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Research Article

Valuing Equity-Linked Death Benefits on Multiple Life with Time until Death following a K_n Distribution

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The purpose of this paper is to investigate the valuation of equity-linked death benefit contracts and the multiple life insurance on two heads based on a joint survival model. Using the exponential Wiener process assumption for the stock price process and a K_n distribution for the time until death, we provide explicit formulas for the expectation of the discounted payment of the guaranteed minimum death benefit products, and we derive closed expressions for some options and numerical illustrations. We investigate multiple life insurance based on a joint survival using the bivariate Sarmanov distribution with K_n (i.e., the Laplace transform of their density function is a ratio of two polynomials of degree at most) marginal distributions. We present analytical results of the joint-life status.

1. Introduction

Most classical insurance and bank products have experienced decrease in interest rates. This situation, due to the financial crisis, has led investors to give prominent attention in high-return products in spite of the high risks involved. Consequently, banks and insurance companies have to innovate by offering attractive products that can yield high rates or allow investors to participate in some underlying asset's benefits. To avoid unwanted market declines, this alternative can be used by stock market investors. As a result, products linked or indexed to a specific value have emerged in the insurance and banking sectors (for instance, variable annuities, guaranteed minimum death benefit (GMDB), and guaranteed minimum living benefit (GMLB)). Although these products are more attractive and meet the expectations of most investors, their valuations are difficult and require an in-depth knowledge of actuarial and financial techniques. In response, [1] proposed a new valuation methodology based on decomposing a liability into two parts (the actuarial or model part and the financial or market part) and then valuing each part individually. Assuming that the underlying stock price follows an exponential Brownian motion, [2] analysed the valuation of GMDB using discounted payments to death. Additionally, they assumed that the time to death follows an exponential distribution. Analytical formulas for options such as lookback options and surrenders based on the assumption of independence between stock price and time of death were developed. Although their results are interesting, they are less attractive from a practical perspective, because the assumptions underlying their model (e.g., the exponential Brownian motion process and exponential distribution assumptions) are merely used to simplify the model rather than to ensure its accuracy. Gerber et al. [3] improved their model by adding a jump in the diffusion process and examining their results for equity-linked death benefits. Liang et al. [4] used the same argument as [2] to estimate guarantee equity-linked contracts. Another study looked at term insurance products with equity-linked or inflation-indexed exercise periods. In addition, an analysis of parameter sensitivities has been incorporated. Deelstra and Hieber [5] approximated the distribution of the remaining lifetime by either a series of Erlang's distributions or a Laguerre series expansion to study death-linked contingent claims paying a random financial return at a random time of death in the general case where financial returns follow a regime-switching model with two-sided phase-type jumps. The literature on GMDB valuation contains several other extensions of the pioneering work of [2, 3] in other direction. For instance, the regime-switching jump volatility was considered in ([6–8]) and the references therein.

Multiple researchers have proposed different distributions due to the difficulty of finding a corresponding distribution to the time until death. For example, [9] addressed this problem by proposing a Laguerre expansion, which was also applied to the valuation of equity-linked death benefits. Results obtained were more accurate when compared to the results of the existing literature. Phase-type distributions to model human lifetimes were used when phase-type jump is incorporated into the diffusion process by [10]. In terms of matrix representation, they derived a closed analytic expression for price. Because dependency modelling is a key concept in financial and actuarial modelling, we are interested in equity-linked death benefits for multiple life scenarios. In Kim et al.'s [11] study, phase-type distributions are applied to joint-life products and to group risk ordering and pricing within a pool of insureds by exploring the properties of phase-type distributions. Moutanabbir and Abdelrahman [12] utilised the bivariate Sarmanov distribution with phase-type marginal distributions to model dependence between lifetimes. The phase-type distributions are used in [13] to model human mortality. Recently, [14] considered mixed exponential distribution and studied the problem of GMDB valuation for married couple.

In thi paper, we study the problem of GMDB by considering the mixture of Erlang's distributions for time until death and model the underlying stock price process by exponential Wiener process, on the one hand, and the problem to valuing equity-linked death benefits on multiple life based on a joint survival using the bivariate Sarmanov distribution with K_n marginal distributions, on the other hand.

The structure of this paper is as follows: the model is presented in Section 2. Section 3 describes the Erlang stopping of a Wiener process. Section 4 provides a valuation of basic options. In Section 5, multiple life insurance is discussed, followed by some numerical results in Section 6.

2. The Model

Consider the problem of GMDB rider that guarantees to the policyholder, max $(S(T_x), K)$, where T_x is the time until death random variable for a life aged x and K is the minimum guaranteed amount. Because max $(S(T_x), K) = S(T_x) +$ max $[K - S(T_x)]_+$, where max $[K - S(T_x)]_+ =$ max $(K - S(T_x), 0)$, the problem of valuing the guarantee becomes the problem of valuing a K-strike put option that is exercised at time T_x. Since T_x is a random variable, the put option is of neither the European style nor the American style. It is a life-contingent put option. Thus, we are interested in evaluating the expectation

$$\mathbb{E}\left[e^{-\delta T_x} b(S(T_x))\right], \quad (1)$$

where δ denotes a constant force of interest and $b(s)$ is an equity-indexed death benefit function. Let f_{T_x} denote the probability density function of T_x. Under the assumption that T_x is independent of the stock price $\{S(t)\}$, the above expectation is

$$\mathbb{E}\left[e^{-\delta T_x} b(S(T_x))\right] = \int_0^{+\infty} \mathbb{E}\left(e^{-\delta t}\right) \mathbb{E}[b(S(t))] f_{T_x}(t) dt. \quad (2)$$

In this paper, T_x is assumed to follow K_n distributions.

The class of K_n, $n \in \mathbb{N}$, distributions is the family of probability distributions whose Laplace transform is given by

$$\tilde{f}(s) = \frac{\lambda_\star + s\beta(s)}{\prod_{i=1}^n (s + \lambda_i)}, \quad (3)$$

where $\lambda_\star = \prod_{i=1}^n \lambda_i$, for $\lambda_i > 0, i = 1, 2, \cdots, n$, and $\beta(s) = \sum_{i=1}^{n-2} \beta_i s^i$ is a polynomial of degree $n-2$ or less. If τ is an arbitrary K_n, random variable, then the mean and variance of the interclaim time random variables are given by

$$\begin{aligned} \mathbb{E}[\tau] &= \sum_{i=1}^n \frac{1}{\lambda_i} - \frac{\beta(0)}{\lambda_\star}, \\ \operatorname{Var}[\tau] &= \sum_{i=1}^n \frac{1}{\lambda_i^2} - \frac{2\beta'(0)\lambda_\star - \beta^2(0)}{\lambda_\star^2}, \end{aligned} \quad (4)$$

respectively. The class of K_n distributions is widely used in applied probability applications (see for instance [15, 16]).

Under the assumption that T_x is independent of the stock price process $\{S(t)\}$, the problem of approximating the expectation (1) reduces to that of evaluating

$$\mathbb{E}\left[e^{-\delta\tau} b(S(\tau))\right], \quad (5)$$

where τ is an arbitrary K_n, random variable independent of $\{S(t)\}$.

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct, then using partial fractions,

$$\tilde{f}_\tau(s) = \sum_{i=1}^n \frac{a_i}{s + \lambda_i}, s \in \mathbb{C}, \quad (6)$$

where

$$a_i = \frac{(\lambda_\star - \lambda_i \beta(-\lambda_i))}{\prod_{j=1, j\neq i}^n (\lambda_j - \lambda_i)}. \quad (7)$$

This gives

$$f_\tau(t) = \sum_{i=1}^n a_i e^{-\lambda_i t} = \sum_{i=1}^n \frac{a_i}{\lambda_i} \lambda_i e^{-\lambda_i t}, t \geq 0, \quad (8)$$

which is the density function of a mixture of exponential distributions, with weights a_i/λ_i, $i = 1, \cdots, n$.

We can use the factorization

$$
\begin{aligned}
\mathbb{E}\left[e^{-\delta\tau}b(S(\tau))\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-\delta\tau}b(S(\tau))\right]\middle|\tau\right] \\
&= \int_0^{+\infty}\mathbb{E}\left[e^{-\delta t}b(S(t))\right]f_\tau(t)dt \\
&= \sum_{i=1}^n a_i\int_0^{+\infty}\mathbb{E}[b(S(t))]e^{-(\delta+\lambda_i)t}dt \\
&= \sum_{i=1}^n \frac{a_i}{\delta+\lambda_i}\int_0^{+\infty}\mathbb{E}[b(S(t))](\delta+\lambda_i)e^{-(\delta+\lambda_i)t}dt \\
&= \sum_{i=1}^n \frac{a_i}{\delta+\lambda_i}\mathbb{E}^\circ[b(S(\tau_i))], \\
f^\circ_{\tau_i}(t) &= (\delta+\lambda_i)e^{-(\delta+\lambda_i)t},\ t\ge 0.
\end{aligned}
\tag{9}
$$

Hence, the derivation formulas for

$$
\mathbb{E}^\circ[b(S(\tau_i))] \tag{10}
$$

are essential.

Let $M(\tau_i)$ denote the running maximum of the Lévy process $\{X(t)\}$ up to time τ_i. As shown in [2, 3] and [17], the random variables $M(\tau_i)$ and $X(\tau_i)-M(\tau_i)$ are independent (which is still true if $\delta=0$ (even though $M(t)$ and $[M(t)-X(t)]$ are not independent)).

The functions

$$
f^\delta_{X(\tau_i),M(\tau_i)}(x,y) = \int_0^\infty e^{-\delta t}f_{X(t),M(t)}(x,y)f_{\tau_i}(t)dt,\ \tau_i\sim T_{x_i} \tag{11}
$$

are referred to as discounted density functions; in the case of negative δ, the adjective inflated might be more appropriate.

Consider the process $\{X(t)=\mu t+\sigma W(t), t\ge 0\}$, where $W(t)$ is a standard Brownian motion and μ and $\sigma>0$ are constants. The process $X(t)$ is stopped at time τ_i. Unless stated otherwise, in this paper, α_i and β_i are two real numbers, which are the solutions of the following quadratic equation:

$$
\eta\rho^2+\mu\rho-(\delta+\lambda_i)=0, \eta=\frac{\sigma^2}{2}, \tag{12}
$$

where σ is defined as the volatility per unit of time of the process $\{X(t), t\ge 0\}$.

