# Dynamics Analysis of a Delayed Crimean-Congo Hemorrhagic Fever Virus Model in Humans 

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#### Abstract

Given that the Crimean and Congo hemorrhagic fever is one of the deadly viral diseases that occur seasonally due to the activity of the carrier "tick," studying and developing a mathematical model simulating this illness are crucial. Due to the delay in the disease's incubation time in the sick individual, the paper involved the development of a mathematical model modeling the transmission of the disease from the carrier to humans and its spread among them. The major objective is to comprehend the dynamics of illness transmission so that it may be controlled, as well as how time delay affects this. The discussion of every one of the solution's qualitative attributes is included. According to the established basic reproduction number, the stability analysis of the endemic equilibrium point and the disease-free equilibrium point is examined for the presence or absence of delay. Hopf bifurcation's triggering circumstance is identified. Using the center manifold theorem and the normal form, the direction and stability of the bifurcating Hopf bifurcation are explored. The next step is sensitivity analysis, which explains the set of control settings that have an impact on how the system behaves. Finally, to further comprehend the model's dynamical behavior and validate the discovered analytical conclusions, numerical simulation has been used.


## 1. Introduction

A critical step in determining the potential development of epidemics and putting preventative and control measures in place is modeling infectious illnesses. Given their enormous economic cost and grave threat to public health, pandemics continue to be a major barrier to humanity's continued survival. As a result, lowering the risk of their spread has been governments', scientists', and health organizations' top goal. Daniel Bernoulli created the first mathematical model in epidemiology for smallpox vaccination in 1760 [1], which was followed by a more comprehensive plan that gained more traction in 1766 [2]. William Budd's thorough examination of typhoid fever in 1918, which covered its traits, modes of transmission, and treatment options, was another important contribution to the study of infectious diseases [3].

The presence of dangerous microorganisms like viruses, bacteria, fungi, parasites, and others causes infectious disor-
ders. The spread of these illnesses within society is dependent on a number of disease-specific criteria, such as the pathogen, mode of transmission, length of incubation and infection, susceptibility, and resistance. The introduction of Kermack and McKendrick's idea [4] in 1927 was a major turning point for infectious disease modeling. Infectious illness modeling has since made significant strides, with models like SIS, SIR, SIRS, SEIS, SEIR, SVIRS, SFIR, and others enabling researchers to gain a thorough grasp of the nature and development of the epidemiological trajectory. Naji and Hussien [5] proposed and studied an epidemic model that describes the dynamics of the spread of infectious diseases with two different types of infectious diseases that spread through both horizontal and vertical transmissions in the host population. Majeed and Naji [6] proposed and investigated a partial temporary immunity SIR epidemic model involving a nonlinear treatment rate. Naji and Thirthar [7] suggested and discussed an SIS epidemic model with a saturated incidence rate and treatment function.


Figure 1: Basic flowchart of epidemic transmission.

Mohsen and Naji [8] carried out a thorough examination of the transmission of HIV/AIDS along with the use of the best control method. Thirthar et al. [9] established a mathematical model of an $\mathrm{SI}_{1} \mathrm{I}_{2} \mathrm{R}$ epidemic disease with saturated incidence and general recovery functions of the first disease $I_{1}$. Kumar et al. evaluated the danger of COVID-19 infection and its effects on public health [10, 11]. Sun et al., for instance, employed the SEQIR model [12] to obtain optimal control when battling epidemic diseases. Many other recent studies are available for those interested, which deal with various infectious disease models in addition to those indicated as approved sources in these studies, for example, [13-21]. To produce a more accurate depiction of system dynamics, there are also important studies aimed at adding the idea of time delay to epidemiological models. In this regard, Thirthar and Naji [22] devised and investigated an SIS epidemic model with two delays; it is assumed that the saturation function represents the incidence rate and treatment rate. Goel et al. [23-25] provided insightful information about the impact of delay in several epidemic models. By using a nonlinear Monod-Haldane infection rate, Hussien and Naji significantly improved our understanding of how media coverage affects the dynamics of a delayed SEIR epidemic model [26]. However, researchers looked at the dynamic behavior of a cancer model in a polluted environment while taking into account the time lag it took for the environment to be cleared of contamination [27]. Many other studies of epidemic models including delay role are available, for example, [28-35].

More than 700,000 people die each year as a result of vector-borne diseases, according to the most recent World Health Organization (WHO) reports, which are shown in [36, 37]. Typically, pathogens (such as ticks, mosquitoes, or livestock) spread among populations by infecting a host, who is frequently a human or an animal; see [38] for more information. Some of the most well-known and widespread vector-borne illnesses are malaria, dengue fever, St. Louis encephalitis, Crimean-Congo hemorrhagic fever, Zika virus, West Nile fever, and plague. The most economically deprived groups of the population are disproportionately affected, and it is most severe in tropical and subtropical areas.

Significant outbreaks of vector-borne diseases have recently harmed communities, resulted in fatalities, and put a strain on the healthcare system in a number of nations. In order to combat and contain these epidemics, scientists and researchers have accepted the responsibility for putting necessary safeguards in place. Mohammadi et al. [39] created a model to study the Crimean-Congo Fever virus's cycle of transmission among people, pets, and ticks. Hoch et al. Reference [40] produced a thorough investigation of the CrimeanCongo fever virus's proliferation in Turkey's Central Anatolia region, along with recommendations for its management. Please see [39, 41-48] for more information regarding the vector-borne diseases. This research seeks to develop a delayed SEIRV epidemiological model that simulates pathogen dynamics and Crimean-Congo hemorrhagic fever virus outbreaks in human populations. Additionally, we determined the fundamental diffusion coefficient $\mathscr{R}_{0}$, examined the model's stability, and examined how the epidemic trajectory behaved.

## 2. Model Construction

In this section, a mathematical model of the Crimean-Congo hemorrhagic fever virus (CCHF) has been developed and is analytically examined to reduce the spread of illnesses and maintain the health and welfare of communities and populations. Both the human (host) population and the vector population (such as ticks or cattle), defined by their densities at time $t$ as $N(t)$ and $V(t)$, respectively, are included in the model. When a viral infection spreads, $N(t)$ will be divided into several compartments, with the susceptible being represented by $S(t)$, which stands for healthy individuals at risk of disease, exposed individuals being those having early symptoms but not yet infectious $(E(t)$ ), infected individuals being those having visible symptoms and can transmit the disease $(I(t))$, and recovered individuals from disease $(R(t))$, and $N(t)=S(t)+E(t)+I(t)+R(t)$. A thorough representation of the dynamic interactions used to create the CCHF virus model is shown in Figure 1.

Based on the above flowchart of epidemic transmission, we assumed that, in the absence of the disease, all members of the host belong to the healthy compartment and that the

Table 1: Symbolization description within the CCHF model.

| Parameters | Description | Units |
| :--- | ---: | :---: |
| $\Lambda$ | Host population recruitment rate | Density $\left(\right.$ day $\left.^{-1}\right)$ |
| $\sigma_{1}$ | Infection rate through exposure to environmental pathogens |  |
| $\sigma_{2}$ | Infection rate through contact with infected individuals | $\left(\right.$ Density $\left.^{-1}\right)\left(\right.$ day $\left.^{-1}\right)$ |
| $\mu_{1}$ and $\mu_{2}$ | Host natural and diseased death rates, respectively | $\left(\right.$ Density $\left.^{-1}\right)\left(\right.$ day $\left.^{-1}\right)$ |
| $\alpha_{1}$ | The rate at which an infected person becomes contagious | Day $^{-1}$ |
| $\alpha_{2}$ | Host recovery rate | Day $^{-1}$ |
| $r_{1}$ | Pathogen growth rate due to infected host | Day $^{-1}$ |
| $r_{2}$ | Pathogen growth rate due to infected vector | Day $^{-1}$ |
| $\mu_{3}$ | Pathogen decay rate | Day $^{-1}$ |
| $\tau$ | Delay rate | Day |

disease, if found, is not transmitted from infected parents to newborns. The disease is transmitted through exposure to environmental pathogens as well as direct contact with infected individuals. Since there is an incubation period for the disease, newly infected individuals move to the cabin of infected individuals by passing through the cabin of exposed individuals, with the assumption that there is a delay period for the transition process and for many reasons, the most important of which is the immunity of people. As a result, the disease cannot transmitted by contact between susceptible individuals and exposed individuals. At the same time, individuals move from the infected compartment to the recovered compartment at a certain percentage because of the treatment used or the body's severe resistance to the host, which eliminates the disease. Finally, the disease multiplies in two ways, the first by the virus released from infected individuals and the second by the vector community such as ticks and cattle, while the disease directly results in decay as a result of the natural death of the virus. Based on the above medically known assumptions, we simulate the process of disease transmission mathematically through the set of nonlinear delayed differential equations below:

$$
\begin{align*}
& \frac{d S}{d t}=\Lambda-\sigma_{1} S(t-\tau) V(t-\tau)-\sigma_{2} S(t-\tau) I(t-\tau)-\mu_{1} S \\
& \frac{d E}{d t}=\sigma_{1} S(t-\tau) V(t-\tau)+\sigma_{2} S(t-\tau) I(t-\tau)-\alpha_{1} E-\mu_{1} E \\
& \frac{d I}{d t}=\alpha_{1} E-\alpha_{2} I-\mu_{1} I-\mu_{2} I \\
& \frac{d R}{d t}=\alpha_{2} I-\mu_{1} R \\
& \frac{d V}{d t}=r_{1} I+r_{2} V-\mu_{3} V \tag{1}
\end{align*}
$$

Concerning system parameters, they are assumed to be positive and can be described in Table 1.

For ecological reasons, it is assumed throughout the work that $\mu_{3}>r_{2}$ is true since if it were false, the virus (and therefore the diseases) would always exist. In addition,
the following beginning conditions for the CCHF model are satisfied:

$$
\left.\begin{array}{l}
S(\theta)=\varphi_{1}(\theta), E(\theta)=\varphi_{2}(\theta), I(\theta)=\varphi_{3}(\theta), R(\theta)=\varphi_{4}(\theta), V(\theta)=\varphi_{5}(\theta)  \tag{2}\\
\varphi_{i}(\theta) \geq 0, \theta \in[-\tau, 0), \varphi_{i}(0)>0 \quad(i=1,2,3,4,5)
\end{array}\right\}
$$

where $\varphi=\left(\varphi_{1}, \varphi_{2}, \varphi_{3}, \varphi_{4}, \varphi_{5}\right)^{T} \in C=C\left([-\tau, 0], \mathbb{R}_{+}^{5}\right)$. Here, $C$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_{+}^{5}$.

## 3. Positivity and Boundedness of Solutions

The model should be well-posted, which implies that the solutions of the system 1 should be nonnegative and bounded, in order to ensure biological correspondence between the reality of the situation and the mathematical structure of the model 1 . That will be shown by the ensuing theorem.

