formulated a mathematical model of the dynamics of CBD. We have calculated the equilibrium points of the system model, derived the basic reproduction number \( R_{\infty} \), and demonstrated that the CBD dies out when \( R_{\infty} < 1 \). We have also demonstrated that CBD persists in the coffee plant population if \( R_{\infty} > 1 \). The findings of the numerical simulation are consistent with theoretical results in stability analysis.

Data Availability

All the data is within the text.

Conflicts of Interest

The authors declare that they have no competing interests.
Authors’ Contributions

This work was carried out in collaboration with all authors. Author DMM designed the study; author HON wrote the first draft of the manuscript and performed mathematical analysis of the study; author GWG managed literature searches; and author WNM performed numerical simulation. All authors read and approved the final manuscript.

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