Let $\Delta_i^2 = 1/\eta(\lambda_i+(\delta+\mu^2/4\eta))$. We have

$$
\begin{aligned}
\alpha_i &= -\Delta_i-\frac{\mu^2}{2\eta}, \\
\beta_i &= \Delta_i-\frac{\mu^2}{2\eta}, \\
\beta_i-\alpha_i &= 2\Delta, \\
i &= 1,2,\cdots,n.
\end{aligned}
\tag{13}
$$

Proposition 1. *As in [2], for each $t>0$,*

$$
\begin{aligned}
f_{X(t)}(x) &= \frac{1}{2\sqrt{\pi\eta t}}e^{-(x-\mu t)^2/4\eta t}, \eta=\frac{\sigma^2}{2}, -\infty<x<\infty, \\
f_{M(t)}(x,y) &= \frac{1}{2\sqrt{\pi\eta t}}e^{-(x-\mu t)^2/4\eta t}-\frac{\mu}{\eta}e^{\mu y/\eta}\Phi\left(\frac{-y-\mu t}{\sqrt{2\pi\eta t}}\right)+\frac{1}{2\sqrt{\pi\eta t}}e^{\mu y/\eta-(x-\mu t)^2/4\eta t}, y\ge 0, \\
f_{X(t),M(t)}(x,y) &= \frac{2y-x}{2\sqrt{\pi\eta^3t^3}}e^{\left(\mu x-(1/2)\mu^2 t-(2y-x)^2/2t\right)/2\eta}, y\ge \max(0,x).
\end{aligned}
\tag{14}
$$

The proof can be found in books such as [18, 19].

The pdf of an inverse Gaussian (IG) random variable W with parameters $b, (b>0)$, and $\nu, (\nu>0)$, i.e., $(W\sim \mathrm{IG}(b,\nu))$, is

$$
f_W(x) = \frac{be^{b\nu}}{\sqrt{2\pi x^3}}\exp\left\{-\frac{1}{2}\left(\frac{b^2}{x}+\nu^2 x\right)\right\}1_{\{x>0\}}, \tag{15}
$$

and its nth moment is

$$
\mathbb{E}(W^n) = \left(\frac{b}{\nu}\right)^n e^{b\nu}\sqrt{\frac{2b\nu}{\pi}}K_{n-1/2}(b\nu), \tag{16}
$$

where K_p is the modified Bessel function of the third kind.

$$
K_{n-(1/2)}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\sum_{k=0}^{n-1}\frac{(n+k-1)!}{k!(n-k-1)!}(2x)^{-k}, \forall n\in\mathbb{N}, \tag{17}
$$

$$K_{-p}(x) = K_p(x). \tag{18}$$

If instead some of the $\lambda_1, \lambda_2, \cdots, \lambda_n$ are not distinct, then using partial fractions

$$\tilde{f}_\tau(s) = \frac{\prod_{i=1}^{k} \lambda_i^{n_i} + s\beta(s)}{\prod_{i=1}^{k} (s+\lambda_i)^{n_i}}, \tag{19}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct, $\lambda_\star = \prod_{i=1}^{k} n_i = n$.

Then using partial fractions,

$$\tilde{f}_\tau(s) = \frac{\prod_{i=1}^{k} \lambda_i^{n_i} + s\beta(s)}{\prod_{i=1}^{k} (s+\lambda_i)^{n_i}} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(s+\lambda_i)^j}, \tag{20}$$

where

$$a_{i,j} = \frac{1}{(n_i - j)!} \frac{\mathrm{d}^{n_i - j}}{\mathrm{d}s^{n_i - j}} \prod_{m=1, m \neq i}^{k} \left. \frac{\lambda_\star + s\beta(s)}{(s+\lambda_m)^{n_m}} \right|_{s=-\lambda_i}. \tag{21}$$

This gives

$$f_\tau(t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{i,j} \frac{t^{j-1} e^{-\lambda_i t}}{(j-1)!} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{i,j}}{\lambda_i^j} \lambda_i^j \frac{t^{j-1} e^{-\lambda_i t}}{(j-1)!}, t \geq 0, i = 1, \cdots, k, j = 1, \cdots, n_i, \tag{22}$$

which is the density function of a mixture of the Erlang distributions, with weights $a_{i,j}/\lambda_i^j$, $i = 1, \cdots, k$ and $j = 1, \cdots, n_i$.

We have

$$\begin{aligned} \mathbb{E}\left[e^{-\delta\tau} b(S(\tau))\right] &= \mathbb{E}\left[\left[e^{-\delta\tau} b(S(\tau)) \middle| \tau\right]\right] = \int_0^{+\infty} \mathbb{E}\left[e^{-\delta t} b(S(t))\right] f_\tau(t) \mathrm{d}t \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(\delta+\lambda_i)^j} \times \int_0^{+\infty} \mathbb{E}[b(S(t))] \frac{(\delta+\lambda_i)^j t^{j-1} e^{-(\delta+\lambda_i)t}}{(j-1)!} \mathrm{d}t \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(\delta+\lambda_i)^j} \mathbb{E}^\circ[b(S(\tau_i))], f_{\tau_i}^\circ(t) = \frac{(\delta+\lambda_i)^j t^{j-1} e^{-(\delta+\lambda_i)t}}{(j-1)!}, t \geq 0. \end{aligned} \tag{23}$$

Hence, this paper will derive formulas for

$$\mathbb{E}^\circ[b(S(\tau_i))], \tag{24}$$

where we will be looking at an Erlang stopping time τ_i.

3. Erlang Stopping of Exponential Wiener Process

Let $S(t)$ denote the time price at time t of a share of stock or unit of a mutual fund. We assume that

$$S(t) = S(0) e^{X(t)}, \tag{25}$$

where $X(t) = \mu t + \sigma W(t)$, where μ represents the drift per unit of time, σ is the volatility per unit of time, and $W(t)$ is the Wiener process.

Theorem 2. *Assuming τ_i is the Erlang distributed, i.e., $\tau_i \sim$ Erlang(n, λ_i), the distribution of the pair $(X(\tau_i), M(\tau_i))$ is*

$$f_{X(\tau_i),M(\tau_i)}^{\delta}(x,y) = \begin{cases} \displaystyle\sum_{k=0}^{n-2} \frac{2\lambda_i^n}{\sigma^2} \frac{(n+k-2)! e^{-\alpha_i x - (\beta_i - \alpha_i) y}}{2^k (n-1)! k! (n-k-2)! \nu^{n+k-1}} \left(\frac{2y-x}{\sigma}\right)^{n-k-1} \times 1_{\{y \geq \max(0,x)\}}, \forall n \in \mathbb{N} \setminus \{0,1\}, \\ \displaystyle\frac{2\lambda_i}{\sigma^2} e^{-\alpha_i x - (\beta_i - \alpha_i) y} 1_{\{y \geq \max(0,x)\}}, \text{if } n = 1, \end{cases} \tag{26}$$

where α_i and β_i are given by (13).

Proof.

$$\begin{aligned} f_{X(\tau_i),M(\tau_i)}^{\delta}(x,y) &= \int_0^{\infty} e^{-\delta t} f_{X(t),M(t)}(x,y) f_{\tau_i}(t) \mathrm{d}t \\ &= \int_0^{\infty} \frac{2y-x}{2\sqrt{\pi\eta^3 t^3}} e^{\left(\mu x - (1/2)\mu^2 t - (2y-x)^2/2t\right)/2\eta} \frac{\lambda_i^n t^{n-1} e^{-(\delta+\lambda_i)t}}{(n-1)!} 1_{\{y \geq \max(0,x)\}} \mathrm{d}t \\ &= \frac{\lambda_i^n e^{\mu x/2\eta}}{\eta(n-1)!} \int_0^{\infty} \frac{(2y-x)/\sqrt{2\eta}}{\sqrt{2\pi t^3}} t^{n-1} e^{-1/2\left[\left(\frac{\mu^2}{2\eta} + 2(\delta+\lambda_i)\right)t + \frac{(2y-x)^2}{2\eta t}\right]} 1_{\{y \geq \max(0,x)\}} \mathrm{d}t. \end{aligned} \tag{27}$$

□

Let $b = (2y-x)/\sqrt{2\eta}$ and $\nu = \sqrt{\mu^2 + 4\eta(\delta+\lambda_i)}/\sqrt{2\eta}$. Then,

$$\begin{aligned} \frac{\mu x}{2\eta} - b\nu &= \frac{\mu x}{2\eta} + \frac{x\sqrt{\mu^2 + 4\eta(\delta+\lambda_i)}}{2\eta} - \frac{2y\sqrt{\mu^2 + 4\eta(\delta+\lambda_i)}}{2\eta} \\ &= -\alpha_i x - (\beta_i - \alpha_i) y. \end{aligned} \tag{28}$$

We have

where $W \sim IG(b, \nu)$. Using Equation (16), for $n \in \mathbb{N} - \{0, 1\}$, we get

$$f^{\delta}_{X(\tau_i),M(\tau_i)}(x, y) = \frac{\lambda_i^n}{\eta(n-1)!} e^{-\alpha_i x-(\beta_i-\alpha_i)y} \left(\frac{b}{\nu}\right)^{n-1} e^{b\nu} \sqrt{\frac{2b\nu}{\pi}} K_{n-(3/2)}(b\nu) 1_{\{y \geq \max(0,x)\}}. \tag{30}$$

Substituting Equation (17) in Equation (30), we get the result for $n \in \mathbb{N} - \{0, 1\}$. For $n = 1$, $\mathbb{E}(W^{n-1}) = 1$, and the result follows.

Theorem 3. *Assuming τ_i is the Erlang distributed, i.e., $\tau_i \sim Erlang(n, \lambda_i)$, $f^{\delta}_{X(\tau_i)}$ and $f^{\delta}_{M(\tau_i)}$ are given, respectively, by the following:*

(1) For $n \in \mathbb{N} - \{0, 1\}$,

$$f^{\delta}_{X(\tau_i)}(x) = \begin{cases} e^{-\beta_i x} \varphi_i^n \sum_{k=0}^{n-2} c(n, k) \sum_{r=0}^{n-k-1} \frac{((\beta_i - \alpha_i/2)x)^r}{r!}, x \geq 0, \\ e^{-\alpha_i x} \varphi_i^n \sum_{k=0}^{n-2} c(n, k) \sum_{r=0}^{n-k-1} \frac{((-(\beta_i - \alpha_i)/2)x)^r}{r!}, x \leq 0, \end{cases}$$
$$f^{\delta}_{M(\tau_i)}(x) = 2\varphi_i^n \sum_{k=0}^{n-2} \sum_{r=0}^{n-k-1} c(n, k) \left(\frac{\beta_i - \alpha_i}{2}\right)^{n-k} \frac{e^{-\beta_i y}}{(-\alpha_i)^{n-k-r}} \frac{y^r}{r!}, \tag{31}$$

where

$$\varphi_i = \frac{\lambda_i}{\sigma^2}; c(n, k) = \frac{2^{2n-1}}{(\beta_i - \alpha_i)^{2n-1}} \left[\binom{n+k-2}{k} \left(1 - \frac{k}{n-1}\right)\right] 2^{-k}. \tag{32}$$

(2) For $n = 1$,

$$f^{\delta}_{X(\tau_i)}(x) = \begin{cases} \frac{2\lambda_i}{(\beta_i - \alpha_i)\sigma^2} e^{-\beta_i x}, x \geq 0, \\ \frac{2\lambda_i}{(\beta_i - \alpha_i)\sigma^2} e^{-\alpha_i x}, x \leq 0, \end{cases}$$
$$f^{\delta}_{M(\tau_i)}(x) = -\frac{2\lambda_i}{\alpha_i \sigma^2} e^{-\beta_i y}. \tag{33}$$

Remark 4. For $n = 1$, the results of Theorem 3 are those obtained in [2]. The mixture of the Erlang distributions is a dense family of distributions, which makes our results more general.

