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Research Article

Tensor z-Transform

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The multi-input multioutput (MIMO) systems involving multirelational signals generated from distributed sources have been emerging as the most generalized model in practice. The existing work for characterizing such a MIMO system is to build a corresponding transform tensor, each of whose entries turns out to be the individual z-transform of a discrete-time impulse response sequence. However, when a MIMO system has a global feedback mechanism, which also involves multirelational signals, the aforementioned individual z-transforms of the overall transfer tensor are quite difficult to formulate. Therefore, a new mathematical framework to govern both feedforward and feedback MIMO systems is in crucial demand. In this work, we define the tensor z-transform to characterize a MIMO system involving multirelational signals as a whole rather than the individual entries separately, which is a novel concept for system analysis. To do so, we extend Cauchy's integral formula and Cauchy's residue theorem from scalars to arbitrary-dimensional tensors, and then, to apply these new mathematical tools, we establish to undertake the inverse tensor z-transform and approximate the corresponding discrete-time tensor sequences. Our proposed new tensor z-transform in this work can be applied to design digital tensor filters including infinite-impulse-response (IIR) tensor filters (involving global feedback mechanisms) and finite-impulse-response (FIR) tensor filters. Finally, numerical evaluations are presented to demonstrate certain interesting phenomena of the new tensor z-transform.

1. Introduction

Tensors or multidimensional arrays are functions of three or more indices $(i, j, k, \cdots)$, which are mathematical objects generalized from matrices (two-dimensional arrays) and vectors (one-dimensional arrays). The development of new tensor theories and tensor-related algorithms has recently drawn the attention of the signal-processing society [1–3], big-data analytics [4–6], and system design [7, 8] as *multirelational characterization* among different attributes and objects is crucial for applications of modern signal and system analysis. It is well-known that the z-transform has been applied as an indispensable tool for the analysis and design of discrete-time signals and systems. Generally speaking, the z-transform converts a real- or complex-valued discrete-time sequence to a complex-valued z-function [9–18]. The existing z-transform is actually a scalar function. Recently, the multi-input multioutput (MIMO) systems involving multirelational signals have been emerging as the most generalized model in practice [1–6]. Our preceding work for characterizing such a MIMO system is to build a corresponding "transform tensor," each of whose entries turns out to be the individual z-transform of a discrete-time impulse response sequence [8]. Take an autoregressive–moving-average (ARMA) tensor-filter for example as illustrated by Figure 1 (the details for implementing an ARMA tensor filter (or a "GARMA" filter) will be presented in Section 5.1 later on). In general, such an ARMA tensor filter involves the "global feedback mechanism" if any of the *feedback coefficient tensors* $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ in Figure 1 is a *nondiagonal* tensor. A tensor $\mathscr{B} \in$

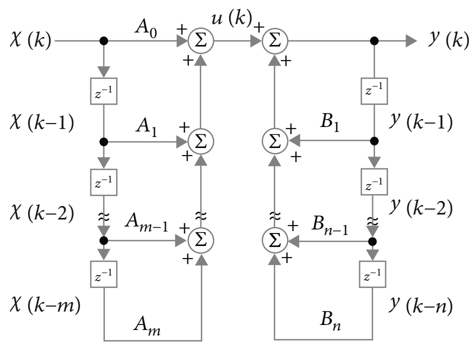


Figure 1: Illustration of the realization of a GARMA-(n, m) filter.

$\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ is a nondiagonal tensor if there exists an entry $b_{i_1,\cdots,i_m;j_1,\cdots,_m}$ in $\mathscr{B}$ such that

$$b_{i_1,\cdots,i_m;j_1,\cdots,_m} \neq 0, \tag{1}$$

where $\sum_{k=1}^{m}(i_k - j_k)^2 \neq 0$. Note that we use semicolons to separate the subscript indices contributed by the row and column parts. The commas between subscript indices are used to separate indices within the same row part or the same column part. For example, the row part of the subscripts of the entry $b_{i_1,\cdots,i_m;j_1,\cdots,_m}$ is $i_1, \cdots, i_m$ while the column part of the subscripts of the entry $b_{i_1,\cdots,i_m;j_1,\cdots,_m}$ is $j_1, \cdots, j_m$. On the other hand, we say that a MIMO system is without any global feedback mechanism if all feedback-coefficient tensors $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ in Figure 1 are diagonal tensors, i.e.,

$$b_{i_1,\cdots,i_m;j_1,\cdots,_m} = 0, \tag{2}$$

where $\sum_{k=1}^{m}(i_k - j_k)^2 \neq 0$. If the condition described by Eq. (2) holds, the transfer tensor (refer to [8] for the introduction of transfer tensor) for a MIMO system (the ARMA tensor filter as illustrated by Figure 1) without any global feedback mechanism can be reduced to a tensor whose entries turn out to be the individual rational z-transforms easily as there exists no coupling relationship [8].

However, when a MIMO system has a global feedback mechanism as described by Eq. (1), which involves the coupling relationship among signals, the aforementioned individual z-transforms of the overall transfer tensor are quite difficult to formulate. Throughout this work, we call a MIMO system without any feedback mechanism a "feedforward MIMO system" where $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ in Figure 1 are all zero tensors (all their entries are zero); we call a MIMO system without any global feedback mechanism (Eq. (2) holds) a "local-feedback MIMO system" where not all of $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ in Figure 1 are zero tensors; at last, we call a MIMO system with global feedback mechanism(s) (Eq. (1) holds) a "global-feedback MIMO system." Therefore, a new mathematical framework to govern all of the feedforward, local-feedback, and global-feedback MIMO systems is in crucial demand. In this work, we propose the novel "tensor z-transform" to characterize an arbitrary MIMO system involving multirelational signals as a whole rather than the individual entries separately.

Some extensions have been made from the conventional z-transform to deal with tensor signals. In [19, 20], the multidimensional z-transformation (referred to as "tensor" in [19, 20]) is introduced to rewrite convolutions and sums of convolutions as products and sums of series. Our proposed tensor z-transform is different from these works since there must involve multiple z-domain variables to characterize a tensor in different ranks in [19, 20]; the multidimensional z-transformation is adopted simply as the generating function for counting the transition probabilities [21] while it is aimed at dealing with the signal and system analysis. To the best of our knowledge, there exists no technique to extend the conventional z-transform from a sequence of complex numbers (scalar) to a sequence of tensors. The obvious difference between the tensor z-transform in this work and the D-transform in [22] is that the tensor z -transform can transform time-domain signals expressed by arbitrary multidimensional arrays (tensors) while the D -transform can only deal with the two-dimensional arrays (matrices). The main difficulty for such an extension is to establish Cauchy's residue theorem for tensors. In this work, we first define the new tensor z-transform and discuss the related properties. Then, we generalize Cauchy's residue theorem from scalars to tensors in order to perform the inverse tensor z-transform. Different from the power-series approach, i.e., long division, the time-domain tensor sequence obtained by taking the inverse z-transform via Cauchy's residue theorem for tensors is not unique since it depends on what contour is chosen in the complex tensor integration. The variability to obtain different time-domain tensor sequences from a tensor z-transform can lead to a new approach to approximate time-domain tensor signal sequences with much less implementational effort from the perspectives of computation and memory complexities. The relationship between the time-domain tensor sequence obtained by applying the power-series approach and that obtained by utilizing Cauchy's residue theorem for tensors to carry out the inverse tensor z -transform will also be established in this work. An application of tensor z-transform for digital tensor-filter design will be discussed. Besides, the frequency responses of the tensor z-transform, namely, the "ensemble magnitude response" and the "ensemble phase response," will be investigated. Finally, numerical evaluations will be presented to demonstrate certain interesting phenomena about the tensor z-transform.

The rest of this paper is organized as follows. The concept about tensor z-transform will be introduced in Section 2. New theories regarding Cauchy's residue theorem for tensors will be presented in Section 3. In Section 4, two approaches to carry out the inverse tensor z-transform, namely, the power-series approach and the integration approach via Cauchy's residue theorem for tensors, will be introduced. An application of the tensor z-transform for digital tensor-filter design will be presented in Section 5. Numerical evaluations about the frequency responses of the tensor z-transform, the tensor-sequence approximation error, and the implementational complexities will be demonstrated in Section 6. The conclusion will be finally drawn in Section 7.

1.1. Nomenclature. The sets of complex and real numbers are denoted by $\mathbb{C}$ and $\mathbb{R}$, respectively. The set of all integers is denoted by $\mathbb{Z}$. The natural-number set is represented by $\mathbb{N}$, and $\mathbb{N}_{\geq 0}$ denotes the set of nonnegative integers. The symbol "$\overset{\text{def}}{=}$" denotes a mathematical definition. The tensor dimensionality discussed in this paper can be expressed by $\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ if it is not specified in the context. Note that ι is reserved to represent $\sqrt{-1}$. Spectrum and eigenvalue share the same meaning throughout this work. Given two invertible tensors $\mathscr{A},\mathscr{B}\in\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathscr{A}/\mathscr{B}$ represents $\mathscr{A}\star_M\mathscr{B}^{-1}$ where "$\star_M$" denotes the "Einstein product" or "tensor product" as defined in [3].

2. Tensor z-Transform

In this section, we would like to introduce a new mathematical framework, namely, the tensor z-transform. The essential definitions and related examples are presented in Section 2.1 while the important properties are described in Section 2.2.

2.1. Definitions and Examples. A *causal tensor sequence* $\mathscr{X}[n]$ is a tensor sequence such that $\mathscr{X}[n]\overset{\text{def}}{=}\mathscr{O}$ for all $n<0$ where $\mathscr{O}$ denotes an *all-zero tensor*. The exponential of a tensor $\mathscr{A}\in\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, denoted by $\exp(\mathscr{A})$, is written as

$$\exp(\mathscr{A})\overset{\text{def}}{=}\sum_{i=0}^{\infty}\mathscr{A}^i, \tag{3}$$

where $\mathscr{A}^i\overset{\text{def}}{=}\underbrace{\mathscr{A}\star_M\mathscr{A}\star_M\cdots\star_M\mathscr{A}}_{\text{there are } i \text{ tensors}}$ and "$\star_M$" denotes the *Einstein product* (a.k.a. *tensor product*) as introduced in [3]. Consider a sequence of tensors indexed by n, denoted by $\mathscr{X}[n]$ where $n=0,1,2,\cdots$. The *tensor* z-transform is given by

$$\boldsymbol{\mathfrak{X}}(z)\overset{\text{def}}{=}\mathscr{Z}(\{\mathscr{X}[n]\})\overset{\text{def}}{=}\sum_{n=-\infty}^{\infty}\mathscr{X}[n]z^{-n}. \tag{4}$$

Lemma 1 below will establish the required convergence radius of z for Eq.(4). We say a sequence of tensors is a *causal tensor sequence* if $\mathscr{X}[n]=\mathscr{O}$ for $n<0$, while a sequence of tensors is called an *anticausal tensor sequence* if $\mathscr{X}[n]=\mathscr{O}$ for $n<0$.

Lemma 1. *Consider a power series in terms of the complex variable z as given by*

$$\sum_{n=0}^{\infty}\mathscr{C}_n(z-z_0)^n, \tag{5}$$

where $\mathscr{C}_n$ denotes the n-th element (tensor) in a tensor sequence and z_0 denotes an arbitrary complex constant. The radius of convergence of the power series stated by Eq. (5) is thus given by

$$R=\frac{1}{\lim_{n\to\infty}\sqrt[n]{\|\mathscr{C}_n\|}}, \tag{6}$$

where $0<R<\infty$ and "$\|\ \|$" represents the tensor norm as introduced in [3].

Proof. Without loss of generality, one may assume $z_0=0$. Now we will prove that the power series $\sum_{n=0}^{\infty}\mathscr{C}_nz^n$ converges for $|z|<R$ while it diverges for $|z|>R$ instead. Given $|z|<R$ and any $\epsilon>0$, there exists only a finite number of indices n such that $\sqrt[n]{\|\mathscr{C}_n\|}\geq 1/R+\epsilon$. Consequently, we may have $\|\mathscr{C}_n\|\leq(1/R+\epsilon)^n$ for all but a finite number of tensors $\mathscr{C}_n$, so the tensor series $\sum_{n=0}^{\infty}\mathscr{C}_nz^n$ converges if $|z|<1/(1/R+\epsilon)$. Therefore, the power series $\sum_{n=0}^{\infty}\mathscr{C}_nz^n$ converges for $|z|<R$. On the other hand, given $\epsilon>0$, if we have an infinite number of tensors $\mathscr{C}_n$ such that $\|\mathscr{C}_n\|\geq(1/R-\epsilon)^n$, then $|z|=1/(1/R-\epsilon)>R$ and the tensor series cannot converge since $\|\mathscr{C}_n\|$ does not converge to 0.

To illustrate the concept of the tensor z-transform, we present the following example. □

Example 1. A sequence of tensors is given by $\{\exp(-\mathscr{A}n), n\in\mathbb{N}_{\geq 0}\}$ where $\mathscr{A}$ is a constant tensor. Thus, the corresponding radius of convergence is given by

$$\left\|z^{-1}\exp(-\mathscr{A})\right\|<1 \text{ or } |z|>\left\|e^{-\mathscr{A}}\right\|. \tag{7}$$

The z-transform of $\{\exp(-\mathscr{A}n), n\in\mathbb{N}_{\geq 0}\}$ can be obtained as

$$\begin{aligned}\mathscr{Z}(\{\exp(-\mathscr{A}n), n\in\mathbb{N}_{\geq 0}\})&=\sum_{n=0}^{\infty}\left[\exp(-\mathscr{A})z^{-1}\right]^n\\&=\frac{\mathscr{I}}{\mathscr{I}-\exp(-\mathscr{A})z^{-1}},\end{aligned} \tag{8}$$

where $\mathscr{I}$ denotes an identity tensor (see [3] for its definition). When $\mathscr{A}$ is a zero tensor, we have

$$\mathscr{Z}(\{\exp(-\mathscr{A}n), n\in\mathbb{N}_{\geq 0}\})=\sum_{n=0}^{\infty}\left(\mathscr{I}z^{-1}\right)^n=\frac{\mathscr{I}}{\mathscr{I}-\mathscr{I}z^{-1}}, \tag{9}$$

where $\mathscr{I}\in\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$.

2.2. Properties. In this subsection, we will introduce several important properties about the tensor z-transform as follows. Due to page limit, we will present the basic properties related to the tensor z-transform, e.g., *linearity*, *time delay*, *upsampling*, and *downsampling*.

