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Research Article

Asymptotic Expansions for Large Degree Tangent and Apostol-Tangent Polynomials of Complex Order

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This paper provides asymptotic expansions for large values of n of tangent $T_n^\mu(z)$ and Apostol-tangent $T_n^\mu(z;\lambda)$ polynomials of complex order. The derivation is done using contour integration with the contour avoiding branch cuts.

1. Introduction

The tangent $T_n^\mu(z)$ and Apostol-tangent $T_n^\mu(z;\lambda)$ polynomials of complex order μ and complex argument z are defined by the relations

$$\left(\frac{2}{e^{2w}+1}\right)^{\mu} e^{wz} = \sum_{n=0}^{\infty} T_n^\mu(z)\frac{w^n}{n!}, |2w| < \pi, \quad (1)$$

$$\left(\frac{2}{\lambda e^{2w}+1}\right)^{\mu} e^{wz} = \sum_{n=0}^{\infty} T_n^\mu(z;\lambda)\frac{w^n}{n!}, |w| < \frac{1}{2}\left|\log\left(\frac{-1}{\lambda}\right)\right|, \quad (2)$$

where $\lambda \in \mathbb{C}\setminus\{0\}$ and log is taken to be the principal branch.

When $\mu = 1$ and $\mu = \lambda = 1$, (1) and (2) reduce to the classical tangent polynomials, respectively (see [1]).

It is worth mentioning that results obtained in [2, 3] may have potential applications in mathematics and physics. More precisely, the numerical values of the zeros of the tangent polynomials may represent important values in engineering and physics while the twisted q-analogue of tangent polynomials may be used in quantum physics, particularly in the study of quantum groups and their representation theory.

Ryoo [4] introduced a variation of tangent numbers and polynomials, known as twisted tangent numbers and polynomials, associated with the p-adic integral on $\mathbb{Z}_p$. Through his work, Ryoo presented intriguing findings and established connections related to these concepts. In addition, Ryoo [5] explored differential equations arising from the generating functions of generalized tangent polynomials and derived explicit identities for them. Furthermore, Ryoo [6] investigated the symmetry property of the deformed fermionic integral on $\mathbb{Z}_p$, which is a mathematical structure defined over a prime field. Specifically, he focused on establishing recurrence identities for tangent polynomials and alternating sums of powers of consecutive even integers within this context. These discoveries expand our knowledge and understanding of this specialized area of mathematics, providing insights into its unique properties and potential applications across different domains. Moreover, a study by Corcino et al. [7] obtained the Fourier expansion of tangent polynomials of integer order.

In this paper, the same method described by López and Temme ([8], p. 4) has been followed in deriving the asymptotic expansion which only gives a first-order approximation. C. Corcino and R. Corcino ([9], p. 2) describe a similar method and provide a first-order and second-order approximations.

2. Asymptotic Expansions

In this section, the asymptotic expansions for large values of n of tangent $T_n^{\mu}(z)$ and Apostol-tangent $T_n^{\mu}(z;\lambda)$ polynomials of complex order are derived.

2.1. Tangent Polynomials of Complex Order μ. Applying Cauchy's integral formula for derivatives to (1), we have

$$T_n^{\mu}(z) = \frac{n!}{2\pi i}\oint_C \left(\frac{2}{e^{2w}+1}\right)^{\mu} e^{wz}\frac{dw}{w^{n+1}}, \tag{3}$$

where C is a circle about 0 with radius $<\pi/2$.

We observe in (3) that the singularities at $\pm\pi i/2$ are the sources for the main asymptotic contributions. We integrate around a circle C_1 about 0 with radius π avoiding the branch cuts running from $\pm\pi i/2$ to $+\infty$ (see Figure 1). Denote the loops by $\mathscr{L}_+$ and $\mathscr{L}_-$ and the remaining part of the circle C_1 by C^*. Then, we have

$$\frac{n!}{2\pi i}\oint_{C_1} f(w)\,dw = \frac{n!}{2\pi i}\left(\int_{C^*} f(w)\,dw + \int_{\mathscr{L}_+} f(w)\,dw + \int_{\mathscr{L}_-} f(w)\,dw\right), \tag{4}$$

where $f(w)$ is the integrand on the right-hand side of (3). By the principle of deformation of paths,

$$\frac{n!}{2\pi i}\oint_{C_1} f(w)\,dw = \frac{n!}{2\pi i}\oint_{C} f(w)\,dw = T_n^{\mu}(z). \tag{5}$$

Then, (4) and (5) yield

$$T_n^{\mu}(z) = \frac{n!}{2\pi i}\left(\int_{C^*} f(w)\,dw + \int_{\mathscr{L}_+} f(w)\,dw + \int_{\mathscr{L}_-} f(w)\,dw\right). \tag{6}$$

The following lemma gives the contribution from the circular arc C^*.

