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Research Article

Effects of Users' Bounded Rationality on a Traffic Network Performance: A Simulation Study

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In this paper, we revisit the principle of bounded rationality applied to dynamic traffic assignment to evaluate its influences on network performance. We investigate the influence of different types of bounded rational user behavior on (i) route flows at equilibrium and (ii) network performance in terms of its internal, inflow, and outflow capacities. We consider the implementation of a bounded rational framework based on Monte Carlo simulation. A Lighthill-Whitham-Richards (LWR) mesoscopic traffic simulator is considered to calculate time-dependent route costs that account for congestion, spillback, and shock-wave effects. Network equilibrium is calculated using the Method of Successive Averages. As a benchmark, the results are compared against both Deterministic and Stochastic User Equilibrium. To model different types of bounded rational user behavior we consider two definitions of user search order (indifferent and strict preferences) and two settings of the indifference band. We also test the framework on a toy Braess network to gain insight into changes in the route flows at equilibrium for both search orders and increasing values of aspiration levels.

1. Introduction

The first notions of traffic assignment were introduced by Wardrop [1]. According to the first Wardrop principle, users aim to minimize their personal route travel times. This leads to a network equilibrium called the Deterministic User Equilibrium (DUE) and it is that most commonly used in dynamic traffic assignment (DTA) problems. Under DUE conditions, no user can decrease his/her own travel time by unilaterally switching routes. However, the first Wardrop principle assumes that users are perfectly rational and perceive all routes and network traffic states perfectly although information on route travel times (i.e., traffic states) is not necessarily perfect. To overcome this problem, Daganzo and Sheffi [2] and Daganzo [3] introduced the Stochastic User Equilibrium (SUE), to take into account the uncertainty of route travel times. The Multinomial Logit and C-Logit are the Random Utility models (RUM) most commonly used in DTA problems. Nonetheless, both these models present several limitations when dealing with correlations between routes. In this study we focus in particular on the Probit model solved using Monte Carlo simulations [4].

Revealed [5] and stated [6] preference surveys show that users tend to choose suboptimal routes instead of optimal ones [7]. We emphasize that a suboptimal route is understood as a route with a longer travel time than the minimum one for the origin-destination (od) pair. In the literature on static traffic assignment, there are other alternative model frameworks that take into account different types of user behavior. One example is the Prospect Theory [8, 9] which considers the users risk-seeking and risk-aversion behavior. It was adapted to the context of route choice by Avineri [10]. In the Prospect Theory, users evaluate the different routes in terms of time prospect and choose the route with the maximum prospect. Users are risk-averse when confronted with prospects of gains and risk-seekers when confronted with prospects of losses and are more sensitive to losses than gains (*loss effect*). Another example is the Regret Theory [11, 12]. The users aim to minimize their regret with respect to the nonselected routes [13, 14]. If the users choose the

route with the minimum travel time, they will feel joy or feel regret otherwise. Another example is the notion of bounded rationality introduced by the seminal works of Simon [15–18]. He stated that users choices are driven by aspiration levels (AL), which represent a set of goal or target variables that should be achieved or exceeded for the users satisfaction. In his original idea, the user searches until a satisfactory alternative is found. This term used to describe this process was coined by Simon as *satisficing*, which stands for the combination of satisfy and suffice. In this study, we focus on the application of the notion of bounded rationality in a dynamic context, by considering distributions of route travel times and a traffic simulator. The goal of this study is to investigate the influence of bounded rational user behavior on individual route flows and network performance. This type of study is very important for decision-making in transportation planning.

Mahmassani and Chang [22] discussed the first notion of bounded rationality applied to traffic assignment, but no mathematical formulation was given. To define users AL, Mahmassani and Chang [22] introduced the concept of *indifference band* (IB), where a route is *satisficing* if the difference between its travel cost and that of the best available route is lower than a given threshold (or IB). The implementation of bounded rationality in traffic assignment is challenging as (i) the calibration of the AL is context dependent [23] and (ii) the BR-UE solutions are not unique [24–26]. Thus, to analyze the BR-UE solutions, some authors have focused on the analysis of the best and worst BR-UE flows of the network [24, 25, 27]. Moreover, the AL can change from user to user. A thorough review of bounded rationality in traffic assignment was provided in Di and Liu [28]. There are two main ingredients that dictate bounded rational network equilibrium: (i) the definition of the AL that dictates whether a route is *satisficing* or not and (ii) the users search order that defines how users are guided in their choice of a *satisficing* route.

For a route to be considered as *satisficing*, its route utility must satisfy

$$U_k \leq AL^{od}, \quad \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi \tag{1}$$

where U_k is the perceived route utility; AL^{od} is the aspiration level we consider in this paper, to be defined at the od level; Ω^{od} is the route choice set for the od pair; and Ξ is the set of all od pairs of the network.

The AL^{od} can be calibrated exogenously by route choice surveys or calibrated endogenously by explicit formulations. The most commonly used definition is based on the concept of indifference band [22, 29, 30]:

$$AL^{od} = \min\left(\overrightarrow{V}\right) + \Delta^{od}, \quad \forall (o,d) \in \Xi \tag{2}$$

where $\overrightarrow{V}$ is a vector containing all deterministic route utilities V_k, $\forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$, and Ω^{od} contains N routes. Δ^{od} is the tolerance or IB at the od level.

Ge and Zhou [20] propose a variable definition of the IB (Δ^{od}):

$$AL^{od} = \min\left(\overrightarrow{V}\right) + \max\left(\left|U_p - U_q\right| \delta_p \delta_q\right), \quad \forall p,q \in \Omega^{od} \wedge p \neq q \wedge \forall (o,d) \in \Xi \tag{3}$$

where δ_p and δ_q are dummy variables that equal 1 if routes p and q belong to Ω^{od}, respectively.

Ge et al. [31] analyzed the BR-DUE equilibrium, considering exogenously fixed AL and fixed and endogenously variable AL. In their model framework, the authors showed that the DUE is a special case of the BR-DUE and discussed the existence conditions of the BR-DUE. However, the uncertainty on the travel times was not considered.

Di and Liu [28] highlighted that a bounded rational behavior can be due to the users habits and inertia or their cognitive costs or individual preferences. In this paper, we focus our attention on the users preferences as a bounded rational behavior to define the search order for the *satisficing* alternatives. Zhao and Huang [19] defined a search order based on a strict preference order for all users sharing the same od pair. This strong assumption allowed obtaining unique BR-UE solutions. To the authors knowledge, the framework of Zhao and Huang [19] has never been tested in a dynamic context, i.e., considering a traffic simulator and time-dependent path costs. In addition, its dynamic implementation using a traffic simulator is highly challenging because it requires solving suboptimization problems to calibrate the AL of the sub-most preferred routes. Thus, a framework capable of solving the global optimization problem is required and discussed further on in this paper. On the other hand, users may also have an indifferent preference for any of the *satisficing* routes (i.e., that satisfy (1)). This is adopted from the notions discussed in [32]. In this case, we consider the fact that all users sharing the same od pair have a similar indifference preference. The choice is modeled by uniform random sampling of any of the *satisficing* routes. Users are then assigned to the *satisficing* route sampled.

Szeto and Lo [30] (the BR-UE [22] and Tolerance-Based Dynamic User Optimum Principle [30] have been used interchangeably in the traffic assignment literature; for the sake of simplicity, we refer to both as BR-UE) discussed an analytical BR-UE dynamic traffic assignment model. The authors proposed a route swapping algorithm, but no clear definition of the users search order was discussed. Instead, the authors targeted certain users on the most congested routes and switched them to less congested ones for each od pair. Moreover, the BR-UE solutions were not unique. Han et al. [33] discussed a dynamic simultaneous departure time and route choice bounded rational framework. However, neither of these frameworks included travel time distributions. In this paper, we revisit the notions of bounded rationality by considering the distribution of travel times rather than deterministic values.

The literature includes a large number of applications of a bounded rational framework to static [25] and dynamic traffic assignment [30, 33], transportation planning [34], traffic policy-making [35], congestion pricing [24], and traffic

safety [36]. However, to the authors knowledge, there is no study in the literature that investigates the influence of users preferences (indifferent and strict) for a bounded rational behavior on individual route flows and network performance in terms of the internal level of congestion and inflow and outflow capacities. The goal of this paper is to fill this gap. We consider time-dependent path costs that account for congestion, shock-waves, and spillback effects calculated using a mesoscopic Lighthill-Whitham-Richards (LWR) model [21]. A spillback effect is the reduction of a link capacity that spreads over other connected links in the network. To model bounded rationality behavior, we relax the definition of the search order of the DUE and SUE frameworks [4]. In both the DUE and SUE cases, users are assigned to the routes with the minimum travel times based on an all-or-nothing procedure. The search order is relaxed to account for the users indifferent and strict preferences. In the case of the indifferent preference search order, users present indifference behavior when choosing any of the *satisficing* routes, whereas in the case of the strict preference search order [19], users are assigned to the most preferred route if this route is perceived as *satisficing* (see (1)), or to the first sub-most preferred route that satisfies (1). We make use of Monte Carlo simulations [4] to account for travel times distribution and consider the classical Method of Successive Averages to calculate the network equilibrium. First, we test the bounded rationality methodology in a toy Braess network and consider a simple linear static and flow dependent utility function. We then consider the two settings of the search order previously mentioned and the AL^{od} defined exogenously. These initial tests allow acquiring insight into how the route flows at equilibrium change according to the two definitions of the search order and increasing values of AL^{od}. Second, for the dynamic implementation, we also consider the two settings of the users search order (i.e., indifferent and strict preferences) and the concept of the IB ((2) and (3)) to define the AL^{od}. The dynamic tests are performed on a Manhattan network. We investigate the influence of the definition of the search order on the individual route flows and analyze the network performance in terms of the internal, inflow, and outflow capacities, given the two search orders and different values of the AL^{od}. The results are compared against both DUE and SUE as benchmarks.