2.2.1. Linearity. Consider a linear combination of two tensor sequences $\mathscr{X}[n]$ and $\mathscr{Y}[n]$. We have

$$\mathscr{A}\star_M\mathscr{X}[n]+\mathscr{B}\star_M\mathscr{Y}[n]\quad\text{for } n\in\mathbb{Z}, \tag{10}$$

where $\mathscr{A},\mathscr{B}\in\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$. According to Eq. (4), we have

$$\mathscr{Z}(\mathscr{A}\star_M\mathscr{X}[n]+\mathscr{B}\star_M\mathscr{Y}[n]) = \sum_{n=-\infty}^{\infty}(\mathscr{A}\star_M\mathscr{X}[n]+\mathscr{B}\star_M\mathscr{Y}[n])z^{-n} = \mathscr{A}\star_M\mathfrak{X}(z)+\mathscr{B}\star_M\mathfrak{Y}(z), \quad (11)$$

where $\mathfrak{X}(z) =^{\text{def}} \mathscr{Z}(\mathscr{X}[n])$ and $\mathfrak{Y}(z)\overset{\text{def}}{=}\mathscr{Z}(\mathscr{Y}[n])$. Therefore,

$$\mathscr{Z}(\mathscr{A}\star_M\mathscr{X}[n]+\mathscr{B}\star_M\mathscr{Y}[n]) = \mathscr{A}\star_M\mathfrak{X}(z)+\mathscr{B}\star_M\mathfrak{Y}(z). \quad (12)$$

2.2.2. Time Delay. The delayed version $\mathscr{X}[n-k]$ of a tensor sequence $\mathscr{X}[n]$ is obtained by delaying k discrete-time instants of $\mathscr{X}[n]$. Thus, we have

$$\begin{aligned}\mathscr{Z}(\mathscr{X}[n-k]) &= \sum_{n=-\infty}^{\infty}\mathscr{X}[n-k]z^{-n} \\ &= \sum_{n'=-\infty}^{\infty}\mathscr{X}\left[n'\right]z^{-(n'+k)} \\ &= \sum_{n'=-\infty}^{\infty}\mathscr{X}\left[n'\right]z^{-n'}z^{-k} \\ &= z^{-k}\sum_{n'=-\infty}^{\infty}\mathscr{X}\left[n'\right]z^{-n'} \\ &= z^{-k}\sum_{n'=-\infty}^{\infty}\mathscr{X}\left[n'\right]z^{-n'} \\ &= z^{-k}\mathfrak{X}(z).\end{aligned} \quad (13)$$

Therefore, we have

$$\mathscr{Z}(\mathscr{X}[n-k]) = z^{-k}\mathfrak{X}(z). \quad (14)$$

2.2.3. Upsampling. Given a tensor sequence $\mathscr{X}[n]$, we define $\mathscr{X}_K[n]$, where $K\in\mathbb{N}$, by

$$\mathscr{X}_K[n] \overset{\text{def}}{=} \begin{cases}\mathscr{X}[i], & \text{if } n = Ki,\\ 0, & \text{if } n \neq Ki,\end{cases} \quad (15)$$

for $n, i \in \mathbb{Z}$. Thus, we have

$$\begin{aligned}\mathfrak{X}_K(z) &= \sum_{n=-\infty}^{\infty}\mathscr{X}_K[n]z^{-n} \\ &= \sum_{i=-\infty}^{\infty}\mathscr{X}_K[i]z^{-iK} \\ &= \sum_{i=-\infty}^{\infty}\mathscr{X}_K[i]\left(z^K\right)^{-i} \\ &= \mathfrak{X}\left(z^K\right).\end{aligned} \quad (16)$$

Therefore, we have

$$\mathscr{Z}(\mathscr{X}_K[n]) = \mathfrak{X}\left(z^K\right). \quad (17)$$

2.2.4. Downsampling. Given a tensor sequence $\mathscr{X}[n]$, we define $\mathscr{Y}[n]$ by

$$\mathscr{Y}[n] \overset{\text{def}}{=} \mathscr{X}[Kn] \text{ for } n\in\mathbb{Z}, \quad \text{where } K\in\mathbb{N}. \quad (18)$$

Thus, we have

$$\begin{aligned}\mathfrak{Y}(z) &= \sum_{n=-\infty}^{\infty}\mathscr{Y}[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty}\mathscr{X}[Kn]z^{-n} \\ &=_1 \sum_{m=-\infty}^{\infty}x[m]\left[\frac{1}{K}\sum_{i=0}^{K-1}\exp\left(\iota\frac{2\pi mi}{K}\right)\right]z^{-m/K} \\ &= \frac{1}{K}\sum_{i=0}^{K-1}\sum_{m=-\infty}^{\infty}x[m]\left[\exp\left(-\frac{\iota 2\pi i}{K}\right)z^{1/M}\right]^{-m} \\ &= \frac{1}{K}\sum_{i=0}^{K-1}\mathfrak{X}\left(\exp\left(-\frac{\iota 2\pi i}{K}\right)z^{1/M}\right),\end{aligned} \quad (19)$$

where the identity "$=_1$" can be attained according to

$$\frac{1}{K}\sum_{i=0}^{K-1}\exp\left(\iota\frac{2\pi mi}{K}\right) = \begin{cases}1, & \text{where } m \text{ is a multiple of } K,\\ 0, & \text{otherwise.}\end{cases} \quad (20)$$

Therefore, we have

$$\mathscr{Z}(\mathscr{X}[Kn]) = \frac{1}{K}\sum_{i=0}^{K-1}\mathfrak{X}\left(\exp\left(-\frac{\iota 2\pi i}{K}\right)z^{1/M}\right). \quad (21)$$

2.2.5. Time Reversal. Given a tensor sequence $\mathscr{X}[n]$, we define $\mathscr{Y}[n]$ by

$$\mathscr{Y}[n] = \mathscr{X}[-n] \quad \text{for } n\in\mathbb{Z}. \quad (22)$$

Thus, we have

$$\begin{aligned}\mathfrak{Y}(z) &= \sum_{n=-\infty}^{\infty}\mathscr{Y}[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty}\mathscr{X}[-n]z^{-n} \\ &= \sum_{m=\infty}^{-\infty}\mathscr{X}[m]\left(z^{-1}\right)^{-m} \\ &= \mathfrak{X}\left(z^{-1}\right).\end{aligned} \quad (23)$$

Therefore, we have

$$\mathscr{Z}(\mathscr{X}[-n]) = \mathfrak{X}\left(z^{-1}\right). \quad (24)$$

2.2.6. Scaling in the z-Domain. Consider a tensor sequence $\{\mathscr{A}^n\star_M\mathscr{X}[n]\}$ where the tensor $\mathscr{A}$ is invertible and commutative for $\mathscr{X}[n]$ with respect to all n. Thus, we have

$$\begin{aligned}\mathscr{Z}(\mathscr{A}^n \star_M \mathscr{X}[n]) &= \sum_{n=-\infty}^{\infty} \mathscr{A}^n \star_M \mathscr{X}[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \mathscr{X}[n] \star_M \left(\mathscr{A}^{-1} z\right)^{-n} \\ &= \mathfrak{X}\left(\mathscr{A}^{-1} z\right).\end{aligned} \tag{25}$$

Therefore, we have

$$\mathscr{Z}(\mathscr{A}^n \star_M \mathscr{X}[n]) = \mathfrak{X}\left(\mathscr{A}^{-1} z\right). \tag{26}$$

2.2.7. Complex Conjugation. Given a tensor sequence $\{\mathscr{X}[n]\}$, $\{\mathscr{X}^*[n]\}$ denotes its complex-conjugated tensor sequence where all elements in $\mathscr{X}^*[n]$ are the complex conjugates of the corresponding elements in $\mathscr{X}[n]$ for all $n \in \mathbb{Z}$. Thus, we have

$$\begin{aligned}\mathscr{Z}(\mathscr{X}^*[n]) &= \sum_{n=-\infty}^{\infty} \mathscr{X}^*[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left[\mathscr{X}[n](z^*)^{-n}\right]^* \\ &= \left[\sum_{n=-\infty}^{\infty} \mathscr{X}[n](z^*)^{-n}\right]^* \\ &= \mathfrak{X}^*(z^*).\end{aligned} \tag{27}$$

Therefore, we have

$$\mathscr{Z}(\mathscr{X}^*[n]) = \mathfrak{X}^*(z^*). \tag{28}$$

2.2.8. Differentiation. Consider a tensor sequence $\{n\mathscr{X}[n]\}$. Thus, we have

$$\begin{aligned}\mathscr{Z}(n\mathscr{X}[n]) &= \sum_{n=-\infty}^{\infty} n\mathscr{X}[n] z^{-n} \\ &= z \sum_{n=-\infty}^{\infty} n\mathscr{X}[n] z^{-n-1} \\ &= -z \sum_{n=-\infty}^{\infty} \mathscr{X}[n]\left(-n z^{-n-1}\right) \\ &= z \sum_{n=-\infty}^{\infty} \mathscr{X}[n] \frac{d}{dz}\left(z^{-n}\right) \\ &= -z \frac{d\mathfrak{X}(z)}{dz}.\end{aligned} \tag{29}$$

Therefore, we have

$$\mathscr{Z}(n\mathscr{X}[n]) = -z \frac{d\mathfrak{X}(z)}{dz}. \tag{30}$$

2.2.9. Convolution. Consider two tensor sequences $\{\mathscr{X}[n]\}$ and $\{\mathscr{Y}[n]\}$ for $n \in \mathbb{Z}$, the convolutional tensor sequence $\{\mathscr{W}[n]\}$ such that $\mathscr{W}[n] \overset{\text{def}}{=} \mathscr{X}[n] \otimes_M \mathscr{Y}[n]$ where "$\otimes_M$" denotes the linear convolution operator and $\mathscr{X}[n], \mathscr{Y}[n], \mathscr{W}[n] \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$. Thus, we have

$$\begin{aligned}\mathscr{Z}(\mathscr{W}[n]) &= \mathscr{Z}(\mathscr{X}[n] \circledast_M \mathscr{Y}[n]) \\ &= \mathscr{Z}\left(\sum_{m=-\infty}^{\infty} \mathscr{X}[m] \star_M \mathscr{Y}[n-m]\right) \\ &= \sum_{n=-\infty}^{\infty}\left(z^{-n} \sum_{m=-\infty}^{\infty} \mathscr{X}[m] \star_M \mathscr{Y}[n-m]\right) \\ &= \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} z^{-n} \mathscr{X}[m] \star_M \mathscr{Y}[n-m]\right) \\ &= \sum_{m=-\infty}^{\infty}\left(\mathscr{X}[m] \star_M \sum_{n=-\infty}^{\infty} z^{-n} \mathscr{Y}[n-m]\right) \\ &= \sum_{m=-\infty}^{\infty} \mathscr{X}[m] \star_M \left(z^{-m} \mathfrak{Y}(z)\right) = \mathfrak{X}(z) \star_M \mathfrak{Y}(z).\end{aligned} \tag{31}$$

Therefore, we have

$$\mathscr{Z}(\mathscr{X}[n] \otimes_M \mathscr{Y}[n]) = \mathfrak{X}(z) \star_M \mathfrak{Y}(z). \tag{32}$$

2.2.10. Accumulation. Consider a tensor sequence $\{\mathscr{Y}[n]\}$ where

$$\mathscr{Y}[n] \overset{\text{def}}{=} \sum_{k=-\infty}^{n} \mathscr{X}[k]. \tag{33}$$

Then, we have

$$\begin{aligned}\mathscr{Z}(\mathscr{Y}[k]) &= \mathscr{Z}\left(\sum_{k=-\infty}^{n} \mathscr{X}[k]\right) \\ &= \sum_{n=-\infty}^{\infty}\left(z^{-n} \sum_{k=-\infty}^{n} \mathscr{X}[k]\right) \\ &= \sum_{n=-\infty}^{\infty} z^{-n}\left(\mathscr{X}[-\infty] + \cdots + \mathscr{X}[n-1] + \mathscr{X}[n]\right) \\ &= \sum_{n=-\infty}^{\infty} z^{-n}\left(\mathscr{X}[n] + \mathscr{X}[n-1] + \cdots + \mathscr{X}[-\infty]\right) \\ &=_1 \mathfrak{X}(z) + z^{-1}\mathfrak{X}(z) + \cdots + z^{-\infty}\mathfrak{X}(z) \\ &= \mathfrak{X}(z)\left(1 + z^{-1} + z^{-2} + \cdots\right) \\ &= \mathfrak{X}(z)\left(\frac{z}{z-1}\right),\end{aligned} \tag{34}$$

where the identity "$=_1$" above can be obtained according to the time delay property. Therefore, we

$$\mathscr{Z}\left(\sum_{k=-\infty}^{n} \mathscr{X}[k]\right) = \mathfrak{X}(z)\left(\frac{z}{z-1}\right). \tag{35}$$

2.2.11. Initial Value Theorem. According to the z-transform definition given by Eq. (4), we have

$$\mathfrak{X}(z) = \mathscr{X}[0] z^0 + \mathscr{X}[1] z^{-1} + \mathscr{X}[2] z^{-2} + \cdots, \tag{36}$$

where $\{\mathcal{X}[n]\}$ is a causal tensor sequence. Taking $z \longrightarrow \infty$ at both sides of Eq. (36), we have $z^{-n} \longrightarrow 0$. Thus, we have

$$\mathcal{X}[0] = \lim_{z\longrightarrow\infty} \boldsymbol{\mathfrak{X}}(z). \tag{37}$$

2.2.12. Final Value Theorem. According to the z-transform definition given by Eq. (4), we have

$$\mathcal{Z}(\mathcal{X}[n+1]) - \mathcal{Z}(\mathcal{X}[n]) = z\boldsymbol{\mathfrak{X}}(z) - z\mathcal{X}[0] - \boldsymbol{\mathfrak{X}}(z), \tag{38}$$

where $\{\mathcal{X}[n]\}$ is a causal tensor sequence. Thus, we have

$$(z-1)\boldsymbol{\mathfrak{X}}(z) - z\mathcal{X}[0] = \sum_{n=0}^{\infty}(\mathcal{X}[n+1] - \mathcal{X}[n])z^{-n}. \tag{39}$$

Taking $z \longrightarrow 1$ at both sides of Eq. (39), we have

$$\lim_{z\longrightarrow 1}[(z-1)\boldsymbol{\mathfrak{X}}(z) - z\mathcal{X}[0]] = \mathcal{X}[1] - \mathcal{X}[0] + \mathcal{X}[2] - \mathcal{X}[1] + \cdots + \mathcal{X}[\infty] - \mathcal{X}[\infty-1]. \tag{40}$$

Therefore, we have

$$\mathcal{X}[\infty] = \lim_{z\longrightarrow 1}[(z-1)\boldsymbol{\mathfrak{X}}(z)]. \tag{41}$$

3. Cauchy's Residue Theorem for Tensor Sequences

It is well-known that the scalar sequence resulting from the inverse z-transform can be determined using Cauchy's residue theorem [23]. We will extend Cauchy's residue theorem to accommodate tensor sequences in this section.

Let us begin with Cauchy's integral formula for tensors in the following subsection.