Lemma 1. *The integral along C^* is $O((\pi)^{-n})$. That is,*

$$\int_{C^*} f(w)\,dw = O((\pi)^{-n}). \tag{7}$$

Proof. Taking the modulus of the integral, we have

$$\left|\int_{C^*} f(w)\,dw\right| \le \left|\left(\frac{2}{e^{2w}+1}\right)^{\mu}\frac{e^{wz}}{w^{n+1}}\right|\cdot\pi = \left|\left(\frac{2}{e^{2w}+1}\right)^{\mu}e^{wz}\right|\cdot\pi^{-n}, \tag{8}$$

for all $w \in C^*$. Since C^* does not pass any singularity, $e^{2w}+1$ is not zero. Thus,

$$\left|\left(\frac{2}{e^{2w}+1}\right)^{\mu}e^{wz}\right| \le A, \tag{9}$$

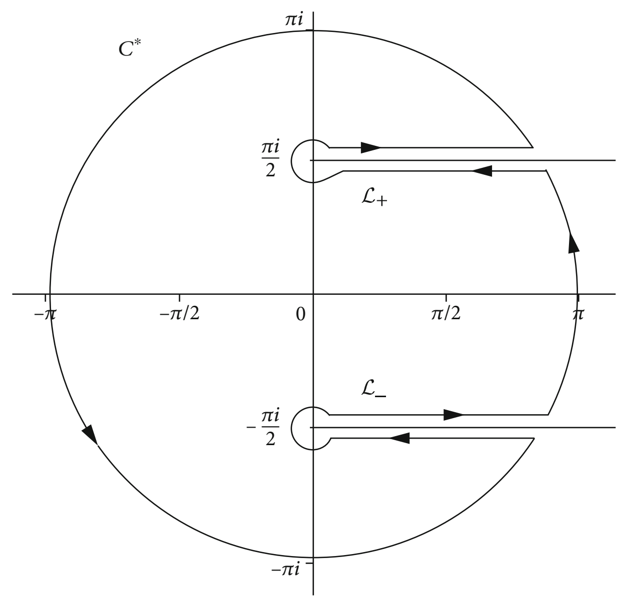

FIGURE 1: Contour for (3).

for some positive number A. So that,

$$\left|\int_{C^*} f(w)\,dw\right| \le A(\pi^{-n}). \tag{10}$$

This proves the lemma. □

Remark 2. Lemma 1 shows that, for large values of n (as $n \longrightarrow \infty$), the contribution from the circular arc C^* is exponentially small with respect to the main contributions.

For the contributions from the loops, let I_+ and I_- be the integrals along $\mathscr{L}_+$ and $\mathscr{L}_-$, respectively. We first compute the integral I_+:

$$I_+ = \frac{n!}{2\pi i}\int_{\mathscr{L}_+}\frac{2^{\mu}e^{wz}}{(e^{2w}+1)^{\mu}}\frac{dw}{w^{n+1}}. \tag{11}$$

Let $w = \pi i/2e^s$. Then, $dw = \pi i/2e^s ds$ and

$$\begin{aligned} I_+ &= \frac{n!}{2\pi i}\int_{C_+}\frac{2^{\mu}e^{\pi i e^s z/2}}{(e^{\pi i e^s}+1)^{\mu}}\frac{2^{n+1}}{(\pi i e^s)^{n+1}}\frac{\pi i e^s}{2}ds \\ &= \frac{n!2^n}{2\pi i}\int_{C_+}\frac{2^{\mu}e^{\pi i e^s z/2}}{(e^{\pi i e^s}+1)^{\mu}}\frac{ds}{(\pi i e^s)^n}, \end{aligned} \tag{12}$$

where C_+ is the image of $\mathscr{L}_+$ under the transformation $w = \pi i/2e^s$. C_+ is the contour that encircles the origin in the clockwise direction. Multiplying the last array by

$e^{-\pi i z/2} e^{\pi i z/2} (\pi i)^{\mu} (\pi i)^{-\mu}$ and since $e^{\pi i} = -1$,

$$\begin{aligned} I_+ &= \frac{n! 2^n}{2\pi i} \int_{C_+} \frac{2^{\mu} e^{\pi i e^s z/2} e^{-\pi i z/2} e^{\pi i z/2} (\pi i)^{\mu} (\pi i)^{-\mu}}{\left(e^{-\pi i}\left(e^{\pi i e^s} e^{-\pi i} - 1\right)\right)^{\mu}} \frac{ds}{\left(\pi i e^s\right)^n} \\ &= \frac{n! e^{\pi i (z/2+\mu)}}{2\pi i} \frac{2^{n+\mu}}{(\pi i)^{n+\mu}} \int_{C_+} \frac{e^{\pi i (e^s - 1) z/2} (\pi i)^{\mu}}{\left(e^{\pi i (e^s-1)} - 1\right)^{\mu}} \frac{ds}{e^{sn}} \\ &= \frac{n! e^{\pi i (z/2+\mu)}}{2\pi i} \left(\frac{2}{\pi i}\right)^{n+\mu} \int_{C_+} \left(\frac{\pi i}{e^{2\eta} - 1}\right)^{\mu} e^{\eta z} e^{-sn} ds, \end{aligned} \tag{13}$$

where $\eta = \pi i (e^s - 1)/2$. Multiplying the last array by $s^{-\mu} s^{\mu}$,

$$I_+ = \frac{n! e^{\pi i (z/2-\mu)}}{2\pi i} \left(\frac{2}{\pi i}\right)^{n+\mu} \int_{C_+} g(s) s^{-u} e^{-ns} ds, \tag{14}$$

where

$$g(s) = \left(\frac{\pi i s}{e^{2\eta} - 1}\right)^{\mu} e^{\eta z}. \tag{15}$$

To obtain an asymptotic expansion, we apply Watson's lemma for loop integrals (see [10], p. 120). We expand

