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Research Article

Tensor Product Technique and Atomic Solution of Fractional Partial Differential Equations

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In this paper, we investigate the atomic solution of a special type of fractional partial differential equations. Tensor product in Banach spaces, some properties of atom operators, and some properties of conformable fractional derivatives are utilized in such process.

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1. Introduction

When using some classical methods, such as separation of variables, it is difficult to find a general solution for some ordinary or partial differential equations of fractional orders [1, 2]. In this case, one can utilize the theory of tensor product of Banach spaces which was introduced by Diestel and Uhl, in 1978 [3]. Later on, it was developed by Light and Cheney in 2006 [4]. Moreover, in 1985, Khalil found, in his work, some of the best results of tensor product theory that was entitled isometries of $Lp^* \widehat{\otimes} Lp$ [5]. Those results have eventually used to obtain the so-called atomic solutions of the differential equation under study [6–9].

In this paper, our main goal is to obtain the atomic solutions of the following fractional partial differential equation:

$$D_x^{\alpha}u(x,y)D_y^{\beta}u(x,y)+D_x^{\alpha-1}u(x,y)D_y^{\beta-1}u(x,y) = x^{2-\alpha}y^{2-\beta}u^2(x,y)$$

where $u(x,y)$ is the unknown function and $D_x^{\alpha}u$ and $D_y^{\beta}u$ are the fractional partial derivatives of $u(x,y)$ with respect to x of order α and the fractional partial derivative of $u(x,y)$ with respect to y of order β, where $\alpha, \beta \in (1,2)$, respectively.

It should be remarked that atomic solution method comes into play when the partial differential equation is not linear or linear but cannot be separated.

Before we introduce the main result, we commence with some definitions and theorems that are related to our work.

2. Atomic Operator

In this section, we introduce some preliminaries related to the main result of this paper.

Let X^* and Y^* be the dual of the two Banach spaces X and Y, respectively. For $(x,y)\in X\times Y$, the linear operator $x\otimes y : X^* \longrightarrow Y$ defined by $x\otimes y(x^*) = x^*(x)y$ is called an atom. It is easy to see that $x\otimes y$ is a bounded linear operator with norm $\|x\otimes y\| = \|x\|\|y\|$. The linear space spanned by the set $\{x\otimes y, (x,y)\in X\times Y\}$ in $L(X^*, Y)$ is denoted by $X\otimes Y$.

One of the most well-known norms that can be defined on $X \otimes Y$ is the injective norm $\|\cdot\|_V$, namely,

$$\|T\|_V = \sup\left\{\sum_{i=1}^{n} |\langle x, x^*\rangle \langle y, y^*\rangle|, x^* \otimes y^* \in X^* \times Y^*, \|x^*\| = \|y^*\| = 1\right\}$$

where $T = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. It should be stated that $(X \otimes Y, \|\cdot\|_V)$ need not to be complete. We let $X \check{\otimes} Y$ denoted the completion of $X \otimes Y$ in $L(X^*, Y)$ with respect to the injective norm.

One of the nice results in tensor product theory is that $C(I, X)$ is isometrically isomorphic to $C(I) \check{\otimes} X$. For more details on tensor product and the use of atoms, we refer the reader to [10–13].

A result that will be used often in this paper is as follows.

Lemma 1 (see [5]). *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two nonzero atoms in $X \otimes Y$ such that $x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$. Then, either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.*

This leads us to the following interesting theorem that lies in the heart of functional analysis as well as approximation theory and guarantees that any continuous function of several variables can be written as a sum of products of continuous separated functions.

Theorem 1 (see [3]). *Let I and J be two compact intervals and $C(I)$, $C(J)$, and $C(I \times J)$ be the spaces of continuous functions on I, J, and $I \times J$, respectively. Then, every $f \in C(I \times J)$ can be written in the form $f(x, y) = \sum_{i=1}^{\infty} u_i(x) v_i(y)$, where $u_i(x) \in C(I)$ and $v_i(y) \in C(J)$.*

3. Fractional Derivative

In [14], a definition of the so-called α-conformable fractional derivative was introduced as follows.

Definition 1. Let $\alpha \in (0, 1)$ and $f : E \subseteq (0,\infty) \longrightarrow \mathbb{R}$. For $x \in E$, let

$$D^{\alpha} f(x) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(x + \varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon} \quad (1)$$

If the limit exists, then it is called the α-conformable fractional derivative of f at x. If f is α-differentiable on $(0, r)$ for some $r > 0$ and $\lim_{\varepsilon \longrightarrow 0^+} D^{\alpha} f(x)$ exists, then we define

$$D^{\alpha} f(0) = \lim_{x \longrightarrow 0} D^{\alpha} f(x) \quad (2)$$

For $\alpha \in (0, 1]$ and f, g are α-differentiable at a point t, and one can easily see that the conformable derivative satisfies

i. $D^{\alpha}(bf + cg) = bD^{\alpha}(f) + cD^{\alpha}(g)$, for all $b, c \in \mathbb{R}$,

ii. $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$,

iii. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$,

iv. $D^{\alpha}(f/g) = (gD^{\alpha}(f) - fD^{\alpha}(g))/g^2, g(t) \neq 0$,

v. If f is differentiable, then $D^{\alpha}(f)(t) = t^{1-\alpha}(df/dt)(t)$.