3.1. Cauchy's Integral Formula for Tensors. Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ be a tensor where

$$\mathcal{A} = \sum_{k=1}^{v} \lambda_k \mathcal{Q}_k \star_1 \mathcal{R}_k^{\mathrm{H}}. \tag{42}$$

Note that

$$v \overset{\text{def}}{=} \prod_{i=1}^{M} I_i. \tag{43}$$

λ_k denotes the k-th eigenvalue of $\mathcal{A}$; $\mathcal{Q}_k$ and $\mathcal{R}_k$ denote the k-th left and right eigentensors (the definitions of left and right eigentensors can be found in [24]) corresponding to λ_k, respectively; and $\mathcal{R}_k^{\mathrm{H}}$ is the Hermitian adjoint (the Hermitian adjoint of a tensor is defined in [3]) of $\mathcal{R}_k$. As we assume that all tensors discussed in this paper are *diagonalizable square tensors*, all tensors throughout this paper can be expressed in the form of Eq. (42). Furthermore, note that $\mathcal{Q}_k, \mathcal{R}_k \in \mathbb{C}^{I_1\times\cdots\times I_M}$,

$$\begin{aligned} \langle \mathcal{Q}_k, \mathcal{R}_j \rangle &= \delta_{k,j}, \\ \mathcal{I} &= \sum_{k=1}^{v} \mathcal{Q}_k \star_1 \mathcal{R}_k^{H}, \end{aligned} \tag{44}$$

where $\langle , \rangle$ denotes the inner product of two tensors whose definition can be found in [3]; $\delta_{k,j} \overset{\text{def}}{=} 0$ if $k \neq j$ while $\delta_{k,j} \overset{\text{def}}{=} 1$ if $k = j$; $\mathcal{I} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ denotes an identity.

Now define $\Lambda(\mathcal{A})$ by the set of all eigenvalues of $\mathcal{A}$, i.e., $\Lambda(\mathcal{A}) \overset{\text{def}}{=} \{\lambda_k : k = 1, 2, \cdots, v\}$. If $f : \blacklozenge \longrightarrow f(\blacklozenge)$ is a function-mapping, where the argument "$\blacklozenge$" can be treated as a scalar, vector, matrix, or a tensor, we can formulate $f(\mathcal{A})$ as

$$f(\mathcal{A}) = \sum_{k=1}^{v} f(\lambda_k) \mathcal{Q}_k \star_1 \mathcal{R}_k^{\mathrm{H}}. \tag{45}$$

Let $\acute{\Lambda}(\mathcal{A})$ represent a subset of $\Lambda(\mathcal{A})$. If $f(\mathcal{A})$ only depends on $\acute{\Lambda}(\mathcal{A})$, we may write $f(\mathcal{A})$ as $f_{\acute{\Lambda}(\mathcal{A})}(\mathcal{A})$ such that

$$f_{\acute{\Lambda}(\mathcal{A})}(\mathcal{A}) = \sum_{\lambda_k \in \acute{\Lambda}(\mathcal{A})} f(\lambda_k) \mathcal{Q}_k \star_1 \mathcal{R}_k^{\mathrm{H}}. \tag{46}$$

We further define the *functional norm* of $f(\mathcal{A})$ by

$$\|f(\mathcal{A})\|_{\sup} \overset{\text{def}}{=} \sup_{\lambda_k \in \Lambda(\mathcal{A})} |f(\lambda_k)|. \tag{47}$$

If $|f(\lambda_k)|$ is bounded for all k, then $\|f(\mathcal{A})\|_{\sup} < \infty$. Motivated from [25], we may define the "resolvent" $G_{\mathcal{A}}(z)$ of $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ with respect to a complex variable $z \notin \Lambda(\mathcal{A})$ such that

$$G_{\mathcal{A}}(z) = \frac{\mathcal{I}}{z\mathcal{I} - \mathcal{A}}. \tag{48}$$

According to Eq. (47), we have

$$\|G_{\mathcal{A}}(z)\|_{\sup} = \sup_{\lambda_k \in \Lambda(\mathcal{A})} \frac{1}{|z - \lambda_k|}. \tag{49}$$

Lemma 2 below provides the alternative expression of $G_{\mathcal{A}}(z)$.

Lemma 2. *Given a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ along with its resolvent $G_{\mathcal{A}}(z)$, we have*

$$G_{\mathcal{A}}(z) = \frac{G_{\mathcal{A}}(z_0)}{\mathcal{I} - (z_0 - z)G_{\mathcal{A}}(z_0)}. \tag{50}$$

Proof. Since we can express $G_{\mathcal{A}}(z)$ by

$$G_{\mathcal{A}}(z) = \sum_{k=1}^{v} \frac{\mathcal{Q}_k \star_1 \mathcal{R}_k^{\mathrm{H}}}{z - \lambda_k}, \tag{51}$$

we have

$$\begin{aligned} G_{\mathscr{A}}(z) - G_{\mathscr{A}}(z_0) &= \sum_{k=1}^{v}\left(\frac{1}{z-\lambda_k} - \frac{1}{z_0-\lambda_k}\right)\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}} \\ &= \sum_{k=1}^{v}\frac{z_0 - z}{(z-\lambda_k)(z_0-\lambda_k)}\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}} \\ &= \frac{z_0 - z}{(z-\mathscr{A})(z_0-\mathscr{A})} \\ &= (z_0 - z)G_{\mathscr{A}}(z)G_{\mathscr{A}}(z_0). \end{aligned} \tag{52}$$

According to Eq. (52), we can derive Eq. (50). □

Lemma 3 below shows that the resolvent $G_{\mathscr{A}}(z)$ is an analytic tensor-valued function of z.

Lemma 3. *Given a tensor $\mathscr{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ along with its resolvent $G_{\mathscr{A}}(z)$, if we have*

$$|z - z_0| < dist(z_0, \Lambda(\mathscr{A})) \overset{def}{=} \min_{\lambda\in\Lambda(\mathscr{A})}|z_0 - \lambda|, \tag{53}$$

then $G_{\mathscr{A}}(z)$ is an analytic function of z.

Proof. To show that the tensor-valued function $G_{\mathscr{A}}(z)$ is an analytic function of z, we can express $G_{\mathscr{A}}(z)$ by a convergent power series. According to Lemma 2, we can express Eq. (50) by

$$G_{\mathscr{A}}(z) = G_{\mathscr{A}}(z_0)\sum_{n=0}^{\infty}[(z_0 - z)G_{\mathscr{A}}(z_0)]^n \tag{54}$$

subject to $\|(z_0 - z)G_{\mathscr{A}}(z_0)\|_{\sup} < 1$. If we assume that $|-z_0| < r$, then

$$\begin{aligned} &\|G_{\mathscr{A}}(z_0)[(z_0 - z)G_{\mathscr{A}}(z_0)]^n\|_{\sup} \\ &\quad\le \left[\frac{r}{\mathrm{dist}(z_0, \Lambda(\mathscr{A}))}\right]^n \times \frac{1}{\mathrm{dist}(z_0, \Lambda(\mathscr{A}))}, \end{aligned} \tag{55}$$

where $\|G_{\mathscr{A}}(z_0)\|_{\sup} = 1/\mathrm{dist}(z_0, \Lambda(\mathscr{A}))$. If the condition given by Eq. (53) holds, then $\|G_{\mathscr{A}}(z_0)[(z_0 - z)G_{\mathscr{A}}(z_0)]^n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

According to Lemma 3, since the resolvent $G_{\mathscr{A}}(z)$ of $\mathscr{A}$ is analytic, we can carry out the contour integrals over it. For example, a contour $\mathfrak{C}_3$ encircles a single eigenvalue, say λ_3, in Figure 2(a) such that

$$\begin{aligned} \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_3} G_{\mathscr{A}}(z)dz &= \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_3}\sum_{k=1}^{v}\frac{\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}}}{z-\lambda_k}dz \\ &= \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_3}\frac{\mathcal{Q}_3 \star_1 \mathscr{R}_3^{\mathrm{H}}}{z-\lambda_3}dz \\ &=_1 \mathcal{Q}_3 \star_1 \mathscr{R}_3^{\mathrm{H}}, \end{aligned} \tag{56}$$

where the identity "$=_1$" above follows from $\oint_{\mathfrak{C}_3} 1/(z-\lambda_3) = 2\pi\iota$.

Now we can present Theorem 4 about Cauchy's integral formula for tensors. □

Theorem 4. *Given a tensor $\mathscr{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and an analytic function $f(z)$ with the region of convergence (ROC) including $\{z : z \in \Lambda(\mathscr{A})\}$, let $\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}$ be a contour where the set of points $\{z : z \in \acute{\Lambda}(\mathscr{A})\}$ are all inside $\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}$.*

Thus, we have

$$f_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A}) = \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(z)}{z\mathscr{I} - \mathscr{A}}dz, \tag{57}$$

where $f_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A})$ is defined by Eq. (46).

Proof. Consider a contour $\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}$ as illustrated by Figure 2(a) or 2(b). Thus, we have

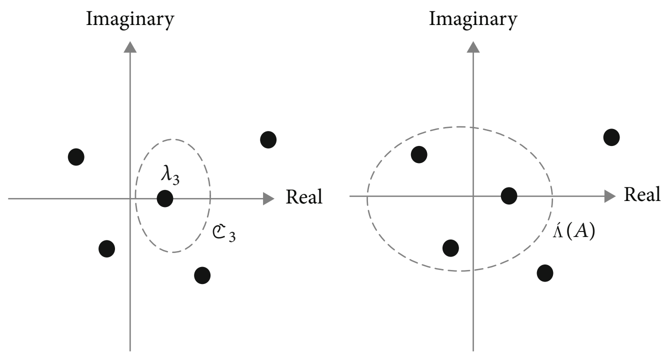
$$\begin{aligned} \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}} G_{\mathscr{A}}(z)dz &= \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\sum_{k=1}^{v}\frac{\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}}}{z-\lambda_k}dz \\ &= \sum_{\lambda_k\in\acute{\Lambda}(\mathscr{A})}\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}}\underbrace{\left(\frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{1}{z-\lambda_k}dz\right)}_{=1} \\ &= \sum_{\lambda_k\in\acute{\Lambda}(\mathscr{A})}\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}}. \end{aligned} \tag{58}$$

Because $f(z)$ is an analytic function with the ROC including $\{z : z \in \Lambda(\mathscr{A})\}$, according to the spectral mapping theorem in [25] and Cauchy's integral formula for scalars in [23], we have

$$\begin{aligned} f_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A}) &= \sum_{\lambda_k\in\acute{\Lambda}(\mathscr{A})} f(\lambda_k)\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}} \\ &= \sum_{\lambda_k\in\acute{\Lambda}(\mathscr{A})}\underbrace{\left(\frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(z)}{z-\lambda_k}dz\right)}_{f(\lambda_k)}\mathcal{Q}_k \star_1 \mathscr{R}_k^{\mathrm{H}} \\ &= \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}} f(z)G_{\acute{\Lambda}(\mathscr{A})}(z)dz \\ &= \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(z)}{z\mathscr{I} - \mathscr{A}}dz. \end{aligned} \tag{59}$$

Define the n-th derivative $f^{(n)}_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A})$ with respect to the tensor $(\mathscr{A})$ by

$$f^{(n)}_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A}) \overset{\mathrm{def}}{=} \lim_{\|\mathscr{X}-\mathscr{A}\|\longrightarrow 0}\frac{f^{(n-1)}_{\acute{\Lambda}(\mathscr{X})}(\mathscr{X}) - f^{(n-1)}_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A})}{\mathscr{X} - \mathscr{A}}, \tag{60}$$



(a) Single eigenvalue in contour (b) Multiple eigenvalues in contour

Figure 2: Illustration for contours encircling eigenvalue(s).

for $n = 1, 2, \cdots$, where $f^{(0)}_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A}) \stackrel{\text{def}}{=} f_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A})$ and "$\| \|$" is the tensor norm as introduced in [3]. Corollary 5 below, which is derived from Theorem 4, presents Cauchy's integral formula for the "tensor derivatives" given by Eq. (60). □

Corollary 5. *Given a tensor $\mathscr{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ together with an analytic function $f(z)$ of z with the ROC including $\{z : z \in \Lambda(\mathscr{A})\}$. Since $\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}$ is a contour where the set of points $\{z : z \in \acute{\Lambda}(\mathscr{A})\}$ are all inside $\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}$, we have*

$$f^{(n)}_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A}) = \frac{n!}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}} \frac{f(z)}{(z\mathscr{I} - \mathscr{A})^{(n+1)}}dz, \tag{61}$$

for $n = 0, 1, 2, \cdots$.

Proof. For $n = 1$ (the first-order derivative), we have

$$\begin{aligned} f^{(1)}_{\acute{\Lambda}(\mathscr{A})}(\mathscr{A}) &=_1 \lim_{\|\mathscr{X}-\mathscr{A}\|\longrightarrow 0} \frac{1}{\mathscr{X}-\mathscr{A}}\left[\frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{X})}}\frac{f(w)}{(w\mathscr{I}-\mathscr{X})}dw - \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(w)}{(w\mathscr{I}-\mathscr{A})}dw\right] \\ &= \frac{1}{2\pi\iota}\lim_{\|\mathscr{X}-\mathscr{A}\|\longrightarrow 0}\frac{1}{\mathscr{X}-\mathscr{A}}\star_M\left[\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{X})}\cup\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(w)(\mathscr{X}-\mathscr{A})}{(w\mathscr{I}-\mathscr{X})(w\mathscr{I}-\mathscr{A})}dw\right] \\ &= \frac{1!}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(z)}{(w\mathscr{I}-\mathscr{A})^2}dw \\ &= \frac{1!}{2\pi\iota}\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathscr{A})}}\frac{f(z)}{(z\mathscr{I}-\mathscr{A})^2}dz, \end{aligned} \tag{62}$$

where "$=_1$" above is deduced from Theorem 4 and Eq. (60). Thus, Corollary 5 can be established by iterating Eq. (62) repeatedly for n times. □

3.2. Generalized Cauchy's Residue Theorem for Tensor Sequences. If $f(\mathscr{X})$ is an analytic tensor-valued function with the argument $\mathscr{X} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ except for an isolated singularity point at $\mathscr{X} = \mathscr{X}_0$, then $f(\mathscr{X})$ has a Laurent series such that

$$f(\mathscr{X}) = \sum_{k=-\infty}^{\infty}\mathscr{A}_k\star_M(\mathscr{X}-\mathscr{X}_0)^k, \tag{63}$$

where the coefficient tensor $\mathscr{A}_{-1} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ is called the "residue" of $f(\mathscr{X})$ at $\mathscr{X} = \mathscr{X}_0$, denoted by $\mathrm{Res}(f\,; \mathscr{X}_0)$. Let us present the following lemmas about the residue properties for analytic tensor-valued functions.