$$g(s) = \sum_{k=0}^{\infty} g_k s^k. \tag{16}$$

Substituting (16)–(14), I_+ becomes

$$\begin{aligned} I_+ &= \frac{n! e^{\pi i (z/2+\mu)}}{2\pi i} \left(\frac{2}{\pi i}\right)^{n+\mu} \int_{C_+} \sum_{k=0}^{\infty} g_k s^k s^{-u} e^{-ns} ds \\ &= \frac{n! e^{\pi i (z/2+\mu)}}{2\pi i} \left(\frac{2}{\pi i}\right)^{n+\mu} \sum_{k=0}^{\infty} g_k \int_{C_+} s^{k-u} e^{-ns} ds \sim n! e^{\pi i (z/2+\mu)} \\ &\quad \cdot \left(\frac{2}{\pi i}\right)^{n+\mu} \sum_{k=0}^{\infty} g_k F_k, \end{aligned} \tag{17}$$

where

$$F_k = \frac{1}{2\pi i} \int_{C_+} s^{k-\mu} e^{-ns} ds, \tag{18}$$

with C_+ extended to $+\infty$. That is, the path of integration starts at $+\infty$ with $\arg s = 2\pi$, encircles the origin in the clockwise direction, and returns to $+\infty$, now with $\arg s = 0$.

Now, we evaluate F_k. First, we turn the path by writing $s = e^{\pi i} t$:

$$F_k = \frac{1}{2\pi i} \int_{D_+} \left(e^{\pi i} t\right)^{k-\mu} e^{nt} (-dt) = e^{\pi i (k-\mu)} \frac{-1}{2\pi i} \int_{D_+} t^{-(\mu-k)} e^{nt} dt, \tag{19}$$

where D_+ is the image of C_+ under the transformation $s = e^{\pi i} t$. D_+ is the contour that starts at $-\infty$ with arg

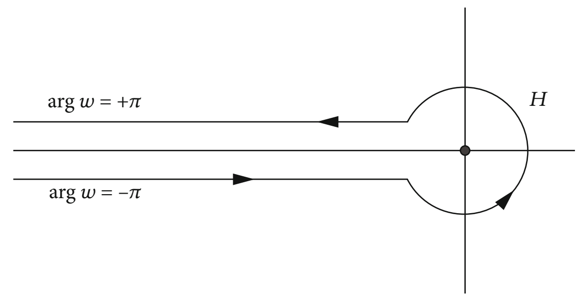

Figure 2: The Hankel contour.

$t = +\pi$, encircles the origin in clockwise direction, and returns to $-\infty$ with $\arg t = -\pi$.

We recall Hankel's loop integral representation for the reciprocal gamma function (see [11, 12], p. 48 and p. 153, respectively):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_H w^{-z} e^w dw, z \in \mathbb{C}, \tag{20}$$

where H is the *Hankel contour* (see Figure 2) that runs from $-\infty$ with $\arg w = -\pi$, encircles the origin in positive direction (that is, counterclockwise), and terminates at $-\infty$, now with $\arg w = +\pi$.

Observe that, by deformation of paths, the contour D_+ is the Hankel contour H traversed in the opposite direction. So that,

$$F_k = e^{\pi i (k-\mu)} \frac{-1}{2\pi i} \int_{-H} t^{-(\mu-k)} e^{nt} dt = e^{\pi i (k-\mu)} \frac{1}{2\pi i} \int_H t^{-(\mu-k)} e^{nt} dt. \tag{21}$$

Let $u = nt$; $du = n\, dt$. Then,

$$\begin{aligned} F_k &= e^{\pi i (k-\mu)} \frac{1}{2\pi i} \int_H \left(\frac{u}{n}\right)^{-(\mu-k)} e^u \frac{du}{n} \\ &= \frac{e^{\pi i (k-\mu)}}{n^{k-\mu+1}} \frac{1}{2\pi i} \int_H u^{-(\mu-k)} e^u du \\ &= n^{\mu-k-1} e^{-\pi i \mu} \frac{(-1)^k}{\Gamma(\mu - k)}. \end{aligned} \tag{22}$$

Moreover,

$$\frac{(-1)^k}{\Gamma(\mu-k)} = \frac{(-1)^k (\mu-1)(\mu-2)\cdots(\mu-k)}{\Gamma(\mu)} = \frac{\langle 1-\mu \rangle_k}{\Gamma(\mu)}, \tag{23}$$

where $\langle x \rangle_k = x(x+1)\cdots(x+k-1)$, the rising factorial of x of degree k. Hence,

$$F_k = n^{\mu-k-1} e^{-\pi i \mu} \frac{\langle 1-\mu \rangle_k}{\Gamma(\mu)}. \tag{24}$$

Applying (24)–(17) and noting that $i^{-1} = e^{-i\pi/2}$, we get

$$\begin{aligned} I_+ &\sim n! e^{\pi i(z/2+\mu)} \left(\frac{2}{\pi i}\right)^{n+\mu} \sum_{k=0}^{\infty} g_k \left(n^{\mu-k-1} e^{-\pi i\mu} \frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)} \right) \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{(\pi i)^{n+\mu}\Gamma(\mu)} e^{\pi i z/2} \sum_{k=0}^{\infty} g_k \frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{iz\pi/2} e^{-i(n+\mu)\pi/2} \sum_{k=0}^{\infty} g_k \frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{i[(z-\mu-n)\pi/2]} \sum_{k=0}^{\infty} g_k \frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{i\beta} \sum_{k=0}^{\infty} g_k \frac{\langle 1-\mu\rangle_k}{n^k}, \end{aligned} \tag{25}$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