Proof. Assume $n \in \mathbb{N} - \{0, 1\}$. According to the expression of $f^{\delta}_{X(\tau_i),M(\tau_i)}$ given by Theorem 2, we have

$$\begin{aligned} f^{\delta}_{X(\tau_i)}(x) &= \int_{\max(0,x)}^{\infty} f^{\delta}_{X(\tau_i),M(\tau_i)}(x, y) dy \\ &= \sum_{k=0}^{n-2} \frac{2\lambda_i^n}{\sigma^{n-k+1}} \frac{(n+k-2)!}{2^k (n-1)! k! (n-k-2)! \nu^{n+k-1}} \\ &= \times \int_{\max(0,x)}^{\infty} e^{-\alpha_i x-(\beta_i-\alpha_i)y} (2y - x)^{n-k-1} dy. \end{aligned} \tag{34}$$

□

By changing the change of variables technique, we have

$$\int_{\max(0,x)}^{\infty} e^{-\alpha_i x-(\beta_i-\alpha_i)y} (2y - x)^{n-k-1} dy = \begin{cases} \frac{2^{n-k-1} e^{-(\alpha_i+\beta_i)(x/2)}}{(\beta_i - \alpha_i)^{n-k}} \int_{(\beta_i-\alpha_i)(x/2)}^{\infty} u^{n-k-1} e^{-u} du, x > 0, \\ \frac{2^{n-k-1} e^{-(\alpha_i+\beta_i)(x/2)}}{(\beta_i - \alpha_i)^{n-k}} \int_{-(\beta_i-\alpha_i)(x/2)}^{\infty} u^{n-k-1} e^{-u} du, x < 0. \end{cases} \tag{35}$$

With the incomplete Gamma function, we have

$$\Gamma\left(n-k, (\beta_i - \alpha_i)\frac{x}{2}\right) = \int_{(\beta_i-\alpha_i)(x/2)}^{\infty} u^{n-k-1} e^{-u} du = \Gamma(n-k) \int_{(\beta_i-\alpha_i)(x/2)}^{\infty} \frac{u^{n-k-1} e^{-u}}{\Gamma(n-k)} du = \Gamma(n-k) \sum_{r=0}^{n-k-1} \frac{e^{-(\beta_i-\alpha_i)(x/2)}((\beta_i - \alpha_i)(x/2))^r}{r!} = (n-k-1)! e^{-(\beta_i-\alpha_i)(x/2)} \sum_{r=0}^{n-k-1} \frac{((\beta_i - \alpha_i/2)x)^r}{r!},$$
$$\int_{-(\beta_i-\alpha_i)(x/2)}^{\infty} u^{n-k-1} e^{-u} du = (n-k-1)! e^{(\beta_i-\alpha_i)(x/2)} \sum_{r=0}^{n-k-1} \frac{(-(\beta_i - \alpha_i/2)x)^r}{r!}. \tag{36}$$

To have (note that $\beta_i - \alpha_i = 2\nu/\sigma$)

$$f^{\delta}_{X(\tau_i)}(x) = \begin{cases} e^{-\beta_i x}\dfrac{\lambda_i^n}{\sigma^{2n}}\displaystyle\sum_{k=0}^{n-2}\dfrac{(n+k-2)!2^{2n-k-1}(n-k-1)}{(n-1)!k!(\beta_i-\alpha_i)^{2n-1}}\sum_{r=0}^{n-k-1}\dfrac{((\beta_i-\alpha_i/2)x)^r}{r!}, x \geq 0, \\ e^{-\alpha_i x}\dfrac{\lambda_i^n}{\sigma^{2n}}\displaystyle\sum_{k=0}^{n-2}\dfrac{(n+k-2)!2^{2n-k-1}(n-k-1)}{(n-1)!k!(\beta_i-\alpha_i)^{2n-1}}\sum_{r=0}^{n-k-1}\dfrac{(-(\beta_i-\alpha_i/2)x)^r}{r!}, x \leq 0. \end{cases} \tag{37}$$

Since

$$\frac{(n+k-2)!}{(n-1)!k!} = \frac{(n+k-2)!}{(n-1)(n-2)!k!} = \frac{1}{n-1}\binom{n+k-2}{k}, \tag{38}$$

we obtain

$$f^{\delta}_{X(\tau_i)}(x) = \begin{cases} e^{-\beta_i x}\varphi_i^n\displaystyle\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\dfrac{((\beta_i-\alpha_i/2)x)^r}{r!}, x \geq 0, \\ e^{-\alpha_i x}\varphi_i^n\displaystyle\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\dfrac{(-(\beta_i-\alpha_i/2)x)^r}{r!}, x \leq 0. \end{cases} \tag{39}$$

φ_i and $c(n,k)$ are given by (32).

We also have

$$\begin{aligned} f^{\delta}_{M(\tau_i)}(y) &= \int_{\max(0,x)}^{\infty} f^{\delta}_{X(\tau_i),M(\tau_i)}(x,y)\,\mathrm{d}x \\ &= \sum_{k=0}^{n-2}\frac{2\lambda_i^n}{\sigma^{n-k+1}}\frac{(n+k-2)!}{2^k(n-1)!k!(n-k-2)!\nu^{n+k-1}} \\ &\quad\times\int_{-\infty}^{y} e^{-\alpha_i x-(\beta_i-\alpha_i)y}(2y-x)^{n-k-1}\,\mathrm{dx}, \end{aligned} \tag{40}$$

with

$$\begin{aligned} \int_{-\infty}^{y} e^{-\alpha_i x-(\beta_i-\alpha_i)y}(2y-x)^{n-k-1}dx &= e^{-(\beta_i-\alpha_i)y}\int_{-\infty}^{y} e^{-\alpha_i x}(2y-x)^{n-k-1}dx = e^{-(\beta_i+\alpha_i)y}\int_{y}^{\infty} e^{\alpha_i t}t^{n-k-1}\mathrm{dt} = \frac{e^{-(\beta_i+\alpha_i)y}}{(-\alpha_i)^{n-k}}\int_{-\alpha_i y}^{\infty} e^{-u}u^{n-k-1}du \\ &= \frac{e^{-(\beta_i+\alpha_i)y}}{(-\alpha_i)^{n-k}}(n-k-1)!e^{\alpha_i y}\sum_{r=0}^{n-k-1}\frac{(-\alpha_i y)^r}{r!} = \frac{e^{-\beta_i y}}{(-\alpha_i)^{n-k}}(n-k-1)!\sum_{r=0}^{n-k-1}\frac{(-\alpha_i y)^r}{r!}. \end{aligned} \tag{41}$$

To finally have

$$\begin{aligned} f^{\delta}_{M(\tau_i)}(y) &= \sum_{k=0}^{n-2}\sum_{r=0}^{n-k-1}\frac{2\lambda_i^n}{\sigma^{n-k+1}}\frac{(n+k-2)!(n-k-1)}{2^k(n-1)!k!\nu^{n+k-1}}\frac{e^{-\beta_i y}}{(-\alpha_i)^{n-k}}\frac{(-\alpha_i y)^r}{r!} \\ &= \sum_{k=0}^{n-2}\sum_{r=0}^{n-k-1}\binom{n+k-2}{k}\left(1-\frac{k}{n-1}\right)\frac{2\lambda_i^n}{2^k\sigma^{n-k+1}\nu^{n+k-1}}\frac{e^{-\beta_i y}}{(-\alpha_i)^{n-k-r}}\frac{y^r}{r!} \\ &= 2\left(\frac{\lambda_i}{\sigma^2}\right)^n\sum_{k=0}^{n-2}\sum_{r=0}^{n-k-1}c(n,k)\left(\frac{\beta_i-\alpha_i}{2}\right)^{n-k}\frac{e^{-\beta_i y}}{(-\alpha_i)^{n-k-r}}\frac{y^r}{r!}. \end{aligned} \tag{42}$$

For $n = 1$,

$$\begin{aligned} f^{\delta}_{X(\tau_i)}(x) &= \frac{2\lambda_i}{\sigma^2}\int_{\max(0,x)}^{\infty} e^{-\alpha_i x-(\beta_i-\alpha_i)y}\mathrm{dy} \\ &= \frac{2\lambda_i}{\sigma^2(\beta_i-\alpha_i)}e^{-\alpha_i x-(\beta_i-\alpha_i)\max(0,x)}, f^{\delta}_{M(\tau_i)}(y) \\ &= \frac{2\lambda_i}{\sigma^2}\int_{-\infty}^{y} e^{-\alpha_i x-(\beta_i-\alpha_i)y}\mathrm{dx} = -\frac{2\lambda_i}{\alpha_i\sigma^2}e^{-\beta_i y}. \end{aligned} \tag{43}$$

4. Valuation of Options

As in Section 3, we denote by $S(t)$ the time t's price of a share of stock or unit of a mutual fund. We assume

$$S(t) = S(0)e^{X(t)}, \tag{44}$$

where $X(t) = \mu t + \sigma W(t)$. It is easy to show that $\mathbb{E}(S(t)) = S(0)e^{\mathrm{vt}}$, $t \geq 0$, and $\nu = \mu + (\sigma^2/2)$.

In this section, we evaluate the expected discounted value of the payoff $b(S(\tau_i))$,

$$\mathbb{E}\left[e^{-\delta\tau_i}b(S(\tau_i))\right], \tag{45}$$

for various payoff or benefit functions $b(s)$. Under the assumption that the random variable τ_i is independent of the process $S(t)$, the expectation (45) is

$$\mathbb{E}\left[e^{-\delta\tau_i}b(S(\tau_i))\right] = \int_{-\infty}^{\infty} b(S(0)e^x)f^{\delta}_{X(\tau_i)}\mathrm{dx}. \tag{46}$$

Since we know that $\tau_i \sim \mathrm{Erlang}(n,\lambda_i)$, we have

$$\mathbb{E}\left[e^{-\delta\tau_i}b(S(\tau_i))\right] = \begin{cases} \varphi_i^n\displaystyle\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\dfrac{(\beta_i-\alpha_i)^r}{2^r r!}\int_0^{\infty} b(S(0)e^x)x^r e^{-\beta_i x}\mathrm{dx}, x \geq 0, \\ \varphi_i^n\displaystyle\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\dfrac{(-(\beta_i-\alpha_i))^r}{2^r r!}\int_{-\infty}^{0} b(S(0)e^x)x^r e^{-\alpha_i x}\mathrm{dx}, x \leq 0. \end{cases} \tag{47}$$

In the special case where $b(s) = s$, Equation (47) becomes

$$\mathbb{E}\left[e^{-\delta\tau_i}b(S(\tau_i))\right] = \begin{cases} S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!}\int_0^\infty x^r e^{-(\beta_i-1)x}\mathrm{d}x, x \ge 0, \\ S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(-(\beta_i-\alpha_i))^r}{2^r r!}\int_{-\infty}^0 x^r e^{-(\alpha_i-1)x}\mathrm{d}x, x \le 0. \end{cases} \tag{48}$$

Remark 5. If $\nu = \delta$, it is straightforward to show that $\mathbb{E}[e^{-\delta\tau_i} S(\tau_i)] = S(0)$ which is the result in the risk-neutral pricing framework, where δ represents the risk-free interest rate in the complete market.