Lemma 6. *If $f(\mathscr{X})$ has a simple pole at $\mathscr{X} = \mathscr{X}_0$ and can be expressed by*

$$f(\mathscr{X}) = \frac{g(\mathscr{X})}{h(\mathscr{X})}, \tag{64}$$

where both $g(\mathscr{X})$ and $h(\mathscr{X})$ are analytic at $\mathscr{X} = \mathscr{X}_0$ such that $g(\mathscr{X}_0) \neq \mathscr{O}$ and $h(\mathscr{X})$ has a simple zero at $\mathscr{X} = \mathscr{X}_0$, then

$$Res(f\,; \mathscr{X}_0) =_1 \lim_{\|\mathscr{X}-\mathscr{X}_0\|\longrightarrow 0} f(\mathscr{X})\star_M(\mathscr{X}-\mathscr{X}_0) =_2 \frac{g(\mathscr{X}_0)}{h^{(1)}(\mathscr{X}_0)}. \tag{65}$$

Proof. Since we have

$$f(\mathscr{X}) = \frac{\mathscr{A}_{-1}}{\mathscr{X}-\mathscr{X}_0} + \mathscr{A}_0 + \mathscr{A}_1\star_M(\mathscr{X}-\mathscr{X}_0) + \cdots, \tag{66}$$

the identity "$=_1$" in Eq. (65) can be obtained by multiplying $(\mathscr{X}-\mathscr{X}_0)$ at both sides of Eq. (66). The identity "$=_2$" in Eq. (65) can be obtained by

$$\begin{aligned} \lim_{\mathscr{X}\longrightarrow\mathscr{X}_0} f(\mathscr{X})\star_M(\mathscr{X}-\mathscr{X}_0) &= \lim_{\|\mathscr{X}-\mathscr{X}_0\|\longrightarrow 0}\frac{g(\mathscr{X})}{h(\mathscr{X})}\star_M(\mathscr{X}-\mathscr{X}_0) \\ &= \lim_{\|\mathscr{X}-\mathscr{X}_0\|\longrightarrow 0}\frac{g(\mathscr{X})}{(h(\mathscr{X})-h(\mathscr{X}_0))/(\mathscr{X}-\mathscr{X}_0)} \\ &= \frac{g(\mathscr{X}_0)}{h^{(1)}(\mathscr{X}_0)}. \end{aligned} \tag{67}$$

□

Lemma 7. *If $f(\mathcal{X})$ has a pole at $\mathcal{X} = \mathcal{X}_0$ with a multiplicity $k \in \mathbb{N}$, then we have*

$$Res(f\,;\mathcal{X}_0) = \frac{d^{(k-1)}}{(k-1)!d\mathcal{X}^{(k-1)}}\left[f(\mathcal{X})\star_M(\mathcal{X}-\mathcal{X}_0)^k\right]\Bigg|_{\mathcal{X}=\mathcal{X}_0}. \tag{68}$$

Proof. We have

$$f(\mathcal{X}) = \mathcal{A}_{-k}\star_M(\mathcal{X}-\mathcal{X}_0)^{-k}+\cdots+\mathcal{A}_{-1}\star_M(\mathcal{X}-\mathcal{X}_0)^{-1} + \mathcal{A}_0 + \mathcal{A}_1\star_M(\mathcal{X}-\mathcal{X}_0)+\cdots. \tag{69}$$

Assume $\tilde{f}(\mathcal{X}) \stackrel{\text{def}}{=} (\mathcal{X}-\mathcal{X}_0)^k\star_M f(\mathcal{X})$. We get

$$\frac{d^{(k-1)}\tilde{f}(\mathcal{X})}{d\mathcal{X}^{(k-1)}} = (k-1)!\mathcal{A}_{-1} + k!\mathcal{A}_0\star_M(\mathcal{X}-\mathcal{X}_0)+\cdots. \tag{70}$$

Thus, according to Eq. (63), Lemma 7 is proven.

Lemma 8 below governs the contour integral of the powers of $(z\mathcal{I}-\mathcal{A})$. □

Lemma 8. *We get*

$$\oint_{\mathfrak{C}}(z\mathcal{I}-\mathcal{A})^n dz = \mathcal{O} \tag{71}$$

for $n \in \mathbb{N}_{\geq 0}$ and any closed contour $\mathfrak{C}$. Moreover, we also have

$$\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathcal{A})}}(z\mathcal{I}-\mathcal{A})^n dz = \mathcal{O} \tag{72}$$

for $n \in \mathbb{Z}$ and $n < -1$.

Proof. For $n \in \mathbb{Z}$ and $n \geq 0$, since we have

$$\oint_{\mathfrak{C}} z^n dz = 0, \tag{73}$$

Eq. (71) is valid by the binomial expansion of $(z\mathcal{I}-\mathcal{A})^n$. For $n \in \mathbb{Z}$ and $n < -1$, since we have

$$(z\mathcal{I}-\mathcal{A})^n = 1\frac{1}{(n+1)}\frac{d}{dz}(z\mathcal{I}-\mathcal{A})^{n+1}, \tag{74}$$

according to Corollary 5, we get

$$\oint_{\mathfrak{C}_{\acute{\Lambda}(\mathcal{A})}}(z\mathcal{I}-\mathcal{A})^n dz = \mathcal{O}. \tag{75}$$

Consequently, Lemma 8 is proven. □

Now we present Cauchy's residue theorem for tensor sequences as Theorem 9 below.

Theorem 9 (Cauchy's residue theorem for tensor sequences). *If $\mathfrak{C}$ is a simple closed contour in the complex z-plane and $f(z)$ is an analytic function with the tensor coefficients except for some tensors $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_m \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ with the corresponding spectra $\acute{\Lambda}(\mathcal{X}_1)$, $\acute{\Lambda}(\mathcal{X}_2)$, …, $\acute{\Lambda}(\mathcal{X}_m)$ inside the contour $\mathfrak{C}$, then we have*

$$\oint_{\mathfrak{C}} f(z)dz = 2\pi\iota\sum_{k=1}^{m}\left[Res(f\,;\mathcal{X}_k)\star_M\sum_{\lambda_{k,j}\in\acute{\Lambda}(\mathcal{X}_k)}\mathcal{Q}_{k,j}\star_1\mathcal{R}_{k,j}^H\right]. \tag{76}$$

Proof. Assume that $m = 1$ and $\mathcal{X}_1$ is a simple pole. Thus, we can express $f(z)$ by

$$f(z) = \frac{g(z)}{z\mathcal{I}-\mathcal{X}_1}, \tag{77}$$

where $g(z)$ is given by

$$g(z) \stackrel{\text{def}}{=} \mathcal{A}_0 + \mathcal{A}_1\star_M(z\mathcal{I}-\mathcal{X}_1) + \mathcal{A}_2\star_M(z\mathcal{I}-\mathcal{X}_1)^2+\cdots, \tag{78}$$

and $\mathcal{A}_i$ denotes the coefficient tensor associated with $(z\mathcal{I}-\mathcal{X}_1)^i$ for $i = 0, 1, 2, \cdots$. Besides, we have $\text{Res}(f\,;\mathcal{X}_1) = \mathcal{A}_0$.

By integrating both sides of the following identity:

$$\frac{g(z)}{z\mathcal{I}-\mathcal{X}_1} = \frac{\mathcal{A}_0}{z\mathcal{I}-\mathcal{X}_1} + \mathcal{A}_1 + \mathcal{A}_2\star_M(z\mathcal{I}-\mathcal{X}_1)+\cdots, \tag{79}$$

we have

$$\oint_{\mathfrak{C}}\frac{g(z)}{z\mathcal{I}-\mathcal{X}_1}dz = \oint_{\mathfrak{C}}\frac{\mathcal{A}_0}{z\mathcal{I}-\mathcal{X}_1}dz + \oint_{\mathfrak{C}}\mathcal{A}_1 dz + \oint_{\mathfrak{C}}\mathcal{A}_2\star_M(z\mathcal{I}-\mathcal{X}_1)dz+\cdots. \tag{80}$$

According to Theorem 4 and Lemma 8, we have

$$g(\mathcal{X}_1) = \frac{1}{2\pi\iota}\oint_{\mathfrak{C}}\frac{g(z)}{z\mathcal{I}-\mathcal{X}_1}dz = \mathcal{A}_0\star_M\sum_{\lambda_{1,j}\in\acute{\Lambda}(\mathcal{X}_1)}\mathcal{Q}_{1,j}\star_1\mathcal{R}_{1,j}^{\text{H}}. \tag{81}$$

In general, let $\mathfrak{R}$ be an ROC in the complex z-plane containing a contour $\mathfrak{C}$ and $\mathfrak{R}'$ be a subregion of $\mathfrak{R}$ without any of the spectra $z \in \Lambda(\mathcal{X}_1)$, $z \in \Lambda(\mathcal{X}_2), \cdots$, $z \in \Lambda(\mathcal{X}_m)$. Consider that these spectra (poles) $z \in \Lambda(\mathcal{X}_1)$, $z \in \Lambda(\mathcal{X}_2)$, $\cdots$, $z \in \Lambda(\mathcal{X}_m)$ are inside the contours (say small circles) $\mathfrak{C}_1$, $\mathfrak{C}_2$, …, $\mathfrak{C}_m$, respectively, such that $\mathfrak{C} - (\mathfrak{C}_1 + \mathfrak{C}_2+\cdots+\mathfrak{C}_m)$ is inside $\mathfrak{R}'$ where "-" and "+" here specify the set difference and union operators, respectively. Theorem 9 is thus proven by applying Theorem 4 and Lemma 8 subject to $\mathfrak{C}_{\acute{\Lambda}(\mathcal{A})} = \mathfrak{C} - (\mathfrak{C}_1 + \mathfrak{C}_2+\cdots+\mathfrak{C}_m)$ in both Eqs. (57) and (72). The *residue tensors* $\text{Res}(f\,;\mathcal{X}_k)$, $k = 1, 2, \ldots, m$, can be obtained using Lemma 6 if $\mathcal{X}_k$ is a simple pole or Lemma 7 if $\mathcal{X}_k$ has a multiplicity larger than one. □

4. Inverse Tensor z-Transform

In this section, we will introduce two approaches to undertake the inverse tensor z-transform. First, the *power-series approach* (or *long-division approach*) will be introduced in Section 4.1 while the contour-integral approach based on our newly derived Cauchy's residue theorem for tensor sequences will be presented in Section 4.2.

4.1. Power-Series (Long-Division) Approach. Suppose that a tensor z-transform $\boldsymbol{\mathfrak{X}}(z)$ is expressed by the following rational (proper fraction) form:

$$\boldsymbol{\mathfrak{X}}(z) \stackrel{\text{def}}{=} \frac{\mathscr{A}_0 + \mathscr{A}_1 z^{-1} + \cdots + \mathscr{A}_m z^{-m}}{\mathscr{B}_0 + \mathscr{B}_1 z^{-1} + \cdots + \mathscr{B}_n z^{-n}}, \tag{82}$$

where we assume that $m < n$ without loss of generality (if $m \geq n$, we can rewrite Eq. (82) as $\boldsymbol{\mathfrak{X}}(z) =^{\text{def}} (\mathscr{A}_0 + \mathscr{A}_1 z^{-1} + \cdots + \mathscr{A}_m z^{-m})/(\mathscr{B}_0 + \mathscr{B}_1 z^{-1} + \cdots + \mathscr{B}_n z^{-n}) = \mathscr{V}(z^{-1}) + (\mathscr{A}'_0 + \mathscr{A}'_1 z^{-1} + \cdots + \mathscr{A}'_{m'} z^{-m'})/(\mathscr{B}_0 + \mathscr{B}_1 z^{-1} + \cdots + \mathscr{B}_n z^{-n})$, where $m' < n$ and $\mathscr{V}(z^{-1})$ denotes a polynomial of z^{-1}). Given a set of invertible tensors, say $\{\mathscr{B}_k, k = 0, 1, 2, \cdots, n\}$ where $\mathscr{B}_k \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ for all k, we want to show that it is always possible to find n tensors $\mathscr{X}_i$, $i = 1, 2, \cdots, n$ satisfying

$$\sum_{i=0}^{n} \mathscr{B}_i z^{-i} = \mathscr{B}_n \prod_{i=0}^{n} \left(z^{-1} \mathscr{I} - \mathscr{X}_i\right). \tag{83}$$

Lemma 10. *Given $\{\mathscr{B}_k, k = 0, 1, 2, \cdots, n\}$ where $\mathscr{B}_k \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ for all k, we have*

$$\prod_{j=1}^{n-1} \binom{(n-j+1)v}{v} \tag{84}$$

solutions totally based on Eq. (83).

Proof. According to the right-hand side of Eq. (83), we have

$$\prod_{i=1}^{n} \left(z^{-1}\mathscr{I} - \mathscr{X}_i\right) = \mathscr{I} z^{-n} + (-1) \frac{\mathscr{B}_{n-1}}{\mathscr{B}_n} z^{-(n-1)} + (-1)^2 \frac{\mathscr{B}_{n-2}}{\mathscr{B}_n} z^{-(n-2)} + \cdots + (-1)^n \frac{\mathscr{B}_0}{\mathscr{B}_n}. \tag{85}$$

According to Chapter 3 in [26], the solution space of $\mathscr{X}_i$'s in Eq. (83) is equivalent to the solution space of the tensor $\mathscr{Z}$ satisfying

$$\mathscr{I} \mathscr{Z}^n + (-1) \frac{\mathscr{B}_{n-1}}{\mathscr{B}_n} \mathscr{Z}^{(n-1)} + (-1)^2 \frac{\mathscr{B}_{n-2}}{\mathscr{B}_n} \mathscr{Z}^{(n-2)} + \cdots + (-1)^n \frac{\mathscr{B}_0}{\mathscr{B}_n} = \mathscr{O}. \tag{86}$$

According to Section 2.2 in [24], the tensors $\mathscr{B}_i$'s and $\mathscr{Z}$ all can be "unfolded" to form the corresponding $v \times v$ *unfolded matrices*, say $\mathbf{B}_i$'s and $\mathbf{Z}$. The solution space of $\mathscr{Z}$ in Eq. (86) is identical to the solution space of $\mathbf{Z}$ in the following:

$$\mathbf{I}_v \mathbf{Z}^n + (-1) \frac{\mathbf{B}_{n-1}}{\mathbf{B}_n} \mathbf{Z}^{(n-1)} + (-1)^2 \frac{\mathbf{B}_{n-2}}{\mathbf{B}_n} \mathbf{Z}^{(n-2)} + \cdots + (-1)^n \frac{\mathbf{B}_0}{\mathbf{B}_n} = \mathbf{O}, \tag{87}$$

where $\mathbf{O}$ denotes an *all-zero matrix*. According to Theorem 4 in [27], there are

$$\binom{nv}{v} \tag{88}$$

solutions of such $\mathscr{Z}$ in Eq. (86). If we know the solution of $\mathscr{X}_i$, $i = 1, 2, \ldots, n-1$, to Eq. (87), then Eq. (83) becomes

$$\sum_{i=1}^{(n-1)} \mathscr{B}'_i z^{-i} = \mathscr{B}'_{n-1} \prod_{i=1}^{(n-1)} \left(z^{-1}\mathscr{I} - \mathscr{X}_i\right), \tag{89}$$

where $\mathscr{B}'_i =^{\text{def}} \mathscr{B}_i / \mathscr{B}_n$, $i = 1, 2, \ldots, n-1$. Thus, by repeatedly applying Theorem 4 in [27] again and again, there are

$$\binom{(n-1)v}{v} \tag{90}$$

solutions of $\mathscr{X}_i$'s to Eq. (89). Finally, we will have

$$\prod_{j=1}^{n-1} \binom{(n-j+1)v}{v} \tag{91}$$

solutions of $\mathscr{X}_i$ for $i = 1, 2, \ldots, n$, to Eq. (83).