This paper is organized as follows. In Section 2, we discuss the bounded rational model framework considered in this paper. In Section 3, we discuss a simple static test scenario on the Braess network, considering both the indifferent and strict preferences search order. In Section 4, we discuss the influence of the bounded rationality behavior on the network performance also considering the two search orders. In Section 5, we outline the conclusions of this paper.

2. Bounded Rational Framework

The analysis of the effect of users' behavior on network performance in terms of its internal inflow and outflow capacities is very important for policy-makers, in particular when determining policies aimed at increasing network performance. In this paper, we focus on two types of bounded rational user behavior.

We start by introducing the general formulation of the route utilities. The perceived route utility, U_k, is

$$U_k = V_k + \epsilon_k, \quad \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi \tag{4}$$

where V_k is the deterministic route utility and ϵ_k is the uncertainty or error term as often referred to in the literature.

The DUE assumes that users are utility minimizers and the error terms ϵ_k are set to 0. Users are assigned based on an all-or-nothing procedure to the route with the minimum travel time. In the case of the SUE, users are also utility minimizers, but they perceive travel times with uncertainties, meaning that the error terms ϵ_k are not 0. Theoretically, the Probit model [2] is the most attractive model for solving the SUE. However, it requires the computation of a covariance matrix and integrating the multivariate normal distribution. The complexity of the computation increases with the number of routes per od pair. An alternative to this is to use Monte Carlo to consider the distributions of route travel times [4]. We consider that the error terms are defined at the link level (i.e., $\epsilon_a, \forall a \in \Gamma$) instead at the route level. This allows capturing existing correlations between different routes sharing the same links. Sheffi [4] proposed to consider link travel times that are normal distributed and to truncate the negative travel times. This skews the distributions and a positive defined travel times distribution is to be preferred. In this paper, we consider that the terms ϵ_a are gamma distributed following Nielsen [37]. The principle of the Monte Carlo simulations is to discretize the error terms ϵ_a into M samples or draws and locally solve DUE problems. For each error draw, the deterministic utility for route k is defined as $U_k^m, m = 1, \ldots, M$. In short, DUE is solved one for all considering the utility U_k for each route. The SUE corresponds to the average of M Monte Carlo trials, where the utilities U_k^m are adjusted for each trial considering the link error terms values. It is sufficient to explain how we extend DUE to account for bounded rational (BR-DUE), as the extension of SUE is exactly similar; i.e., we have to solve a BR-DUE problem for each Monte Carlo trial of link error terms and then average the results to get the BR-SUE solutions. Mahmassani and Chang [22] introduced the first notions of bounded rationality applied to route choice. Lou et al. [24] and Di et al. [25] formulated the BR-DUE mathematically, but without defining preference rules among *satisficing* routes. Under BR-DUE, all users are satisfied with their choices and no longer consider switching routes. It should be noted that the DUE is an extreme case of the BR-DUE (i.e., when $\Delta^{od} = 0$). The idea is that users are assigned to *satisficing* routes instead of optimal routes (i.e., routes with the minimum perceived travel times). A *satisficing* route should satisfy (1); i.e., $U_k \leq AL^{od}, \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$. Let us define ω^{od} as the set of *satisficing* routes. The users are then assigned to one route depending on the preference rule. In this paper, we consider that the search order is defined according to the user's preferences (indifferent and strict preferences, see below). The previous description of BR-SUE was given based on Monte Carlo trials as it helps to make the connection with

BR-DUE. We can also provide a formal definition of the BR-SUE independently of the solution method. Let P_k be the probability that route k is chosen for one od pair:

$$\begin{aligned} P_k &= P\left(\text{"}U_k \le AL^{od}\text{"} \right. \\ &\quad \left. \cap \text{choosing route k according to the preference rule}\right) \\ &= P\left(U_k \le AL^{od}\right) \\ &\quad \times P\left(\text{choosing route k according to the preference rule} \setminus \text{"}U_k \right. \\ &\quad \left. \le AL^{od}\text{"}\right) \end{aligned} \tag{5}$$

The second part of the equation is the conditional probability of route k being chosen when k is perceived as *satisficing*. Its value depends on the preference rule. We consider two behavioral rules to represent the user's search order. We consider indifferent preferences, where users are uniformly assigned to *satisficing* routes. In this case, P(choosing route k according to the preference rule\"$U_k \le AL^{od}$") $= 1/|\omega^{od}|$, where $|\cdot|$ is the number of *satisficing* routes listed in ω^{od}. This number is at least equal to 1, as route k is *satisficing*. A close form for this probability cannot be provided as the number of other *satisficing* routes than k is a random variable whose distribution is intractable in the general case. Interestingly, Monte Carlo trials for link error terms make the calculation of $|\omega^{od}|$ straightforward, for each Monte Carlo iteration step since all *satisficing* paths are then known. We also consider a strict preference order [19], where users are selecting a *satisficing* route according to a predefined list of preferences Υ^{od}. The users are always choosing the *satisficing* route of highest rank in this list Υ^{od}. Again, the probability P(choosing route k according to the preference rule\"$U_k \le AL^{od}$") does not have a closed form, but Monte Carlo trials permit determining the unique assignment at each iteration.

The idea of the strict preference order was introduced by Zhao and Huang [19], to deal with the nonuniqueness of the equilibrium solution. However, we highlight two main differences between our methodology and that discussed in Zhao and Huang [19]. First, we consider the fact that routes are *satisficing* if and only if their perceived utility satisfies $U_k = V_k \le AL^{od}$, $\forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$ for the BR-DUE or $U_k^m \le AL^{od}$, $\forall m = 1,\ldots,M \wedge \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$ for the BR-SUE. Zhao and Huang [19] consider that routes are *satisficing* according to the strict preference order; i.e., the users are first assigned to the most preferred route and then consecutively to the sub-preferred routes, until all the users are assigned. Second, we consider that AL is defined at the od level (i.e., AL^{od}), while Zhao and Huang [19] consider its definition at the route level. We also assume that all users sharing the same od pair have the same AL^{od}. We consider that it is more realistic from the user's perspective to set a global AL^{od} instead of establishing AL for the sub-preferred routes based on the most preferred ones.

In this paper, we consider the two definitions of AL^{od} as defined in (2) and (3).

To reach a solution for the BR-SUE and simulate the probability term P(route k is satisficing) (see (5)), we consider Monte Carlo simulations as discussed in Sheffi [4] and the classical Method of Successive Averages (MSA). The MSA solves a fixed point problem and is commonly used in traffic assignment to solve both the DUE and SUE [4]. The Monte Carlo simulations consist in discretizing the distributions of the link travel times into M samples or draws and solving BR-DUE problems locally. For each discretization, we identify the *satisficing* routes and assign the users based on an all-or-nothing assignment following the search order established. If the search order is considered to be the indifferent preferences, all the users are assigned randomly to any of the *satisficing* routes. On the other hand, if the search order follows a strict user preference order, all users are assigned to the first *satisficing* route found on this strict sequence of preferences. The new temporary route flows, Q_k^*, correspond to averaging all the local BR-DUE solutions. This corresponds to the temporary route flows Q_k^*, which will be used to update the new route flows Q_k^{j+1} at iteration $j+1$, as

$$Q_k^{j+1} = Q_k^j + \alpha_j \left\{Q_k^* - Q_k^j\right\}, \quad \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi \tag{6}$$

where Q_k^j represent the route flows at iteration j of the MSA and α_j is the descent step. This process is repeated at every descent step of the MSA algorithm.

The sequence of descent steps α_j guarantees the convergence of the MSA. For the theoretical convergence of the algorithm, the following two conditions must be satisfied [4]:

$$\sum_{j=1}^{\infty} \alpha_j = \infty \tag{7}$$

$$\lim_{j \longrightarrow \infty} \alpha_j \longrightarrow 0 \tag{8}$$

One definition of α_j that satisfies both of the previous conditions is $\alpha_j = 1/j$. We consider this definition of α_j in this paper. Other definitions of the descent step size are discussed in the literature [38–41].