4.1. Out-of-the-Money All-or-Nothing Call Option. The payoff function is

$$b(s) = s^m 1_{\{s>K\}}. \tag{49}$$

Here, m is a real number; $m = 0$ and $m = 1$ are two special cases of particular interest. The constant K is greater than $S(0)$; the term "out-of-the-money" means that the option, if exercised now, is worth nothing. Let

$$\theta = \ln\left(\frac{K}{S(0)}\right), \tag{50}$$

which is positive since $K > S(0)$.

Theorem 6. *If $\beta_i \ge m$, then*

$$\mathbb{E}\left[e^{-\delta\tau_i}S^m(\tau_i)1_{\{S(\tau_i)>K\}}\middle| S(0)<K\right] = S^m(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1}\sum_{p=0}^{r} \frac{(\beta_i-\alpha_i)^r}{2^r(\beta_i-m)^{r-p+1}} \times \frac{1}{p!}\left(\frac{S(0)}{K}\right)^{\beta_i-m}\ln\left(\frac{K}{S(0)}\right)^p. \tag{51}$$

Proof.

$$\mathbb{E}\left[e^{-\delta\tau_i}S^m(\tau_i)1_{\{S(\tau_i)>K\}}\middle| S(0)<K\right] = \varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!} \times \int_\theta^\infty b(S(0)e^x)x^r e^{-\beta_i x}\mathrm{d}x, \tag{52}$$

with

$$\begin{aligned}\int_\theta^\infty b(S(0)e^x)x^r e^{-\beta_i x}\mathrm{d}x &= S^m(0)\int_\theta^\infty x^r e^{-(\beta_i-m)x}\mathrm{d}x \\ &= \frac{S^m(0)}{(\beta_i-m)^{r+1}}\int_{(\beta_i-m)\theta}^\infty u^r e^{-u}\mathrm{d}u, \text{ if } \beta_i \ge m \\ &= \frac{S^m(0)}{(\beta_i-m)^{r+1}} r! e^{-(\beta_i-m)\theta}\sum_{p=0}^{r}\frac{(\beta_i-m)^p\theta^p}{p!}, \text{ if } \beta_i \ge m.\end{aligned} \tag{53}$$

□

4.2. At-the-Money All-or-Nothing Call Option. For $K = S(0)$, we have

$$\mathbb{E}\left[e^{-\delta\tau_i}S^m(\tau_i)1_{\{S(\tau_i)>K\}}\middle| S(0)=K\right] = S^m(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r(\beta_i-m)^{r+1}}. \tag{54}$$

4.3. Out-of-the-Money Call Option. The payoff function is

$$b(s) = (s-K)_+ = s1_{\{s>K\}} - K1_{\{s>K\}}. \tag{55}$$

Here, $K > S(0)$ because the option is out-of-the-money. By applying (51) with $m = 1$ and $m = 0$, we have

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta\tau_i}(S(\tau_i)-K)1_{\{S(\tau_i)>K\}}\middle| S(0)<K\right] \\ &= \varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1}\sum_{p=0}^{r} \frac{(\beta_i-\alpha_i)^r}{2^r}\frac{S^{\beta_i}(0)}{K^{\beta_i-1}} \\ &\quad\times \frac{1}{p!}\ln\left(\frac{K}{S(0)}\right)^p\left[\frac{1}{(\beta_i-1)^{r-p+1}} - \frac{1}{\beta_i^{r-p+1}}\right].\end{aligned} \tag{56}$$

4.4. At-the-Money Call Option. The payoff function is

$$b(s) = (s - S(0))_+, \tag{57}$$

which is (55) with $K = S(0)$. Thus, it follows from (54) that

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta\tau_i}(S(\tau_i)-S(0))_+\middle| S(0)=K\right] \\ &= S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r} \times \left[\frac{1}{(\beta_i-1)^{r+1}} - \frac{1}{\beta_i^{r+1}}\right].\end{aligned} \tag{58}$$

4.5. Out-of-the-Money All-or-Nothing Put Option. The payoff function is

$$b(s) = s^m 1_{\{s<K\}}. \tag{59}$$

Here, m is the real number, and $K < S(0)$ because the option is out-of-the-money. Since $\theta = \ln(K/S(0)) < 0$, it follows from the following.

Theorem 7. *If $\alpha_i \le m$, then,*

$$\begin{aligned}&\mathbb{E}\left[e^{-\delta\tau_i}S^m(\tau_i)1_{\{S(\tau_i)<K\}}\middle| S(0)>K\right] \\ &= -S^m(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1}\sum_{p=0}^{r} \frac{(-(\beta_i-\alpha_i))^r}{2^r(\alpha_i-m)^{r-p+1}} \\ &\quad\times \frac{1}{p!}\left(\frac{S(0)}{K}\right)^{\alpha_i-m}\ln\left(\frac{K}{S(0)}\right)^p.\end{aligned} \tag{60}$$

Proof.

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}S^m(\tau_i)1_{\{S(\tau_i)<K\}}\middle|S(0)>K\right] &= S^m(0)\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\frac{(-(\beta_i-\alpha_i))^r}{2^r r!}\\ &\quad\times\int_{-\infty}^{\theta}x^r e^{-(\alpha_i-m)x}\mathrm{dx},\end{aligned}\tag{61}$$

with

$$\begin{aligned}\int_{-\infty}^{\theta}x^r e^{-(\alpha_i-m)x}\mathrm{dx} &= -\frac{1}{(\alpha_i-m)^{r+1}}\int_{(\alpha_i-m)\theta}^{\infty}u^r e^{-u}du, \alpha_i\\ &\leq m = -\frac{r!}{(\alpha_i-m)^{r+1}}\left(\frac{S(0)}{K}\right)^{\alpha_i-m}\\ &\quad\cdot\sum_{p=0}^{r}\frac{((\alpha_i-m)\theta)^p}{p!}, \alpha_i\leq m.\end{aligned}\tag{62}$$

□

4.6. At-the-Money Put Option. For $K=S(0)$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}S^m(\tau_i)1_{\{S(\tau_i)<K\}}\middle|S(0)=K\right] &= -S^m(0)\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\frac{(-(\beta_i-\alpha_i))^r}{2^r(\alpha_i-m)^{r+1}}.\end{aligned}\tag{63}$$

4.7. Out-of-the-Money Put Option. The payoff function is

$$b(s)=(K-s)_+=K1_{\{s<K\}}-s1_{\{s<K\}}.\tag{64}$$

By applying (60) with $m=0$ and $m=1$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}(K-S(\tau_i))1_{\{S(\tau_i)<K\}}\middle|S(0)>K\right] &= -\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\sum_{p=0}^{r}\frac{(-(\beta_i-\alpha_i))^r}{2^r}\frac{S^{\alpha_i}(0)}{K^{\alpha_i-1}}\\ &\quad\times\frac{1}{p!}\ln\left(\frac{K}{S(0)}\right)^p\left[\frac{1}{\alpha_i^{r-p+1}}-\frac{1}{(\alpha_i-1)^{r-p+1}}\right].\end{aligned}\tag{65}$$

4.8. At-the-Money Put Option. The payoff function is

$$b(s)=(S(0)-s)_+.\tag{66}$$

By (63), we have

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}(S(0)-S(\tau_i))1_{\{S(\tau_i)<K\}}\middle|S(0)=K\right] &= -S(0)\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\frac{(-(\beta_i-\alpha_i))^r}{2^r}\\ &\quad\times\left[\frac{1}{\alpha_i^{r+1}}-\frac{1}{(\alpha_i-1)^{r+1}}\right].\end{aligned}\tag{67}$$

4.9. In-the-Money Put and Call Options

Theorem 8.

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}(K-S(\tau_i))_+\middle|S(0)<K\right] &= -S(0)\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\frac{(-(\beta_i-\alpha_i))^r}{2^r}\left[\frac{1}{\alpha_i^{r+1}}-\frac{1}{(\alpha_i-1)^{r+1}}\right]\\ &\quad+\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\sum_{p=0}^{r}(-1)^{r-p}\frac{(\beta_i-\alpha_i)^r}{2^r}\frac{1}{p!}\left(\ln\left(\frac{K}{S(0)}\right)\right)^p\\ &\quad\times\frac{K^{1-\beta_i}}{(S(0))^{-\beta_i}}\left[\frac{1}{(-\beta_i)^{r-p+1}}-\frac{1}{(1-\beta_i)^{r-p+1}}\right],\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}(S(\tau_i)-K)_+\middle|S(0)>K\right] &= S(0)\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\frac{(\beta_i-\alpha_i)^r}{2^r}\left[\frac{1}{(\beta_i-1)^{r+1}}-\frac{1}{\beta_i^{r+1}}\right]\\ &\quad+\varphi_i^n\sum_{k=0}^{n-2}c(n,k)\sum_{r=0}^{n-k-1}\sum_{p=0}^{r}(-1)^{r-p}\frac{(-(\beta_i-\alpha_i))^r}{2^r}\frac{1}{p!}\\ &\quad\times\left(\ln\left(\frac{K}{S(0)}\right)\right)^p\frac{K^{1-\alpha_i}}{(S(0))^{-\alpha_i}}\left[\frac{1}{(-\alpha_i)^{r-p+1}}-\frac{1}{(1-\alpha_i)^{r-p+1}}\right].\end{aligned}\tag{68}$$

Proof.

$$\begin{aligned}\mathbb{E}\left[e^{-\delta\tau_i}(K-S(\tau_i))_+\middle|S(0)<K\right] &= \mathbb{E}\left[e^{-\delta\tau_i}(K-S(\tau_i))1_{\{S(\tau_i)<K\}}\middle|S(0)<K\right]\\ &= \int_{-\infty}^{0}(K-S(0)e^x)f_{X(\tau_i)}^{\delta}\mathrm{dx}+\int_{0}^{\theta}(K-S(0)e^x)f_{X(\tau_i)}^{\delta}\mathrm{dx}\\ &= \mathbb{E}\left[e^{-\delta\tau_i}(K-S(\tau_i))_+\middle|S(0)=K\right]+\int_{0}^{\theta}(K-S(0)e^x)f_{X(\tau_i)}^{\delta}\mathrm{dx}.\end{aligned}\tag{69}$$