Suppose that we have the following tensor-root decomposition for the denominator polynomial in Eq. (82):

$$\mathscr{B}_0 + \mathscr{B}_1 z^{-1} + \cdots + \mathscr{B}_n z^{-n} = \mathscr{B}_n \prod_{i=1}^{\rho} \left(z^{-1}\mathscr{I} - \mathscr{P}_i\right)^{\kappa_i}. \tag{92}$$

Consequently, the rational tensor z-function $\boldsymbol{\mathfrak{X}}(z)$ given by Eq. (82) can be expressed by

$$\boldsymbol{\mathfrak{X}}(z) = \frac{\mathscr{A}_m}{\mathscr{B}_n} \sum_{i=1}^{\rho} \left[\frac{\mathscr{D}_{i,1}}{z^{-1}\mathscr{I} - \mathscr{P}_i} + \cdots + \frac{\mathscr{D}_{i,\kappa_i}}{\left(z^{-1}\mathscr{I} - \mathscr{P}_i\right)^{\kappa_i}} \right], \tag{93}$$

where $\mathscr{P}_i$ (associated with the multiplicity κ_i), $i = 1, 2, \ldots, \rho$, are ρ-distinct roots (in terms of tensors) of the polynomial with tensor coefficients: $\mathscr{B}_0 + \mathscr{B}_1 z^{-1} + \cdots + \mathscr{B}_n z^{-n}$. We call the expansion given by Eq. (93) "tensor partial fraction decomposition (T-PDF)." Note that the coefficient tensors $\mathscr{D}_{i,j}$ for $j = 1, 2, \ldots, \kappa_i$ and $i = 1, 2, \ldots, \rho$ in Eq. (93) can be determined using Lemmas 6 and 7 by substituting $\mathscr{Z}$ for $z^{-1}\mathscr{I}$ such that

$$\mathscr{D}_{i,\kappa_i-\ell}=\frac{\mathscr{B}_n}{\mathscr{A}_m}\star_M\frac{1}{\ell!}\left[\frac{d^\ell\boldsymbol{\mathfrak{X}}(\mathscr{Z})}{d\mathscr{Z}^\ell}\star_M(\mathscr{Z}-\mathscr{P}_i)^{\kappa_i}\bigg|_{\mathscr{Z}=\mathscr{P}_i}\right], \tag{94}$$

where $\ell=0,1,\ldots,\kappa_i-1$ and $i=1,2,\ldots,\rho$.

Furthermore, we can rewrite Eq. (93) to split $\boldsymbol{\mathfrak{X}}(z)$ into the causal and anticausal parts as the following

$$\begin{aligned}\boldsymbol{\mathfrak{X}}(z)=&\underbrace{\frac{\mathscr{A}_m}{\mathscr{B}_n}\sum_{i_c=1}^{\rho_c}\left[\frac{\mathscr{D}_{i_c,1}}{z^{-1}\mathscr{I}-\mathscr{P}_{i_c}}+\cdots+\frac{\mathscr{D}_{i_c,k_{i_c}}}{\left(z^{-1}\mathscr{I}-\mathscr{P}_{i_c}\right)^{k_{i_c}}}\right]}_{\text{causal part}}\\&+\underbrace{\frac{\mathscr{A}_m}{\mathscr{B}_n}\sum_{i_a=1}^{\rho_a}\left[\frac{\mathscr{D}_{i_a,1}}{z^{-1}\mathscr{I}-\mathscr{P}_{i_a}}+\cdots+\frac{\mathscr{D}_{i_a,k_{i_a}}}{\left(z^{-1}\mathscr{I}-\mathscr{P}_{i_a}\right)^{k_{i_a}}}\right]}_{\text{anticausal part}},\end{aligned} \tag{95}$$

where i_c and i_a represent the indices for the causal and anticausal parts of $\boldsymbol{\mathfrak{X}}(z)$ with ρ_c- and ρ_a-distinct roots (in terms of tensors), respectively, and $\rho_c+\rho_a=\rho$. Then, the ROC of the tensor z-transform $\boldsymbol{\mathfrak{X}}(z)$ given by Eq. (95), denoted by $\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z))$, can be written as

$$\begin{aligned}\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z))=&\bigcap_{i_c=1}^{\rho_c}\bigcap_{\kappa_{i_c}=1}^{v}\left\{z\in\mathbb{C}:|z|>\frac{1}{\left|\lambda_{i_c,\kappa_{i_c}}\right|}\right\}\cap\bigcap_{i_a=1}^{\rho_a}\bigcap_{\kappa_{i_a}=1}^{v}\\&\cdot\left\{z\in\mathbb{C}:|z|<\left|\lambda_{i_c,\kappa_{i_c}}\right|\right\},\end{aligned} \tag{96}$$

where $\{\lambda_{i_c,\kappa_{i_c}}\in\Lambda(\mathscr{P}_{i_c})\}$ and $\{\lambda_{i_a,\kappa_{i_a}}\in\Lambda(\mathscr{P}_{i_a})\}$ denote the set of eigenvalues of the pole tensors $\mathscr{P}_{i_c}$ and $\mathscr{P}_{i_a}$, respectively. Because the power-series approach has to consider all pole tensors, one needs to involve all eigenvalues of each pole tensor for determining $\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z))$. Given a rational tensor z-function $\boldsymbol{\mathfrak{X}}(z)$ with the ROC specified by Eq. (96), we can find the corresponding inverse tensor z-transform using the power-series (long-division) approach such that

$$\mathscr{X}[n]\overset{\text{def}}{=}\mathscr{Z}^{-1}(\boldsymbol{\mathfrak{X}}(z)). \tag{97}$$

An example to illustrate how to apply the power-series approach, i.e., long division, to carry out the inverse tensor z-transform is presented below. □

Example 2. Here, we will present an example to illustrate how to apply the power-series approach, i.e., long division, to carry out the inverse tensor z-transform. Suppose that $\boldsymbol{\mathfrak{X}}(z)$ is a rational tensor z-function, which can be expressed by

$$\boldsymbol{\mathfrak{X}}(z)\overset{\text{def}}{=}\frac{\boldsymbol{\mathfrak{F}}(z^{-1})}{\boldsymbol{\mathfrak{G}}(z^{-1})}, \tag{98}$$

where $\boldsymbol{\mathfrak{F}}(z^{-1})$ and $\boldsymbol{\mathfrak{G}}(z^{-1})$ denote the numerator and denominator polynomials of the variable z^{-1}, respectively, such that

$$\begin{aligned}\boldsymbol{\mathfrak{F}}(z^{-1})&\overset{\text{def}}{=}-\left[\begin{array}{cc|cc}4&0&0&3\\0&0&0&0\\\hline0&0&0&0\\3&0&0&4\end{array}\right]+\left[\begin{array}{cc|cc}2&0&0&3\\0&0&0&0\\\hline0&0&0&0\\3&0&0&2\end{array}\right]z^{-1}\\&=\underbrace{\left[\begin{array}{cc|cc}2&0&0&3\\0&0&0&0\\\hline0&0&0&0\\3&0&0&2\end{array}\right]}_{\mathscr{A}_1}\star_2\left(\mathscr{I}z^{-1}-\underbrace{\left[\begin{array}{cc|cc}2&0&0&1\\0&0&0&0\\\hline0&0&0&0\\1&0&0&2\end{array}\right]}_{\mathscr{C}}\right)\end{aligned} \tag{99}$$

$$\begin{aligned}\boldsymbol{\mathfrak{G}}(z^{-1})&\overset{\text{def}}{=}-\left[\begin{array}{cc|cc}4&1&0&18\\0&0&0&0\\\hline0&0&0&0\\18&0&0&4\end{array}\right]+\left[\begin{array}{cc|cc}8&3&0&21\\0&0&0&0\\\hline0&0&0&0\\21&0&0&8\end{array}\right]z^{-1}-\left[\begin{array}{cc|cc}5&3&0&8\\0&0&0&0\\\hline0&0&0&0\\8&0&0&5\end{array}\right]z^{-2}+\mathscr{I}z^{-3}\\&=\left(\mathscr{I}z^{-1}-\underbrace{\left[\begin{array}{cc|cc}2&1&0&3\\0&0&0&0\\\hline0&0&0&0\\3&0&0&2\end{array}\right]}_{\mathscr{P}_1}\right)^2\star_2\left(\mathscr{I}z^{-1}-\underbrace{\left[\begin{array}{cc|cc}1&1&0&2\\0&0&0&0\\\hline0&0&0&0\\2&0&0&1\end{array}\right]}_{\mathscr{P}_2}\right).\end{aligned} \tag{100}$$

According to Eq. (93), we can express $\mathfrak{X}(z)$ by

$$\mathfrak{X}(z) = \frac{\mathscr{A}_1}{\mathscr{I}} \star_2 \left[\frac{\mathscr{D}_{1,1}}{\mathscr{I}z^{-1} - \mathscr{P}_1} + \frac{\mathscr{D}_{1,2}}{\left(\mathscr{I}z^{-1} - \mathscr{P}_1\right)^2} + \frac{\mathscr{D}_{2,1}}{\mathscr{I}z^{-1} - \mathscr{P}_2} \right]. \tag{101}$$

Note that we can obtain the coefficient tensors $\mathscr{D}_{1,1}$, $\mathscr{D}_{1,2}$, and $\mathscr{D}_{2,1}$ as given by Eq. (94) such that

$$\begin{aligned} \mathscr{D}_{1,1} &= \frac{\mathscr{A}_1}{\mathscr{I}} \star_2 \frac{\mathscr{C} - \mathscr{P}_2}{(\mathscr{P}_1 - \mathscr{P}_2)^2} = \left[\begin{array}{cc:cc} 2 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 2 \end{array}\right], \\ \mathscr{D}_{1,2} &= \frac{\mathscr{A}_1}{\mathscr{I}} \star_2 \frac{\mathscr{P}_1 - \mathscr{C}}{\mathscr{P}_1 - \mathscr{P}_2} = \left[\begin{array}{cc:cc} 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{array}\right], \\ \mathscr{D}_{2,1} &= \frac{\mathscr{A}_1}{\mathscr{I}} \star_2 \frac{\mathscr{P}_2 - \mathscr{C}}{(\mathscr{P}_2 - \mathscr{P}_1)^2} = \left[\begin{array}{cc:cc} -2 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -2 \end{array}\right]. \end{aligned} \tag{102}$$

According to the power-series approach to undertake the inverse tensor z-transform, we have

$$\mathscr{Z}^{-1}\left(\frac{\mathscr{D}_{1,1}}{\mathscr{I}z^{-1} - \mathscr{P}_1}\right) = -\mathscr{D}_{1,1} \star_2 \left(\mathscr{P}_1^{-1}\right)^{k+1} u[k], \tag{103}$$

where

$$u[k] \overset{\text{def}}{=} \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k \geq 0 \end{cases} \tag{104}$$

represents the *unit-step function* with the discrete-time index k and the tensor $\mathscr{P}_1^{-1}$ is given by

$$\mathscr{P}_1^{-1} = \left[\begin{array}{cc:cc} 1/2 & -1/6 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \end{array}\right]. \tag{105}$$

Thus, according to the power-series method, we have

$$\mathscr{Z}^{-1}\left(\frac{\mathscr{D}_{1,2}}{\left(\mathscr{I}z^{-1} - \mathscr{P}_1\right)^2}\right) = -(k+1)\mathscr{D}_{1,2} \star_2 \left(\mathscr{P}_1^{-1}\right)^{k+2} u[k]. \tag{106}$$

Again, according to the power-series approach to undertake the inverse tensor z-transform, we have

$$\mathscr{Z}^{-1}\left(\frac{\mathscr{D}_{2,1}}{\mathscr{I}z^{-1} - \mathscr{P}_2}\right) = -\mathscr{D}_{2,1} \star_2 \left(\mathscr{P}_2^{-1}\right)^{k+1} u[k], \tag{107}$$

where the tensor $\mathscr{P}_2^{-1}$ is given by

$$\mathscr{P}_2^{-1} = \left[\begin{array}{cc:cc} 1 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \end{array}\right]. \tag{108}$$

Finally, by the linearity of the z-transform, we have

$$\begin{aligned} \mathscr{Z}^{-1}(\mathfrak{X}(z)) &= \frac{\mathscr{A}_1}{\mathscr{I}} \star_2 \left[\mathscr{Z}^{-1}\left(\frac{\mathscr{D}_{1,1}}{\mathscr{I}z^{-1} - \mathscr{P}_1}\right) + \mathscr{Z}^{-1}\left(\frac{\mathscr{D}_{1,2}}{(\mathscr{I}z^{-1} - \mathscr{P}_1)^2}\right) + \mathscr{Z}^{-1}\left(\frac{\mathscr{D}_{2,1}}{\mathscr{I}z^{-1} - \mathscr{P}_2}\right) \right] \\ &= \left[\begin{array}{cc:cc} -4 & 9 & 0 & 9 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 4 \end{array}\right] \star_2 \left(\mathscr{P}_1^{-1}\right)^{k+1} u[k] - (k+1) \left[\begin{array}{cc:cc} 0 & 9 & 0 & 18 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 18 & 0 & 0 & 4 \end{array}\right] \star_2 \left(\mathscr{P}_1^{-1}\right)^{k+2} u[k] \\ &+ \left[\begin{array}{cc:cc} 4 & 9 & 0 & -9 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ -9 & 0 & 0 & 4 \end{array}\right] \star_2 \left(\mathscr{P}_2^{-1}\right)^{k+1} u[k], \end{aligned} \tag{109}$$

where $\mathscr{P}_1^{-1}$ and $\mathscr{P}_2^{-1}$ are given by Eqs. (105) and (108), respectively.