A commonly used convergence or stopping criterion is based on the comparison between the current and the previous descent step of the MSA that must be lower to a predefined threshold. Instead we consider the number of violations $N(\lambda)$ and the relative Gap [42]. $N(\lambda)$ represents the number of cases where $|Q_k^{j+1} - Q_k^j|$ is higher than a predefined path convergence threshold Φ. Note that Φ represents an upper bound. The convergence of the algorithm is achieved if $N(\lambda) \le \Phi$. The relative Gap for the DUE is as follows [42]:

$$Gap = \frac{\sum_o \sum_d \sum_{k \in \Omega^{od}} Q_k^{od} \left(V_k^{od} - \min\left(V^{od}\right)\right)}{\sum_o \sum_d Q^{od} \min\left(V^{od}\right)} \tag{9}$$

where Q^{od} is the total demand for the od pair; and V_k^{od} is the average travel time of route k; and $\min(V^{od})$ is the minimum route travel time for the od pair.

The Gap function (see (9)) represents the difference between the travel costs and the equilibrium travel costs. Thus, under perfect DUE conditions, $Gap = 0$. This means that all users choose the routes with the minimum travel

Input the AL^{od} (if they are set exogenously).
Input the network, demand scenario and simulation duration T.
Calculate the route choice set Ω^{od} for each od pair.
Perform an initial network loading.
Set $N(\lambda) > \Phi$ and $Gap^{BRUE} > tol$. Initialize $j = 1, \eta, \zeta, \alpha_j = 1$ and $Q_k^{j=1} = 0$.
Set the MSA stopping criterion *tol*.
while $Gap \geq tol$ **or** $N(\lambda) \geq \Phi$ **or** $j \leq N_{max}$ **do**
  Set $Q_k^j = Q_k^{j+1}$.
  If set endogenously, update the AL^{od} based on Eq. (2) or Eq. (3).
  Perform M error samplings at the link level, based on η and ζ.
  for m=1 to M **do**
    Compute the route utilities.
    Determine the *satisficing* routes and update ω^{od}, $\forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$.
    Based on the defined search order, perform an all-or-nothing assignment. If the search order is the indifferent preferences, repeat A times the all-or-nothing assignment, for the BR-SUE case.
    If $\omega^{od} = \emptyset$, all users are assigned to the minimum utility route.
  end
  Update the new route flows $Q_k^*, \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$, based on an averaging of the users choices over all error samples.
  Update the route flows according to Eq. (6).
  Run the LWR mesoscopic simulator [21].
  Based on the link travel times, fit a gamma distribution to update η and ζ.
  Calculate the Gap (Eq. (9)) and the number of violations $N(\lambda)$.
  Update $\alpha_j = 1/j$.
  Set $j = j + 1$.
end
Save the route flows: $Q_k^{j+1}, \forall k \in \Omega^{od} \wedge \forall (o,d) \in \Xi$.

Algorithm 1: Dynamic implementation algorithm of the *satisficing* model.

times. Under SUE conditions, the Gap is higher than 0, however small. In this case not all users choose the routes with the minimum travel times. In the case of bounded rationality, the Gap value increases as AL^{od} increases. The Gap function is also a measure of how close users are to the equilibrium route travel times (or T^{UE}). The definition of the Gap as defined in (9) is valid for DUE and SUE and informs on how far we are from the DUE. For both the BR-DUE and BR-SUE convergence, we modify the Gap function as follows:

$$Gap^{BRUE} = \frac{\sum_o \sum_d \sum_{k \in \omega^{od}} Q_k^{od} \max\left(V_k^{od} - AL^{od}, 0\right)}{\sum_o \sum_d Q^{od} AL^{od}} \quad (10)$$

Thus, under BR-DUE or BR-SUE conditions, the Gap is about 0 if $\min(V^{od}) \leq V_k^{od} \leq AL^{od}$, $\forall (o,d) \in \Xi$ and the equilibrium condition is fulfilled. Note that, throughout the paper, we use the definition of the Gap as in (9) as an indicator that measures how far the bounded rational equilibria are from the DUE and (10) as the equilibrium convergence criterion for the MSA.

We present the solution algorithm of this framework in Algorithm 1. Note that the difference between Algorithm 1 and that proposed by Sheffi [4] is that we assign the users to *satisficing* routes instead of routes with the minimal travel times. They are assigned to these *satisficing* routes according to one of the search orders discussed previously (i.e., indifferent or strict preferences) at every descent step of the MSA. The first step before entering the MSA loop consists in calculating the route choice set Ω^{od}, for each od pair. It defines the set of routes for the users choices. We then perform an initial loading on these routes and consider the number of violations, the Gap (see (10)), and the maximum number of iterations for the MSA convergence criteria. The corresponding *tol*, Φ, and N_{max} are set. It is also necessary to define the input scale (η) and shape (ζ) parameters of the link travel time gamma distributions for the first Monte Carlo simulations. We then enter in the MSA loop and the AL^{od} is first updated based on the average route travel times (see (2) or (3)). The next step consists in performing the link error sampling considering the η and ζ parameters. This is done through Monte Carlo simulations. The algorithm then loops over all the M error samples and locally solves the BR-DUE problems. For each sample, the route utilities are computed to identify the *satisficing* routes based on AL^{od}. This defines the *satisficing* set of routes ω^{od}. Users are assigned based on the predefined search order (indifferent or strict preferences) based on an all-or-nothing procedure to one route in ω^{od}. It should be noted that in the case of solving the BR-SUE and taking the indifferent preferences into account, it is necessary to repeat the all-or-nothing assignment on the *satisficing* routes A times. The users choices for the local BR-DUE correspond to averaging the previous choices. By applying the law of large numbers, when A is large, we converge to the same average values. The new temporary route flows Q_k^*

Table 1: Notations used in this paper.

k	Route.
a	Link.
o	Origin.
d	Destination.
Q^{od}	Total demand for the od pair.
Ξ	Set of all od pairs of the network.
U_k	Perceived route utility.
V_k	Deterministic route utility.
ϵ_k	Uncertainty or error term.
ϵ_a	Link error term.
Ω^{od}	Route choice set for od pair.
N	Number of routes listed in the route choice set per each od pair.
Q_k	Flow of route k.
q_a	Link flow.
δ_{ak}	Dummy variable that equals 1 if link a belongs to route k.
Γ_a	Set of links that define the graph.
AL^{od}	Aspiration level defined at the od level.
$\overrightarrow{V}$	Vector containing all deterministic route utilities.
Δ^{od}	Tolerance or Indifference Band.
δ_p	Dummy variable that equals 1 if route p is listed on the choice set.
δ_q	Dummy variable that equals 1 if route q is listed on the choice set.
ω^{od}	Set of *satisficing* routes.
T^{UE}	User Equilibrium route travel time.
Υ^{od}	Users' strict preference order for the *od* pair.
M	Number of draws for the error term discretization.
$N(\lambda)$	Number of violations.
Φ	Pre-defined path convergence threshold.
η and ζ	Scale and shape parameters of a gamma distribution.
j	Iterative counter of the MSA method.
tol	Tolerance for the stopping criterium of the MSWA-I.
N_{max}	Maximum number of descent step iterations.
T_{ff}^{a}	Free-flow travel time of link a.
T	Simulation period.
A	Number of assignment repetitions for BR-SUE and indifferent preferences search order.

correspond to the average of all the local BR-DUE choices. The new route flows Q_k^{j+1} are updated according to (6) and the network loading is updated. To determine time-dependent link costs that consider congestion, shock-waves, and spillback effects, we run an LWR mesoscopic traffic simulator [21]. The link travel time distributions are obtained based on the simulated vehicle travel times. To update η and ζ, we fit a gamma distribution to each link travel time distribution. The updated values of η and ζ will be used to perform the error samplings in the next MSA descent step. The Gap^{BRUE} (see (10)) and number of violations $N(\lambda)$ are updated based on the new average route travel times through the individual vehicles travel times. This process is repeated until convergence is achieved. Note that Algorithm 1 also allows solving the BR-DUE by setting $\eta = 0$ and $\zeta = 0$.

In Table 1 we summarize the notations of all symbols and variables used in this paper.

3. First Tests on a Toy Network

We first test the bounded rational model framework discussed in the previous section, on a toy Braess network, and consider a static flow dependent utility function. The goal of these simple initial tests is to assess and acquire insight into how the route flows at equilibrium change according to the two definitions of the search order (i.e., indifferent and strict preference search order) and increasing values of AL^{od}. The AL^{od} are defined exogenously.