Theorem 1. The entire model 1's solutions initiated in the interior of $\mathbb{R}_{+}^{5}$ are always positive and bounded.

Proof. According to the exposed equation of model 1, it is obtained that

$$
\begin{equation*}
E^{\prime}(t)=E\left[\left(\frac{\sigma_{1} V(t-\tau)+\sigma_{2} I(t-\tau)}{E(t)}\right) S(t-\tau)-\left(\alpha_{1}+\mu_{1}\right)\right] . \tag{3}
\end{equation*}
$$

By integration, the result is

$$
\begin{align*}
E(t)= & E(0) \exp \left\{\int _ { 0 } ^ { t } \left[\left(\frac{\sigma_{1} V(\varepsilon-\tau)+\sigma_{2} I(\varepsilon-\tau)}{E(\varepsilon)}\right)\right.\right.  \tag{4}\\
& \left.\left.\cdot S(\varepsilon-\tau)-\left(\alpha_{1}+\mu_{1}\right)\right] d(\varepsilon)\right\}>0
\end{align*}
$$

Likewise, we have
$I(t)=I(0) \exp \left\{-\int_{0}^{t}\left(\alpha_{2}+\mu_{1}+\mu_{2}\right) d(\varepsilon)\right\}>0, \quad$ for $I(0)>0$,
$R(t)=R(0) \exp \left\{-\int_{0}^{t} \mu_{1} d(\varepsilon)\right\}>0, \quad$ for $R(0)>0$,
$V(t)=V(0) \exp \left\{-\int_{0}^{t}\left(\mu_{3}-r_{2}\right) d(\varepsilon)\right\}>0, \quad$ for $V(0)>0$.

Moreover, to show the positivity of the susceptible equation solution for all $t \geq 0$, we assume otherwise. Then, there exists a first-time $t_{1}>0$ such that $S\left(t_{1}\right)=0$. Thus, by Eq. (1) of the CCHF model, we have $S^{\prime}\left(t_{1}\right)=\Lambda>0$, and hence, $S(t)<0$ for $t \in\left(t_{1}-\varepsilon, t_{1}\right)$, where $\varepsilon>0$ is sufficiently small. This contradicts $S(t)>0$ for $t \in\left[0, t_{1}\right)$. Hence, $S(t)>$ 0 for $t>0$.

Now, in terms of ensuring the solution's boundedness, we define

$$
\begin{equation*}
N(t)=S(t)+E(t)+I(t)+R(t) \tag{6}
\end{equation*}
$$

Subsequently, the following is obtained:

$$
\begin{equation*}
\frac{d N}{d t}=\Lambda-\left(\mu_{1} S+\mu_{1} E+\left(\mu_{1}+\mu_{2}\right) I+\mu_{1} R\right) \leq \Lambda-\mu_{1} N \tag{7}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup N(t) \leq \frac{\Lambda}{\mu_{1}} \tag{8}
\end{equation*}
$$

Furthermore, the virus population equation of model 1 gives

$$
\begin{equation*}
\frac{d V}{d t} \leq r_{1}\left(\frac{\Lambda}{\mu}\right)-\left(\mu_{3}-r_{2}\right) V \tag{9}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\lim _{t \longrightarrow+\infty} \sup V(t) \leq r_{1}\left(\frac{\Lambda}{\mu}\right)\left(\mu_{3}-r_{2}\right)^{-1} \tag{10}
\end{equation*}
$$

Therefore, the solution $(S(t), E(t), I(t), R(t), V(t))$ of model 1 remains bounded. Hence,
$\Omega=\left\{(S, E, I, R, V) \in \mathbb{R}_{+}^{5}: 0 \leq S+E+I+R \leq \frac{\Lambda}{\mu}, 0 \leq V \leq r_{1}\left(\frac{\Lambda}{\mu}\right)\left(\mu_{3}-r_{2}\right)^{-1}\right\}$
is positively invariant set for model 1.
It is important to note that model 1 includes population dynamics for all relevant parameters and maintains nonnegative state variables. It also becomes clear that the first three equations in model 1, together with the fifth equation, appear to be independent of the fourth equation. As a result,
by focusing on the following subsystem, model 1 can be examined without sacrificing generality:

$$
\begin{align*}
& \frac{d S}{d t}=\Lambda-\sigma_{1} S(t-\tau) V(t-\tau)-\sigma_{2} S(t-\tau) I(t-\tau)-\mu_{1} S \\
& \frac{d E}{d t}=\sigma_{1} S(t-\tau) V(t-\tau)+\sigma_{2} S(t-\tau) I(t-\tau)-\alpha_{1} E-\mu_{1} E \\
& \frac{d I}{d t}=\alpha_{1} E-\alpha_{2} I-\mu_{1} I-\mu_{2} I \\
& \frac{d V}{d t}=r_{1} I+r_{2} V-\mu_{3} V \tag{12}
\end{align*}
$$

The fourth equation of system 1 can thus be solved directly by inserting the value of $I$ from the solution of (2) in the fourth equation, which is a part of model 1.

## 4. Equilibrium States and Basic Reproduction Number

Examining the existence of equilibrium states is crucial for understanding the qualitative dynamics of the CCHF model 2. Thus, it has been demonstrated in this section that model 2 displays two unique nonnegative equilibrium points:
(1) Disease-free equilibrium point (DFEP): $E^{0}=\left(S^{0}, 0,0\right.$ , 0 ), where $S^{0}=\Lambda / \mu_{1}$
(2) Endemic equilibrium point (EEP): $E^{1}=\left(S^{*}, E^{*}, I^{*}\right.$, $V^{*}$ ), where

$$
\begin{align*}
& S^{*}=\frac{\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)}{\alpha_{1}\left(r_{1} \sigma_{1}+\sigma_{2}\left(\mu_{3}-r_{2}\right)\right)}>0, \\
& E^{*}=\frac{\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right) V^{*}}{\alpha_{1} r_{1}}, \\
& I^{*}=\frac{\left(\mu_{3}-r_{2}\right) V^{*}}{r_{1}}, V^{*}=\frac{\alpha_{1} r_{1} \mu_{1}\left(S^{0}-S^{*}\right)}{\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)} . \tag{13}
\end{align*}
$$

The reproduction number $\mathscr{R}_{0}$, which measures the typical number of secondary infections resulting from a single infected individual in a population that is fully susceptible to infection, is a scale used in epidemiology to evaluate the potential spread of infectious diseases in a population. Consider

$$
\begin{equation*}
\frac{d X}{d t}=\mathscr{F}_{i}(X)-\mathscr{H}_{i}(X) \tag{14}
\end{equation*}
$$

where $X=[E, I, V]^{T}, \mathscr{F}_{i}(X)$ signifies the matrix of total incoming inflows from new infections within compartment
$i$, while $\mathscr{H}_{i}(X)$ encompasses the remaining items across compartments, giving

$$
\begin{align*}
& \mathscr{F}_{i}(X)=\left[\begin{array}{c}
\sigma_{1} S V+\sigma_{2} S I \\
0 \\
0
\end{array}\right]  \tag{15}\\
& \mathscr{H}_{i}(X)=\left[\begin{array}{c}
\left(\alpha_{1}+\mu_{1}\right) E \\
\left(\alpha_{2}+\mu_{1}+\mu_{2}\right) I-\alpha_{1} E \\
\mu_{3} V-r_{1} I-r_{2} V
\end{array}\right] .
\end{align*}
$$

Subsequently, the Jacobian matrix of $\mathscr{F}_{i}(X)$ and $\mathscr{H}_{i}(X)$ at $E^{0}=\left(S^{0}, 0,0,0\right)$ can be expressed as

$$
\begin{align*}
& \mathscr{D} \mathscr{F}=\left[\begin{array}{ccc}
0 & \sigma_{2} S^{0} & \sigma_{1} S^{0} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \\
& \mathscr{D} \mathscr{H}=\left[\begin{array}{ccc}
\left(\alpha_{1}+\mu_{1}\right) & 0 & 0 \\
-\alpha_{1} & \left(\alpha_{2}+\mu_{1}+\mu_{2}\right) & 0 \\
0 & -r_{1} & \left(\mu_{3}-r_{2}\right)
\end{array}\right] . \tag{16}
\end{align*}
$$

Thus, $\mathscr{R}_{0}$ is the spectral radius of a square matrix ( $\mathscr{D}$ $\mathscr{F})(\mathscr{D} \mathscr{H})^{-1}$, which means $\mathscr{R}_{0}$ is the maximum of the absolute values of its eigenvalues, where

$$
(\mathscr{D} \mathscr{F})(\mathscr{D} \mathscr{H})^{-1}=\left[\begin{array}{ccc}
\frac{\alpha_{1} S^{0}\left(\sigma_{1} r_{1}+\sigma_{2}\left(\mu_{3}-r_{2}\right)\right)}{\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)} & \frac{S^{0}\left(\sigma_{1} r_{1}+\sigma_{2}\left(\mu_{3}-r_{2}\right)\right)}{\left(\mu_{3}-r_{2}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)} & \frac{\sigma_{1} S^{0}}{\left(\mu_{3}-r_{2}\right)}  \tag{17}\\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] .
$$

This gives

$$
\begin{equation*}
\mathscr{R}_{0}=\frac{\alpha_{1} S^{0}\left(\sigma_{1} r_{1}+\sigma_{2}\left(\mu_{3}-r_{2}\right)\right)}{\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)} . \tag{18}
\end{equation*}
$$

Consequently, we may determine the reproduction number by translating the endemic equilibrium $E^{1}$ into that form.
$S^{*}=\frac{S^{0}}{\mathscr{R}_{0}}>0, E^{*}=\frac{\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right) V^{*}}{\alpha_{1} r_{1}}, I^{*}=\frac{\left(\mu_{3}-r_{2}\right) V^{*}}{r_{1}}$, $V^{*}=\frac{\alpha_{1} r_{1} \mu_{1} S^{0}}{\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)}\left(1-\frac{1}{\mathscr{R}_{0}}\right)$.

The reproduction number $\mathscr{R}_{0}$ governs the existence of the CCHF model's equilibrium points as follows:
(i) If $\mathscr{R}_{0} \leq 1 \Longrightarrow E^{0}$ is the only equilibrium in $\Omega$
(ii) If $\mathscr{R}_{0}>1 \Longrightarrow E^{0}$ remains, and an additional distinct endemic equilibrium $E^{1}$ emerges within the set $\Omega$

## 5. Analysis of Stability

A CCHF model's equilibrium responses are examined as part of stability analysis to see how resilient they are to disturbances. In this section, we describe in more detail how to move the equilibria to the origin and linearize model 2 so that it revolves around it as follows. Define $\widehat{E}=(\widehat{S}, \widehat{E}, \widehat{I}, \widehat{V})$ that denotes any arbitrary equilibrium point of model 2.