Because the eigenvalues of the tensor $\mathscr{P}_1^{-1}$ are 1/2, 1/2, 1/3, and 1/3 while the eigenvalues of the tensor $\mathscr{P}_2^{-1}$ are 1, 1, 1/2, and 1/2, we can determine $\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z))$ according to Eq. (96) such that

$$\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z)) = \bigcap_{i_c=1}^{2} \bigcap_{\kappa_{i_c}=1}^{4} \left\{ z \in \mathbb{C} : |z| > \frac{1}{\left|\lambda_{i_c,\kappa_{i_c}}\right|} \right\} = \{|z| > 1\}. \tag{110}$$

4.2. Generalized Contour-Integral Approach. Given an analytic function $\boldsymbol{\mathfrak{X}}(z)$ and a contour $\mathfrak{C}$, according to Theorem 4, we have

$$\mathscr{X}_{\mathfrak{C}}[n] \overset{\mathrm{def}}{=} \mathscr{Z}_{\mathfrak{C}}^{-1}(\boldsymbol{\mathfrak{X}}(z)) \overset{\mathrm{def}}{=} \frac{1}{2\pi\iota}\oint_{\mathfrak{C}} \boldsymbol{\mathfrak{X}}(z) z^{n-1} dz =_1 \sum_{k=1}^{m} \cdot \left[\mathrm{Res}\left(\boldsymbol{\mathfrak{X}}(z) z^{n-1}; \mathscr{P}_k\right) \star_M \sum_{\lambda_{k,j} \in \acute{\Lambda}(\mathscr{P}_k)} \mathscr{Q}_{k,j} \star_1 \mathscr{R}_{k,j}^{\mathrm{H}} \right], \tag{111}$$

where "$=_1$" is induced from Theorem 9, $\mathfrak{C}$ is a simple closed contour in the complex plane, and $\boldsymbol{\mathfrak{X}}(z)$ is analytic except for the positions at the pole tensors: $z = \mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m \in \mathbb{C}^{I_1\times\cdots\times I_M \times I_1 \times\cdots\times I_M}$ with their corresponding spectra: $\lambda_{k,j} \in \acute{\Lambda}(\mathscr{P}_k)$ for $j = 1, 2, \ldots, J_k$ and $k = 1, 2, \ldots, m$ inside the contour $\mathfrak{C}$. Note that $J_k \le \upsilon$ and $J_k = \upsilon$ if the closed contour $\mathfrak{C}$ includes all spectra $\Lambda(\mathscr{P}_k)$. In addition, $\mathscr{Z}(\mathscr{X}_{\mathfrak{C}}[n]) \neq \boldsymbol{\mathfrak{X}}(z)$ if the contour $\mathfrak{C}$ does not include all eigenvalues of the pole tensors $\mathscr{P}_k$ for k=1, 2, …, m while $\mathscr{Z}(\mathscr{X}_{\mathfrak{C}}[n]) = \boldsymbol{\mathfrak{X}}(z)$ if the contour $\mathfrak{C}$ does include all eigenvalues of the pole tensors $\mathscr{P}_k$ for $k = 1, 2, \ldots, m$. Different from the power-series approach (which requires all eigenvalues of all pole tensors to be inside $\mathfrak{C}$), we can involve only those eigenvalues of the pole tensors inside $\mathfrak{C}$ in determining $\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z))$ (some eigenvalues of any pole tensor out of the contour $\mathfrak{C}$ are allowed) in our proposed new contour-integral approach. In summary, it is possible to have $\boldsymbol{\mathfrak{X}}(z) = \mathscr{Z}(\mathscr{X}[n]) = \mathscr{Z}(\mathscr{X}_{\mathfrak{C}}[n])$ even though $\mathscr{X}[n] \neq \mathscr{X}_{\mathfrak{C}}[n]$. Therefore, the so-called inverse tensor z-transform is not unique, or precisely speaking, it is contour $\mathfrak{C}$ dependent. Theorem 11 below will discuss such a contour-dependence property of the inverse tensor z-transform.

Theorem 11. *Let $\boldsymbol{\mathfrak{X}}(z)$ be an analytic function except for the positions at the pole tensors $z = \mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m \in \mathbb{C}^{I_1\times\cdots\times I_M \times I_1 \times\cdots\times I_M}$ with the corresponding spectra: $\{\lambda_{k,j} \in \Lambda(\mathscr{P}_k)\}$ for $j = 1, 2, \ldots, J_k$ and $k = 1, 2, \ldots, m$. We have τ contours; i.e., $\mathfrak{C}_i$, $i = 1, 2, \ldots, \tau$, and all pole tensors' eigenvalues inside $\mathfrak{C}_i$ are collected as the set $\beta_{\mathfrak{C}_i}$ such that*

$$\bigcup_{i=1}^{\tau} \beta_{\mathfrak{C}_i} = \bigcup_{k=1}^{m} \Lambda(\mathscr{P}_k), \beta_{\mathfrak{C}_i} \cap \beta_{\mathfrak{C}_{i'}} = \varnothing, \quad \text{for any } i \neq i'. \tag{112}$$

If we assume that

$$\mathscr{X}[n] = \sum_{i=1}^{\tau} \mathscr{X}_{\mathfrak{C}_i}[n], \tag{113}$$

where $\mathscr{X}[n] \overset{\mathrm{def}}{=} \mathscr{Z}^{-1}(\boldsymbol{\mathfrak{X}}(z))$ according to Eq. (97) and $\mathscr{X}_{\mathfrak{C}_i}[n] =^{def} \mathscr{Z}_{\mathfrak{C}_i}^{-1}(\boldsymbol{\mathfrak{X}}(z))$ according to Eq. (111).

Proof. We select a large contour $\mathfrak{C}_\infty$ that includes all eigenvalues belonging to $\bigcup_{k=1}^{m} \Lambda(\mathscr{P}_k)$. Thus, in extension of the inverse (scalar) z-transform study in Section 3.4.1 of [28], we obtain

$$\begin{aligned} \mathscr{X}[n] = \mathscr{X}_{\mathfrak{C}_\infty}[n] &= \frac{1}{2\pi\iota}\oint_{\mathfrak{C}_\infty} \boldsymbol{\mathfrak{X}}(z) z^{n-1} dz \\ &= \sum_{k=1}^{m} \left[\mathrm{Res}\left(\boldsymbol{\mathfrak{X}}(z) z^{n-1}; \mathscr{P}_k\right) \star_M \sum_{\lambda_{k,j} \in \Lambda(\mathscr{P}_k)} \mathscr{Q}_{k,j} \star_1 \mathscr{R}_{k,j}^{\mathrm{H}} \right] \\ &=_1 \sum_{i=1}^{\tau} \left\{ \sum_{k=1}^{m} \left[\mathrm{Res}\left(\boldsymbol{\mathfrak{X}}(z) z^{n-1}; \mathscr{P}_k\right) \star_M \sum_{\lambda_{k,j} \in \beta_{\mathfrak{C}_i}} \mathscr{Q}_{k,j} \star_1 \mathscr{R}_{k,j}^{\mathrm{H}} \right] \right\} \\ &= \sum_{i=1}^{\tau} \mathscr{X}_{\mathfrak{C}_i}[n], \end{aligned} \tag{114}$$

where "$=_1$" arises from Eq. (112).

Suppose that we want to carry out the inverse tensor z-transform of $\boldsymbol{\mathfrak{X}}(z)$ given by Eq. (95) using the generalized contour-integral approach subject to a closed contour $\mathfrak{C}$ such that $\acute{\Lambda}(\mathscr{P}_k)$, $k = 1, 2, \ldots, m$, denote the spectra inside $\mathfrak{C}$. If there are J_k eigenvalues included in the spectrum $\acute{\Lambda}(\mathscr{P}_k)$, then the ROC of the tensor z-transform given by Eq. (95), denoted by $\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z))$, can be expressed by

$$\mathrm{ROC}(\boldsymbol{\mathfrak{X}}(z)) = \bigcap_{i_c=1}^{\rho_c} \bigcap_{\kappa_{i_c}=1}^{J_{i_c}} \left\{ z \in \mathbb{C} : |z| > \frac{1}{\left|\lambda_{i_c,\kappa_{i_c}}\right|} \right\} \cap \bigcap_{i_a=1}^{\rho_a} \bigcap_{\kappa_{i_a}=1}^{J_{i_a}} \cdot \left\{ z \in \mathbb{C} : |z| < \left|\lambda_{i_c,\kappa_{i_c}}\right| \right\}, \tag{115}$$

where $\{\lambda_{i_c,\kappa_{i_c}} \in \acute{\Lambda}_{\mathscr{P}_{i_c}}\}$ and $\{\lambda_{i_a,\kappa_{i_a}} \in \acute{\Lambda}_{\mathscr{P}_{i_a}}\}$ are the eigenvalues of the pole tensors $\mathscr{P}_{i_c}$ and $\mathscr{P}_{i_a}$, respectively. Note that the numbers of eigenvalues in the spectrum $\acute{\Lambda}_{\mathscr{P}_{i_c}}$ and the spectrum $\acute{\Lambda}_{\mathscr{P}_{i_a}}$ are represented by J_{i_c} and J_{i_a}, respectively. Different from the power-series approach stated in Section 4.1, the contour-integral approach here only involves the eigenvalues of all pole tensors inside $\mathfrak{C}$.

The inverse tensor z-transform via the contour-integral approach can help us to establish the "generalized Parseval's relation," given two tensor-sequences, say $\{\mathscr{X}[n]\}$ and $\{\mathscr{Y}[n]\}$ with the corresponding tensor z-transforms as given by

$$\mathfrak{X}(z) \stackrel{\text{def}}{=} \mathscr{Z}(\mathscr{X}[n]), \mathfrak{Y}(z) \stackrel{\text{def}}{=} \mathscr{Z}(\mathscr{Y}[n]). \tag{116}$$

According to the generalized contour-integral approach stated in this subsection, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathscr{X}[n] \star_M \mathscr{Y}^*[n] &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi\iota} \oint_{\mathfrak{C}_\infty} \mathfrak{X}(z) z^{n-1} dz \right) \star_M \mathscr{Y}^*[n] \\ &= \frac{1}{2\pi\iota} \oint_{\mathfrak{C}_\infty} \mathfrak{X}(z) \star_M \left[\sum_{n=0}^{\infty} \mathscr{Y}^*[n] z^{n-1} \right] dz \\ &= \frac{1}{2\pi\iota} \oint_{\mathfrak{C}_\infty} \mathfrak{X}(z) \star_M \left[\sum_{n=0}^{\infty} \mathscr{Y}[n] \left(\frac{1}{z^*} \right)^{-n} \right]^* z^{-1} dz \\ &= \frac{1}{2\pi\iota} \oint_{\mathfrak{C}_\infty} \mathfrak{X}(z) \star_M \mathfrak{Y}^* \left(\frac{1}{z^*} \right) z^{-1} dz.\end{aligned} \tag{117}$$

Therefore, we establish the generalized Parseval's relation below for tensor sequences:

$$\sum_{n=0}^{\infty} \mathscr{X}[n] \star_M \mathscr{Y}^*[n] = \frac{1}{2\pi\iota} \oint_{\mathfrak{C}_\infty} \mathfrak{X}(z) \star_M \mathfrak{Y}^* \left(\frac{1}{z^*} \right) z^{-1} dz. \tag{118}$$

An example to illustrate how to apply the generalized contour-integral approach to undertake the inverse tensor z-transform is presented below. □

Example 3. In this example, we will illustrate how to employ the generalized contour-integral approach to undertake the inverse tensor z-transform. A rational tensor z-function $\mathfrak{X}(z)$ is given by

$$\mathfrak{X}(z) \stackrel{\text{def}}{=} \frac{\overbrace{\left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]}^{}_{\stackrel{\text{def}}{=} \mathscr{D}_{1,1}}}{\mathscr{I} - \underbrace{\left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right]}_{\stackrel{\text{def}}{=} \mathscr{P}_1} z^{-1}} + \frac{\underbrace{\left[\begin{array}{cc|cc} 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{array}\right]}_{\stackrel{\text{def}}{=} \mathscr{D}_{2,1}}}{\mathscr{I} - \underbrace{\left[\begin{array}{cc|cc} 3 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 3 \end{array}\right]}_{\stackrel{\text{def}}{=} \mathscr{P}_2} z^{-1}}. \tag{119}$$

According to Eq. (119), the eigenvalues and the eigentensors of the pole tensor $\mathscr{P}_1$ are given by

$$\begin{aligned}\lambda_{1,1} &= 1, \quad \mathscr{Q}_{1,1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{R}_{1,1} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \\ \lambda_{1,2} &= 1, \quad \mathscr{Q}_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{R}_{1,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ \lambda_{1,3} &= 2, \quad \mathscr{Q}_{1,3} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathscr{R}_{1,3} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ \lambda_{1,4} &= 2, \quad \mathscr{Q}_{1,4} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{R}_{1,4} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned} \tag{120}$$

Similarly, the eigenvalues and the eigentensors of the pole tensor $\mathscr{P}_2$ are given by

$$\begin{aligned}\lambda_{2,1} &= 3, \quad \mathscr{Q}_{2,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{R}_{2,1} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \\ \lambda_{2,2} &= 3, \quad \mathscr{Q}_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathscr{R}_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ \lambda_{2,3} &= 4, \quad \mathscr{Q}_{2,3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{R}_{2,3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ \lambda_{2,4} &= 4, \quad \mathscr{Q}_{2,4} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{R}_{2,4} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.\end{aligned} \tag{121}$$

If we consider the inverse tensor z-transform as a causal tensor-sequence subject to a contour $\mathfrak{C}$ which is any simple closed curve in the region: $|z| > 4$, and includes the spectrum

$\{1, 1, 2, 2\}$ of the pole tensor $\mathscr{P}_1$ and the spectrum $\{3, 3, 4, 4\}$ of the pole tensor $\mathscr{P}_2$, we have

$$\begin{aligned}\mathscr{X}_{\mathfrak{C}}[n] = \mathscr{Z}_{\mathfrak{C}}^{-1}(\mathfrak{X}(z)) = & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \star_2 \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right]^k u[n] \\ & + \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right] \star_2 \left[\begin{array}{cc|cc} 3 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 3 \end{array}\right]^k u[n].\end{aligned} \tag{122}$$

Furthermore, if we consider the inverse tensor z-transform as a causal tensor sequence subject to another contour $\overline{\mathfrak{C}}$ which is any simple closed curve in the region: $3 < |z| < 4$, and includes the spectrum $\{1, 1, 2, 2\}$ of the pole tensor $\mathscr{P}_1$ and a partial spectrum $\{3, 3\}$ of the pole tensor $\mathscr{P}_2$, we have

$$\begin{aligned}\mathscr{X}_{\overline{\mathfrak{C}}}[n] = \mathscr{Z}_{\overline{\mathfrak{C}}}^{-1}(\mathfrak{X}(z)) = & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \star_2 \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right]^k u[n] \\ & + \mathscr{D}_{2,1} \star_2 \left(3\mathscr{Q}_{2,1} \star_1 \mathscr{R}_{2,1}^{\mathrm{H}} + 3\mathscr{Q}_{2,2} \star_1 \mathscr{R}_{2,2}^{\mathrm{H}}\right)^k u[n] \\ = & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \star_2 \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right]^k u[n] \\ & + \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right] \star_2 \left[\begin{array}{cc|cc} 3 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]^k u[n].\end{aligned} \tag{123}$$

Note that we only involve the eigenvalues 3, 3 here because the contour $\overline{\mathfrak{C}}$ does not include the eigenvalues 4, 4.

5. Applications of Tensor z-Transform

In this section, we will introduce two applications of our proposed tensor z-transform, namely, "tensor filters." Tensor filters can be widely adopted to characterize massive MIMO (multi-input multioutput) linear-time-invariant (LTI) systems [29–33].