3.1. Definition of the Test Network. For the first test, we consider that the perceived travel times (i.e., route utility) depend only on route flows and route free-flow travel times. We resort to the following definition of the perceived route utility, $U_k(Q_k)$:

$$U_k(Q_k) = \sum_{a \in \Gamma_a} (V_a(q_a) + \epsilon_a)\,\delta_{ak}, \quad \forall k \in \Omega^{od} \tag{11}$$

Table 2: BR-DUE route flows for different values of the AL^{od}. The Gap values represent average values based on 1000 repetitions of the BR-DUE calculations.

Δ^{od}	AL^{od}	Q_1/Q^{od}	Q_2/Q^{od}	Q_3/Q^{od}	U_1	U_2	U_3	$\overline{Gap}$
0	46.6	0.17 ± 0.00	0.00 ± 0.00	0.83 ± 0.00	46.7	58.3	46.7	0.00
1	47.0	0.20 ± 0.00	0.00 ± 0.00	0.80 ± 0.00	47.0	58.0	46.0	0.00
2	47.3	0.23 ± 0.00	0.00 ± 0.00	0.77 ± 0.00	47.3	57.7	45.4	0.01
3	47.7	0.27 ± 0.00	0.00 ± 0.00	0.73 ± 0.00	47.7	57.3	44.7	0.02
4	48.0	0.30 ± 0.00	0.00 ± 0.00	0.70 ± 0.00	48.0	57.0	44.0	0.03
5	50.0	0.22 ± 0.16	0.00 ± 0.00	0.78 ± 0.16	47.2	57.8	45.6	0.06
15	60.0	0.41 ± 0.19	0.01 ± 0.01	0.58 ± 0.19	49.0	56.0	41.6	0.12
20	65.0	0.34 ± 0.14	0.17 ± 0.07	0.49 ± 0.21	46.7	58.3	39.8	0.19
30	66.3	0.37 ± 0.26	0.25 ± 0.15	0.37 ± 0.26	46.2	58.8	37.5	0.34
40	76.3	0.33 ± 0.27	0.35 ± 0.28	0.32 ± 0.26	44.8	60.2	36.5	0.46
50	86.3	0.34 ± 0.27	0.33 ± 0.26	0.33 ± 0.26	45.1	59.9	36.5	0.44
75	105.0	0.33 ± 0.27	0.33 ± 0.27	0.33 ± 0.27	45.0	60.0	36.6	0.45
100	136.5	0.33 ± 0.26	0.33 ± 0.26	0.34 ± 0.27	45.1	59.9	36.8	0.43

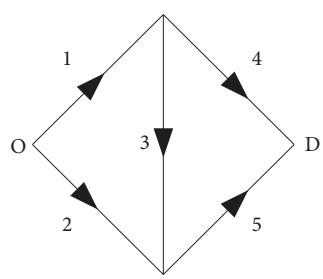

Figure 1: Braess network.

where $V_a(q_a) = \theta_1 T^a_{ff} + \theta_2 q_a$ and $q_a = \sum_{l\in a} \delta_{al} Q_l$; T^a_{ff} is the link free-flow travel time of link a; q_a is the flow of link a; δ_{al} is a dummy variable that equals 1 if route l uses link a, or 0 otherwise; Q_l is the total flow of route l; and ϵ_a is the error term associated with link a. θ_1 and θ_2 are parameters set to 1.

For the static tests, we consider the Braess network (Figure 1). The choice set is $\Omega_k = \{1,4;2,5;1,3,5\}$, $\forall k = 1, 2, 3$. The following sets of link free-flow travel times are considered: $T^1_{ff} = 5$, $T^2_{ff} = 45$, $T^3_{ff} = 10$, $T^4_{ff} = 30$, and $T^5_{ff} = 5$.

For the MSA convergence, we set $tol = 10^{-2}$, $\Phi = 0$, and $N_{max} = 10000$. A total demand of $Q^{od} = 10$ is considered. As a reference of user perfect rationality, we consider the DUE. Note that the ϵ_a terms are set to 0 for both the DUE and the BR-DUE calculations. For the BR-SUE calculations, we consider a gamma distribution with a shape parameter set to $\eta = 1$ and a scale parameter set to $\zeta = 4$; and a total of $M = 2000$ error samples. The BR-DUE and BR-SUE are calculated based on Algorithm 1, except that the link travel times are not updated considering the LWR mesoscopic traffic simulator [21]. Instead, we consider the route utility as defined in (11). The link travel times are only sampled at the initiation of the MSA descent procedure.

3.2. Indifferent Preference Search Order and Exogenous AL^{od}. In this section, we analyze both the BR-DUE and BR-SUE results considering the users indifferent preference search order and the AL^{od} defined as in (2). We also analyze the algorithms convergence towards the equilibrium solution through the Gap function. To define the search order, we consider a uniform distribution to simulate the users choices among the set of *satisficing* routes ω^{od}. This procedure must be repeated many times to reach convergence by the law of large numbers (on average) with the same solution of route flows. Then, for each value of AL^{od}, we repeat the assignment procedure 1000 times and calculate the average route flows and corresponding standard deviation. We do so for both the BR-DUE and BR-SUE calculations. First, under DUE conditions, only routes 1 and 3 are used. This means that $T^{UE} = U_1 = U_3 < U_2$ and corresponds to the route flows: $Q_1 = 1.7$, $Q_2 = 0$, and $Q_3 = 8.3$. Note that T^{UE} is the route travel times at the User Equilibrium. Under SUE conditions, only routes 1 and 3 are *satisficing*. But, due to the users perception of travel times, there is a residual flow on route 2. The route flows are $Q_1 = 3.5$, $Q_2 = 0.3$, and $Q_3 = 6.2$.

We first analyze the BR-DUE results, calculated for increasing values of Δ^{od}. These results are listed in Table 2. The first test consists in reproducing the perfect rationality behavior, by setting $\Delta^{od} = 0 \Longrightarrow AL^{od} = T^{UE} = 46.7$. The route flows under BR-DUE are similar to the DUE. Then, to analyze the equilibrium results for increasing values of $\Delta^{od} \in [0, +\infty[$, we must first identify the critical points for the BR-DUE, that is to say the utility values of each route when the total demand Q^{od} is assigned to each of the routes. We first consider $Q_1 = 10$, $Q_2 = 0$, and $Q_3 = 0$, which yields $U_1 = 55$. Similarly for route 2, $U_2(Q_1 = 0, Q_2 = 10, Q_3 = 0) = 60$, and for route 3, $U_3(Q_1 = 0, Q_2 = 0, Q_3 = 10) = 50$. These critical points play an important role in analyzing the equilibrium solutions. The minimum of the critical points indicates the value of AL^{od} from which the objective function is no longer convex. We analyze the BR-DUE route flows for increasing values of AL^{od} in more detail. For $AL^{od} \in [T^{UE}, 50[$, the users switch from route 3 to route 1. Note that the users switch from the *satisficing* routes with higher route flows to the ones with