Now, the integral I_- along the loop $\mathscr{L}_-$,

$$I_- = \frac{n!}{2\pi i} \int_{\mathscr{L}_-} \frac{2^\mu e^{wz}}{(e^{2w}+1)^\mu} \frac{dw}{w^{n+1}} \tag{26}$$

can be obtained similarly. After the substitution $w = (-\pi i/2)e^s$, we obtain

$$I_- = \frac{n! e^{\pi i\mu - (\pi i/2)z}}{2\pi i} \left(\frac{2}{-\pi i}\right)^{n+\mu} \int_{C_-} \bar{g}(s) s^{-u} e^{-ns} ds, \tag{27}$$

where

$$\begin{aligned} \bar{g}(s) &= \left(\frac{-\pi i s}{e^{-2\eta}-1}\right)^{\mu} e^{-\eta z}, \\ \eta &= \frac{\pi i}{2}(e^s - 1). \end{aligned} \tag{28}$$

We expand $\bar{g}(s) = \sum_{k=0}^{\infty} \bar{g}_k s^k$ and interchange the summation and integration in (27) and get

$$I_- \sim n! e^{\pi i\mu - (\pi i/2)z} \left(\frac{2}{-\pi i}\right)^{n+\mu} \sum_{k=0}^{\infty} \bar{g}_k F_k, \tag{29}$$

where $F_k{}'s$ are the integrals in (18). Applying (24) and noting that $-i^{-1} = e^{\pi i/2}$, we obtain

$$\begin{aligned} I_- &\sim \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{-i[(z-\mu-n)\pi/2]} \sum_{k=0}^{\infty} \bar{g}_k \frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{-i\beta} \sum_{k=0}^{\infty} \bar{g}_k \frac{\langle 1-\mu\rangle_k}{n^k}, \end{aligned} \tag{30}$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

We observe that $\bar{g}(s)$ is just the complex conjugate of $g(s)$ (not considering z and μ as complex numbers). So that, if we write $g_k = g_k^{(r)} + i g_k^{(i)}$ (with $g_k^{(r)}, g_k^{(i)}$ real when z and μ are real), then $\bar{g}_k = g_k^{(r)} - i g_k^{(i)}$. Hence, by Remark 2 and applying (25) and (30), we obtain

$$\begin{aligned} T_n^\mu(z) \sim I_+ + I_- &= \left(\frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{i\beta} \sum_{k=0}^{\infty} g_k \frac{\langle 1-\mu\rangle_k}{n^k} \right) \\ &\quad + \left(\frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} e^{-i\beta} \sum_{k=0}^{\infty} \bar{g}_k \frac{\langle 1-\mu\rangle_k}{n^k} \right) \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left(\sum_{k=0}^{\infty} e^{i\beta} g_k \frac{\langle 1-\mu\rangle_k}{n^k} + \sum_{k=0}^{\infty} e^{-i\beta} \bar{g}_k \frac{\langle 1-\mu\rangle_k}{n^k} \right) \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \sum_{k=0}^{\infty} \left(e^{i\beta} g_k + e^{-i\beta} \bar{g}_k \right) \frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \sum_{k=0}^{\infty} \left[\left(e^{i\beta} + e^{-i\beta} \right) g_k^{(r)} + i \left(e^{i\beta} - e^{-i\beta} \right) g_k^{(i)} \right] \frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n! 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \sum_{k=0}^{\infty} \left(2\cos\beta g_k^{(r)} - 2\sin\beta g_k^{(i)} \right) \frac{\langle 1-\mu\rangle_k}{n^k}. \end{aligned} \tag{31}$$

Consequently, we have the following theorem.

Theorem 3. *As $n \longrightarrow \infty$, μ and z are fixed complex numbers.*

$$T_n^\mu(z) \sim \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left[\cos\beta \sum_{k=0}^{\infty} \frac{\langle 1-\mu\rangle_k g_k^{(r)}}{n^k} - \sin\beta \sum_{k=0}^{\infty} \frac{\langle 1-\mu\rangle_k g_k^{(i)}}{n^k} \right], \tag{32}$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

Compute the first few values of $g_k^{(r)}$ and $g_k^{(i)}$ using Mathematica:

$$\begin{aligned} g_0^{(r)} &= 1, \\ g_0^{(i)} &= 0, \\ g_1^{(r)} &= -\frac{\mu}{2}, \\ g_1^{(i)} &= \alpha, \\ g_2^{(r)} &= \frac{1}{24}\left(-12\alpha^2 - \left(1-\pi^2\right)\mu + 3\mu^2\right), \\ g_2^{(i)} &= \frac{1}{2}(1-\mu)\alpha, \\ g_3^{(r)} &= \frac{1}{48}\left(-24\alpha^2 + \left(12\alpha^2 + 2\pi^2\right)\mu + \left(1-\pi^2\right)\mu^2 - \mu^3\right), \\ g_3^{(i)} &= -\frac{1}{24}\alpha\left(-4 + 4\alpha^2 + \left(7-\pi^2\right)\mu - 3\mu^2\right). \end{aligned} \tag{33}$$

A first-order approximation is obtained by taking $g_0^{(r)}$ and $g_0^{(i)}$ for $g_k^{(r)}$ and $g_k^{(i)}$, respectively, and taking the first term of the sum. This is given in the following theorem.