We list here the fractional derivatives of certain functions:

i. $D^{\alpha}(t^p) = pt^{p-\alpha}$,

ii. $D^{\alpha}(\sin((1/\alpha)t^{\alpha})) = \cos((1/\alpha)t^{\alpha})$,

iii. $D^{\alpha}(\cos((1/\alpha)t^{\alpha})) = -\sin((1/\alpha)t^{\alpha})$,

iv. $D^{\alpha}(e^{(1/\alpha)t^{\alpha}}) = e^{(1/\alpha)t^{\alpha}}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives. Further, one should notice that a function could be α-conformable differentiable at a point but not differentiable. For example, take $f(t) = 2\sqrt{t}$. Then, $D^{1/2}(f)(0) = 1$. This is not the case for the known classical fractional derivatives, since $D^1(f)(0)$ does not exist. For more on fractional calculus and its applications, we refer to [15, 16]. Many differential equations can be transformed to fractional form and can have many applications in many branches of science.

Let us write $D_s^{\alpha} u$ and $D_t^{\alpha} u$ to denote the partial α-conformable fractional derivative with respect to s and t, respectively. For more details on the conformable differential calculus of functions of several real variables, we refer the reader to [17].

Definition 2 (see [14]). The α-fractional integral of a function f starting from $a \geq 0$ is denoted by $I_{\alpha}^{a}(f)(t)$ such that

$$I_{\alpha}^{\mathbf{a}}(f)(t) = I_1^{\mathbf{a}}\left(t^{\alpha-1} f\right) = \int_{\mathbf{a}}^{t} \frac{f(x)}{x^{1-\alpha}}\, dx \quad (3)$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1)$.

4. Atomic Solution of Fractional Partial Differential Equation

Consider the following fractional partial differential equation:

$$D_x^{\alpha} u(x, y) D_y^{\beta} u(x, y) + D_x^{\alpha-1} u(x, y) D_y^{\beta-1} u(x, y) = x^{2-\alpha} y^{2-\beta} u^2(x, y) \quad (4)$$

where $u(x, y)$ is the unknown function and α, $\beta \in (1, 2)$.

It should be remarked that in [14], the definition of $D_x^{\alpha} u$ was handled when $1 < \alpha$. So, we can write $\alpha = 1 + \alpha_1$, where $0 < \alpha_1 < 1$, and similarly for $\beta = 1 + \beta_1$, where $0 < \beta_1 < 1$. Thus, $\alpha_1 = \alpha - 1$ and $\beta_1 = \beta - 1$ such that α_1, $\beta_1 \in (0, 1)$. Therefore, Equation (4) can be written in the form

$$D_x^{1+\alpha_1} u D_y^{1+\beta_1} u + D_x^{\alpha_1} u D_y^{\beta_1} u = x^{1-\alpha_1} y^{1-\beta_1} u^2 \quad (5)$$

Now, by using the result in [14], Equation (5) can be written in the form

$$D_x^{\alpha_1} D_x u D_y^{\beta_1} D_y u + D_x^{\alpha_1} u D_y^{\beta_1} u = x^{1-\alpha_1} y^{1-\beta_1} u^2 \quad (6)$$

Clearly, Equation (6) is far from being linear. Hence, the techniques of separation of variables and fractional Fourier series [7] are not applicable. Consequently, the concept of atomic solution is inevitable in such case. To do so, we, first, assume the following initial conditions to be imposed on $u(x, y)$:

$$u(0,0)=1, u_x(0,0)=1, \text{and } u_y(0,0)=1. \tag{7}$$

4.1. Procedure. According to Theorem (1), we start our approach with assuming that

$$u(x,y)=P(x)Q(y) \tag{8}$$

Moreover, from Equation (7), we can assume, without loss of generality, that

$$P(0)=P'(0)=1 \text{ and } Q(0)=Q'(0)=1 \tag{9}$$

Now, we substitute Equation (8) into the partial differential equation (Equation (6)) taking into account $D^{\alpha}(f)(t)=D^{\alpha-1}Df(t)K1$, to get

$$\begin{aligned}(D_x^{\alpha_1}D_xP(x))\left(D_y^{\beta_1}D_yQ(y)\right)P(x)Q(y)\\ +(D_x^{\alpha_1}P(x))\left(D_y^{\beta_1}Q(y)\right)P(x)Q(y)\\ =x^{1-\alpha_1}y^{1-\beta_1}P^2(x)Q^2(y)\end{aligned} \tag{10}$$

Consequently,

$$\begin{aligned}\left(x^{1-\alpha_1}D_xD_xP(x)\right)\left(y^{1-\beta_1}D_yD_yQ(y)\right)P(x)Q(y)\\ +\left(x^{1-\alpha_1}D_xP(x)\right)\left(y^{1-\beta_1}D_yQ(y)\right)P(x)Q(y)\\ =x^{1-\alpha_1}y^{1-\beta_1}P^2(x)Q^2(y)\end{aligned} \tag{11}$$