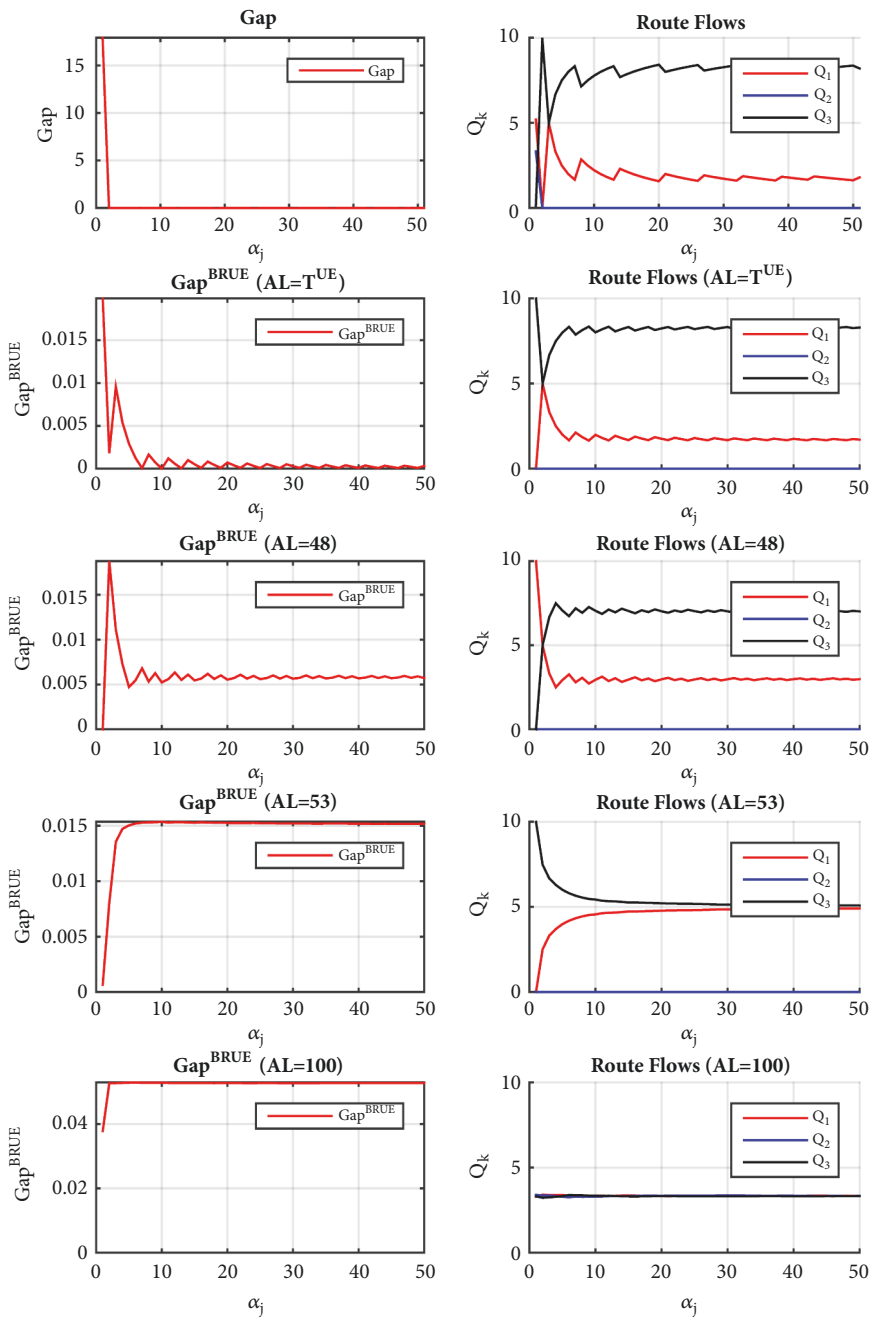

FIGURE 2: Gap (left) and route flows (right) as a function of the increasing number of MSA descent steps j for the DUE and several values of $AL^{od} = T^{UE}, 48, 53, 100$.

lower route flows. For $AL^{od} \in [50, 55[$, the algorithm does not converge to the same solution as evidenced by the standard deviation values listed in Table 2. For example, for $AL^{od} = 50$, two feasible solutions are found: $(Q_1 = 5, Q_2 = 0, Q_3 = 5)$ which yields $(U_1 = 50, U_2 = 55, U_3 = 40)$; and $(Q_1 = 0, Q_2 = 0, Q_3 = 10)$ that yields $(U_1 = 45, U_2 = 60, U_3 = 50)$. The convergence of the algorithm to any of these feasible solutions depends on the initial loading of the network for the MSA algorithm. This explains why we do not converge to the same set of route flows for $AL^{od} \geq 50$. For $AL^{od} \in [55, 60[$, route 2 becomes *satisficing* and users switch from routes 3 and 1 to route 2. For $AL^{od} \geq 60$, the route flows will converge to 1/3 as the value of AL^{od} increases. This represents the users indifference for choosing any of the *satisficing* routes. Under SUE conditions the values in Table 2 are obtained.

We investigate the algorithms convergence for different values of Δ^{od}, which corresponds to different values of AL^{od}, as shown in Figure 2. To do this, we consider a total of 50 descent steps j of the MSA algorithm, despite the convergence criterion of $Gap \leq 10^{-2}$ being verified for a

Table 3: BR-SUE route flows for different values of the AL^{od}.

Δ^{od}	AL^{od}	Q_1/Q^{od}	Q_2/Q^{od}	Q_3/Q^{od}	U_1	U_2	U_3	$\overline{Gap}$
0	42.6	0.34 ± 0.00	0.03 ± 0.00	0.63 ± 0.00	48.1	56.9	42.5	0.06
1	43.5	0.34 ± 0.00	0.03 ± 0.00	0.63 ± 0.00	48.1	56.9	42.5	0.06
2	44.6	0.34 ± 0.00	0.03 ± 0.00	0.63 ± 0.00	48.1	56.9	42.5	0.06
3	45.5	0.34 ± 0.00	0.03 ± 0.00	0.63 ± 0.00	48.1	56.9	42.5	0.06
4	46.5	0.34 ± 0.00	0.03 ± 0.00	0.63 ± 0.00	48.1	56.9	42.5	0.06
5	47.6	0.34 ± 0.00	0.03 ± 0.00	0.63 ± 0.00	48.1	56.9	42.5	0.06
15	56.4	0.40 ± 0.00	0.03 ± 0.00	0.57 ± 0.00	48.7	56.3	41.3	0.08
20	59.7	0.42 ± 0.00	0.09 ± 0.00	0.49 ± 0.00	48.3	56.7	39.8	0.13
30	67.7	0.39 ± 0.01	0.23 ± 0.00	0.39 ± 0.00	46.6	58.4	37.7	0.22
40	76.9	0.35 ± 0.00	0.32 ± 0.01	0.35 ± 0.00	45.4	59.6	36.9	0.27
50	86.8	0.33 ± 0.00	0.33 ± 0.00	0.34 ± 0.00	45.1	59.9	36.7	0.29
75	111.9	0.33 ± 0.00	0.33 ± 0.00	0.33 ± 0.00	45.0	60.0	36.7	0.29
100	136.7	0.33 ± 0.00	0.33 ± 0.00	0.33 ± 0.00	45.0	60.0	36.7	0.29

lower number of j. This allows observing that the solution no longer changes after the convergence criterion is satisfied. In Figure 2, we show the evolution of the Gap and route flows for increasing values of j, for the DUE and $AL^{od} = T^{UE}, 48, 53, 100$. For all four cases, the Gap value converges to a constant value for increasing values of j as well as the route flows.

We analyze the BR-SUE results for increasing values of Δ^{od}. These results are listed in Table 3. We first confirm that the route flows under BR-SUE and SUE are similar, by setting $\Delta^{od} = 0$. For $AL^{od} \in [42.6, 55]$, the users change from route 3 to route 1. For $AL^{od} \geq 57$, route 2 becomes *satisficing* and the users also start choosing this route. The comparison results of the BR-DUE and BR-SUE for $AL^{od} \geq 65$ are of particular interest. In both cases, the route flows converge to 1/3 when the value of AL^{od} is sufficiently large. This represents the users indifference for choosing any of the *satisficing* behaviors, since all the routes comply with the condition defined by (1). The effect of the perception of the route travel times explains the small differences verified in the route flows between the BR-DUE and BR-SUE for the same value of AL^{od}. For example, it is interesting to consider the case where $\Delta^{od} = 40$. The BR-DUE route flows are ($Q_1 = 3.3; Q_2 = 3.5; Q_3 = 3.2$) and the BR-SUE route flows are ($Q_1 = 3.5; Q_2 = 3.2; Q_3 = 3.5$). It can also be seen from the BR-SUE results shown in Table 3 that the MSA algorithm converges to the same solution, as evidenced by the standard deviation of the route flows. To solve the BR-SUE, we solve M BR-DUE problems locally, and since the search order is the indifference preferences, the users are assigned A times for each BR-DUE problem. By applying the law of large numbers, we converge on average to the same solution when A is sufficiently large.

We also observe that, for both the BR-DUE and BR-SUE results, the average $\overline{Gap}$ values increase as we increase AL^{od}, as expected. This represents the effect of the *satisficing* behavior, where users choose *satisficing* routes instead of the routes with the shortest travel times.

3.3. Strict Preference Search Order and Exogenous AL^{od}. In this section, we analyze the BR-DUE results calculated considering an exogenous definition of AL^{od} and a strict preference search order [19]. We calculate the BR-DUE results for the Braess network (Figure 1), considering the six possible strict preference search orders (Υ^{od}, $\forall k = 1, 2, 3$). For the calculations, we consider our bounded rational model framework and the model discussed by Zhao and Huang [19]. The mathematical methodology discussed by Zhao and Huang [19] is not suitable for a dynamic implementation considering a traffic simulator. This is because it requires solving suboptimization problems to calibrate the AL of the sub-preferred routes. In this section, we compare the route flows calculated considering the two frameworks and the six preference search orders. The AL^{od} are set exogenously according to each strict preference order. We apply the model discussed by Zhao and Huang [19], considering that the utility of the most preferred route is equal to the value of the aspiration level AL^{od}; and the route flows of the remaining routes correspond to the result obtained by solving the sub-UE problem, as done by Zhao and Huang [19]. The results are listed in Table 4.

It can be seen that when $AL^{od} = T^{UE} = 46.6$, the route flows obtained for both models are equivalent to the DUE result. Thus, both models are able to reproduce the users perfect rationality whatever the preference order.

Consider the first strict preference order $\Upsilon^{od} = 1, 2, 3$. For $47.5 \leq AL^{od} < 50$, in both models the users switch directly from route 3 to the most preferred route 1. The flows on these two routes are equal for both models. But in our model, route 2 is not selected because it is not considered as *satisficing* whereas in the model of Zhao and Huang [19], the assignment problem solved is $U_1 = A_1$ and $U_2 = U_3$. Under sub-User Equilibrium (sub-UE) conditions, $U_2 > U_3$. This means that the remaining users that have not chosen the most preferred route 1 will choose route 3. In our model, the users are assigned to the most preferred route until the *satisficing* condition (see (1)) is satisfied. Users are then assigned to the

Table 4: Route flow distribution for the Braess network for different values of the AL and under BR-DUE conditions. A set of strict preferences is considered for the search order.