□

We have

$$\begin{aligned}
\int_0^\theta S(0)e^x f^\delta_{X(\tau_i)}\mathrm{dx} &= S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!}\int_0^\theta x^r e^{-(\beta_i-1)x}\mathrm{dx} \\
&= S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!} \\
&\quad \cdot \frac{1}{(1-\beta_i)^{r+1}}\int_0^{(1-\beta_i)\theta} u^r e^u \mathrm{du} \\
&= S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!}\frac{1}{(1-\beta_i)^{r+1}} \\
&\quad \times e^{(1-\beta_i)\theta}\sum_{p=0}^{r}(-1)^{r-p}\frac{r!}{p!}((1-\beta_i)\theta)^p \\
&= S(0)\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!}\frac{1}{(1-\beta_i)^{r+1}} \\
&\quad \times \left(\frac{K}{S(0)}\right)^{1-\beta_i}\sum_{p=0}^{r}(-1)^{r-p}\frac{r!}{p!}\left((1-\beta_i)\ln\left(\frac{K}{S(0)}\right)\right)^p,
\end{aligned}$$

$$\begin{aligned}
K\int_0^\theta f^\delta_{X(\tau_i)}\mathrm{dx} &= K\varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1} \frac{(\beta_i-\alpha_i)^r}{2^r r!}\frac{1}{(-\beta_i)^{r+1}} \\
&\quad \times \left(\frac{K}{S(0)}\right)^{-\beta_i}\sum_{p=0}^{r}(-1)^{r-p}\frac{r!}{p!}\left((-\beta_i)\ln\left(\frac{K}{S(0)}\right)\right)^p.
\end{aligned} \tag{70}$$

Hence,

$$\begin{aligned}
\int_0^\theta (K-S(0)e^x) f^\delta_{X(\tau_i)}\mathrm{dx} &= \varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1}\sum_{p=0}^{r}(-1)^{r-p} \\
&\quad \cdot \frac{(\beta_i-\alpha_i)^r}{2^r}\frac{1}{p!}\left(\ln\left(\frac{K}{S(0)}\right)\right)^p \\
&\quad \times \frac{k^{1-\beta_i}}{(S(0))^{-\beta_i}}\left[\frac{1}{(-\beta_i)^{r-p+1}}-\frac{1}{(1-\beta_i)^{r-p+1}}\right],
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta\tau_i}(S(\tau_i)-K)_+ \middle| S(0)>K\right] &= \int_\theta^0 (S(0)e^x-K) f^\delta_{X(\tau_i)}\mathrm{dx} \\
&\quad + \int_0^\infty (S(0)e^x-K) f^\delta_{X(\tau_i)}\mathrm{dx} \\
&= \mathbb{E}\left[e^{-\delta\tau_i}(S(\tau_i)-K)_+ \middle| S(0)=K\right] \\
&\quad + \int_\theta^0 (S(0)e^x-K) f^\delta_{X(\tau_i)}\mathrm{dx}.
\end{aligned} \tag{71}$$

To finally have

$$\begin{aligned}
\int_0^\theta (S(0)e^x-K) f^\delta_{X(\tau_i)}\mathrm{dx} &= \varphi_i^n \sum_{k=0}^{n-2} c(n,k) \sum_{r=0}^{n-k-1}\sum_{p=0}^{r}(-1)^{r-p}\frac{(-(\beta_i-\alpha_i))^r}{2^r}\frac{1}{p!} \\
&\quad \times \left(\ln\left(\frac{K}{S(0)}\right)\right)^p \frac{K^{1-\alpha_i}}{(S(0))^{-\alpha_i}}\left[\frac{1}{(1-\alpha_i)^{r-p+1}}-\frac{1}{(-\alpha_i)^{r-p+1}}\right].
\end{aligned} \tag{72}$$

5. Multiple Life Insurance on Two Heads

In this section, we apply K_n distributions in the context of joint-life modelling. The survival of the two lives is referred to as the status of interest or simply the status. There are two common types of status: the joint-life and the last survival status. Consider two random variables T_x and T_y which are assumed to be dependent. The random variables denote the future lifetimes of a life aged x and y, respectively. The dependence can be introduced using copulas or a common shock model. In this paper, we use the bivariate Sarmanov distribution which is given by

$$h(s,t) = f_x(s) f_y(t)\left[1+\omega\Psi(s)\Psi(t)\right], \tag{73}$$

where f_x and f_y are the marginal probability distribution functions of the future life random variables T_x and T_y, respectively. The kernel function $\{\Psi, i=x,y\}$ is assumed to be bounded and nonconstant such that $\mathbb{E}[\Psi_i(T_i)]=0$. The dependence parameter ω is a real number such that

$$1+\omega\Psi(s)\Psi(t)\geq 0, \tag{74}$$

for all $s,t\in\mathbb{R}\setminus\{0\}$. If $\omega=0$, then we have achieved independence. The choice of a suitable kernel function is very important. In the literature, the most commonly used kernel functions are as follows (see [20] for details):

(i) Farlie-Gumbel-Morgenstern (FGM) copula case: $\Psi_i(t)=1-F_i(t)$, where $F_i(t)$ is the cumulative distribution function associated to T_i

(ii) Exponential kernel case: $\Psi_i(t)=e^{\gamma_i t}-\mathbb{E}[e^{\gamma_i t}]$

(iii) The marginal kernel case: $\Psi_i(t)=e^{f_i(t)}-\mathbb{E}[e^{f_i(t)}]$

Define $\nu_i=\int_0^{+\infty} s\Psi_i()f_i(s)\mathrm{ds}$ for $i=x,y$; then, the covariance and correlation coefficient are given by

$$\begin{aligned}
\mathrm{Cov}(T_x,T_y) &= \omega\nu_x\nu_y, \\
\mathrm{Cor}(T_x,T_y) &= \frac{\omega\nu_x\nu_y}{\mathrm{Var}(T_x)\mathrm{Var}(T_y)}.
\end{aligned} \tag{75}$$

The maximum attainable correlation for a bivariate Sarmanov distribution is discussed in [21] for the different marginal distributions. In this paper, it is assumed that both T_x and T_y are following K_n with

$$\begin{aligned}
f_{T_x}(t) &= \sum_{i=1}^{k_x}\sum_{j=1}^{l_i^x} a_{i,j}^x \frac{t^{j-1}e^{-\lambda_i^x t}}{(j-1)!}, \\
f_{T_y}(t) &= \sum_{i=1}^{k_y}\sum_{j=1}^{l_i^y} a_{i,j}^y \frac{t^{j-1}e^{-\lambda_i^y t}}{(j-1)!}.
\end{aligned} \tag{76}$$

In the rest of the paper, we will be using the Erlang-type kernel function.

$$g_q(t) = \sum_{r=0}^{m_q-1} \frac{\left(\gamma_q t\right)^r}{r!} e^{-\gamma_q t}, q = x, y. \quad (77)$$

Then, the joint distribution of T_x and T_y is given as

$$\begin{aligned} h(s,t) &= f_x(s) f_y(t)\left(1+\omega c_x c_y\right) - \omega c_y f_x(s) g_x(s) f_y(t) \\ &\quad - \omega c_x f_x(s) g_y(t) f_y(t) + \omega f_x(s) g_x(s) f_y(t) g_y(t), \end{aligned} \quad (78)$$

or in a compact form

$$h(s,t) = f_x(s) f_y(t) + \omega \sum_{k,l}^{1} C_{kl} f_x(s) g_x^k(s) f_y(t) g_y^l(t), \quad (79)$$

with $g_i^0(s) = 1$ for $i = x, y$ and for all s with

$$C_{kl} = (-1)^{k+1} C_1^{1-k} C_2^{1-l} \text{ for } l \text{ and } k \text{ in } \{0, 1\}. \quad (80)$$

If both T_x and T_y follow a bivariate Sarmanov distribution, we have the following:

$$\begin{aligned} h(u,v) &= \sum_{i_x=1}^{k_x} \sum_{j_x=1}^{l_i^x} \sum_{i_y=1}^{k_y} \sum_{j_y=1}^{l_i^y} a_{i_x,j_x}^x a_{i_y,j_y}^y \frac{u^{j_x-1} v^{j_y-1} e^{-\lambda_i^y v - \lambda_i^x u}}{(j_x-1)!\left(j_y-1\right)!} \\ &\quad + \omega \sum_{k,l}^{1} C_{kl} \sum_{i_x=1}^{k_x} \sum_{j_x=1}^{l_i^x} \sum_{i_y=1}^{k_y} \sum_{j_y=1}^{l_i^y} \sum_{r_x=0}^{m_x-1} \sum_{r_y=0}^{m_y-1} \\ &\quad \times \gamma_x^{r_x} \gamma_y^{r_y} a_{i_x,j_x}^x a_{i_y,j_y}^y \frac{u^{r_x+j_x-1} v^{r_y+j_y-1} e^{-\left(\gamma_y+\lambda_i^y\right) v - \left(\gamma_x+\lambda_i^x\right) u}}{r_x! r_y! (j_x-1)!\left(j_y-1\right)!}. \end{aligned} \quad (81)$$

Theorem 9. *The CDF and survival functions follow*

$$\begin{aligned} H(s,t) &= \sum_{i_x=1}^{k_x} \sum_{j_x=1}^{l_i^x} \sum_{i_y=1}^{k_y} \sum_{j_y=1}^{l_i^y} \frac{a_{i_x,j_x}^x a_{i_y,j_y}^y}{\left(\lambda_i^x\right)^{j_x} \left(\lambda_i^y\right)^{j_y}} \left[1 - e^{-\lambda_i^x s} \sum_{q_x=0}^{j_x-1} \frac{\left(\lambda_i^x s\right)^{q_x}}{q_x!}\right] \\ &\quad \cdot \left[1 - e^{-\lambda_i^y t} \sum_{q_y=0}^{j_y-1} \frac{\left(\lambda_i^y t\right)^{q_y}}{q_y!}\right] + \omega \sum_{k,l}^{1} C_{kl} \sum_{i_x=1}^{k_x} \sum_{j_x=1}^{l_i^x} \sum_{i_y=1}^{k_y} \sum_{j_y=1}^{l_i^y} \sum_{r_x=0}^{m_x-1} \sum_{r_y=0}^{m_y-1} \\ &\quad \cdot \frac{\gamma_x^{r_x} \gamma_y^{r_y} a_{i_x,j_x}^x a_{i_y,j_y}^y}{\left(\gamma_x+\lambda_i^x\right)^{r_x+j_x} \left(\gamma_y+\lambda_i^y\right)^{r_y+j_y}} \times \begin{pmatrix} r_x+j_x-1 \\ r_x \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} r_y+j_y-1 \\ r_y \end{pmatrix} \left[1 - e^{-\left(\gamma_x+\lambda_i^x\right) s} \sum_{q_x=0}^{r_x+j_x-1} \frac{\left[\left(\gamma_x+\lambda_i^x\right) s\right]^{q_x}}{q_x!}\right] \\ &\quad \times \left[1 - e^{-\left(\gamma_y+\lambda_i^y\right) t} \sum_{q_y=0}^{r_y+j_y-1} \frac{\left[\left(\gamma_y+\lambda_i^y\right) t\right]^{q_y}}{q_y!}\right], \end{aligned} \quad (82)$$