Moreover, let $S(t)=U_{1}(t)+\widehat{S}, E(t)=U_{2}(t)+\widehat{E}, I(t)=U_{3}(t)$ $+\widehat{I}$, and $V(t)=U_{4}(t)+\widehat{V}$. Then, we have
$\dot{U}_{1}(t)=a_{11} U_{1}(t)+b_{11} U_{1}(t-\tau)+b_{13} U_{3}(t-\tau)+b_{14} U_{4}(t-\tau)$,
$\dot{U}_{2}(t)=b_{21} U_{1}(t-\tau)+a_{22} U_{22}(t)+b_{23} U_{3}(t-\tau)+b_{24} U_{4}(t-\tau)$,
$\dot{U}_{3}(t)=a_{32} U_{2}(t)+a_{33} U_{3}(t)$,
$\dot{U}_{4}(t)=a_{43} U_{3}(t)+a_{44} U_{4}(t)$.

This constitutes a transfer form, where

$$
\begin{align*}
& a_{11}=-\mu_{1}, a_{22}=-\left(\alpha_{1}+\mu_{1}\right), a_{32}=\alpha_{1} \\
& a_{33}=-\left(\alpha_{2}+\mu_{1}+\mu_{2}\right), a_{43}=r_{1}, a_{44}=-\left(\mu_{3}-r_{2}\right) \\
& b_{11}=-\left(\sigma_{1} \widehat{V}+\sigma_{2} \widehat{I}\right), b_{13}=-\sigma_{2} \widehat{S}, b_{14}=-\sigma_{1} \widehat{S}  \tag{21}\\
& b_{21}=\left(\sigma_{1} \widehat{V}+\sigma_{2} \widehat{I}\right), b_{23}=\sigma_{2} \widehat{S}, b_{24}=\sigma_{1} \widehat{S}
\end{align*}
$$

As a result, the Jacobian matrix $J_{\widehat{E}}$ of the model (2) is given as

$$
J_{\widehat{E}}=\left[\begin{array}{cccc}
\left(a_{11}+b_{11} e^{-\lambda \tau}\right) & 0 & b_{13} e^{-\lambda \tau} & b_{14} e^{-\lambda \tau}  \tag{22}\\
b_{21} e^{-\lambda \tau} & a_{22} & b_{23} e^{-\lambda \tau} & b_{24} e^{-\lambda \tau} \\
0 & a_{32} & a_{33} & 0 \\
0 & 0 & a_{43} & a_{44}
\end{array}\right]
$$

which leads to
$\lambda^{4}+A_{1} \lambda^{3}+A_{2} \lambda^{2}+A_{3} \lambda+A_{4}+\left(B_{1} \lambda^{3}+B_{2} \lambda^{2}+B_{3} \lambda+B_{4}\right) e^{-\lambda \tau}=0$,
where

$$
\begin{align*}
A_{1}= & \mu_{1}+\left(\alpha_{1}+\mu_{1}\right)+\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)+\left(\mu_{3}-r_{2}\right)>0, \\
A_{2}= & \left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{1}+\left(\alpha_{1}+\mu_{1}\right)+\left(\mu_{3}-r_{2}\right)\right) \\
& +\left(\mu_{3}-r_{2}\right)\left(\mu_{1}+\left(\alpha_{1}+\mu_{1}\right)\right)+\mu_{1}\left(\alpha_{1}+\mu_{1}\right)>0, \\
A_{3}= & -\left[\mu _ { 1 } \left(\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)+\left(\alpha_{1}+\mu_{1}\right)\left(\mu_{3}-r_{2}\right)\right.\right. \\
& \left.+\left(\mu_{3}-r_{2}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\right)+\left(\alpha_{1}+\mu_{1}\right) \\
& \left.\cdot\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)\right]<0, \\
A_{4}= & \mu_{1}\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)>0, \\
B_{1}= & \sigma_{1} \widehat{V}+\sigma_{2} \widehat{I}>0, \\
B_{2}= & \left(\sigma_{1} \widehat{V}+\sigma_{2} \widehat{I}\right)\left[\left(\alpha_{1}+\mu_{1}\right)+\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)+\left(\mu_{3}-r_{2}\right)\right] \\
& -\alpha_{1} \sigma_{2} \widehat{S}, \\
B_{3}= & \left(\sigma_{1} \widehat{V}+\sigma_{2} \widehat{I}\right)\left[( \alpha _ { 2 } + \mu _ { 1 } + \mu _ { 2 } ) \left(\left(\alpha_{1}+\mu_{1}\right)\right.\right. \\
& \left.\left.+\left(\mu_{3}-r_{2}\right)\right)+\left(\alpha_{1}+\mu_{1}\right)\left(\mu_{3}-r_{2}\right)\right] \\
& -\left(r_{1} \alpha_{1} \sigma_{1} \widehat{S}+\alpha_{1} \mu_{1} \sigma_{2} \widehat{S}+\left(\mu_{3}-r_{2}\right) \alpha_{1} \sigma_{2} \widehat{S}\right), \\
B_{4}= & \left(\mu_{3}-r_{2}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\sigma_{1} \widehat{V}+\sigma_{2} \widehat{I}\right)  \tag{24}\\
& -\left(\left(\mu_{3}-r_{2}\right) \alpha_{1} \mu_{1} \sigma_{2} \widehat{S}+r_{1} \alpha_{1} \mu_{1} \sigma_{1} \widehat{S}\right) .
\end{align*}
$$

Next, we elaborate on the main result, namely, the local stability of both the disease-free equilibrium $E^{0}$ and the endemic equilibrium $E^{1}$ in the context of model 2 for all $\tau \geq 0$, through the following theorems.

Theorem 2. The DFEP is locally asymptotically stable in $\Omega$ for $\mathscr{R}_{0}<1$ and unstable for $\mathscr{R}_{0}>1$.

Proof. Substituting the value of $E^{0}$ in Eqs. (22) and (23) gives, respectively, that

$$
\begin{align*}
& J_{E^{0}}=\left[\begin{array}{cccc}
-\mu_{1} & 0 & -\sigma_{2} S^{0} e^{-\lambda \tau} & -\sigma_{1} S^{0} e^{-\lambda \tau} \\
0 & -\left(\alpha_{1}+\mu_{1}\right) & \sigma_{2} S^{0} e^{-\lambda \tau} & \sigma_{1} S^{0} e^{-\lambda \tau} \\
0 & \alpha_{1} & -\left(\alpha_{2}+\mu_{1}+\mu_{2}\right) & 0 \\
0 & 0 & r_{1} & -\left(\mu_{3}-r_{2}\right)
\end{array}\right],  \tag{25}\\
& \left(-\mu_{1}-\lambda\right)\left[\lambda^{3}+A_{10} \lambda^{2}+A_{20} \lambda+A_{30}-\left(B_{10} \lambda+B_{20}\right) e^{-\lambda \tau}\right]=0,
\end{align*}
$$

where
$A_{10}=\left(\alpha_{1}+\mu_{1}\right)+\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)+\left(\mu_{3}-r_{2}\right)>0$,
$A_{20}=\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\left(\alpha_{1}+\mu_{1}\right)+\left(\mu_{3}-r_{2}\right)\right)+\left(\alpha_{1}+\mu_{1}\right)\left(\mu_{3}-r_{2}\right)>0$,
$A_{30}=\left(\alpha_{1}+\mu_{1}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\mu_{3}-r_{2}\right)>0$,
$B_{10}=\alpha_{1} \sigma_{2} S^{0}>0$,
$B_{20}=\alpha_{1}\left[r_{1} \sigma_{1}+\left(\mu_{3}-r_{2}\right) \sigma_{2}\right] S^{0}>0$.

Obviously, the first eigenvalue is always negative and given by $\lambda_{10}=-\mu_{1}<0$, while the other three eigenvalues can be obtained from the third-degree polynomial in Eq. (26) in two cases.

For $\tau=0$, the third-degree polynomial in Eq. (26) transfers to

$$
\begin{equation*}
\lambda^{3}+A_{10} \lambda^{2}+\left(A_{20}-B_{10}\right) \lambda+A_{30}\left(1-\mathscr{R}_{0}\right)=0 . \tag{28}
\end{equation*}
$$

According to the Routh-Hurwitz ( $\mathrm{R}-\mathrm{H}$ ) criterion, all roots of Eq. (28) are negative or have negative real parts if and only if $A_{10}>0, A_{30}\left(1-\mathscr{R}_{0}\right)>0$, and $A_{10}\left(A_{20}-B_{10}\right)-$ $A_{30}\left(1-\mathscr{R}_{0}\right)>0$. Direct computation shows that the R-H requirements follow if $\mathscr{R}_{0}<1$ and violate if $\mathscr{R}_{0}>1$. Therefore, when $\tau=0$, the DFEP is locally asymptotically stable if $\mathscr{R}_{0}<1$ and saddle point if $\mathscr{R}_{0}>1$.

Now, for $\tau>0$, the third-degree polynomial in Eq. Eq. (26) can be rewritten in the form

$$
\begin{equation*}
\mathscr{F}(\lambda)=\mathscr{G}(\lambda), \tag{29}
\end{equation*}
$$

where $\mathscr{F}(\lambda)=\lambda^{3}+A_{10} \lambda^{2}+A_{20} \lambda$ and $\mathscr{G}(\lambda)=A_{30}\left[\left(B_{10} / A_{30} \lambda\right.\right.$ $\left.\left.+\mathscr{R}_{0}\right) e^{-\lambda \tau}-1\right]$; then, the following is obtained.

When $\mathscr{R}_{0}>1$, it is observed that $\mathscr{F}(0)=0$ and $\lim _{\lambda \longrightarrow+\infty}$ $\mathscr{F}(\lambda)=\infty$, while $\mathscr{G}(0)=A_{30}\left(\mathscr{R}_{0}-1\right)>0$ and $\lim _{\lambda \rightarrow+\infty} \mathscr{G}(\lambda)$ $=-A_{30}<0$, which means that $\mathscr{G}(\lambda)$ is decreasing. Therefore, $\mathscr{F}(\lambda)$ and $\mathscr{G}(\lambda)$ must intersect for some $\lambda^{*}>0$. Hence, Eq. (29) has a positive real solution, and that makes $E^{0}$ unstable.