Theorem 4. *As $n \longrightarrow \infty$, μ and z are fixed complex numbers.*

$$T_n^{\mu}(z) \sim \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)}\left[\cos\beta + O\left(\frac{1}{n}\right)\right], \tag{34}$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

A second-order approximation is given as follows.

Theorem 5. *As $n \longrightarrow \infty$, μ and z are fixed complex numbers.*

$$\begin{aligned} T_n^{\mu}(z) &\sim \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)}\left\{\left[1+\frac{(1-\mu)g_1^{(r)}}{n}\right]\cos\beta - \left[\frac{(1-u)g_1^{(i)}}{n}\right]\sin\beta + O\left(\frac{1}{n^2}\right)\right\} \\ &= \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)}\left\{\left[1-\frac{(1-\mu)\mu}{2n}\right]\cos\beta - \left[\frac{(1-u)\alpha}{n}\right]\sin\beta + O\left(\frac{1}{n^2}\right)\right\}, \end{aligned} \tag{35}$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

2.2. Apostol-Tangent Polynomials of Complex Order μ. We apply the same method as in the previous subsection.

For convenience, we take $\lambda = e^{2\xi\pi i}$, where $\xi \in \mathbb{R}$ and $|\xi| < 1/2$. Then, (2) reduces to

$$\left(\frac{2}{e^{2w+2\xi\pi i}+1}\right)^{\mu} e^{wz} = \sum_{n=0}^{\infty} T_n^{\mu}\left(z; e^{2\xi\pi i}\right)\frac{w^n}{n!}, |w| < \frac{\pi}{2} - |\xi|\pi. \tag{36}$$

Applying Cauchy's integral formula for derivative to (36), we have

$$T_n^{\mu}\left(z; e^{2\xi\pi i}\right) = \frac{n!}{2\pi i}\oint_C \left(\frac{2}{e^{2w+2\xi\pi i}+1}\right)^{\mu} e^{wz}\frac{dw}{w^{n+1}}, \tag{37}$$

where C is a circle about 0 with radius $<(\pi/2) - |\xi|\pi$.

We consider (37) and observe that the singularities at $w_0 = (\pi i/2) - \xi\pi i$ and $w_{-1} = (-\pi i/2) - \xi\pi i$ are the source for the main asymptotic contribution. We integrate around a circle C_2 about 0 with radius π avoiding the branch cuts running from $(\pi i/2) - \xi\pi i$ to $+\infty$ and $(-\pi i/2) - \xi\pi i$ to $+\infty$ (see Figure 3). Denote the loops by $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$ and the remaining part of the circle C_2 by C^{**}. Then, we have

$$\frac{n!}{2\pi i}\oint_{C_2} f(w)dw = \frac{n!}{2\pi i}\left(\int_{C^{**}} f(w)dw + \int_{\mathcal{L}_+^*} f(w)dw + \int_{\mathcal{L}_-^*} f(w)dw\right), \tag{38}$$

where $f(w)$ is the integrand on the right-hand side of (37). By the principle of deformation of paths,

$$\frac{n!}{2\pi i}\oint_{C_2} f(w)dw = \frac{n!}{2\pi i}\int_C f(w)dw = T_n^{\mu}\left(z; e^{2\xi\pi i}\right). \tag{39}$$

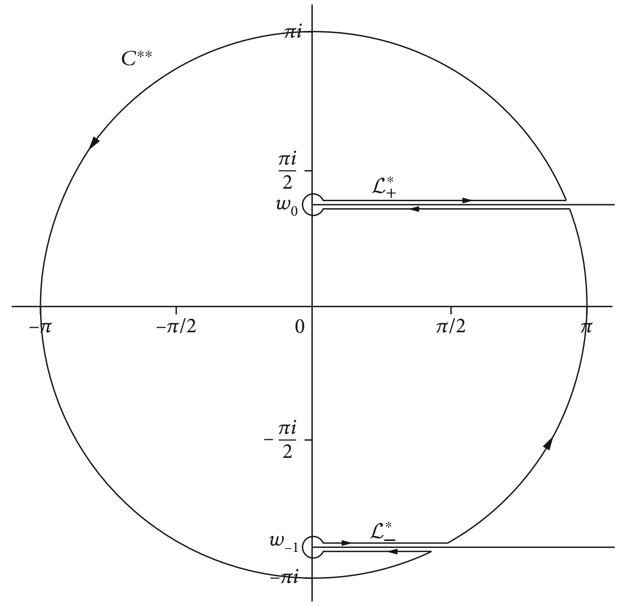

Figure 3: Contour for (37) when $\xi > 0$.

Then, (38) and (39) yield

$$T_n^{\mu}\left(z; e^{2\xi\pi i}\right) = \frac{n!}{2\pi i}\left(\int_{C^{**}} f(w)dw + \int_{\mathcal{L}_+^*} f(w)dw + \int_{\mathcal{L}_-^*} f(w)dw\right). \tag{40}$$

Remark 6. It follows from Lemma 1 that the contribution from the circular arc C^{**} is also $O((\pi)^{-n})$, so that, for large values of n (as $n \longrightarrow \infty$), it is exponentially small with respect to the main contributions.