	Our model						Zhao and Huang [19]					
AL^{od}	Q_1/Q^{od}	Q_2/Q^{od}	Q_3/Q^{od}	U_1	U_2	U_3	Q_1/Q^{od}	Q_2/Q^{od}	Q_3/Q^{od}	U_1	U_2	U_3
Preference order $\Upsilon^{od} = 1\ 2\ 3$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.5	0.25	0.00	0.75	47.5	57.5	45.0	0.25	0.00	0.75	47.5	57.5	45.0
50.0	0.50	0.00	0.50	50.0	55.0	40.0	0.50	0.00	0.50	50.0	55.0	40.0
52.5	0.87	0.13	0.00	52.5	52.5	30.0	0.75	0.00	0.25	52.5	52.5	35.0
55.0	1.00	0.00	0.00	55.0	50.0	30.0	1.00	0.00	0.00	55.0	50.0	30.0
Preference order $\Upsilon^{od} = 1\ 3\ 2$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.5	0.25	0.00	0.75	47.5	57.5	45.0	0.25	0.00	0.75	47.5	57.5	45.0
50.0	0.50	0.00	0.50	50.0	55.0	40.0	0.50	0.00	0.50	50.0	55.0	40.0
52.5	0.75	0.00	0.25	52.5	52.5	35.0	0.75	0.00	0.25	52.5	52.5	35.0
55.0	1.00	0.00	0.00	55.0	50.0	30.0	1.00	0.00	0.00	55.0	50.0	30.0
Preference order $\Upsilon^{od} = 2\ 1\ 3$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	~	~	~	~	~	~
47.5	0.25	0.00	0.75	47.5	57.5	45.0	~	~	~	~	~	~
50.0	0.50	0.00	0.50	50.0	55.0	40.0	~	~	~	~	~	~
55.0	0.75	0.25	0.00	50.0	55.0	30.0	0.50	0.00	0.50	50.0	55.0	40.0
60.0	0.50	0.50	0.00	45.0	60.0	30.0	0.125	0.125	0.75	45.0	60.0	45.0
65.0	0.25	0.75	0.00	40.0	65.0	30.0	0.00	0.50	0.50	40.0	65.0	40.0
70.0	0.00	1.00	0.00	35.0	70.0	30.0	0.00	1.00	0.00	35.0	70.0	30.0
Preference order $\Upsilon^{od} = 2\ 3\ 1$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	~	~	~	~	~	~
47.5	0.13	0.00	0.87	46.3	58.7	47.5	~	~	~	~	~	~
50.0	0.00	0.00	1.00	45.0	60.0	50.0	~	~	~	~	~	~
55.0	0.00	0.00	1.00	45.0	60.0	50.0	0.50	0.00	0.50	50.0	55.0	40.0
60.0	0.00	0.00	1.00	45.0	60.0	49.9	0.125	0.125	0.75	45.0	60.0	45.0
65.0	0.00	0.50	0.50	40.0	65.0	40.0	0.00	0.50	0.50	40.0	65.0	40.0
70.0	0.00	1.00	0.00	35.0	70.0	30.0	0.00	1.00	0.00	35.0	70.0	30.0
Preference order $\Upsilon^{od} = 3\ 1\ 2$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.0	0.15	0.00	0.85	46.5	58.5	47.0	0.15	0.00	0.85	46.5	58.5	47.0
48.0	0.10	0.00	0.90	46.0	59.0	48.0	0.10	0.00	0.90	46.0	59.0	48.0
49.0	0.05	0.00	0.95	45.5	59.5	49.0	0.05	0.00	0.95	45.5	59.5	49.0
50.0	0.00	0.00	1.00	45.0	60.0	50.0	0.00	0.00	1.00	45.0	60.0	50.0
Preference order $\Upsilon^{od} = 3\ 2\ 1$												
46.6	0.17	0.00	0.83	46.7	58.3	46.7	0.17	0.00	0.83	46.7	58.3	46.6
47.0	0.15	0.00	0.85	46.5	58.5	47.0	0.15	0.00	0.85	46.5	58.5	47.0
48.0	0.10	0.00	0.90	46.0	59.0	48.0	0.10	0.00	0.90	46.0	59.0	48.0
49.0	0.05	0.00	0.95	45.5	59.5	49.0	0.05	0.00	0.95	45.5	59.5	49.0
50.0	0.00	0.00	1.00	45.0	60.0	50.0	0.00	0.00	1.00	45.0	60.0	50.0

sub-preferred routes if and only if they are *satisficing*. Since route 2 is not *satisficing*, the remaining users choose route 3. Thus, both models yield similar route flows for these two values of AL^{od}. For $AL^{od} = 52.5$, route 2 becomes *satisficing* for our model. Thus the users will switch according to the strict preference order Υ^{od}. Thus, the users will first switch from route 3 to 2 and then from 2 to 1. In the case of the model of Zhao and Huang [19], the sub-UE solution corresponds to $U_2 > U_3$. Thus, no user chooses route 2. For $AL^{od} \geq 55$, all the users choose the most preferred route 1 for both models.

Consider the second strict preference order $\Upsilon^{od} = 1, 3, 2$. In this case, the users switch directly from route 3 to the most preferred route 1 as AL^{od} increases. In the case of our model, the users are assigned to the most preferred route until it is considered as *satisficing*. Then, the remaining users are assigned to the sub-preferred route 3. In the case of the

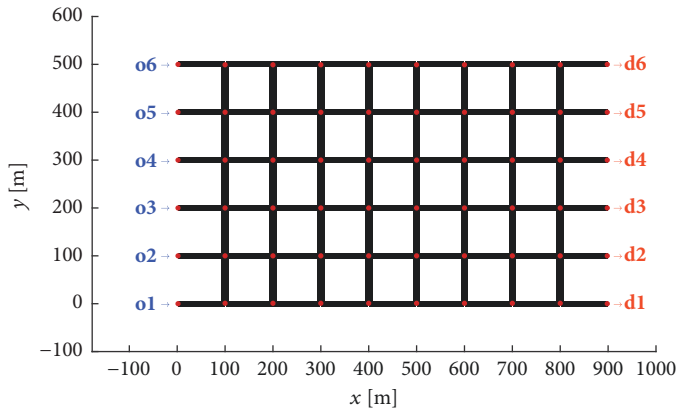

Figure 3: Manhattan network.

model of Zhao and Huang [19], the sub-UE solution also corresponds to $U_2 > U_3$ and thus no users choose route 2. So, for this strict preference order, both models give similar route flows as AL^{od} increases.

Consider the third strict preference order $\Upsilon^{od} = 2, 1, 3$. Given this preference order and for $AL^{od} \leq 50$, route 2 is not *satisficing* for our model. Therefore, the users will switch from route 3 (the less preferred) to the second most preferred route 1. The model of Zhao and Huang [19] cannot be applied for $AL^{od} < 55$, since $AL^{od} \neq U_2$. This would lead to a violation of the strict preference order assumption. We make this assumption more flexible. Although route 2 is the most preferred route, it is not considered as *satisficing* and the users choose other most preferred routes that are *satisficing*. This is the case of route 1. In the case of our model, for $AL^{od} \geq 55$, all the users switch from route 1 to route 2 (the most preferred one). In the case of the model of Zhao and Huang [19], the users will switch from route 1 to routes 2 and 3 (for $AL^{od} = 55$ and $AL^{od} = 60$) and then from 3 to 2 (for $AL^{od} = 65$ and $AL^{od} = 70$). Although route 3 is that least preferred, the users switch from route 1 to route 3. Note that we solve the following assignment problem for the model of Zhao and Huang [19]: $AL^{od} = U_2$ and $U_1 = U_3$.

Consider the fourth strict preference order $\Upsilon^{od} = 2, 3, 1$. In the case of our model, for $AL^{od} \leq 60$, route 2 is not *satisficing*. Therefore, all the users switch from route 1 to route 3 in accordance with the strict preference order. For $AL^{od} \geq 60$, route 2 becomes *satisficing* and the users switch from route 3 to route 2. In the case of the model of Zhao and Huang [19], the condition $AL^{od} = U_2$ is satisfied only for $AL^{od} \geq 55$. For $AL^{od} \geq 55$, we observe a route flow pattern similar to that in the previous strict preference order ($\Upsilon^{od} = 2, 1, 3$) for the same reasons discussed previously.