However, when $\mathscr{R}_{0}<1$, it is obtained that for $\lambda \geq 0$, $\mathscr{F}(\lambda)$ becomes an increasing function while $\mathscr{G}(\lambda)$ remains a decreasing function of $\lambda$, with $\mathscr{G}(0)=A_{4}\left(\mathscr{R}_{0}-1\right)<0$. So, there is no guarantee to have a positive intersection point as in the above case. Consequently, to verify that, let us assume that $\lambda=i \omega_{1}\left(\omega_{1}>0\right)$, which satisfies Eq. (29). Then, after some mathematical manipulation, it is obtained that

$$
\begin{equation*}
\omega_{1}^{6}+\left(A_{10}^{2}-2 A_{20}\right) \omega_{1}^{4}+\left(A_{20}^{2}-2 A_{10} A_{30}-A_{10}^{2}\right) \omega_{1}^{2}+A_{30}^{2}-B_{20}^{2}=0 . \tag{30}
\end{equation*}
$$

$\operatorname{Set} \omega_{1}^{2}=Z_{1}$ yields

$$
\begin{equation*}
Z_{1}^{3}+\left(A_{10}^{2}-2 A_{20}\right) Z_{1}^{2}+\left(A_{20}^{2}-2 A_{10} A_{30}-A_{10}{ }^{2}\right) Z_{1}+A_{30}^{2}-B_{20}^{2}=0, \tag{31}
\end{equation*}
$$

where

$$
\begin{gather*}
A_{10}^{2}-2 A_{20}>0, \\
A_{20}^{2}-2 A_{10} A_{30}-A_{10}^{2},  \tag{32}\\
A_{30}^{2}-B_{20}^{2}=A_{30}^{2}\left(1-\mathscr{R}_{0}^{2}\right)>0 .
\end{gather*}
$$

According to Descartes' Rule of Signs, if $A_{20}{ }^{2}-2 A_{10}$ $A_{30}-A_{10}{ }^{2}>0$, then there is no positive root for Eq. (31), and hence, Eq. (26) does not have positive eigenvalues. However, if $A_{20}{ }^{2}-2 A_{10} A_{30}-A_{10}{ }^{2}<0$, then Eq. (31) has either two or no positive roots, which means there is no firm decision to have a positive root that leads to a periodic solution around the DFEP. Suppose such a periodic solution exists, since the existence of a periodic solution around the DFEP requires that at least one of the axes that represent solutions be cut, this contradicts the uniqueness of the solution. Therefore, there is no positive root for Eq. (31), so the DFEP is stable.

Theorem 3. The EEP is locally asymptotically stable in $\Omega$ for $\mathscr{R}_{0}>1$ with $\tau \in\left[0, \tau_{0}\right)$ provided that

$$
\left.\begin{array}{l}
B_{3}^{*}>A_{3}  \tag{33}\\
\sigma_{1} V^{*}+\sigma_{2} I^{*}>\mu_{1}\left(\alpha_{1}+\mu_{1}\right) \\
\left(A_{1}+B_{1}^{*}\right)\left(A_{2}+B_{2}^{*}\right)\left(A_{3}+B_{3}^{*}\right)>\left(A_{3}+B_{3}^{*}\right)^{2}+\left(A_{1}+B_{1}^{*}\right)^{2}\left(A_{4}+B_{4}^{*}\right)
\end{array}\right\} .
$$

It is an unstable for $\tau>\tau_{0}$, provided that

$$
\begin{equation*}
A_{4}^{2}-B_{4}^{*^{2}}<0 \tag{34}
\end{equation*}
$$

Proof. Substituting $E^{1}=\left(S^{*}, E^{*}, I^{*}, V^{*}\right)$ instead of $\widehat{E}=(\widehat{S}, \widehat{E}$, $\widehat{I}, \widehat{V}$ ) in Eqs. (22) and (23) leads after some algebraic computation to the following characteristic equation.

$$
\begin{equation*}
\lambda^{4}+A_{1} \lambda^{3}+A_{2} \lambda^{2}+A_{3} \lambda+A_{4}+\left(B_{1}^{*} \lambda^{3}+B_{2}^{*} \lambda^{2}+B_{3}^{*} \lambda+B_{4}^{*}\right) e^{-\lambda \tau}=0 \tag{35}
\end{equation*}
$$

where $A_{i}, i=1,2,3,4$, is given in Eq. (23) with

$$
\begin{align*}
B_{1}^{*}= & \sigma_{1} V^{*}+\sigma_{2} I^{*} \\
B_{2}^{*}= & \left(\sigma_{1} V^{*}+\sigma_{2} I^{*}\right)\left[\left(\alpha_{1}+\mu_{1}\right)+\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)+\left(\mu_{3}-r_{2}\right)\right] \\
& -\alpha_{1} \sigma_{2} S^{*}, \\
B_{3}^{*}= & \left(\sigma_{1} V^{*}+\sigma_{2} I^{*}\right)\left[( \alpha _ { 2 } + \mu _ { 1 } + \mu _ { 2 } ) \left(\left(\alpha_{1}+\mu_{1}\right)\right.\right. \\
& \left.\left.+\left(\mu_{3}-r_{2}\right)\right)+\left(\alpha_{1}+\mu_{1}\right)\left(\mu_{3}-r_{2}\right)\right] \\
& -\left(r_{1} \alpha_{1} \sigma_{1} S^{*}+\alpha_{1} \mu_{1} \sigma_{2} S^{*}+\left(\mu_{3}-r_{2}\right) \alpha_{1} \sigma_{2} S^{*}\right) \\
B_{4}^{*}= & \left(\mu_{3}-r_{2}\right)\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(\sigma_{1} V^{*}+\sigma_{2} I^{*}\right)  \tag{36}\\
& -\left(\left(\mu_{3}-r_{2}\right) \alpha_{1} \mu_{1} \sigma_{2} S^{*}+r_{1} \alpha_{1} \mu_{1} \sigma_{1} S^{*}\right) .
\end{align*}
$$

Now, for $\tau=0$, Eq. (35) transfers to
$\lambda^{4}+\left(A_{1}+B_{1}^{*}\right) \lambda^{3}+\left(A_{2}+B_{2}^{*}\right) \lambda^{2}+\left(A_{3}+B_{3}^{*}\right) \lambda+\left(A_{4}+B_{4}^{*}\right)=0$.

Thus, according to the R-H criterion, all roots of Eq. (37) have negative real parts provided that $\left(A_{1}+B_{1}^{*}\right)>0,\left(A_{3}+\right.$ $\left.B_{3}^{*}\right)>0,\left(A_{4}+B_{4}^{*}\right)>0$, and $\left(A_{1}+B_{1}^{*}\right)\left(A_{2}+B_{2}^{*}\right)\left(A_{3}+B_{3}^{*}\right)>$ $\left(A_{3}+B_{3}^{*}\right)^{2}+\left(A_{1}+B_{1}^{*}\right)^{2}\left(A_{4}+B_{4}^{*}\right)$. Direct computation shows that these conditions are satisfied if the condition set (33) holds. Hence, the EEP is locally asymptotically stable.

The transfer from the stable case to unstable required that the real part of at least one of the eigenvalues intersect the imaginary axis. Therefore, to check the possibility of transferring the behavior of EEP from stable to unstable for the case $\tau>0, \lambda=i \omega_{2}\left(\omega_{2}>0\right)$ is substituted into Eq. (35); then, after doing some algebraic steps, the following are obtained:

$$
\begin{align*}
\omega_{2}^{4}-A_{2} \omega_{2}^{2}+A_{4}= & \left(B_{2}^{*} \omega_{2}^{2}-B_{4}^{*}\right) \cos \left(\omega_{2} \tau\right)  \tag{38}\\
& +\left(B_{1}^{*} \omega_{2}^{3}-B_{3}^{*} \omega_{2}\right) \sin \left(\omega_{2} \tau\right) \\
A_{3} \omega_{2}-A_{1} \omega_{2}^{3}= & \left(B_{1}^{*} \omega_{2}^{3}-B_{3}^{*} \omega_{2}\right) \cos \left(\omega_{2} \tau\right) \\
& -\left(B_{2}^{*} \omega_{2}^{2}-B_{4}^{*}\right) \sin \left(\omega_{2} \tau\right) . \tag{39}
\end{align*}
$$

Now, by squaring and adding, the following is obtained:

$$
\begin{align*}
\omega_{2}^{8} & +\left(A_{1}^{2}-\left(2 A_{2}+B_{1}^{*^{2}}\right)\right) \omega_{2}^{6} \\
& +\left(2\left(A_{4}+B_{1}^{*} B_{3}^{*}\right)-\left(A_{2}^{2}+2 A_{1} A_{3}+B_{2}^{*^{2}}\right)\right) \omega_{2}^{4} \\
& +\left(A_{3}^{2}+2 B_{2}^{*} B_{4}^{*}-\left(2 A_{2} A_{4}+B_{3}^{*^{2}}\right)\right) \omega_{2}^{2}+\left(A_{4}^{2}-B_{4}^{*^{2}}\right)=0 \tag{40}
\end{align*}
$$

Taking $\omega_{2}^{2}=Z_{2}$, Eq. (40) becomes

$$
\begin{equation*}
Z_{2}^{4}+q_{1} Z_{2}^{3}+q_{2} Z_{2}^{2}+q_{3} Z_{2}+q_{4}=0 \tag{41}
\end{equation*}
$$

where

$$
\begin{align*}
& q_{1}=A_{1}^{2}-\left(2 A_{2}+B_{1}^{*^{2}}\right) \\
& q_{2}=2\left(A_{4}+B_{1}^{*} B_{3}^{*}\right)-\left(A_{2}^{2}+2 A_{1} A_{3}+B_{2}^{*^{2}}\right)  \tag{42}\\
& q_{3}=A_{3}^{2}+2 B_{2}^{*} B_{4}^{*}-\left(2 A_{2} A_{4}+B_{3}^{*^{2}}\right) \\
& q_{4}=A_{4}^{2}-B_{4}^{*^{2}}
\end{align*}
$$

Obviously, condition (34) guarantees the existence of at least one positive root for Eq. (41). Therefore, $Z_{20}$ denotes the positive root of Eq. (41), leading to Eq. (35) having two imaginary roots $\pm i \omega_{20}= \pm i \sqrt{Z_{20}}$. That leads to the fact that there is a vital point $\tau_{0}>0$ so that EEP becomes unstable for $\tau>\tau_{0}>0$.