$$\begin{aligned} \bar{H}(s,t) &= \sum_{i_x=1}^{k_x} \sum_{j_x=1}^{l_i^x} \sum_{i_y=1}^{k_y} \sum_{j_y=1}^{l_i^y} \frac{a_{i_x,j_x}^x a_{i_y,j_y}^y}{\left(\lambda_i^x\right)^{j_x} \left(\lambda_i^y\right)^{j_y}} e^{-\lambda_i^x s - \lambda_i^y t} \sum_{q_x=0}^{j_x-1} \sum_{q_y=0}^{j_y-1} \\ &\quad \cdot \frac{\left(\lambda_i^x s\right)^{q_x}}{q_x!} \frac{\left(\lambda_i^y t\right)^{q_y}}{q_y!} + \omega \sum_{k,l}^{1} C_{kl} \sum_{i_x=1}^{k_x} \sum_{j_x=1}^{l_i^x} \sum_{i_y=1}^{k_y} \sum_{j_y=1}^{l_i^y} \sum_{r_x=0}^{m_x-1} \sum_{r_y=0}^{m_y-1} \\ &\quad \cdot \frac{\gamma_x^{r_x} \gamma_y^{r_y} a_{i_x,j_x}^x a_{i_y,j_y}^y}{\left(\gamma_x+\lambda_i^x\right)^{r_x+j_x} \left(\gamma_y+\lambda_i^y\right)^{r_y+j_y}} \\ &\quad \times \begin{pmatrix} r_x+j_x-1 \\ r_x \end{pmatrix} \begin{pmatrix} r_y+j_y-1 \\ r_y \end{pmatrix} e^{-\left(\gamma_x+\lambda_i^x\right) s - \left(\gamma_y+\lambda_i^y\right) t} \\ &\quad \times \sum_{q_x=0}^{r_x+j_x-1} \sum_{q_y=0}^{r_y+j_y-1} \frac{\left[\left(\gamma_x+\lambda_i^x\right) s\right]^{q_x}}{q_x!} \frac{\left[\left(\gamma_y+\lambda_i^y\right) t\right]^{q_y}}{q_y!}. \end{aligned} \quad (83)$$

Proof.

$$H(s,t) = \mathbb{P}\left[T_x < s, T_y < t\right] = \int_0^s \int_0^t h(u,v) \mathrm{dudv}. \quad (84)$$

(i) Computing of $h_1(s,t) = \int_0^s \int_0^t u^{j_x-1} v^{j_y-1} e^{-\lambda_i^y v - \lambda_i^x u} \mathrm{dudv}$

By the Fubini theorem, we have

$$h_1(s,t) = \int_0^s u^{j_x-1} e^{-\lambda_i^x u} du \int_0^t v^{j_y-1} e^{-\lambda_i^y v} \mathrm{dv}. \quad (85)$$

Let $w = \lambda_i^x u$; we have

$$\begin{aligned} \int_0^s u^{j_x-1} e^{-\lambda_i^x u} \mathrm{du} &= \frac{1}{\left(\lambda_i^x\right)^{j_x}} \int_0^{\lambda_i^x s} w^{j_x-1} e^{-w} \mathrm{dw} \\ &= \frac{1}{\left(\lambda_i^x\right)^{j_x}} \left[\Gamma(j_x) - \Gamma(j_x, \lambda_i^x s)\right] \\ &= \frac{(j_x-1)!}{\left(\lambda_i^x\right)^{j_x}} \left[1 - e^{-\lambda_i^x s} \sum_{q_x=0}^{j_x-1} \frac{\left(\lambda_i^x s\right)^{q_x}}{q_x!}\right]. \end{aligned} \quad (86)$$

Hence,

$$\begin{aligned} h_1(s,t) &= \frac{(j_x-1)!\left(j_y-1\right)!}{\left(\lambda_i^x\right)^{j_x} \left(\lambda_i^y\right)^{j_y}} \left[1 - e^{-\lambda_i^x s} \sum_{q_x=0}^{j_x-1} \frac{\left(\lambda_i^x s\right)^{q_x}}{q_x!}\right] \\ &\quad \cdot \left[1 - e^{-\lambda_i^y t} \sum_{q_y=0}^{j_y-1} \frac{\left(\lambda_i^y t\right)^{q_y}}{q_y!}\right]. \end{aligned} \quad (87)$$

(ii) Computing of $h_2(s,t) = \int_0^s \int_0^t u^{r_x+j_x-1} v^{r_y+j_y-1} e^{-\left(\gamma_y+\lambda_i^y\right) v - \left(\gamma_x+\lambda_i^x\right) u} \mathrm{dudv}$

$$h_2(s,t)=\int_0^s u^{r_x+j_x-1}e^{-(\gamma_x+\lambda_i^x)u}\mathrm{du}\int_0^t v^{r_y+j_y-1}e^{-(\gamma_y+\lambda_i^y)v}\mathrm{dv}. \quad (88)$$

Put $w=(\gamma_x+\lambda_i^x)u$; we have

$$\begin{aligned}\int_0^s u^{r_x+j_x-1}e^{-(\gamma_x+\lambda_i^x)u}\mathrm{du} &= \frac{1}{(\gamma_x+\lambda_i^x)^{r_x+j_x}}\int_0^{(\gamma_x+\lambda_i^x)s} w^{r_x+j_x-1}e^{-w}\mathrm{dw} \\ &= \frac{1}{(\gamma_x+\lambda_i^x)^{r_x+j_x}}\left[\Gamma(r_x+j_x)-\Gamma(r_x+j_x,(\gamma_x+\lambda_i^x)s)\right] \\ &= \frac{(r_x+j_x-1)!}{(\gamma_x+\lambda_i^x)^{r_x+j_x}}\left[1-e^{-(\gamma_x+\lambda_i^x)s}\sum_{q_x=0}^{r_x+j_x-1}\frac{[(\gamma_x+\lambda_i^x)s]^{q_x}}{q_x!}\right].\end{aligned} \quad (89)$$

Thus,

$$\begin{aligned}h_2(s,t) &= \frac{(r_x+j_x-1)!\left(r_y+j_y-1\right)!}{(\gamma_x+\lambda_i^x)^{r_x+j_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y}} \\ &\cdot\left[1-e^{-(\gamma_x+\lambda_i^x)s}\sum_{q_x=0}^{r_x+j_x-1}\frac{[(\gamma_x+\lambda_i^x)s]^{q_x}}{q_x!}\right] \\ &\times\left[1-e^{-(\gamma_y+\lambda_i^y)t}\sum_{q_y=0}^{r_y+j_y-1}\frac{\left[\left(\gamma_y+\lambda_i^y\right)t\right]^{q_y}}{q_y!}\right].\end{aligned} \quad (90)$$

(iii) Computing of $H(s,t)$

$$\begin{aligned}H(s,t) &= \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y} a_{i_x,j_x}^x a_{i_y,j_y}^y\frac{1}{(j_x-1)!\left(j_y-1\right)!} \\ &\cdot\int_0^s\int_0^t u^{j_x-1}v^{j_y-1}e^{-\lambda_i^y v-\lambda_i^x u}\mathrm{dudv} \\ &+\omega\sum_{k,l}^{1}C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a_{i_x,j_x}^x a_{i_y,j_y}^y \\ &\cdot\frac{1}{r_x!r_y!(j_x-1)!\left(j_y-1\right)!} \\ &\times\int_0^s\int_0^t u^{r_x+j_x-1}v^{r_y+j_y-1}e^{-\left(\gamma_y+\lambda_i^y\right)v-(\gamma_x+\lambda_i^x)u}\mathrm{dudv} \\ &= \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y} a_{i_x,j_x}^x a_{i_y,j_y}^y\frac{1}{(j_x-1)!\left(j_y-1\right)!}h_1(s,t) \\ &+\omega\sum_{k,l}^{1}C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a_{i_x,j_x}^x a_{i_y,j_y}^y \\ &\cdot\frac{1}{r_x!r_y!(j_x-1)!\left(j_y-1\right)!}\times h_2(s,t),\end{aligned} \quad (91)$$

which gives (82). For Equation (83), we have

$$\bar{H}(s,t)=\mathbb{P}\left[T_x>s,T_y>t\right]=\int_s^{+\infty}\int_t^{+\infty}h(u,v)\mathrm{dudv}. \quad (92)$$

(iv) Computing of $\bar{h}_1(s,t)=\int_s^{+\infty}\int_t^{+\infty}u^{j_x-1}v^{j_y-1}e^{-\lambda_i^y v-\lambda_i^x u}$ dudv

By the Fubini theorem, we have

$$\bar{h}_1(s,t)=\int_s^{+\infty}u^{j_x-1}e^{-\lambda_i^x u}\mathrm{du}\int_t^{+\infty}v^{j_y-1}e^{-\lambda_i^y v}\mathrm{dv}. \quad (93)$$

Let $w=\lambda_i^x u$; we have

$$\begin{aligned}\int_s^{+\infty}u^{j_x-1}e^{-\lambda_i^x u}\mathrm{du} &= \frac{1}{(\lambda_i^x)^{j_x}}\int_{\lambda_i^x s}^{+\infty}w^{j_x-1}e^{-w}\mathrm{dw} \\ &= \frac{1}{(\lambda_i^x)^{j_x}}\Gamma(j_x,\lambda_i^x s) \\ &= \frac{(j_x-1)!}{(\lambda_i^x)^{j_x}}e^{-\lambda_i^x s}\sum_{q_x=0}^{j_x-1}\frac{(\lambda_i^x s)^{q_x}}{q_x!}.\end{aligned} \quad (94)$$

Hence,

$$\bar{h}_1(s,t)=\frac{(j_x-1)!\left(j_y-1\right)!}{(\lambda_i^x)^{j_x}\left(\lambda_i^y\right)^{j_y}}e^{-\lambda_i^x s}e^{-\lambda_i^y t}\sum_{q_x=0}^{j_x-1}\frac{(\lambda_i^x s)^{q_x}}{q_x!}\sum_{q_y=0}^{j_y-1}\frac{(\lambda_i^y t)^{q_y}}{q_y!}. \quad (95)$$