We proceed to compute the contributions from the loops $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$. Let I_+^* be the integral along the loop $\mathcal{L}_+^*$. Then,

$$I_+^* = \frac{n!}{2\pi i}\int_{\mathcal{L}_+^*} \frac{2^{\mu} e^{wz}}{\left(e^{2w+2\xi\pi i}+1\right)^{\mu}}\frac{dw}{w^{n+1}}. \tag{41}$$

Let $w = ((\pi i/2) - \xi\pi i)e^s = 2^{-1}(\pi i - 2\xi\pi i)e^s$. Then, $dw = 2^{-1}(\pi i - 2\xi\pi i)e^s ds$ and

$$\begin{aligned} I_+^* &= \frac{n!}{2\pi i}\int_{C_+^*} \frac{2^{\mu} e^{((\pi i/2)-\xi\pi i)e^s z}}{\left(e^{2((\pi i/2)-\xi\pi i)e^s + 2\xi\pi i}+1\right)^{\mu}}\frac{2^{-1}(\pi i - 2\xi\pi i)e^s ds}{\left(2^{-1}(\pi i - 2\xi\pi i)e^s\right)^{n+1}} \\ &= \frac{n! 2^{n+\mu}}{2\pi i(\pi i - 2\xi\pi i)^n}\int_{C_+^*} \frac{e^{(\pi i/2)(1-2\xi)e^s z}}{\left(e^{\pi i e^s - 2\xi\pi i(e^s-1)}+1\right)^{\mu}}\frac{ds}{e^{sn}}, \end{aligned} \tag{42}$$

where C_+^* is the image of $\mathcal{L}_+^*$ under the transformation $w = ((\pi i/2) - \xi\pi i)e^s$. C_+^* is the contour that encircles the origin in the clockwise direction. Multiplying the last array by $e^{(\pi i/2)(1-2\xi)z}e^{(-\pi i/2)(1-2\xi)z}(\pi i-2\xi\pi i)^{\mu}(\pi i-2\xi\pi i)^{-\mu}$ and since $e^{-\pi i}=-1$,

$$\begin{aligned} I_+^* &= \frac{n!2^{n+\mu}e^{(\pi i/2)(1-2\xi)z}}{2\pi i(\pi i-2\xi\pi i)^{n+\mu}}\int_{C_+^*}\frac{e^{(\pi i/2)(1-2\xi)e^s z}e^{(-\pi i/2)(1-2\xi)z}(\pi i-2\xi\pi i)^{\mu}}{\left(e^{-\pi i}\left(e^{\pi i(e^s-1)-2\xi\pi i(e^s-1)}-1\right)\right)^{\mu}}\frac{ds}{e^{sn}} \\ &= \frac{n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}}{2\pi i}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\int_{C_+^*}\frac{e^{(\pi i/2)(1-2\xi)(e^s-1)z}(\pi i-2\xi\pi i)^{\mu}}{\left(e^{\pi i(1-2\xi)(e^s-1)}-1\right)^{\mu}}\frac{ds}{e^{sn}} \\ &= \frac{n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}}{2\pi i}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\int_{C_+^*}\left(\frac{\pi i(1-2\xi)}{e^{2\eta}-1}\right)^{\mu}e^{\eta z}e^{-sn}ds, \end{aligned} \tag{43}$$

where $\eta=(\pi i/2)(1-2\xi)(e^s-1)$. Multiplying the last array by $s^{\mu}s^{-\mu}$,

$$I_+^* = \frac{n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}}{2\pi i}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\int_{C_+^*}h(s)s^{-\mu}e^{-sn}ds, \tag{44}$$

where

$$h(s)=\left(\frac{\pi i(1-2\xi)s}{e^{2\eta}-1}\right)^{\mu}e^{\eta z}. \tag{45}$$

We expand $h(s)=\sum_{k=0}^{\infty}h_k s^k$; (49) becomes

$$\begin{aligned} I_+^* &= \frac{n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}}{2\pi i}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\int_{C_+^*}\sum_{k=0}^{\infty}h_k s^k s^{-\mu}e^{-sn}ds \\ &= \frac{n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}}{2\pi i}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\sum_{k=0}^{\infty}h_k\int_{C_+^*}s^{k-\mu}e^{-sn}ds \\ &\sim n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\sum_{k=0}^{\infty}h_k F_k, \end{aligned} \tag{46}$$

where $F_k{}'s$ are the integrals in (18).

Applying (24)–(51) and noting that $i^{-1}=e^{-i\pi/2}$, we get

$$\begin{aligned} I_+^* &\sim n!e^{\pi i\mu+(\pi i/2)(1-2\xi)z}\left(\frac{2}{\pi i(1-2\xi)}\right)^{n+\mu}\sum_{k=0}^{\infty}h_k\left(n^{\mu-k-1}e^{-\pi i\mu}\frac{\langle 1-\mu\rangle_k}{\Gamma(\mu)}\right) \\ &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi i(1-2\xi))^{n+\mu}\Gamma(\mu)}e^{i(\pi z/2)}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1-2\xi))^{n+\mu}\Gamma(\mu)}e^{i\pi z/2}e^{-i\pi/2(n+\mu)}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1-2\xi))^{n+\mu}\Gamma(\mu)}e^{i(z-\mu-n)\pi/2}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1-2\xi))^{n+\mu}\Gamma(\mu)}e^{i\beta}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{n^k}, \end{aligned} \tag{47}$$

where $\beta=\alpha-(n\pi/2)$ and $\alpha=(z-\mu)\pi/2$.