5.1. Infinite-Impulse-Response (IIR) Tensor Filters. Without loss of generality, we consider a discrete-time MIMO LTI system (filter) characterized by a higher-order difference equation with the coefficient tensors $\mathcal{A}_i \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{B}_i \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ such that

$$\mathcal{Y}[k] - \sum_{i=1}^{n} \mathcal{B}_i \star_M \mathcal{Y}[k-i] = \sum_{i=0}^{m} \mathcal{A}_i \star_M \mathcal{X}[k-i], \tag{124}$$

where $\mathcal{X}[k] \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{Y}[k] \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ denote the discrete-time input tensor sequence and the discrete-time output tensor sequence, respectively. The tensor-based MIMO system formulated by Eq. (124) is actually a "generalized autoregressive (AR) and moving average (MA) filter," denoted by a GARMA-(n, m) filter, which can accommodate arbitrary-dimensional input and output tensor sequences and thus can have much broader applicability than the conventional (single-dimensional) ARMA filters. When there exists a nonzero coefficient tensor among $\mathcal{B}_i$'s, we call Eq. (124) an *infinite-impulse-response* (IIR) *tensor filter*. When there exists a nonzero coefficient tensor among $\mathcal{A}_i$'s and all of $\mathcal{B}_i$'s are all-zero tensors, we call Eq. (124) a *finite-impulse-response* (FIR) *tensor filter*. When there exists a nonzero coefficient tensor $\mathcal{B}_n$ and all of $\mathcal{A}_i$'s are all-zero tensors, we call Eq. (124) a GAR-(n) filter. When there exists a nonzero coefficient tensor $\mathcal{A}_m$ and all of $\mathcal{B}_i$'s are all-zero tensors, we call Eq. (124) a GMA-(m) filter.

Taking the tensor z-transform of both sides of Eq. (124), we have the "*transfer tensor*" (the definition of a transfer tensor can be found in [8]) $\mathfrak{H}(z)$ such that

$$\mathfrak{H}(z) \overset{\text{def}}{=} \frac{\mathfrak{Y}(z)}{\mathfrak{X}(z)} = \frac{\sum_{i=0}^{m} \mathcal{A}_i z^{-i}}{\mathcal{I} - \sum_{i=1}^{n} \mathcal{B}_i z^{-i}}. \tag{125}$$

Let us define $\mathcal{U}[k]$ by

$$\mathcal{U}[k] \overset{\text{def}}{=} \sum_{i=0}^{m} \mathcal{A}_i \star_M \mathcal{X}[k-i]. \tag{126}$$

Then, the output tensor sequence $\mathcal{Y}[k]$ is given by

$$\mathcal{Y}[k] = \sum_{i=1}^{n} \mathcal{B}_i \star_M \mathcal{Y}[k-i] + \mathcal{U}[k]. \tag{127}$$

The block diagram given by Figure 1 illustrates how to realize a GARMA-(n, m) filter.

Now consider a special case of $m = n$. We can decompose Eq. (125) into the following two equations:

$$\mathfrak{W}(z) \overset{\text{def}}{=} \frac{\mathfrak{X}(z)}{\mathcal{I} - \sum_{i=1}^{n} \mathcal{B}_i z^{-i}}, \tag{128}$$

$$\mathfrak{Y}(z) = \left(\sum_{i=0}^{n} \mathcal{A}_i z^{-i} \right) \mathfrak{W}(z). \tag{129}$$

Taking the inverse tensor z-transform of both sides of Eqs. (128) and (129), we can get

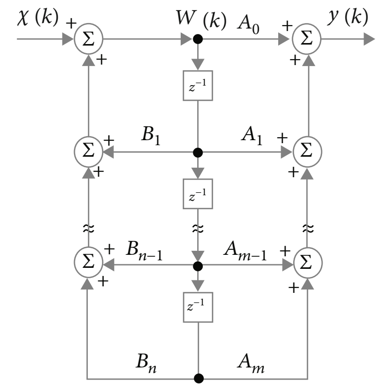


Figure 3: Illustration of the block diagram of the GARMA-(n, m) filter for $n = m$. Note that the input and output signal sequences and all coefficients involve tensors.

$$\mathcal{W}[k] = \sum_{i=1}^{n} \mathcal{B}_i \mathcal{W}[k-i] + \mathcal{X}[k], \tag{130}$$

$$\mathcal{Y}[k] = \sum_{i=0}^{n} \mathcal{A}_i \mathcal{W}[k-i]. \tag{131}$$

The block diagram illustrated by Figure 3 demonstrates how to implement the aforementioned GARMA-(n, n) filter formulated by Eqs. (130) and (131).

5.2. Finite-Impulse-Response (FIR) Tensor Filters. Given the input tensors $\mathcal{X}[k] \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, the output tensors $\mathcal{Y}[k] \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, and the coefficient tensors $\mathcal{A}_i \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{B}_i \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, the general formula of the finite-impulse-response (FIR) tensor filter can be written as

$$\mathcal{Y}[k] = \sum_{i=0}^{m} \mathcal{A}_i \star_M \mathcal{X}[k-i], \tag{132}$$

where m specifies the maximum tap number. The filter impulse response for Eq. (132) can thus be given by

$$\mathcal{H}[i] \overset{\text{def}}{=} \begin{cases} \mathcal{A}_i, & i = 0, 1, 2, \cdots, m, \\ 0, & \text{otherwise}. \end{cases} \tag{133}$$

The block diagrams (for the direct forms I and II) illustrated by Figure 4 demonstrate how to implement the aforementioned m-th order FIR tensor filter formulated by Eq. (132).

6. Numerical Evaluation

In this section, we will present the numerical evaluation for an IIR tensor filter. The corresponding frequency and phase responses will be discussed in Section 6.1 while the approximation of discrete-time tensor signals through our proposed spectral-selection technique along with the implementation-

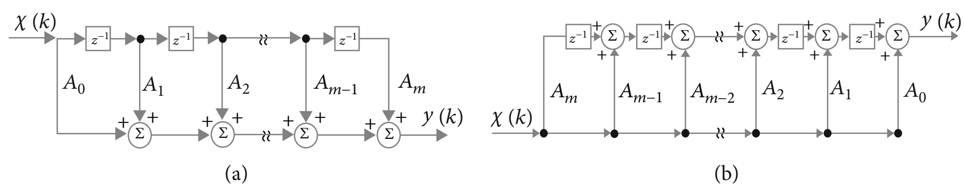


FIGURE 4: Illustration of the block diagrams of the m-th order FIR tensor filter: (a) the direct form I and (b) the direct form II.

complexity analysis will be investigated in Sections 6.2, and Section 6.3, respectively.

Let us consider the following:

$$\mathfrak{H}(z) \stackrel{\text{def}}{=} \frac{\underbrace{\left[\begin{array}{cc|cc} -0.35-0.08\iota & -0.07+0.21\iota & -0.45+0.74\iota & 0.224-0.13\iota \\ 0.38+0.01\iota & 0.67-0.07\iota & -0.12-0.02\iota & 0.56-0.23\iota \\ \hline -0.59-0.49\iota & 0.39-0.14\iota & 0.26-0.08\iota & 0.15+0.36\iota \\ -0.59-0.49\iota & 0.39-0.14\iota & 0.26-0.08\iota & 0.15+0.36\iota \end{array}\right]}_{\stackrel{\text{def}}{=}\mathscr{D}_{1,1}}}{\mathscr{I}-\underbrace{\left[\begin{array}{cc|cc} 1.74+0.0\iota & -0.77-0.65\iota & -0.77+0.65\iota & 2.03+0.0\iota \\ 0.96+0.11\iota & 0.18+0.25\iota & 0.66-0.38\iota & -0.13+0.07\iota \\ \hline 0.95-0.11\iota & 0.66+0.38\iota & 0.18-0.25\iota & -0.13-0.07\iota \\ 2.25+0.0\iota & -0.11-0.28\iota & -0.11+0.28\iota & 2.08+0.0\iota \end{array}\right]}_{\stackrel{\text{def}}{=}\mathscr{P}_1} z^{-1}} + \frac{\underbrace{\left[\begin{array}{cc|cc} -0.14-0.06\iota & 0.36+0.42\iota & -0.59+0.02\iota & 0.35+0.44\iota \\ 0.27-0.17\iota & -0.04+0.32\iota & 0.61-0.12\iota & 0.62+0.13\iota \\ \hline -0.34+0.36\iota & 0.22-0.51\iota & 0.22-0.43\iota & 0.14+0.44\iota \\ -0.34+0.36\iota & 0.22-0.51\iota & 0.22-0.43\iota & 0.14+0.44\iota \end{array}\right]}_{\stackrel{\text{def}}{=}\mathscr{D}_{2,1}}}{\mathscr{I}-\underbrace{\left[\begin{array}{cc|cc} 1.99+0.0\iota & -0.86-1.04\iota & -0.86+1.04\iota & 2.31+0.0\iota \\ 0.94-0.11\iota & 0.39+0.32\iota & 0.89-0.25\iota & -0.89-0.09\iota \\ \hline 0.94+0.11\iota & 0.89+0.25\iota & 0.39-0.32\iota & -0.89+0.09\iota \\ 2.38+0.0\iota & -0.27-0.37\iota & -0.27+0.37\iota & 0.61+0.0\iota \end{array}\right]}_{\stackrel{\text{def}}{=}\mathscr{P}_2} z^{-1}}. \quad (134)$$

6.1. Frequency and Phase Responses. Given a tensor z-transform $\mathcal{H}(z) \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, the frequency response is given by

$$\mathfrak{H}(z)\big|_{z=e^{\iota\omega}} \stackrel{\text{def}}{=} \mathfrak{H}(e^{\iota\omega}) \stackrel{\text{def}}{=} \mathfrak{H}_{\text{Re}}(\omega) + \iota\mathfrak{H}_{\text{Im}}(\omega), \quad (135)$$

where $-\pi \le \omega < \pi$; $\mathcal{H}_{\text{Re}}(\omega)$ and $\mathcal{H}_{\text{Im}}(\omega)$ denote the real and imaginary parts of the frequency response $\mathcal{H}(e^{\iota\omega})$ (a complex-valued tensor ω-function). If an IIR tensor filter, or a GARMA-(n, m) filter, is realized according to Eq. (125), both $\mathcal{H}_{\text{Re}}(\omega)$ and $\mathcal{H}_{\text{Im}}(\omega)$ are real-valued tensor ω-functions. There can be two different definitions about the magnitude response of $\mathcal{H}(e^{\iota\omega})$. Let us write $\mathfrak{H}(e^{\iota\omega}) \stackrel{\text{def}}{=} [\mathfrak{h}_{i_1,\cdots,i_M;j_1,\cdots,j_M}(e^{\iota\omega})]$. The first is called the "entry-wise magnitude response" $|\mathfrak{h}_{i_1,\cdots,i_M;j_1,\cdots,j_M}(e^{\iota\omega})|$ for each entry indexed by $i_1,\cdots,i_M$; $j_1,\cdots,j_M$. The second is called the "ensemble magnitude response" $\|\mathfrak{H}(e^{\imath\omega})\|$ as defined by

$$\|\mathfrak{H}(e^{\imath\omega})\| \stackrel{\text{def}}{=} \sqrt{\|\mathfrak{H}_{\text{Re}}(\omega)\|^2 + \|\mathfrak{H}_{\text{Im}}(\omega)\|^2}, \tag{136}$$

where "$\|\ \|$" represents the tensor norm as introduced in [3]. Figure 5 depicts the magnitude responses for the IIR tensor filter characterized by Eq. (134). Two entry-wise magnitude responses $|\mathfrak{h}_{1,1;1,1}(e^{\imath\omega})|$ and $|\mathfrak{h}_{2,1;2,1}(e^{\imath\omega})|$ are delineated. The ensemble magnitude response $\|\mathfrak{H}(e^{\imath\omega})\|$ (denoted by "Ensemble" in the figure) is also delineated in Figure 5. Since the ensemble magnitude response includes the absolute values of all entries according to Eq. (136), it should be larger than any entry-wise magnitude response as illustrated in Figure 5. Besides, we have two definitions for the phase response of the IIR tensor filter having the tensor z-transform $\mathfrak{H}(z) \stackrel{\text{def}}{=} [\mathfrak{h}_{i_1,\cdots,i_M;j_1,\cdots,j_M}(e^{\imath\omega})] \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$. First, we define the "entry-wise phase spectrum" of $\mathfrak{H}(e^{\imath\omega})$ by $\angle\mathfrak{h}_{i_1,\cdots,i_M;j_1,\cdots,j_M}(e^{\imath\omega})$, which represents the phase response of the entry index $i_1,\cdots,i_M;j_1,\cdots,j_M$. Second, we define the "ensemble phase response" of $\mathfrak{H}(e^{\imath\omega})$ by $\angle\mathfrak{H}(e^{\imath\omega})$ as given by

$$\angle\mathfrak{H}(e^{\imath\omega}) = \sum_{i_1,\cdots,i_M,j_1,\cdots,j_M} \phi_{i_1,\cdots,i_M,j_1,\cdots,j_M} \times \angle\mathfrak{h}_{i_1,\cdots,i_M,j_1,\cdots,j_M}(e^{\imath\omega}), \tag{137}$$

where $\phi_{i_1,\cdots,i_M,j_1,\cdots,j_M} \geq 0$ for all $i_1, i_2, \ldots, i_M$ and $j_1, j_2, \ldots, j_M$ and $\sum_{i_1,\cdots,i_M,j_1,\cdots,j_M} w_{i_1,\cdots,i_M,j_1,\cdots,j_M} = 1$.

Figure 6 plots the phase responses of the IIR tensor filter characterized by Eq. (134). Two entry-wise phase responses $\mathfrak{h}_{1,1;1,1}(e^{\imath\omega})$ and $\mathfrak{h}_{2,1;2,1}(e^{\imath\omega})$ together with the ensemble phase response, denoted by "Ensemble," are depicted by Figure 6.

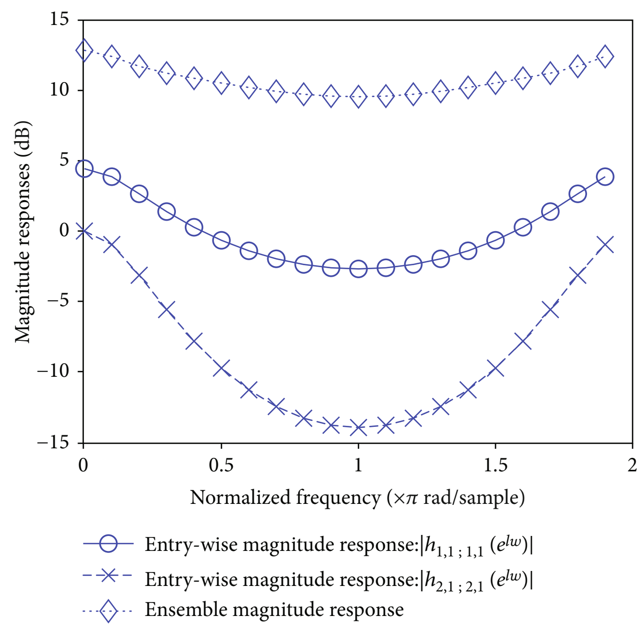


Figure 5: The entry-wise and ensemble magnitude responses of the IIR tensor filter characterized by Eq. (134).