Now, to make life easy, let us assume that $P(x)$ and $Q(y)$ are twice differentiable functions. Hence, Equation (11) can be reformed to be

$$P''(x)Q''(y)+P'(x)Q'(y)=P(x)Q(y) \tag{12}$$

Clearly, each term of Equation (12) is just a product of two functions, one of them is pure in x and the other is pure in y. Therefore, in tensor product form, Equation (12) can be written in the form

$$P''(x)\otimes Q''(y)+P'(x)\otimes Q'(y)=P(x)\otimes Q(y) \tag{13}$$

This implies that the sum of two atoms is an atom. By Lemma (1), we have two cases to be considered:

i. $P''(x)=P'(x)=P(x)$,

ii. $Q''(y)=Q'(y)=Q(y)$.

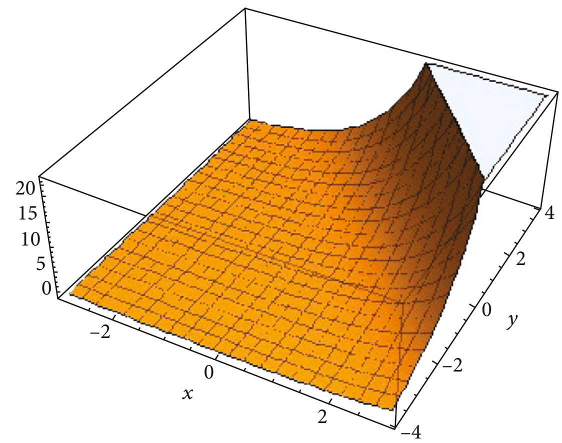

Figure 1: The first atomic solution $u_1(x, y)$ (Equation (17)).

Case i. This case has the following three situations:

a. $P''(x)=P'(x)$,

b. $P''(x)=P(x)$,

c. $P'(x)=P(x)$.

Now, from the above three situations and condition (9), we have

$$P(x)=e^x \tag{14}$$

Now, we proceed by substituting Equation (14) into Equation (13) to obtain

$$Q''(y)+Q'(y)-Q(y)=0 \tag{15}$$

which can be solved together with condition (9) as

$$Q(y)=\cosh\left(\frac{\sqrt{5}}{2}y\right)e^{-(1/2)y}+\sinh\left(\frac{\sqrt{5}}{2}y\right)e^{-(1/2)y} \tag{16}$$

Therefore, the first atomic solution (see Figure 1) corresponding to the Case i is given by

$$u_1(x,y)=e^x\left[\cosh\left(\frac{\sqrt{5}}{2}y\right)e^{-(1/2)y}+\sinh\left(\frac{\sqrt{5}}{2}y\right)e^{-(1/2)y}\right] \tag{17}$$

For the second atomic solution (see Figure 2), by symmetry, one can deduce that

$$u_2(x,y)=e^y\left[\cosh\left(\frac{\sqrt{5}}{2}x\right)e^{-(1/2)x}+\sinh\left(\frac{\sqrt{5}}{2}x\right)e^{-(1/2)x}\right] \tag{18}$$

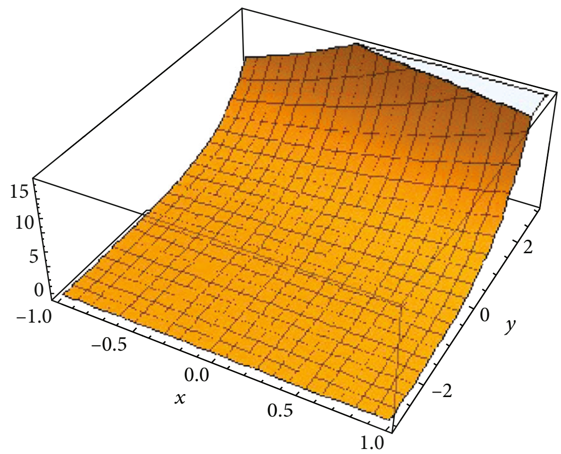

FIGURE 2: The second atomic solution $u_2(x, y)$ (Equation (18)).

5. Conclusions

The use of classical methods to solve some fractional differential equations, usually, leads to dead ends with no possible solutions. Hence, in this paper, we have successfully introduced a new analytic method for solving such problems. To do so, we propose a nonlinear and nonhomogeneous fractional partial differential equation together with the so-called atomic solution process. The theory of tensor product of Banach spaces coupled with some properties of atom operators has been utilized for achieving such a notion. The proposed fractional partial differential equation admits two solutions, namely, two atomic solutions. Some other kinds of partial differential equations are left to the future for further consideration.

Data Availability Statement

There is no underlying data used to support the results and conclusion in the paper.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

The manuscript was written with contributions from all authors. All authors have given their approval to the final version of the manuscript. Waseem Ghazi Alshanti and Roshdi Khalil: methodology, formal analysis, supervision, and writing of the original draft. Ma'mon Abu Hammad and Ahmad Alshanty: methodology, analysis and interpretation of atomic solution procedure, manuscript writing, reviewing, and editing.

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