Next, let I_-^* be the integral along loop $\mathscr{L}_-^*$. Then,

$$I_-^* = \frac{n!}{2\pi i}\int_{\mathscr{L}_-^*}\frac{2^{\mu}e^{wz}}{\left(e^{2w+2\xi\pi i}+1\right)^{\mu}}\frac{dw}{w^{n+1}}. \tag{48}$$

After the substitution $w=((-\pi i/2)-\xi\pi i)e^s=2^{-1}(-\pi i-2\xi\pi i)e^s$, we obtain

$$I_-^* = \frac{n!e^{\pi i\mu+(\pi i/2)(-1-2\xi)z}}{2\pi i}\left(\frac{2}{\pi i(-1-2\xi)}\right)^{n+\mu}\int_{C_-^*}f(s)s^{-\mu}e^{-sn}ds, \tag{49}$$

where

$$\begin{aligned} f(s) &= \left(\frac{\pi i(-1-2\xi)s}{e^{2\eta}-1}\right)^{\mu}e^{\eta z}, \\ \eta &= \frac{\pi i}{2}(-1-2\xi)(e^s-1). \end{aligned} \tag{50}$$

We expand $f(s)=\sum_{k=0}^{\infty}f_k s^k$ and interchange the summation and integration in (49) and get

$$I_-^* \sim n!e^{\pi i\mu+(\pi i/2)(-1-2\xi)z}\left(\frac{2}{\pi i(-1-2\xi)}\right)^{n+\mu}\sum_{k=0}^{\infty}f_k F_k, \tag{51}$$

where $F_k{}'s$ are the integrals in (18). Applying (24) to (51) and noting that $-i^{-1}=e^{i\pi/2}$, we get

$$\begin{aligned} I_-^* &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1+2\xi))^{n+\mu}\Gamma(\mu)}e^{-i(z-\mu-n)\pi/2}\sum_{k=0}^{\infty}f_k\frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1+2\xi))^{n+\mu}\Gamma(\mu)}e^{-i\beta}\sum_{k=0}^{\infty}f_k\frac{\langle 1-\mu\rangle_k}{n^k}, \end{aligned} \tag{52}$$

where $\beta=\alpha-(n\pi/2)$ and $\alpha=(z-\mu)\pi/2$.

Then, by Remark 6 and applying (47) and (52), we obtain

$$\begin{aligned} T_n^{\mu}\left(z;e^{2\xi\pi i}\right) &\sim I_+^*+I_-^* = \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1-2\xi))^{n+\mu}\Gamma(\mu)}e^{i\beta}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{n^k} \\ &\quad+\frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{(\pi(1+2\xi))^{n+\mu}\Gamma(\mu)}e^{-i\beta}\sum_{k=0}^{\infty}f_k\frac{\langle 1-\mu\rangle_k}{n^k} \\ &= \frac{n!e^{-\xi\pi iz}2^{n+\mu}n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \\ &\quad\times\left(\frac{e^{i\beta}}{(1-2\xi)^{n+\mu}}\sum_{k=0}^{\infty}h_k\frac{\langle 1-\mu\rangle_k}{n^k}\right. \\ &\quad\left.+\frac{e^{-i\beta}}{(1+2\xi)^{n+\mu}}\sum_{k=0}^{\infty}f_k\frac{\langle 1-\mu\rangle_k}{n^k}\right). \end{aligned} \tag{53}$$

Hence, we have the following theorem.

Theorem 7. *As $n \longrightarrow \infty$, μ and z are fixed complex numbers.*

$$T_n^{\mu}\left(z; e^{2\xi\pi i}\right) \sim \frac{n! e^{-\xi\pi i z} 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \times \left(\frac{e^{i\beta}}{(1-2\xi)^{n+\mu}} \sum_{k=0}^{\infty} h_k \frac{\langle 1-\mu\rangle_k}{n^k} + \frac{e^{-i\beta}}{(1+2\xi)^{n+\mu}} \sum_{k=0}^{\infty} f_k \frac{\langle 1-\mu\rangle_k}{n^k}\right), \quad (54)$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

Remark 8. When $\xi = 0$, Theorem 7 reduces to Theorem 3.

Compute for the first few values of h_k and f_k using Mathematica:

$$\begin{aligned} h_0 &= 1, \\ h_1 &= -\frac{\mu}{2} + i\alpha(1-2\xi), \\ h_2 &= \frac{1}{24}\left(-12\alpha^2(1-2\xi)^2 + \mu\left(-1 + (1-2\xi)^2\pi^2 + 3\mu\right)\right) \\ &\quad + \frac{i}{2}\alpha(-1+2\xi)(-1+\mu), \end{aligned} \quad (55)$$

$$\begin{aligned} f_0 &= 1, \\ f_1 &= -\frac{\mu}{2} + i\alpha(-1-2\xi), \\ f_2 &= \frac{1}{24}\left(-12\alpha^2(1+2\xi)^2 + \mu\left(-1 + (1+2\xi)^2\pi^2 + 3\mu\right)\right) \\ &\quad + \frac{i}{2}\alpha(1+2\xi)(-1+\mu). \end{aligned} \quad (56)$$

A first-order approximation is obtained by taking h_0 and f_0 for h_k and f_k, respectively, and taking the first term of the sum. This is given in the following theorem.

Theorem 9. *As $n \longrightarrow \infty$, μ and z are fixed complex numbers.*

$$T_n^{\mu}\left(z; e^{2\xi\pi i}\right) \sim \frac{n! e^{-\xi\pi i z} 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left[\frac{e^{i\beta}(1+2\xi) + e^{-i\beta}(1-2\xi)}{\left(1-4\xi^2\right)^{n+\mu}} + O\left(\frac{1}{n}\right)\right], \quad (57)$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

Remark 10. When $\xi = 0$, Theorem 9 reduces to Theorem 4.