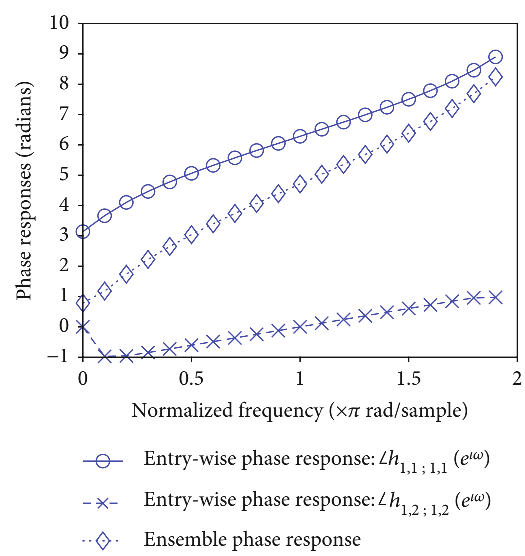


Figure 6: The entry-wise and ensemble phase responses of the IIR tensor filter characterized by Eq. (134).

6.2. Approximation of Discrete-Time Tensor Signals via Spectral Selection. According to Section 4.2, we can obtain different inverse tensor z-transforms by selecting different eigenvalues within an integration contour. The norm (the tensor norm is defined in Section 2 of [3]) of the coefficient-tensor $\mathscr{H}_{\mathfrak{C}}[n]$ at the n-th time index (tap) produced by the inverse tensor z-transform (the transfer tensor) of the IIR tensor filter characterized by Eq. (134) (such that $\mathscr{H}_{\mathfrak{C}}[n] = \mathscr{Z}_{\mathfrak{C}}^{-1}(\mathfrak{H}(z))$) is depicted in Figure 7. Let $\Lambda(\mathscr{P}_1) \stackrel{\text{def}}{=} \{0.09, 2.01, 2.98, 3.01\}$ represent the eigenvalues of the pole tensor $\mathscr{P}_1$ as highlighted in Eq. (134) enclosed by a contour $\mathfrak{C}$ selected for the inverse tensor z-transform.

Similarly, let $\Lambda(\mathscr{P}_2) \stackrel{\text{def}}{=} \{0.09, 0.21, 2.98, 4.01\}$ represent the eigenvalues of the pole tensor $\mathscr{P}_2$ in Eq. (134) enclosed by a contour C for the inverse tensor z-transform. The red bar-graph in Figure 7 depicts the tensor norms of the inverse tensor z-transform subject to a contour $\mathfrak{C}$ enclosing the spectra $\Lambda(\mathscr{P}_1)$ and $\Lambda(\mathscr{P}_2)$. Now let us change $\Lambda(\mathscr{P}_2)$ to $\acute{\Lambda}(\mathscr{P}_2) \stackrel{\text{def}}{=} \{0.09, 0.21, 2.98\}$ and the green bar-graph in Figure 7 depicts the tensor norms of the inverse tensor z-transform subject to a contour $\mathfrak{C}$ enclosing the spectra $\Lambda(\mathscr{P}_1)$ and $\acute{\Lambda}(\mathscr{P}_2)$. Note that the tensor norms marked in green are always less than those marked in red with respect to discrete-time indices n since the former tensor norms result from the spectrum of the pole tensor $\mathscr{P}_2$ without an eigenvalue 4 in comparison with the latter tensor norms. The legend of Figure 7 denotes the eigenvalues of the two pole tensors $\mathscr{P}_1$ and $\mathscr{P}_2$ inside a selected contour $\mathfrak{C}$.

Given $\mathfrak{H}(z)$, $\mathscr{H}_{\mathfrak{C}}[n] = \mathscr{Z}_{\mathfrak{C}}^{-1}(\mathfrak{H}(z))$, and $\mathscr{H}_{\mathfrak{C}'}[n] = \mathscr{Z}_{\mathfrak{C}'}^{-1}(\mathfrak{H}(z))$, define the "approximation-error-norm tensor sequence" $\{\|\mathscr{H}_{\mathfrak{C}}[n] - \mathscr{H}_{\mathfrak{C}'}[n]\|\}$, where $\mathfrak{C}$ encloses all eigenvalues of all pole tensors but $\mathfrak{C}'$ leaves out some of them.

Figure 8 depicts the "approximation-error norm" $\|\mathscr{H}_{\mathfrak{C}}[n] - \mathscr{H}_{\mathfrak{C}'}[n]\|$ with respect to the discrete-time index n

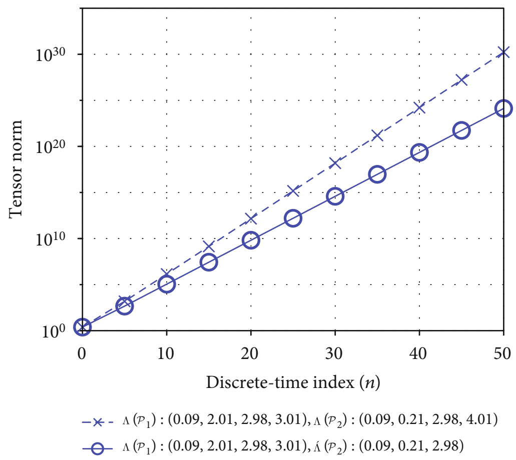


Figure 7: The tensor norms of the inverse tensor z-transforms subject to the contours $\mathfrak{C}$ enclosing two different combinations of spectra, namely, $\Lambda(\mathscr{P}_1)$, $\Lambda(\mathscr{P}_2)$, and $\Lambda(\mathscr{P}_1)$, $\acute{\Lambda}(\mathscr{P}_2)$.

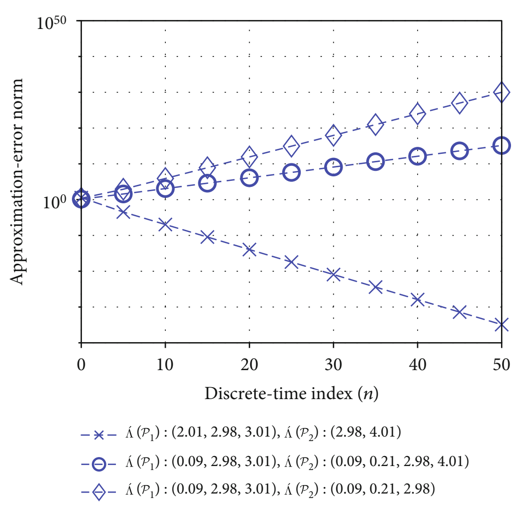


Figure 8: Comparison of the approximation-error norms with respect to the discrete-time index n subject to three different spectra. Note that the complete spectrum of the pole tensor $\mathscr{P}_1$ is $\Lambda(\mathscr{P}_1) \overset{\text{def}}{=} \{0.09, 2.01, 2.98, 3.01\}$ and the complete spectrum of the pole tensor $\mathscr{P}_2$ is $\Lambda(\mathscr{P}_2) \overset{\text{def}}{=} \{0.09, 0.21, 2.98, 4.01\}$.

subject to three different contours $\mathfrak{C}'$, where the symbol "⋄" in the figure indicates that $\mathfrak{C}'$ encloses the spectra $\acute{\Lambda}(\mathscr{P}_1) \overset{\text{def}}{=} \{0.09, 2.98, 3.01\}$ and $\acute{\Lambda}(\mathscr{P}_2) \overset{\text{def}}{=} \{0.09, 0.21, 2.98\}$, the symbol "∘" in the figure indicates that $\mathfrak{C}'$ encloses the spectra $\acute{\Lambda}(\mathscr{P}_1) \overset{\text{def}}{=} \{0.09, 2.98, 3.01\}$ and $\acute{\Lambda}(\mathscr{P}_2) \overset{\text{def}}{=} \{0.09, 0.21, 2.98, 4.01\}$, and the symbol "×" in the figure indicates that $\mathfrak{C}'$ encloses the spectra $\acute{\Lambda}(\mathscr{P}_1) \overset{\text{def}}{=} \{2.01, 2.98, 3.01\}$ and $\acute{\Lambda}(\mathscr{P}_2) \overset{\text{def}}{=} \{2.98, 4.01\}$.

Now we define a "principal eigenvalue" λ such that $|\lambda| > 1$ where $\lambda \in \Lambda(\mathscr{P}_1)$ or $\Lambda(\mathscr{P}_2)$. Similarly, we define "minor eigenvalues" λ such that $|\lambda| < 1$ where $\lambda \in \Lambda(\mathscr{P}_1)$ or $\Lambda(\mathscr{P}_2)$. According to Figure 8, we observe that the approximation-error norm delineated by the symbol "⋄" is the largest at every k because two repeated principal eigenvalues 2 of the

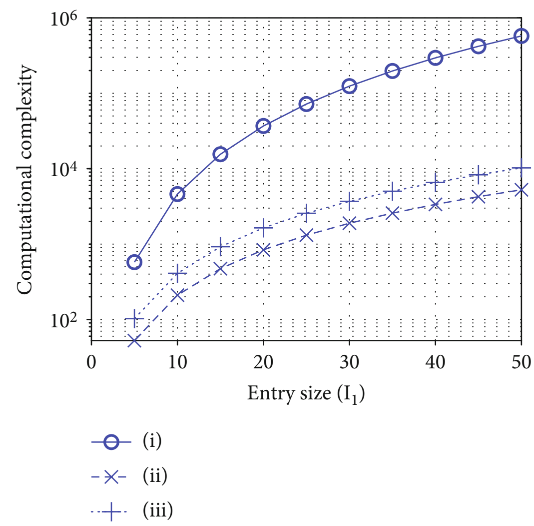


Figure 9: The computational complexities required for calculating $\mathscr{X}_{\mathfrak{C}}[10]$ using our proposed generalized contour-integral approach subject to three different contours $\mathfrak{C}$ enclosing (i) all eigenvalues of all pole tensors, (ii) two eigenvalues of all pole tensors, and (iii) four eigenvalues of all tensors.

pole tensor $\mathscr{P}_1$ and a single principal eigenvalue 4 of the pole tensor $\mathscr{P}_2$ are not involved in the contour integral for the inverse tensor z-transform. On the other hand, the approximation-error norm delineated by the symbol "×" is the smallest at every n because all principal eigenvalues (except some minor eigenvalues) of the pole tensors $\mathscr{P}_1$ and $\mathscr{P}_2$ are involved in the contour integral for the inverse tensor z-transform. As a matter of fact, the approximation-error norm delineated by the symbol "×" will converge to zero as the discrete-time index n approaches ∞. This phenomenon implies the flexibility of our proposed tensor z-transform for the application of approximating an IIR tensor filter characterized by Eq. (125) by another IIR or FIR tensor filter.

6.3. Computational-Complexity Study of Inverse Tensor z-Transform. In this subsection, we will investigate the computational complexity for undertaking the inverse tensor z-transform. Suppose

$$\mathscr{X}_{\mathfrak{C}}[n] \overset{\text{def}}{=} \mathscr{Z}_{\mathfrak{C}}^{-1}(\boldsymbol{\mathfrak{X}}(z)) = \sum_{i=1}^{v} \mathscr{D}_i \star_M \mathscr{P}_i^n, \tag{138}$$

where $\mathscr{D}_i, \mathscr{P}_i \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathscr{P}_i$ denotes the i-th involved pole tensor with the associated coefficient tensor $\mathscr{D}_i$. If we apply the recursion procedure to carry out Eq. (138), the required computational complexity is given by $\mathscr{O}(v^4 \log(n))$ in terms of the Big-O notation. If we just involve ζ principal eigenvalues inside the contour $\mathbb{C}$ to approximate a pole tensor $\mathscr{P}_i$, the required computational complexity is given by $\mathscr{O}(\zeta v^2)$ since there are v elements in each eigentensor. Figure 9 depicts the computational complexities required for carrying out Eq. (138) with respect to the entry-size I_1 (note that $I_1 = I_2 = \cdots I_M$) using the contour-integral subject

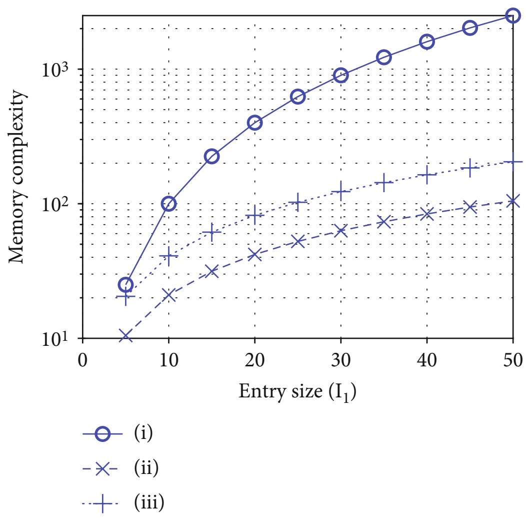


FIGURE 10: The memory complexities required for calculating $\mathscr{X}_{\mathfrak{C}}$ [10] using our proposed generalized contour-integral approach subject to three different contours $\mathfrak{C}$ enclosing (i) all eigenvalues of all pole tensors, (ii) two eigenvalues of all pole tensors, and (iii) four eigenvalues of all tensors.

to the contours enclosing (i) all eigenvalues of all pole tensors (denoted by "○" in the figure), (ii) two eigenvalues of all pole tensors (denoted by " × " in the figure), and (iii) four eigenvalues of all tensors (denoted by "+" in the figure), respectively. The approximation approach can reduce computational complexity significantly according to Figure 9.

For the memory-complexity study, the exact computation of Eq. (138) requires the memory with the size $\mathcal{O}(v^2)$ in terms of the Big-O notation. If we just involve ζ principal eigenvalues inside the contour $\mathbb{C}$ to approximate a pole tensor $\mathscr{P}_i$, the required memory complexity is given by $\mathcal{O}(\zeta v)$ since there are v elements in each eigentensor. Figure 10 depicts the memory complexities required to carry out Eq. (138) with respect to the entry-size I_1 (note that $I_1 = I_2 = \cdots I_M$) using the contour-integral subject to the contours enclosing (i) all eigenvalues of all pole tensors (denoted by "○" in the figure), (ii) two eigenvalues of all pole tensors (denoted by " × " in the figure), and (iii) four eigenvalues of all tensors (denoted by "+" in the figure). According to both Figures 9 and 10, our approximation approach by use of the contour integral involving fewer eigenvalues of the pole tensors inside a contour can greatly reduce the computational and memory complexities required for the inverse tensor z-transform.

7. Conclusion

In this work, a new arbitrary-dimensional transform, namely, the tensor z-transform, is established to characterize multirelational signals and multi-input multioutput linear-time-invariant systems. The definition of the tensor z-transform is first introduced, and then, the essential mathematical properties are discussed. We extend the conventional Cauchy's integral formula and Cauchy's residue theorem for dealing with scalar functions to the new generalized Cauchy's integral formula and the new generalized Cauchy's residue theorem for tensor functions. Thus, we propose a new generalized contour-integral approach for undertaking the inverse tensor z-transform. The applications of our proposed new tensor z-transform in this work include the design and approximation of infinite-impulse-response (IIR) and finite-impulse-response (FIR) tensor filters. By selecting the eigenvalues of the pole tensors of a tensor z-transform inside a contour for the inverse tensor z-transform, we can control the corresponding approximation-error norm and the required computational/memory complexity. Our proposed novel tensor z-transform framework can facilitate a promising analysis tool for signal and information processing over networks in the future.

Data Availability

No underlying data was collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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