Now, by utilizing Eqs. (38) and (39), the relevant critical delay value for $\omega_{20}$ can be determined as

$$
\begin{equation*}
\tau_{0}=\frac{1}{\omega_{20}} \cos ^{-1}\left(\frac{h_{1} \omega_{2}^{6}+h_{2} \omega_{2}^{4}+h_{3} \omega_{2}^{2}+h_{4}}{\boldsymbol{g}_{1} \omega_{2}^{6}+\boldsymbol{g}_{2} \omega_{2}^{4}+\boldsymbol{g}_{3} \omega_{2}^{2}+\boldsymbol{g}_{4}}\right) \tag{43}
\end{equation*}
$$

where

$$
\begin{align*}
& h_{1}=B_{2}^{*}-B_{1}^{*} A_{1} \\
& h_{2}=B_{1}^{*} A_{3}+B_{3}^{*} A_{1}-\left(B_{2}^{*} A_{2}+B_{4}^{*}\right) \\
& h_{3}=B_{2}^{*} A_{4}+B_{4}^{*} A_{2}-B_{3}^{*} A_{3} \\
& h_{4}=-B_{4}^{*} A_{4} \\
& \boldsymbol{g}_{1}=B_{1}^{*^{2}}  \tag{44}\\
& \boldsymbol{g}_{2}=B_{2}^{*^{2}}-2 B_{1}^{*} B_{3}^{*} \\
& \boldsymbol{g}_{3}=B_{3}^{*^{2}}-B_{2}^{*} B_{4}^{*} \\
& \boldsymbol{g}_{4}=B_{4}^{*^{2}}
\end{align*}
$$

In the next section, we will delve into the condition when $\tau=\tau_{0}$, and its impact on the dynamic equilibrium of the CCHF model will be examined.

## 6. Hopf Bifurcation Analysis

In this section, the conditions under which Hopf bifurcation arises are established, employing the time lag $\tau$ as the bifurcation parameter while considering the stipulation of $\mathscr{R}_{0}>1$ , to signify the existence of EEP, through the next theorem.

Theorem 4. The EEP of model 2 becomes unstable, and a Hopf bifurcation at $\tau=\tau_{0}$ is born provided that

$$
\begin{equation*}
\mathscr{C}_{1} \mathscr{M}-\mathscr{C}_{2} \mathscr{N} \neq 0 \tag{45}
\end{equation*}
$$

where $\mathscr{M}, \mathcal{N}, \mathscr{C}_{1}$, and $\mathscr{C}_{2}$ are defined in the proof.
Proof. According to Theorem 3, model 2 has pure imaginary eigenvalues at $\tau=\tau_{0}$, and hence, the EEP became unstable for $\tau>\tau_{0}$. Now, to prove that a Hopf bifurcation is born, it is sufficient to show that $\left.(d / d \tau) \operatorname{Re}\{\lambda(\tau)\}\right|_{\tau=\tau_{0}} \neq 0$. Substituting $\lambda(\tau)=\alpha_{2}(\tau)+i \omega_{2}(\tau)$ into Eq. (35) consequently yields the following result:

$$
\begin{align*}
\mathscr{M}(\tau) \frac{d \alpha_{2}}{d \tau}+\mathscr{N}(\tau) \frac{d \omega_{2}}{d \tau} & =\mathscr{C}_{1}(\tau)  \tag{46}\\
-\mathscr{N}(\tau) \frac{d \alpha_{2}}{d \tau}+\mathscr{M}(\tau) \frac{d \omega_{2}}{d \tau} & =\mathscr{C}_{2}(\tau)
\end{align*}
$$

where

$$
\begin{aligned}
\mathscr{M}(\tau)= & A_{3}-3 A_{1} \omega_{2}^{2}+\left(\tau\left(B_{2}^{*} \omega_{2}^{2}-B_{4}^{*}\right)-\left(3 B_{1}^{*} \omega_{2}^{2}-B_{3}^{*}\right)\right) \\
& \cdot \cos \left(\omega_{2} \tau\right)+\left(\tau\left(B_{1}^{*} \omega_{2}^{3}-B_{3}^{*} \omega_{2}\right)+2 B_{2}^{*} \omega_{2}^{2}\right) \sin \left(\omega_{2} \tau\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{N}(\tau)= & 4 \omega_{2}^{4}-2 A_{2} \omega_{2}+\left(\tau\left(B_{3}^{*} \omega_{2}-B_{1}^{*} \omega_{2}^{3}\right)-2 B_{2}^{*} \omega_{2}\right) \cos \left(\omega_{2} \tau\right) \\
& +\left(\tau\left(B_{2}^{*} \omega_{2}^{2}-B_{4}^{*}\right)-\left(3 B_{1}^{*} \omega_{2}^{2}-B_{3}^{*}\right)\right) \sin \left(\omega_{2} \tau\right)
\end{aligned}
$$

$\mathscr{C}_{1}(\tau)=\omega_{2}\left(\left(B_{2}^{*} \omega_{2}^{2}-B_{4}^{*}\right) \sin \left(\omega_{2} \tau\right)-\left(B_{1}^{*} \omega_{2}^{3}-B_{3}^{*} \omega_{2}\right) \cos \left(\omega_{2} \tau\right)\right)$,

$$
\begin{equation*}
\mathscr{C}_{2}(\tau)=\omega_{2}\left(\left(B_{1}^{*} \omega_{2}^{3}-B_{3}^{*} \omega_{2}\right) \sin \left(\omega_{2} \tau\right)+\left(B_{2}^{*} \omega_{2}^{2}-B_{4}^{*}\right) \cos \left(\omega_{2} \tau\right)\right) \tag{47}
\end{equation*}
$$

Following Cramer's rule, we obtain

$$
\begin{equation*}
\left.\frac{\mathrm{d} \alpha_{2}}{\mathrm{~d} \tau}\right|_{\substack{\tau=\tau_{0} \\ \omega_{2}=\omega_{20}}}=\frac{\mathscr{C}_{1} \mathscr{M}-\mathscr{C}_{2} \cdot \mathcal{N}}{\mathscr{M}^{2}+\mathscr{N}^{2}} \tag{48}
\end{equation*}
$$

Thus, $\left.(d / d \tau) \operatorname{Re}\{\lambda(\tau)\}\right|_{\substack{\tau=\tau_{0} \\ \omega_{2}=\omega_{20}}} \neq 0$, if precondition
(45) is met. This concludes the proof.

## 7. Direction and Stability of the Hopf Bifurcation

This section presents in detail the precise formulas that, using $\tau_{0}$ as the bifurcation parameter, determine the direction, stability, and period of periodic solutions bifurcating in a CCHF model 2 around $E^{1}$.

Let $\quad U_{1}(t)=S(t)-S^{*}, U_{2}(t)=E(t)-E^{*}, U_{3}(t)=I(t)-$ $I^{*}, U_{4}(t)=V(t)-V^{*}$, and $\gamma=\tau-\tau_{0}$, where $\gamma \in \mathbb{R}$ and at $\gamma$ $=0$ gives the parameter of the Hopf bifurcation. By normalizing time as $t \longrightarrow t / \tau$, model 2 may be represented as

$$
\begin{equation*}
\dot{U}(t)=T_{\gamma}\left(U_{t}\right)+F\left(\gamma, U_{t}\right) \tag{49}
\end{equation*}
$$

where $U(t)=\left(U_{1}(t), U_{2}(t), U_{3}(t), U_{4}(t)\right)^{T} \in \mathbb{R}^{4}, \quad T_{\gamma}: C$ $\longrightarrow \mathbb{R}^{4}, F: \mathbb{R} \times C \longrightarrow \mathbb{R}^{4}$, and $C=C\left([-1,0], \mathbb{R}_{+}^{4}\right)$. Thus, for $\varphi=\left(\varphi_{1}, \varphi_{2}, \varphi_{3}, \varphi_{4}\right)^{T} \in C$, we obtain

$$
\begin{align*}
T_{\gamma}(\varphi) & =\left(\tau_{0}+\gamma\right)(\mathscr{A} \varphi(0)+\mathscr{B} \varphi(-1)) \\
F(\gamma, \varphi) & =\left(\tau_{0}+\gamma\right)\left(\begin{array}{c}
F_{1} \\
F_{2} \\
F_{3} \\
F_{4}
\end{array}\right) \tag{50}
\end{align*}
$$

with

$$
\begin{align*}
\mathscr{A} & =\left(\begin{array}{cccc}
f_{1000}^{(1)} & 0 & 0 & 0 \\
0 & f_{1000}^{(2)} & 0 & 0 \\
0 & f_{0100}^{(3)} & f_{0010}^{(3)} & 0 \\
0 & 0 & f_{0010}^{(4)} & f_{0001}^{(4)}
\end{array}\right) \\
& =\left(\begin{array}{cccc}
a_{11} & 0 & 0 & 0 \\
0 & a_{22} & 0 & 0 \\
0 & a_{32} & a_{33} & 0 \\
0 & 0 & a_{43} & a_{44}
\end{array}\right), \\
\mathscr{B} & =\left(\begin{array}{cccc}
f_{0100}^{(1)} & 0 & f_{0010}^{(1)} & f_{0001}^{(1)} \\
f_{0100}^{(2)} & 0 & f_{0010}^{(2)} & f_{0001}^{(2)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)  \tag{51}\\
& =\left(\begin{array}{cccc}
b_{11} & 0 & b_{13} & b_{14} \\
b_{21} & 0 & b_{23} & b_{24} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{align*}
$$

In this context, the notations $a_{i j}$ and $b_{i j}$ represent the elements in Eq. (20). Moreover,

$$
\begin{align*}
& F_{1}=\sum_{a+j+k+l \geq 2} \frac{1}{a!j!k!!!} f_{a j k l}^{(1)} \varphi_{1}^{a}(0) \varphi_{1}^{j}(-1) \varphi_{3}^{k}(-1) \varphi_{4}^{l}(-1) \\
& F_{2}=\sum_{b+j+k+l \geq 2} \frac{1}{b!j!k!l!} f_{b j k l}^{(2)} \varphi_{2}^{b}(0) \varphi_{1}^{j}(-1) \varphi_{3}^{k}(-1) \varphi_{4}^{l}(-1) \\
& F_{3}=\sum_{b+c \geq 2} \frac{1}{b!c!} f_{b c}^{(3)} \varphi_{2}^{b}(0) \varphi_{3}^{c}(0) \\
& F_{4}=\sum_{c+d \geq 2} \frac{1}{c!d!} f_{c d}^{(4)} \varphi_{3}^{c}(0) \varphi_{4}^{d}(0) \tag{52}
\end{align*}
$$

where $a, b, c, d, j, k$, and $l \geq 0$ are integer numbers. Next, we define

$$
\begin{aligned}
f^{(1)}\left(\varphi_{1}, \tilde{\varphi}_{1}, \tilde{\varphi}_{3}, \tilde{\varphi}_{4}\right)= & \Lambda-\left(\sigma_{1} \tilde{\varphi}_{1} \tilde{\varphi}_{4}+\sigma_{1} V^{*} \tilde{\varphi}_{1}\right. \\
& +\sigma_{1} S^{*} \tilde{\varphi}_{4}+\sigma_{1} S^{*} V^{*}+\sigma_{2} \tilde{\varphi}_{1} \tilde{\varphi}_{3} \\
& +\sigma_{2} I^{*} \tilde{\varphi}_{1}+\sigma_{2} S^{*} \tilde{\varphi}_{3}+\sigma_{2} S^{*} I^{*} \\
& \left.+\mu_{1}\left(S^{*}+\varphi_{1}\right)\right),
\end{aligned}
$$