(v) Computing of $\bar{h}_2(s,t)=\int_s^{+\infty}\int_t^{+\infty}u^{r_x+j_x-1}v^{r_y+j_y-1}e^{-(\gamma_y+\lambda_i^y)v-(\gamma_x+\lambda_i^x)u}$ dudv

$$\bar{h}_2(s,t)=\int_s^{+\infty}u^{r_x+j_x-1}e^{-(\gamma_x+\lambda_i^x)u}\mathrm{du}\int_t^{+\infty}v^{r_y+j_y-1}e^{-\left(\gamma_y+\lambda_i^y\right)v}\mathrm{dv}. \quad (96)$$

Put $w=(\gamma_x+\lambda_i^x)u$; we have

$$\begin{aligned}\int_s^{+\infty}u^{r_x+j_x-1}e^{-(\gamma_x+\lambda_i^x)u}\mathrm{du} &= \frac{1}{(\gamma_x+\lambda_i^x)^{r_x+j_x}}\int_{(\gamma_x+\lambda_i^x)s}^{+\infty}w^{r_x+j_x-1}e^{-w}\mathrm{dw} \\ &= \frac{1}{(\gamma_x+\lambda_i^x)^{r_x+j_x}}\Gamma(r_x+j_x,(\gamma_x+\lambda_i^x)s) \\ &= \frac{(r_x+j_x-1)!}{(\gamma_x+\lambda_i^x)^{r_x+j_x}}e^{-(\gamma_x+\lambda_i^x)s}\sum_{q_x=0}^{r_x+j_x-1}\frac{[(\gamma_x+\lambda_i^x)s]^{q_x}}{q_x!}.\end{aligned} \quad (97)$$

Thus,

$$\bar{h}_2(s,t) = \frac{(r_x+j_x-1)!\left(r_y+j_y-1\right)!}{(\gamma_x+\lambda_i^x)^{r_x+j_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y}} e^{-(\gamma_x+\lambda_i^x)s} e^{-\left(\gamma_y+\lambda_i^y\right)t} \times \sum_{q_x=0}^{r_x+j_x-1} \frac{[(\gamma_x+\lambda_i^x)s]^{q_x}}{q_x!} \sum_{q_y=0}^{r_y+j_y-1} \frac{\left[\left(\gamma_y+\lambda_i^y\right)t\right]^{q_y}}{q_y!} q. \tag{98}$$

(vi) Computing of $\bar{H}(s,t)$

$$\bar{H}(s,t) = \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y} a_{i_x,j_x}^x a_{i_y,j_y}^y \frac{1}{(j_x-1)!\left(j_y-1\right)!}\bar{h}_1(s,t) + \omega\sum_{k,l}^{1} C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\gamma_x^{r_x}\gamma_y^{r_y} a_{i_x,j_x}^x a_{i_y,j_y}^y \times \frac{\bar{h}_2(s,t)}{r_x!r_y!(j_x-1)!\left(j_y-1\right)!}. \tag{99}$$

□

5.1. Joint Status. The joint-life status is one that requires the survival of both lives. Accordingly, the status terminates upon the first death of one of the two lives. The joint-life status of two lives x and y will be denoted by (xy), and the moment of death random variable is given by $T_{(xy)} = \min\,(T_x, T_y)$.

Theorem 10. *The survival function for $T_{(xy)}$ is given by*

$$\bar{F}_{T_{(xy)}}(t) == \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{q_x=0}^{j_x-1}\sum_{q_y=0}^{j_y-1}\frac{a_{i_x,j_x}^x a_{i_y,j_y}^y}{(\lambda_i^x)^{j_x-q_x}\left(\lambda_i^y\right)^{j_y-q_y}}\frac{1}{q_x!}\frac{1}{q_y!}t^{q_x+q_y}e^{-\left(\lambda_i^x+\lambda_i^y\right)t} + \omega\sum_{k,l}^{1} C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\sum_{q_x=0}^{r_x+j_x-1}\sum_{q_y=0}^{r_y+j_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a_{i_x,j_x}^x a_{i_y,j_y}^y \times \frac{1}{(\gamma_x+\lambda_i^x)^{r_x+j_x-q_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y-q_y}}\begin{pmatrix} r_x+j_x-1 \\ r_x\end{pmatrix} \cdot \begin{pmatrix} r_y+j_y-1 \\ r_y\end{pmatrix}\frac{1}{q_x!}\frac{1}{q_y!}\times t^{q_x+q_y}e^{-\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)t}. \tag{100}$$

Using the survival function, we get the following pdf:

$$f_{T_{(xy)}}(t) == \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{q_x=0}^{j_x-1}\sum_{q_y=0}^{j_y-1}\frac{a_{i_x,j_x}^x a_{i_y,j_y}^y}{(\lambda_i^x)^{j_x-q_x}\left(\lambda_i^y\right)^{j_y-q_y}}\frac{1}{q_x!}\frac{1}{q_y!}\Gamma\left(q_x+q_y+1\right) \times\left[-\frac{t^{q_x+q_y-1}e^{-\left(\lambda_i^x+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y\right)} + \left(\lambda_i^x+\lambda_i^y\right)\frac{t^{\left(q_x+q_y+1\right)-1}e^{-\left(\lambda_i^x+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y+1\right)}\right] + \omega\sum_{k,l}^{1} C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\sum_{q_x=0}^{r_x+j_x-1}\sum_{q_y=0}^{r_y+j_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a_{i_x,j_x}^x a_{i_y,j_y}^y \times\frac{\Gamma\left(q_x+q_y+1\right)}{(\gamma_x+\lambda_i^x)^{r_x+j_x-q_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y-q_y}}\begin{pmatrix} r_x+j_x-1 \\ r_x\end{pmatrix} \cdot \begin{pmatrix} r_y+j_y-1 \\ r_y\end{pmatrix}\frac{1}{q_x!}\frac{1}{q_y!}\times\left[-\frac{t^{q_x+q_y-1}e^{-\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y\right)} + \left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)\frac{t^{\left(q_x+q_y+1\right)-1}e^{-\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y+1\right)}\right]. \tag{101}$$

Proof.

$$\bar{F}_{T_{(xy)}}(t) = \mathbb{P}\left[T_x > t, T_y > t\right] = \int_t^{+\infty}\int_t^{+\infty} h(u,v)\mathrm{dudv} = \bar{H}(t,t), \quad f_{T_{(xy)}}(t) = -\bar{F}'_{T_{(xy)}}(t). \tag{102}$$

□

Remark 11. Clearly, the above distribution is a combination of mixture of the Erlang distribution, since

$$f_{T_{(xy)}}(t) == \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{q_x=0}^{j_x-1}\sum_{q_y=0}^{j_y-1}\frac{a_{i_x,j_x}^x a_{i_y,j_y}^y}{(\lambda_i^x)^{j_x-q_x}\left(\lambda_i^y\right)^{j_y-q_y}} \cdot\frac{1}{q_x!}\frac{1}{q_y!}\Gamma\left(q_x+q_y+1\right) \times\left[-\mathrm{Erlang}\left(q_x+q_y,\lambda_i^x+\lambda_i^y\right) + \left(\lambda_i^x+\lambda_i^y\right)\mathrm{Erlang}\left(q_x+q_y+1,\lambda_i^x+\lambda_i^y\right)\right] + \omega\sum_{k,l}^{1} C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\sum_{q_x=0}^{r_x+j_x-1}\sum_{q_y=0}^{r_y+j_y-1} \cdot\gamma_x^{r_x}\gamma_y^{r_y}a_{i_x,j_x}^x a_{i_y,j_y}^y\times\frac{\Gamma\left(q_x+q_y+1\right)}{(\gamma_x+\lambda_i^x)^{r_x+j_x-q_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y-q_y}} \cdot\begin{pmatrix} r_x+j_x-1 \\ r_x\end{pmatrix}\begin{pmatrix} r_y+j_y-1 \\ r_y\end{pmatrix}\frac{1}{q_x!}\frac{1}{q_y!} \times\left[-\mathrm{Erlang}\left(q_x+q_y,\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right) + \left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)\mathrm{Erlang} \cdot\left(q_x+q_y+1,\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)\right]. \tag{103}$$

Equation (47) can be generalized as follows:

For $x \geq 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta T_{(xy)}} b\left(S\left(T_{(xy)}\right)\right)\right] &= \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{q_x=0}^{j_x-1}\sum_{q_y=0}^{j_y-1} \\
&\cdot \frac{a^x_{i_x,j_x} a^y_{i_y,j_y}}{(\lambda_i^x)^{j_x-q_x}(\lambda_i^y)^{j_y-q_y}} \frac{1}{q_x!}\frac{1}{q_y!}\Gamma\left(q_x+q_y+1\right) \\
&\times \left[(\lambda_i^x+\lambda_i^y)\Phi_{1,i}^{N+1}\sum_{k=0}^{N-1} c(N+1,k)\sum_{r=0}^{N-k}\frac{(\beta_{1,i}-\alpha_{1,i})^r}{2^r r!}\right. \\
&\left. -\Phi_{1,i}^{N}\sum_{k=0}^{N-2} c(N,k)\sum_{r=0}^{N-k-1}\frac{(\beta_{1,i}-\alpha_{1,i})^r}{2^r r!}\right]\int_0^\infty b(S(0)e^x)x^r e^{-\beta_{1,i}x}\mathrm{dx} \\
&+\omega\sum_{k,l}^{1} C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\sum_{q_x=0}^{r_x+j_x-1}\sum_{q_y=0}^{r_y+j_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a^x_{i_x,j_x}a^y_{i_y,j_y} \\
&\times \frac{\Gamma\left(q_x+q_y+1\right)}{(\gamma_x+\lambda_i^x)^{r_x+j_x-q_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y-q_y}}\binom{r_x+j_x-1}{r_x} \\
&\cdot \binom{r_y+j_y-1}{r_y}\frac{1}{q_x!}\frac{1}{q_y!} \\
&\times\times\left[\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)\Phi_{2,i}^{N+1}\sum_{k=0}^{N-1}c(N+1,k)\sum_{r=0}^{N-k}\frac{(\beta_{2,i}-\alpha_{2,i})^r}{2^r r!}\right. \\
&\left. -\Phi_{2,i}^{N}\sum_{k=0}^{N-2}c(N,k)\sum_{r=0}^{N-k-1}\frac{(\beta_{2,i}-\alpha_{2,i})^r}{2^r r!}\right]\int_0^\infty b(S(0)e^x)x^r e^{-\beta_{2,i}x}\mathrm{dx}.
\end{aligned} \tag{104}$$