3. Summary

This paper derives asymptotic expansions for the tangent polynomials $T_n^{\mu}(z)$ and Apostol-tangent $T_n^{\mu}(z;\lambda)$ with complex orders. The primary objective is to approximate these polynomials effectively when n takes on large values. To accomplish this, the authors have employed a mathematical technique known as contour integration. This approach entails integrating the polynomials along specific paths in the complex plane, carefully avoiding branch cuts. By utilizing contour integration, the authors have derived expressions that offer valuable approximations for the tangent and Apostol-tangent polynomials as n becomes increasingly large. More precisely, as $n \longrightarrow \infty$, μ and z are fixed complex numbers; the tangent polynomials have the following asymptotic expansion:

$$T_n^{\mu}(z) \sim \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left[\cos\beta \sum_{k=0}^{\infty} \frac{\langle 1-\mu\rangle_k g_k^{(r)}}{n^k} - \sin\beta \sum_{k=0}^{\infty} \frac{\langle 1-\mu\rangle_k g_k^{(i)}}{n^k}\right], \quad (58)$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$. Consequently, the first-order approximation is obtained given as follows:

$$T_n^{\mu}(z) \sim \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left[\cos\beta + O\left(\frac{1}{n}\right)\right], \quad (59)$$

and the second-order approximation is given by

$$\begin{aligned} T_n^{\mu}(z) &\sim \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left\{\left[1 + \frac{(1-\mu) g_1^{(r)}}{n}\right] \cos\beta - \right. \\ &\quad \left. \times \left[\frac{(1-u) g_1^{(i)}}{n}\right] \sin\beta + O\left(\frac{1}{n^2}\right)\right\} \\ &= \frac{n! 2^{n+\mu+1} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \left\{\left[1 - \frac{(1-\mu)\mu}{2n}\right] \cos\beta - \right. \\ &\quad \left. \times \left[\frac{(1-u)\alpha}{n}\right] \sin\beta + O\left(\frac{1}{n^2}\right)\right\}. \end{aligned} \quad (60)$$

On the other hand, the asymptotic expansion for the Apostol-tangent polynomials is given as follows:

$$T_n^{\mu}\left(z; e^{2\xi\pi i}\right) \sim \frac{n! e^{-\xi\pi i z} 2^{n+\mu} n^{\mu-1}}{\pi^{n+\mu}\Gamma(\mu)} \times \left(\frac{e^{i\beta}}{(1-2\xi)^{n+\mu}} \sum_{k=0}^{\infty} h_k \frac{\langle 1-\mu\rangle_k}{n^k} + \frac{e^{-i\beta}}{(1+2\xi)^{n+\mu}} \sum_{k=0}^{\infty} f_k \frac{\langle 1-\mu\rangle_k}{n^k}\right), \quad (61)$$

where $\beta = \alpha - (n\pi/2)$ and $\alpha = (z-\mu)\pi/2$.

These findings contribute to our comprehension of the behaviors exhibited by these polynomials and can prove beneficial in various applications that necessitate approximations for significant values of n.

Data Availability

The articles used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] C. S. Ryoo, "A note on the tangent numbers and polynomials," *Advanced Studies in Theoretical Physics*, vol. 7, no. 9, pp. 447–454, 2013.

[2] C. S. Ryoo, "A numerical investigation on the zeros of the tangent polynomials," *Journal of Applied Mathematics & Informatics*, vol. 32, no. 3_4, pp. 315–322, 2014.

[3] C. S. Ryoo, "On the twisted q-tangent numbers and polynomials," *Applied Mathematical Sciences*, vol. 7, no. 99, pp. 4935–4941, 2013.

[4] C. S. Ryoo, "On the analogues of tangent numbers and polynomials associated with p-adic integral on Zp," *Applied Mathematical Sciences*, vol. 7, no. 61-64, pp. 3177–3183, 2013.

[5] C. S. Ryoo, "Explicit identities for the generalized tangent polynomials," *Nonlinear Analysis and Differential Equations*, vol. 6, no. 1, pp. 43–51, 2018.

[6] C. S. Ryoo, "A note on the symmetric properties for the tangent polynomials," *International Journal of Mathematical Analysis*, vol. 7, no. 52, pp. 2575–2581, 2013.

[7] C. B. Corcino, R. B. Corcino, and J. Casquejo, "Fourier expansion, integral representation and explicit formula at rational arguments of the tangent polynomials of higher-order," *European Journal of Pure and Applied Mathematics*, vol. 14, no. 4, pp. 1457–1466, 2021.

[8] J. L. López and N. M. Temme, "Large degree asymptotics of generalized Bernoulli and Euler polynomials," *Journal of Mathematical Analysis and Applications*, vol. 363, no. 1, pp. 197–208, 2010.

[9] C. B. Corcino and R. B. Corcino, "Approximations of Genocchi polynomials of complex order," *Asian-European Journal of Mathematics*, vol. 14, no. 5, article 2150083, 2021.

[10] F. Olver, *Asymptotics and Special Functions*, AK Peters/CRC Press, 1997.

[11] B. Simon, *A Comprehensive Course in Analysis, Part 2B*, American Mathematical Society, Providence, Rhode Island, 2015.

[12] N. M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, John Wiley & Sons, 1996.