$$
\begin{align*}
f^{(2)}\left(\varphi_{2}, \tilde{\varphi}_{1}, \tilde{\varphi}_{3}, \tilde{\varphi}_{4}\right)= & \sigma_{1} \tilde{\varphi}_{1} \tilde{\varphi}_{4}+\sigma_{1} V^{*} \tilde{\varphi}_{1} \\
& +\sigma_{1} S^{*} \tilde{\varphi}_{4}+\sigma_{1} S^{*} V^{*}+\sigma_{2} \tilde{\varphi}_{1} \tilde{\varphi}_{3} \\
& +\sigma_{2} I^{*} \tilde{\varphi}_{1}+\sigma_{2} S^{*} \tilde{\varphi}_{3}+\sigma_{2} S^{*} I^{*} \\
& -\left(\alpha_{1}+\mu_{1}\right)\left(E^{*}+\varphi_{2}\right), \\
f^{(3)}\left(\varphi_{2}, \varphi_{3}\right)= & \alpha_{1}\left(E^{*}+\varphi_{2}\right)-\left(\alpha_{2}+\mu_{1}+\mu_{2}\right)\left(I^{*}+\varphi_{3}\right), \\
f^{(4)}\left(\varphi_{3}, \varphi_{4}\right)= & r_{1}\left(I^{*}+\varphi_{3}\right)-\left(\mu_{3}-r_{2}\right)\left(V^{*}+\varphi_{4}\right), \\
f_{a j k l}^{(1)}= & \left.\frac{\partial^{a+j+k+l} f^{(1)}}{\partial \varphi_{1}^{a} \tilde{\varphi}_{1}^{j} \tilde{\varphi}_{3}^{k} \tilde{\varphi}_{4}^{l}}\right|_{\left(\varphi_{1}, \tilde{\varphi}_{1}, \tilde{\varphi}_{3}, \tilde{\varphi}_{4}\right)=(0,-1,-1,-1)}, \\
f_{b j k l}^{(2)}= & \left.\frac{\partial^{b+j+k+l} f^{(2)}}{\partial \varphi_{2}^{b} \tilde{\varphi}_{1}^{j} \tilde{\varphi}_{3}^{k} \tilde{\varphi}_{4}^{l}}\right|_{\left(\varphi_{2}, \tilde{\varphi}_{1}, \tilde{\varphi}_{3}, \tilde{\varphi}_{4}\right)=(0,-1,-1,-1)}, \\
f_{b c}^{(3)}= & \left.\frac{\partial^{b+c} f^{(3)}}{\partial \varphi_{2}^{b} \varphi_{3}^{c}}\right|_{\left(\varphi_{2}, \varphi_{3}\right)=(0,0,0,0)}, \\
f_{c d}^{(4)}= & \left.\frac{\partial^{c+d} f^{(4)}}{\partial \varphi_{3}^{c} \varphi_{4}^{d}}\right|_{\left(\varphi_{3}, \varphi_{4}\right)=(0,0,0,0)} \tag{53}
\end{align*}
$$

A $4 \times 4$ matrix function $\psi(\chi, \gamma)$ with entries having a bounded variation for $\chi \in[-1,0]$ can be found by using the Riesz representation theorem [49], allowing for the following:

$$
\begin{equation*}
T_{\gamma}(\varphi)=\int_{-1}^{0} d \psi(\chi, \gamma) \varphi(\chi), \quad \text { for } \varphi \in C \tag{54}
\end{equation*}
$$

In fact, we have a choice.

$$
\begin{equation*}
\psi(\chi, \gamma)=\left(\tau_{0}+\gamma\right)(\mathscr{A} \delta(\chi)-\mathscr{B} \delta(\chi+1)) \tag{55}
\end{equation*}
$$

In this case, $\delta(\chi)$ stands for the Dirac delta function. For $\varphi \in C^{1}\left([-1,0], \mathbb{R}_{+}^{4}\right)$, define that

$$
\begin{aligned}
& Q(\gamma) \varphi(\chi)= \begin{cases}\frac{d \varphi(\chi)}{d \chi}, & -1 \leq \chi<0, \\
\int_{-1}^{0} d \psi(\chi, \gamma) \varphi(\chi), & \chi=0,\end{cases} \\
& R(\gamma) \varphi(\chi)= \begin{cases}0, & -1 \leq \chi<0, \\
F(\gamma, \varphi), & \chi=0 .\end{cases}
\end{aligned}
$$

Then, model 2 can be expressed as

$$
\begin{equation*}
\dot{U}(t)=Q(\gamma) U_{t}+R(\gamma) U_{t} \tag{57}
\end{equation*}
$$

where $U_{t}(\chi)=U(t+\chi)$ for $\chi \in[-1,0]$.

Furthermore, for $\Psi \in C^{1}\left([0,1],\left(\mathbb{R}_{+}^{4}\right)^{*}\right)$, the adjoint operator $Q^{*}$ of $Q(0)$ is given by

$$
Q^{*} \Psi(\kappa)= \begin{cases}-\frac{d \Psi(\kappa)}{d \kappa}, & 0<\kappa \leq 1  \tag{58}\\ \int_{-1}^{0} d \psi^{T}(t, 0) \Psi(-t), & \kappa=0\end{cases}
$$

That is connected to a bilinear form

$$
\begin{equation*}
\langle\Psi(\kappa), \varphi(\chi)\rangle=\bar{\Psi}(0) \varphi(0)-\int_{\chi=-1}^{0} \int_{v=0}^{\chi} \bar{\psi}(\omega-\chi) d \psi(\chi) \varphi(v) d v \tag{59}
\end{equation*}
$$

where $\psi(\chi)=\psi(\chi, 0)$.
Obviously, model 2 contains $\pm i \omega_{20} \tau_{0}$, which are defined as the eigenvalues of $Q(0)$ and $Q^{*}$. Therefore, by conducting a straightforward calculation similar to the one shown in [50], we can derive the following:

$$
\left.\begin{array}{l}
q(\chi)=\left(1, q_{2}, q_{3}, q_{4}\right)^{T} e^{i \omega_{20} \tau_{0} \chi}  \tag{60}\\
q^{*}(\kappa)=D\left(1, q_{2}^{*}, q_{3}^{*}, q_{4}^{*}\right)^{T} e^{-i \omega_{20} \tau_{0} \kappa}
\end{array}\right\}
$$

where

$$
\begin{aligned}
& q_{2}=\frac{-\left(f_{0010}^{(3)}-i \omega_{20}\right) q_{3}}{f_{0100}^{(3)}}, \\
& q_{3}=\frac{-\left(f_{0001}^{(4)}-i \omega_{20}\right) q_{4}}{f_{0010}^{(4)}}, \\
& q_{4}=\frac{i \omega_{20} f_{0010}^{(4)}-\left(f_{1000}^{(1)} f_{0010}^{(4)}+f_{0100}^{(1)} f_{0010}^{(4)} e^{-i \omega_{20} \tau_{0}}\right)}{\left(f_{0001}^{(1)} f_{0010}^{(4)}+i \omega_{20} f_{0010}^{(1)}-f_{0010}^{(1)} f_{0001}^{(4)}\right) e^{-i \omega_{20} \tau_{0}}}, \\
& q_{2}^{*}=\frac{-\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{-i \omega_{20} \tau_{0}}+i \omega_{20}\right)}{f_{0100}^{(2)} e^{-i \omega_{20} \tau_{0}}}, \\
& q_{3}^{*}=\frac{\left(f_{1000}^{(2)}+i \omega_{20}\right)\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{-i \omega_{20} \tau_{0}}+i \omega_{20}\right)}{f_{0100}^{(2)} f_{0100}^{(3)} e^{-i \omega_{20} \tau_{0}}}, \\
& q_{4}^{*}=\frac{\left(f_{0001}^{(1)}+f_{0001}^{(2)}\right)\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{-i \omega_{20} \tau_{0}}+i \omega_{20}\right)}{\left(f_{0001}^{(4)}+i \omega_{20}\right) f_{0100}^{(2)}} .
\end{aligned}
$$

Derived from Eq. (59), one can deduce the following:

$$
\begin{align*}
\left\langle q^{*}(\kappa), q(\chi)\right\rangle= & \bar{D}\left[\left(1+q_{2} \overline{q_{2}^{*}}+q_{3} \overline{q_{3}^{*}}+q_{4} \overline{q_{4}^{*}}\right)\right. \\
& +\tau_{0}\left(f_{0100}^{(1)}+q_{3} f_{0010}^{(1)}+q_{4} f_{0001}^{(1)}\right. \\
& \left.\left.+\overline{q_{2}^{*}}\left(f_{0100}^{(2)}+q_{3} f_{0010}^{(2)}+q_{4} f_{0001}^{(2)}\right)\right) e^{-i \omega_{20} \tau_{0}}\right] \tag{62}
\end{align*}
$$

which gives

$$
\begin{align*}
\bar{D}= & {\left[\left(1+q_{2} \overline{q_{2}^{*}}+q_{3} \overline{q_{3}^{*}}+q_{4} \overline{q_{4}^{*}}\right)\right.} \\
& +\tau_{0}\left(f_{0100}^{(1)}+q_{3} f_{0010}^{(1)}+q_{4} f_{0001}^{(1)}\right.  \tag{63}\\
& \left.\left.+\overline{q_{2}^{*}}\left(f_{0100}^{(2)}+q_{3} f_{0010}^{(2)}+q_{4} f_{0001}^{(2)}\right)\right) e^{-i \omega_{20} \tau_{0}}\right]^{-1}
\end{align*}
$$

such that $\left\langle q^{*}(\kappa), q(\chi)\right\rangle=1$ and $\left\langle q^{*}(\kappa), \bar{q}(\chi)\right\rangle=0$.
Using the same methodology described in [51], we can then reach the following conclusions:

$$
\begin{align*}
g(\hbar, \bar{\hbar}) & =\overline{q^{*}}(0) F_{0}(\hbar, \bar{\hbar})=\tau_{0} \bar{D}\left(1, \overline{q_{2}^{*}}, \overline{q_{3}^{*}}, \overline{q_{4}^{*}}\right) \\
& \cdot\left(\begin{array}{c}
\mathscr{J}_{1} \hbar^{2}+\mathscr{J}_{2} \hbar \bar{\hbar}+\mathscr{J}_{3} \bar{\hbar}^{2}+\mathscr{J}_{4} \hbar^{2} \bar{\hbar}+\cdots \\
\mathscr{J}_{5} \hbar^{2}+\mathscr{J}_{6} \hbar \bar{\hbar}+\mathscr{J}_{7} \bar{\hbar}^{2}+\mathscr{J}_{8} \hbar^{2} \bar{\hbar}+\cdots \\
0 \\
0
\end{array}\right) \\
& =P_{1} \hbar^{2}+P_{2} \hbar \bar{\hbar}+P_{3} \bar{\hbar}^{2}+P_{4} \hbar^{2} \bar{\hbar}+\text { h.o.i. } \tag{64}
\end{align*}
$$