Consider the fifth and sixth strict preference orders $\Upsilon^{od} = 3, 1, 2$ and $\Upsilon^{od} = 3, 2, 1$. In the case of our model, route 2 is not *satisficing* for these two strict preference orders. Thus, users switch directly from route 1 to route 3, as AL^{od} increases. For $AL^{od} \geq 50$, all users choose the most preferred route 3. In the case of the model of Zhao and Huang [19], we solve the following assignment problem: $AL^{od} = U_3$ and $U_1 < U_2$ (sub-UE problem). Since $U_1 < U_2$ for all the listed values of AL^{od}, the remaining users that have not chosen route 3 choose route 1 for both strict preference orders. This is why we observe similar route flows for both models and both strict preference orders.

In summary, we validate our bounded rational framework considering a strict users preference search order, by comparing the route flows at equilibrium with the results obtained by the model of Zhao and Huang [19]. Moreover, our bounded rational framework is suitable for dynamic implementation with a traffic simulator and will be tested in the next section.

4. Dynamic Implementation on a Manhattan Network

In this section, we investigate the influence of different types of bounded rational user behavior on (i) individual route flows and (ii) network performance in terms of its internal, inflow, and outflow capacities. To do this, we consider the implementation of the bounded rational framework described in Algorithm 1. To determine the time-dependent cost paths that account for congestion, shock-waves, and spillback effects, we consider a mesoscopic LWR traffic simulator [21]. The tests are performed on a Manhattan network. We consider the indifferent and strict preference search orders and both definitions of the AL^{od}, as in (2) and (3).

4.1. Test Scenario Definition. For the dynamic implementation, we consider the Manhattan network composed of 134 links, as shown in Figure 3. All the links have the same length of 100 meters. Traffic lights regulate all the intersections. A green light duration of 45 seconds is considered for the traffic lights of the horizontal links whereas a duration of 15 seconds is considered for the traffic lights of the vertical links. Green times are set in the West-East and in the North-South directions. The offsets considered are of 10 and 20 seconds.

A triangular fundamental diagram is considered for each lane of the network, with the following parameters: $u = 15$ (m/s) for the free-flow speed; $w = 5$ (m/s) for the wave speed;

Table 5: The Gap value and the average travel times per route $\overline{TT_k}$ [in s] calculated from the distributions shown in Figure 6 are also listed. These values are listed for the DUE, SUE, and different settings of the indifference band.

Model	ID Model	Preference	Gap	$\overline{TT_k}$ [s]
DUE	1	~	0.3	315
SUE	2	~	0.3	295
IB ($\Delta^{od} = 0$)	3	Indifferent	0.3	294
IB ($\Delta^{od} = 100$)	4	Indifferent	7.3	337
IB ($\Delta^{od} = 500$)	5	Indifferent	14.7	332
IB [20]	6	Indifferent	16.2	350
IB ($\Delta^{od} = 0$)	7	Strict	68.0	312
IB ($\Delta^{od} = 100$)	8	Strict	68.8	297
IB ($\Delta^{od} = 500$)	9	Strict	68.8	297
IB [20]	10	Strict	69.1	297

and $k_{jam} = 0.2$ (veh/m/lane) for the jam density. The entry links (i.e., from O1 to 6) have two lanes. The total link flow is assigned equally on each lane.

The Manhattan network shown in Figure 3 has six entries (identified by O1 to O6 in Figure 3) and exits (identified by D1 to D6 in Figure 3). For each of the six entries, we consider a constant inflow (demand) of 0.5 (veh/s). There is no capacity restriction at the exits. There is a total of 36 possible od pairs. To define the choice set Ω^{od}, we consider 3 paths per od pair. These paths are calculated using a K-shortest path algorithm. This gives a total of 108 routes, considering the 36 possible od pairs.

For the dynamic tests and the bounded rational route choice model, we consider the endogenous definitions of the indifference band for AL^{od} as defined by (2) and (3) and two settings of the search order:

(i) an *indifferent* preference search order, where users randomly choose any of the *satisficing* alternatives; or, the least worst if there are no *satisficing* alternatives.

(ii) a *strict preference* search order, where users have a strict preference for the routes with the most reliable travel times. We consider the variances of the route travel times as the time reliability indicator. Then, the set of strict preference is built by ordering the routes from the lowest to the highest variance value for each od pair. This set of preferences is updated at every descent step of the MSA, based on the route travel time distributions of the previous simulation.

Considering this search order, users seek *satisficing* alternatives based on this set of strict preferences and on AL^{od}. Similarly, for the strict preference search order discussed in Section 3.3, the users choose only the most preferred route if it is perceived as *satisficing*, i.e., that conforms to (1). Then, if the most preferred route is not perceived as *satisficing*, the users consider the other most preferred routes until they find one that is *satisficing*. If none of the routes are *satisficing*, the users choose the route with the minimal travel time.

As a reference, we consider the DUE and SUE. To solve the SUE, we consider the Probit model with gamma distributed error terms and use Monte Carlo simulations [4]. For the indifference band defined by (2), we consider three exogenous values for Δ^{od}: 0; 100; and 500. We have a total of 10 simulation scenarios, considering both search orders defined above. The total simulation period is $T = 3000$ seconds. For the convergence, we set $tol = 10^{-2}$, $\Phi = 0$, and $N_{max} = 250$.

4.2. Analysis of the Individual Route Flows. In this section, we analyze the individual route flows for the 10 simulation scenarios. Each scenario is identified by one ID number, as listed in Table 5. We also list the Gap values in Table 5, that are calculated using (9). In Figures 4 and 5, we show the route flow distributions for each od pair of the network and all ten scenarios. In Figure 6, we show the distributions of the average route travel times for the 10 simulation scenarios. Note that the average travel time per route for each scenario (i.e., the average of these distributions) is also listed in Table 5.

We analyze the individual route flows shown in Figures 4 and 5. By setting $\Delta^{od} = 0$ (Model 3), we observe that for the indifferent preference search order, we obtain similar route flows compared to the SUE (Model 2). However, this is not observed for the strict preference search order, when comparing $\Delta^{od} = 0$ (Model 7) and the SUE (Model 2). This is due to the specific definition of the search order, where the routes with the most reliable travel times (i.e., with the lowest variances) may not correspond to the routes with the lowest travel times. This is also evidenced by the Gap values listed in Table 5, for the settings of the strict preference search order. Also note that in the case of the indifferent preference search order, setting $\Delta^{od} = 0$, only the lowest travel time route per od pair is considered as *satisficing* at each descent step of the MSA. For the indifferent preference search order, the users indifference increases as we increase Δ^{od} from 0 to 500. The route flows will then converge to 1/3 for all the od pairs (Model 5, in Figures 4 and 5). For $\Delta^{od} = 500$, the indifference band is sufficiently high with the result that all the routes in Ω^{od} for all od pairs are *satisficing*. Thus, the users can choose any of the routes. Since the users indifference increases, they will choose routes with higher travel times and consequently the