For $x \leq 0$,

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta T_{(xy)}} b\left(S\left(T_{(xy)}\right)\right)\right] &= \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{q_x=0}^{j_x-1}\sum_{q_y=0}^{j_y-1} \\
&\cdot \frac{a^x_{i_x,j_x} a^y_{i_y,j_y}}{(\lambda_i^x)^{j_x-q_x}(\lambda_i^y)^{j_y-q_y}} \frac{1}{q_x!}\frac{1}{q_y!}\Gamma\left(q_x+q_y+1\right) \\
&\times \left[(\lambda_i^x+\lambda_i^y)\Phi_{1,i}^{N+1}\sum_{k=0}^{N-1} c(N+1,k)\sum_{r=0}^{N-k}\frac{(-(\beta_{1,i}-\alpha_{1,i}))^r}{2^r r!}\right. \\
&\left. -\Phi_{1,i}^{N}\sum_{k=0}^{N-2} c(N,k)\sum_{r=0}^{N-k-1}\frac{(-(\beta_{1,i}-\alpha_{1,i}))^r}{2^r r!}\right]\int_0^\infty b(S(0)e^x)x^r e^{-\alpha_{1,i}x}\mathrm{dx} \\
&+\omega\sum_{k,l}^{1} C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\sum_{q_x=0}^{r_x+j_x-1}\sum_{q_y=0}^{r_y+j_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a^x_{i_x,j_x}a^y_{i_y,j_y} \\
&\times \frac{\Gamma\left(q_x+q_y+1\right)}{(\gamma_x+\lambda_i^x)^{r_x+j_x-q_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y-q_y}}\binom{r_x+j_x-1}{r_x} \\
&\cdot \binom{r_y+j_y-1}{r_y}\frac{1}{q_x!}\frac{1}{q_y!} \\
&\times\times\left[\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)\Phi_{2,i}^{N+1}\sum_{k=0}^{N-1}c(N+1,k)\right. \\
&\left.\cdot\sum_{r=0}^{N-k}\frac{(-(\beta_{2,i}-\alpha_{2,i}))^r}{2^r r!}-\Phi_{2,i}^{N}\sum_{k=0}^{N-2}c(N,k)\sum_{r=0}^{N-k-1}\frac{(-(\beta_{2,i}-\alpha_{2,i}))^r}{2^r r!}\right] \\
&\cdot\int_0^\infty b(S(0)e^x)x^r e^{-\alpha_{2,i}x}\mathrm{dx},
\end{aligned} \tag{105}$$

where

$$\begin{aligned}
N &= q_x+q_y, \\
\Phi_{1,i} &= \frac{\lambda_i^x+\lambda_i^y}{\sigma^2}, \\
\Phi_{2,i} &= \frac{\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y}{\sigma^2}.
\end{aligned} \tag{106}$$

$c(N,k)$ is given by (32); $\alpha_{1,i}$ and $\beta_{1,i}$ are solutions of Equation (12), with λ_i replaced by $\lambda_i^x+\lambda_i^y$; $\alpha_{2,i}$ and $\beta_{2,i}$ are also solutions of Equation (12), with λ_i replaced by $\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y$.

5.2. The Last-Survivor Status. The other common status is the last-survivor status. The last-survivor status is one that ends upon the death of both lives. That is, the status survives as long as at least one of the component members remains alive. The last-survivor status of two lives x and y will be denoted by $(\overline{xy})$, and the moment of death random variable is given by $T_{(\overline{xy})}=\max(T_x,T_y)$.

Theorem 12. *The CDF and survival functions follow*

$$\begin{aligned}
F_{T_{(\overline{xy})}}(t) &= \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\frac{a^x_{i_x,j_x}a^y_{i_y,j_y}}{(\lambda_i^x)^{j_x}(\lambda_i^y)^{j_y}} \\
&\cdot\left[1-e^{-\lambda_i^x t}\sum_{q_x=0}^{j_x-1}\frac{(\lambda_i^x t)^{q_x}}{q_x!}\right]\left[1-e^{-\lambda_i^y t}\sum_{q_y=0}^{j_y-1}\frac{(\lambda_i^y t)^{q_y}}{q_y!}\right] \\
&+\omega\sum_{k,l}^{1}C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum \\
&\cdot_{r_y=0}^{m_y-1}\frac{\gamma_x^{r_x}\gamma_y^{r_y}a^x_{i_x,j_x}a^y_{i_y,j_y}}{(\gamma_x+\lambda_i^x)^{r_x+j_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y}} \\
&\times\binom{r_x+j_x-1}{r_x}\binom{r_y+j_y-1}{r_y} \\
&\cdot\left[1-e^{-(\gamma_x+\lambda_i^x)t}\sum_{q_x=0}^{r_x+j_x-1}\frac{[(\gamma_x+\lambda_i^x)t]^{q_x}}{q_x!}\right] \\
&\times\left[1-e^{-(\gamma_y+\lambda_i^y)t}\sum_{q_y=0}^{r_y+j_y-1}\frac{\left[\left(\gamma_y+\lambda_i^y\right)t\right]^{q_y}}{q_y!}\right],
\end{aligned} \tag{107}$$

Table 1: Numerical results for call and put option 1.

	$n=4$	$\sigma=0.18$	$\mu=0.001$	$\delta=0.02$
λ_i	0.011	0.014	0.017	0.015
			$\mathbb{E}\left[e^{-\delta\tau}b(S(\tau))\right]$	
Out-of-the-money call option	$S(0)=100$	$K=120$	19.31578	
In-the-money call option	$S(0)=120$	$K=100$	23.69608	
In-the-money put option	$S(0)=100$	$K=120$	19.7145	

Table 2: Numerical results for call and put option 2.

	$n=4$	$\sigma=0.18$	$\mu=0.001$	$\delta=0.02$
λ_i	0.015	0.012	0.018	0.017
			$\mathbb{E}\left[e^{-\delta\tau}b(S(\tau))\right]$	
Out-of-the-money call option	$S(0)=100$	$K=120$	20.59528	
In-the-money call option	$S(0)=120$	$K=100$	25.32719	
In-the-money put option	$S(0)=100$	$K=120$	21.06781	

Table 3: Numerical results for call and put option 3.

	$n=4$	$\sigma=0.19$	$\mu=0.001$	$\delta=0.02$
λ_i	0.015	0.012	0.018	0.017
			$\mathbb{E}\left[e^{-\delta\tau}b(S(\tau))\right]$	
Out-of-the-money call option	$S(0)=100$	$K=120$	32.4068	
In-the-money call option	$S(0)=120$	$K=100$	39.51472	
In-the-money put option	$S(0)=100$	$K=120$	32.89137	

and the pdf is also given by

$$\begin{aligned} f_{T_{(xy)}}(t) = f_{T_{(\bar{xy})}}(t) = & \sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{q_x=0}^{j_x-1}\sum_{q_y=0}^{j_y-1}\frac{a_{i_x,j_x}^x a_{i_y,j_y}^y}{(\lambda_i^x)^{j_x-q_x}(\lambda_i^y)^{j_y-q_y}} \\ & \cdot\frac{1}{q_x!}\frac{1}{q_y!}\Gamma\left(q_x+q_y+1\right) \\ & \times\left[-\frac{t^{q_x+q_y-1}e^{-\left(\lambda_i^x+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y\right)}+\left(\lambda_i^x+\lambda_i^y\right)\frac{t^{\left(q_x+q_y+1\right)-1}e^{-\left(\lambda_i^x+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y+1\right)}\right] \\ & +\omega\sum_{k,l}^{1}C_{kl}\sum_{i_x=1}^{k_x}\sum_{j_x=1}^{l_i^x}\sum_{i_y=1}^{k_y}\sum_{j_y=1}^{l_i^y}\sum_{r_x=0}^{m_x-1}\sum_{r_y=0}^{m_y-1}\sum_{q_x=0}^{r_x+j_x-1}\sum_{q_y=0}^{r_y+j_y-1}\gamma_x^{r_x}\gamma_y^{r_y}a_{i_x,j_x}^x a_{i_y,j_y}^y \\ & \times\frac{\Gamma\left(q_x+q_y+1\right)}{(\gamma_x+\lambda_i^x)^{r_x+j_x-q_x}\left(\gamma_y+\lambda_i^y\right)^{r_y+j_y-q_y}}\binom{r_x+j_x-1}{r_x} \\ & \cdot\binom{r_y+j_y-1}{r_y}\frac{1}{q_x!}\frac{1}{q_y!}\times\left[-\frac{t^{q_x+q_y-1}e^{-\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y\right)}\right. \\ & \left.+\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)\frac{t^{\left(q_x+q_y+1\right)-1}e^{-\left(\gamma_x+\lambda_i^x+\gamma_y+\lambda_i^y\right)t}}{\Gamma\left(q_x+q_y+1\right)}\right]. \end{aligned} \tag{108}$$

Proof.

$$\begin{aligned} F_{T_{(\bar{xy})}}(t) &= \mathbb{P}\left[\max\left(T_x, T_y\right) < t\right] = \mathbb{P}\left[T_x < t, T_y < t\right] \\ &= \int_0^t\int_0^t h(u,v)\mathrm{dudv} = H(t,t), \end{aligned}$$

$$f_{T_{(\bar{xy})}}(t) = F'_{T_{(\bar{xy})}}(t) = -\bar{F}'_{T_{(xy)}}(t) = f_{T_{(xy)}}(t). \tag{109}$$

From Theorem 10 and Theorem 12, we can easily notice that the distributions of $T_{(xy)}$ and $T_{(\bar{xy})}$ have the same form just with different parameters, and one can deduce $\mathbb{E}[e^{-\delta T_{(\bar{xy})}}b(S(T_{(\bar{xy})}))]$ similarly as $\mathbb{E}[e^{-\delta T_{(xy)}}b(S(T_{(xy)}))]$ in Remark 11. □

6. Some Numerical Results

This section presents some numerical results for call and put options.

6.1. Comments. The average age of death calculated with the values of parameters λ_i in Table 1 is approximately 71 years. This age is around 67 in Tables 2–4. Clearly, the higher the

Table 4: Numerical results for call and put option 4.

	$n = 4$	$\sigma = 0.18$	$\mu = 0.002$	$\delta = 0.02$
λ_i	0.015	0.012	0.018	0.017
			$\mathbb{E}\left[e^{-\delta\tau}b(S(\tau))\right]$	
Out-of-the-money call option	$S(0) = 100$	$K = 120$	26.12719	
In-the-money call option	$S(0) = 120$	$K = 100$	31.94692	
In-the-money put option	$S(0) = 100$	$K = 120$	26.57115	

average age of death, the lower the premium to be paid. This remains true with the modification of other parameters such as the expectation μ and the volatility σ. Tables 2 and 3 show that the premium increases with a slight increase in the volatility. This is similar to that of the expectation μ, but less sensitive than that of the volatility σ (see Tables 3 and 4).

Therefore, parameter values play an important role in the applicability of the results.

7. Concluding Remarks

It has provided a contribution to the study of the valuation of equity-linked death benefits. Under the exponential Lévy process assumption for the stock price process and K_n distribution for the time until death, explicit formulas are derived for the discounted payment of the guaranteed minimum death benefit products. A closed expression is established for both call and put options. Using a bivariate Sarmanov distribution with K_n marginal distributions, we analyze multiple life insurance based on joint survival. Calls and puts are illustrated numerically. In future work, we plan to investigate the case of death following a matrix exponential distribution.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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