Here, "h.o.i." denotes higher-order items, which gives the following:

$$
\left.\begin{array}{l}
g_{20}=2 P_{1}  \tag{65}\\
g_{11}=P_{2} \\
g_{02}=2 P_{3} \\
g_{21}=2 P_{4}
\end{array}\right\}
$$

where

$$
\begin{align*}
& P_{1}=\tau_{0} \bar{D}\left(\mathscr{J}_{1}+\overline{q_{2}^{*}} \mathscr{J}_{5}\right), \\
& P_{2}=\tau_{0} \bar{D}\left(\mathscr{J}_{2}+\overline{q_{2}^{*}} \mathscr{J}_{6}\right), \\
& P_{3}=\tau_{0} \bar{D}\left(\mathscr{J}_{3}+\overline{q_{2}^{*}} \mathscr{J}_{7}\right),  \tag{66}\\
& P_{4}=\tau_{0} \bar{D}\left(\mathscr{J}_{4}+\overline{q_{2}^{*}} \mathscr{J}_{8}\right),
\end{align*}
$$

with

$$
\begin{align*}
& \mathscr{J}_{1}=\left(q_{3} f_{0110}^{(1)}+q_{4} f_{0101}^{(1)}\right) e^{-2 i \omega_{20} \tau_{0}}, \\
& \mathscr{F}_{2}=\left(q_{3}+\overline{q_{3}}\right) f_{0110}^{(1)}+\left(q_{4}+\overline{q_{4}}\right) f_{0101}^{(1)}, \\
& \mathcal{J}_{3}=\left(\overline{q_{3}} f_{0110}^{(1)}+\overline{q_{4}} f_{0101}^{(1)}\right) e^{2 i \omega_{20} \tau_{0}},  \tag{69}\\
& \mathcal{J}_{4}=\xi_{1} f_{0110}^{(1)}+\xi_{2} f_{0101}^{(1)},  \tag{67}\\
& \mathcal{J}_{5}=\left(q_{3} f_{0110}^{(2)}+q_{4} f_{0101}^{(2)}\right) e^{-2 i \omega_{20} \tau_{0}}, \\
& \mathcal{F}_{6}=\left(q_{3}+\overline{q_{3}}\right) f_{0110}^{(2)}+\left(q_{4}+\overline{q_{4}}\right) f_{0101}^{(2)}, \\
& \mathcal{J}_{7}=\left(\overline{q_{3}} f_{0110}^{(2)}+\overline{q_{4}} f_{0101}^{(2)}\right) e^{2 i \omega_{20} \tau_{0}}, \\
& \mathscr{F}_{8}=\xi_{1} f_{0110}^{(2)}+\xi_{2} f_{0101}^{(2)} .
\end{align*}
$$

can be computed as

$$
\left.\begin{array}{l}
U_{20}(\chi)=\frac{i g_{20}}{\omega_{20} \tau_{0}} q(0) e^{i \omega_{20} \tau_{0} \chi}+\frac{i \bar{g}_{20}}{3 \omega_{20} \tau_{0}} \bar{q}(0) e^{-i \omega_{20} \tau_{0} \chi}+\mathscr{R}_{1} e^{2 i \omega_{20} \tau_{0} \chi} \\
U_{11}(\chi)=-\frac{i g_{11}}{\omega_{20} \tau_{0}} q(0) e^{i \omega_{20} \tau_{0} \chi}+\frac{i \bar{g}_{11}}{\omega_{20} \tau_{0}} \bar{q}(0) e^{-i \omega_{20} \tau_{0} \chi}+\mathscr{R}_{2} \tag{70}
\end{array}\right\},
$$

It is stated that computing $U_{20}(\chi)$ and $U_{11}(\chi)$ must be prioritized in order to determine the value $g_{21}$. This indicates that

$$
\begin{aligned}
& U_{20}(\chi)=\left(U_{20}^{(1)}(\chi), U_{20}^{(2)}(\chi), U_{20}^{(3)}(\chi), U_{20}^{(4)}(\chi)\right)^{T} \\
& U_{11}(\chi)=\left(U_{11}^{(1)}(\chi), U_{11}^{(2)}(\chi), U_{11}^{(3)}(\chi), U_{11}^{(4)}(\chi)\right)^{T}
\end{aligned}
$$

Here,

$$
\begin{align*}
& \xi_{1}=\left(q_{3} U_{11}^{(1)}(-1)+U_{11}^{(3)}(-1)\right) e^{-i \omega_{20} \tau_{0}} \\
& \xi_{2}=\frac{1}{2}\left(\overline{q_{3}} U_{20}^{(1)}(-1)+U_{20}^{(3)}(-1)\right) e^{i \omega_{20} \tau_{0}} \tag{68}
\end{align*}
$$

,
where $\mathscr{R}_{\mathrm{i}}=\left(\mathscr{R}_{i}^{(1)}, \mathscr{R}_{i}^{(2)}, \mathscr{R}_{i}^{(3)}, \mathscr{R}_{i}^{(4)}\right)^{T} \in \mathbb{R}_{+}^{4}$ for $i=1,2$ represents constant vectors, obtained from the following equations:

$$
\begin{gather*}
\left(\begin{array}{cccc}
2 i \omega_{20}-\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi}\right) & 0 & -f_{0010}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} \\
-f_{0100}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & 2 i \omega_{20}-f_{1000}^{(2)} & -f_{0010}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} \\
0 & -f_{0100}^{(3)} & 2 i \omega_{20}-f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & 2 i \omega_{20}-f_{0001}^{(4)}
\end{array}\right)\left(\begin{array}{c}
\mathscr{R}_{1}^{(1)} \\
\mathscr{R}_{1}^{(2)} \\
\mathscr{R}_{1}^{(3)} \\
\mathscr{R}_{1}^{(4)}
\end{array}\right)=2\left(\begin{array}{c}
\mathcal{J}_{1} \\
\mathscr{J}_{5} \\
0 \\
0
\end{array}\right),  \tag{71}\\
\left(\begin{array}{cccc}
-f_{1000}^{(1)} & 0 & 0 & 0 \\
0 & -f_{1000}^{(2)} & 0 & 0 \\
0 & -f_{0100}^{(3)} & -f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & -f_{0001}^{(4)}
\end{array}\right)\left(\begin{array}{c}
\mathscr{R}_{2}^{(1)} \\
\mathscr{R}_{2}^{(2)} \\
\mathscr{R}_{2}^{(3)} \\
\mathscr{R}_{2}^{(4)}
\end{array}\right)=\left(\begin{array}{c}
\mathscr{J}_{2} \\
\mathscr{J}_{6} \\
0 \\
0
\end{array}\right)
\end{gather*}
$$

Applying Cramer's rule, we have the following:

$$
\begin{equation*}
\mathscr{R}_{1}^{(i)}=\frac{\left|\mathscr{D}_{1 i}\right|}{\left|\mathscr{D}_{1}\right|}, \mathscr{R}_{2}^{(i)}=\frac{\left|\mathscr{D}_{2 i}\right|}{\left|\mathscr{D}_{2}\right|} \quad \text { for } i=1,2,3,4 \tag{72}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathscr{D}_{1}=\left(\begin{array}{cccc}
2 i \omega_{20}-\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi}\right) & 0 & -f_{0010}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} \\
-f_{0100}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & 2 i \omega_{20}-f_{1000}^{(2)} & -f_{0010}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} \\
0 & -f_{0100}^{(3)} & 2 i \omega_{20}-f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & 2 i \omega_{20}-f_{0001}^{(4)}
\end{array}\right), \\
& \mathscr{D}_{2}=\left(\begin{array}{cccc}
-f_{1000}^{(1)} & 0 & 0 & 0 \\
0 & -f_{1000}^{(2)} & 0 & 0 \\
0 & -f_{0100}^{(3)} & -f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & -f_{0001}^{(4)}
\end{array}\right), \\
& \mathscr{D}_{11}=\left(\begin{array}{cccc}
\mathscr{J}_{1} & 0 & -f_{0010}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} \\
\mathscr{J}_{5} & 2 i \omega_{20}-f_{1000}^{(2)} & -f_{0010}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} \\
0 & -f_{0100}^{(3)} & 2 i \omega_{20}-f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & 2 i \omega_{20}-f_{0001}^{(4)}
\end{array}\right), \\
& \mathscr{D}_{12}=\left(\begin{array}{cccc}
2 i \omega_{20}-\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi}\right) & \mathcal{J}_{1} & -f_{0010}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} \\
-f_{0100}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & \mathscr{J}_{5} & -f_{0010}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & -f_{0001}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} \\
0 & 0 & 2 i \omega_{20}-f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & 2 i \omega_{20}-f_{0001}^{(4)}
\end{array}\right),  \tag{73}\\
& \mathscr{D}_{13}=\left(\begin{array}{cccc}
2 i \omega_{20}-\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi}\right) & 0 & \mathcal{J}_{1} & -f_{0001}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} \\
-f_{0100}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & 2 i \omega_{20}-f_{1000}^{(2)} & \mathcal{J}_{5} & -f_{0001}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} \\
0 & -f_{0100}^{(3)} & 0 & 0 \\
0 & 0 & 0 & 2 i \omega_{20}-f_{0001}^{(4)}
\end{array}\right), \\
& \mathscr{D}_{14}=\left(\begin{array}{cccc}
2 i \omega_{20}-\left(f_{1000}^{(1)}+f_{0100}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi}\right) & 0 & -f_{0010}^{(1)} e^{2 i \omega_{20} \tau_{0} \chi} & \mathscr{J}_{1} \\
-f_{0100}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & 2 i \omega_{20}-f_{1000}^{(2)} & -f_{0010}^{(2)} e^{2 i \omega_{20} \tau_{0} \chi} & \mathscr{J}_{5} \\
0 & -f_{0100}^{(3)} & 2 i \omega_{20}-f_{0010}^{(3)} & 0 \\
0 & 0 & -f_{0010}^{(4)} & 0
\end{array}\right) .
\end{align*}
$$

Similarly, $\mathscr{D}_{2 i}$ can be calculated for $i=1,2,3,4$. As an outcome, it is now possible to compute both $U_{20}(\chi)$ and $U_{11}(\chi)$, as described in (70), using the information provided in (72). Additionally, it becomes simple to identify the $g_{21}$

$$
\begin{equation*}
C_{1}(0)=\frac{i}{2 \omega_{20} \tau_{0}}\left(g_{11} g_{20}-2\left|g_{11}\right|^{2}-\frac{1}{3}\left|g_{02}\right|^{2}\right)+\frac{1}{2} g_{21} \tag{74}
\end{equation*}
$$



FIgURe 2: The trajectory of system 1 using the set (76). (a) Approaches to EEP when $\Lambda=1$ with $\mathscr{R}_{0}=3.333$. (b) Approaches to DFEP when $\Lambda=0.2$ with $\mathscr{R}_{0}=0.6667$.

Moreover, we have

$$
\left.\begin{array}{l}
\mathscr{M}_{2}=-\frac{\operatorname{Re}\left\{C_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}}  \tag{75}\\
\mathscr{Y}_{2}=2 \operatorname{Re}\left\{C_{1}(0)\right\} \\
\mathscr{T}_{2}=-\frac{\operatorname{Im}\left\{C_{1}(0)\right\}+\mathscr{M}_{2} \operatorname{Im}\left\{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}\right\}}{\omega_{20} \tau_{0}}
\end{array}\right\} .
$$

The dynamical properties of the manifold periodic solutions will be determined at the bifurcation parameter $\tau_{0}$ based on the sign of the variables $\mathscr{M}_{2}, \mathscr{Y}_{2}$, and $\mathscr{T}_{2}$. Further, according to the methodology described in [51], the following theorem is obtained directly.

Theorem 5. For CCHF model 2,
(1) If $\mathscr{M}_{2}>0\left(\mathscr{M}_{2}<0\right)$, then the Hopf bifurcation is supercritical (subcritical)
(2) If $\mathscr{Y}_{2}<0\left(\mathscr{Y}_{2}>0\right)$, then the bifurcation periodic trajectories are stable (unstable)
(3) If $\mathscr{T}_{2}>0\left(\mathscr{T}_{2}<0\right)$, then the bifurcating periodic trajectories increase (decrease)

## 8. Numerical Simulations

To support the theoretical analysis presented above in this study, a numerical simulation is used in this section. To comprehend the impact of these parameters on the spread of disease, the sensitivity analysis of system 1 is also included using the following estimated set of parameter values.


Figure 3: Sensitivity of basic reproduction number using data set given Eq. (76).

$$
\begin{gather*}
\Lambda=1, \sigma_{1}=0.1, \sigma_{2}=0.1, \mu_{1}=0.1, \alpha_{1}=0.05, \alpha_{2}=0.5 \\
\mu_{2}=0.5, r_{1}=0.1, r_{2}=0.1, \mu_{3}=0.11 \tag{76}
\end{gather*}
$$

It is observed that for this set of data, the value of the basic reproduction number is $\mathscr{R}_{0}=3.3333$, and hence, the system approaches asymptotically to EEP which is given by $E^{1}=(3,4.66,0.21,1.06,2.12)$ as shown in the Figure 2(a), while it approaches DFEP which is given by $E^{0}=(2,0,0,0$, 0 ) when the value of $\Lambda<0.3$, due to the falling value of $\mathscr{R}_{0}$ in the range $\mathscr{R}_{0} \leq 1$, as presented in Figure 2(b).


FIgure 4: The trajectory of system 1 using data (76) with $\tau=10.5$. (a) Time series of the solution approaches to periodic dynamics. (b) The projection on SEV space. (c) The projection on SIV space. (d) The projection on EIV space.

According to Figure 2, it is concluded that the obtained analytical results are verified, and increasing the value of $\Lambda$ increases the spread of disease. Now that it understands the influence of other parameters, the sensitivity analysis of the basic reproduction number is used.

When researching infectious illness models, one of the most crucial factors to consider is the fundamental reproduction number. The fundamental reproduction number of the model was discovered using Eq. (18). A sensitivity analysis is currently being done on the fundamental reproduction number. We can learn from such studies the importance of each variable in the transmission of disease. Sensitivity analysis is widely used to evaluate how parameter values affect model forecasts since errors in data collection and expected parameter values are frequent. It is used to pinpoint variables that should be the subject of intervention efforts because they significantly affect $\mathscr{R}_{0}$. To put it more precisely, sensitivity measures allow you to quantify the proportional alteration in a variable when a parameter is altered. This is accomplished using the normalized forward sensitivity index of a variable, which is the ratio of the relative
change in the variable to the relative change in the parameter. If the variable in question is differentiable with respect to the parameter, the sensitivity index is constructed using partial derivatives as follows.

The standardized forward sensitivity index of $\mathscr{R}_{0}$, calculated by [52], is differentiable for a parameter.

$$
\begin{equation*}
\operatorname{SEN}\left(\mathscr{R}_{0}, \Gamma\right)=\frac{\partial \mathscr{R}_{0}}{\partial \Gamma} \cdot \frac{\Gamma}{\mathscr{R}_{0}} \tag{77}
\end{equation*}
$$

As a result, the following can be used to compute the standardized forward sensitivity index of $\mathscr{R}_{0}$ with regard to model 2's parameters:

$$
\begin{aligned}
& \operatorname{SEN}\left(\mathscr{R}_{0}, \Lambda\right)=1 \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \sigma_{1}\right)=\frac{r_{1} \sigma_{1}}{r_{1} \sigma_{1}+\left(\mu_{3}-r_{2}\right) \sigma_{2}} \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \sigma_{2}\right)=1-\frac{r_{1} \sigma_{1}}{r_{1} \sigma_{1}+\left(\mu_{3}-r_{2}\right) \sigma_{2}}
\end{aligned}
$$

$$
\begin{align*}
& \operatorname{SEN}\left(\mathscr{R}_{0}, \mu_{1}\right)=-2+\frac{\alpha_{1}}{\alpha_{1}+\mu_{1}}-\frac{\mu_{1}}{\alpha_{2}+\mu_{1}+\mu_{2}}, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \alpha_{1}\right)=\frac{\mu_{1}}{\alpha_{1}+\mu_{1}}, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \alpha_{2}\right)=-\frac{\alpha_{2}}{\alpha_{2}+\mu_{1}+\mu_{2}}, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \mu_{2}\right)=-\frac{\mu_{2}}{\alpha_{2}+\mu_{1}+\mu_{2}}, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, r_{1}\right)=\frac{r_{1} \sigma_{1}}{r_{1} \sigma_{1}+\left(\mu_{3}-r_{2}\right) \sigma_{2}}, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, r_{2}\right)=\frac{r_{1} r_{2} \sigma_{1}}{\left(r_{2}-\mu_{3}\right)\left(-r_{1} \sigma_{1}+\left(r_{2}-\mu_{3}\right) \sigma_{2}\right)}, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \mu_{3}\right)=-\frac{r_{1} \mu_{3} \sigma_{1}}{\left(r_{2}-\mu_{3}\right)\left(-r_{1} \sigma_{1}+\left(r_{2}-\mu_{3}\right) \sigma_{2}\right)} \tag{78}
\end{align*}
$$

Consequently, utilizing the data set given by Eq. (76), it is obtained that

$$
\begin{align*}
& \operatorname{SEN}\left(\mathscr{R}_{0}, \Lambda\right)=1, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \sigma_{1}\right)=0.9090909090909093, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \sigma_{2}\right)=0.09090909090909083, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \mu_{1}\right)=-1.7575757575757576, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \alpha_{1}\right)=0.6666666666666666,  \tag{79}\\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \alpha_{2}\right)=-0.45454545454545453, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \mu_{2}\right)=-0.45454545454545453, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, r_{1}\right)=0.9090909090909093, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, r_{2}\right)=9.090909090909097, \\
& \operatorname{SEN}\left(\mathscr{R}_{0}, \mu_{3}\right)=-10 .
\end{align*}
$$

Therefore, in the following Figure 3, the sensitivity analysis is represented.

According to Figure 3, the set of parameters that are positively proportional with $\mathscr{R}_{0}$ is given by $\Lambda, \sigma_{1}, \sigma_{2}, \alpha_{1}, r_{1}, r_{2}$, while the set of parameters that are negatively proportional with $\mathscr{R}_{0}$ is given by $\mu_{l}, \alpha_{2}, \mu_{2}, \mu_{3}$. However, the parameters that do not affect $\mathscr{R}_{0}$ are $\tau$.

To comprehend the effect of the delay parameter $\tau$, which has no effect on $\mathscr{R}_{0}$ as illustrated in Figure 3, as seen in the aforementioned sections, the value of $\tau$ is gradually increased in order to find the critical value $\tau_{0}$. It is noted that for the data (76) with $\tau>\tau_{0}=10.45$, the system becomes unstable, and a Hopf bifurcation takes place; see Figure 4 for the value of $\tau=10.5$.

Obviously, Figure 4 indicates the occurrence of Hopf bifurcation as $\tau>\tau_{0}=10.45$, which confirms the analytical results.

## 9. Discussion

In this paper, an epidemic model of the Crimean-Congo hemorrhagic fever virus has been suggested and analyzed, with direct transmission resulting through contact with infected individuals as well as indirect transmission via pathogens. It was considered that the delay was brought on by the system's incubation period for the infection. The goal of the study is to comprehend how illness transmission can be managed as well as the impact of delay on model dynamics. Two equilibrium points in the system have been observed. The basic reproduction number stability requirements for both places were determined with and without a time delay. The DFEP is seen to be asymptotically stable for $\tau \geq 0$ for $\mathscr{R}_{0}<1$, but unstable when $\mathscr{R}_{0}>1$. However, the EEP is unstable with the occurrence of periodic Hopf bifurcation for $\tau \geq \tau_{0}$ and asymptotically stable for $\tau<\tau_{0}$ with $\mathscr{R}_{0}>1$. The central manifold theory is used to explore the characteristics of the bifurcating Hopf bifurcation.

In order to validate the obtained analytical results and comprehend the control set of characteristics that affect the transmission of disease, a numerical simulation has lastly been applied utilizing an estimated data set (76) on the sample population. To pinpoint the impact of the parameters on the illness outbreak, the sensitivity analysis has been used with the same set of data. It is discovered that all of the analytical findings are reliable. While the EEP is asymptotically stable when $\mathscr{R}_{0}>1$ and $\tau<10.45$ and Hopf bifurcation occurs for $\tau=10.45$ with an increase in the magnitude of the period as $\tau>10.45$, the DFEP is asymptotically stable when $\mathscr{R}_{0}<1$ for all values of $\tau$. The host population recruitment rate, infection rates, the rate at which an infected person becomes contagious, and pathogen growth rates are found to be the characteristics that increase the outbreak of disease as their value increases. The host natural and diseased death rates, the host recovery rate, and the pathogen decay rate, on the other hand, are the variables that regulate the spread of illness as their values rise.

## Data Availability

All the data is within the text.

## Conflicts of Interest

The authors declare that there are no competing interests regarding the publication of this paper.

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