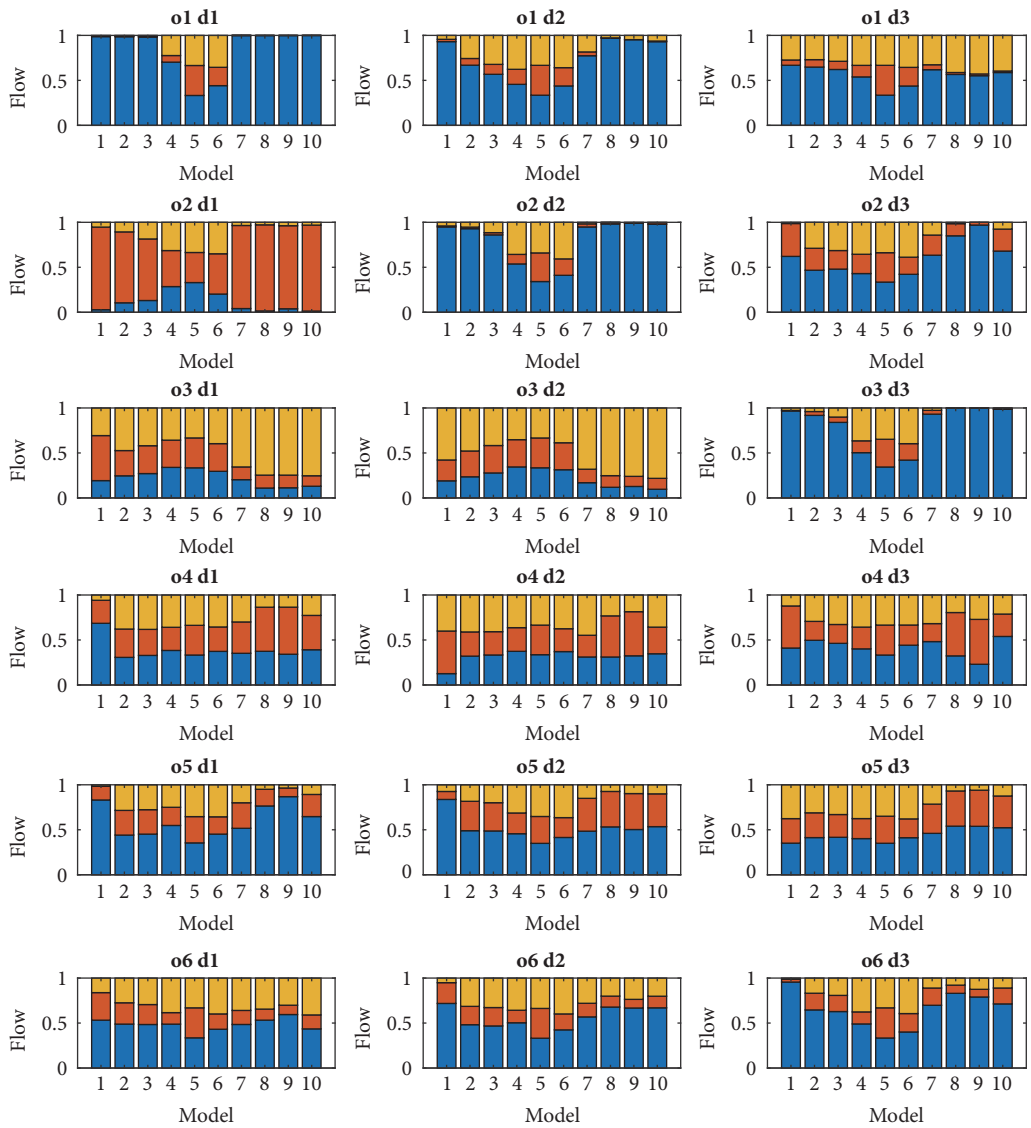

Figure 4: Route flow distributions for the 10 simulation scenarios and for the od pairs: $o = 1, \ldots, 6$ and $d = 1, 2, 3$. Each simulation scenario is identified by the Model ID equivalent to the ID values listed in Table 5.

Gap value also increases. Note that, here, the Gap indicates how far the simulation results are from the DUE; also it is calculated as in (9). On the other hand, the distributions of the average route travel times (Figure 6) also shift towards longer travel times due to an increase in user indifference. The average travel times per route also increase from 295 seconds for $\Delta^{od} = 0$ to 332 seconds for $\Delta^{od} = 500$. The strict preference search order reduces the variances of the distributions of the average route travel times compared to the indifferent preference search order.

4.3. Analysis of the Aggregated Traffic Stats of the Network. In this section, we analyze the network performance in terms of its inflow and outflow capacities and internal accumulation of vehicles, through the macroscopic fundamental diagram (MFD). We investigate the critical accumulation of vehicles n_c and the critical production P_c of the MFD obtained for the different settings of bounded rationality, compared against the values for the DUE and SUE, i.e., the reference models. To better highlight these differences, we define three criteria that represent

(i) the relative difference between the average TTD of the different PT settings ($\overline{TTD^*}$) and of the reference models ($\overline{TTD^{ref}}$):

$$\alpha_{TTD} = \frac{\overline{TTD^*} - \overline{TTD^{ref}}}{\overline{TTD^{ref}}} \times 100 \tag{12}$$

(ii) the relative difference between the average TTT of the different PT settings ($\overline{TTT^*}$) and of the reference models ($\overline{TTT^{ref}}$):

$$\alpha_{TTT} = \frac{\overline{TTT^*} - \overline{TTT^{ref}}}{\overline{TTT^{ref}}} \times 100 \tag{13}$$

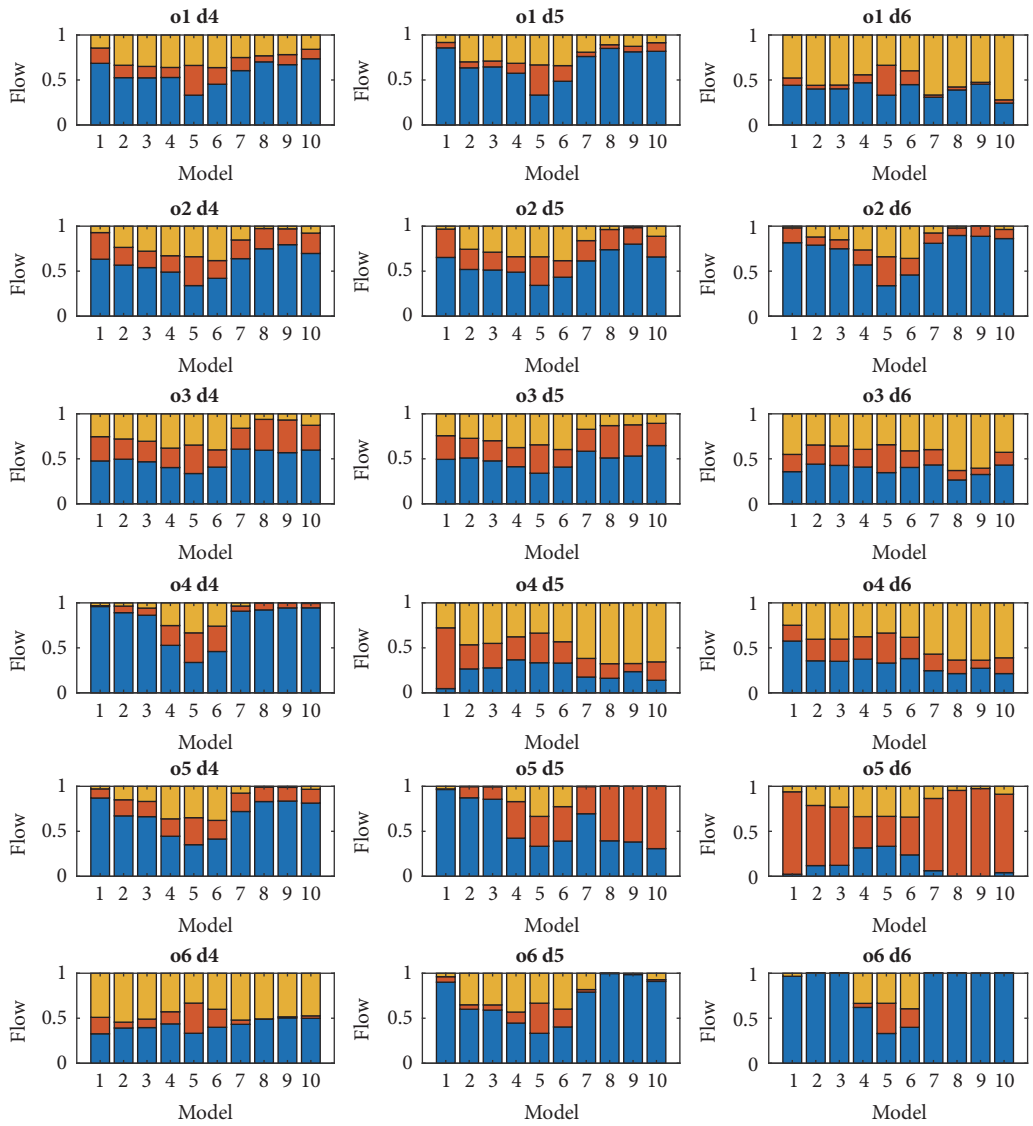

FIGURE 5: Same as in Figure 4, but for the od pairs: $o = 1, \ldots, 6$ and $d = 4, 5, 6$.

(iii) the relative difference between the average TTD of the different PT settings ($\overline{Q^*_{out}}$) and of the reference models ($\overline{Q^{ref}_{out}}$):

$$\alpha_{Q_{out}} = \frac{\overline{Q^*_{out}} - \overline{Q^{ref}_{out}}}{\overline{Q^{ref}_{out}}} \times 100 \tag{14}$$

The analysis of the three criteria is simple. If $\alpha_{TTD} < 0$, the network capacity decreases compared with the reference model. The vehicles accumulation inside the network is higher and congestion might spread backwards, increasing the average waiting times for vehicles to enter the network. The network inflow capacity decreases. Moreover, if the accumulation inside the network increases, the outflow performance of the network might also decrease. If $\alpha_{TTT} < 0$, the mean speed of vehicles inside the network is higher than the reference model. If $\alpha_{Q_{out}} > 0$, the outflow performance of the network is higher compared against the reference model.

We show the evolution of the total traveled distance (TTD) as well as the outgoing flow Q_{out} as a function of the total travel time (TTT) for the indifferent (Figure 7) and strict (Figure 8) preference orders. The average values for TTD, TTT, and Q_{out} are calculated for the simulation interval between 500 and 2500 seconds, for the 10 model settings. We then calculate α_{TTD}, α_{TTT}, and $\alpha_{Q_{out}}$ and estimate confidence intervals for these three criteria. The results are shown in Figure 7 for the indifferent preference order and in Figure 8 for the strict preference order. Our results show that the network capacity is higher for the strict preference search order case. This is observed by comparing the $\Delta^{od} = 500$ for both search orders, where the TTD is much lower for the indifferent preference search order compared to the strict preference search order. It can also be seen that the network capacity is approximately similar for the strict preference

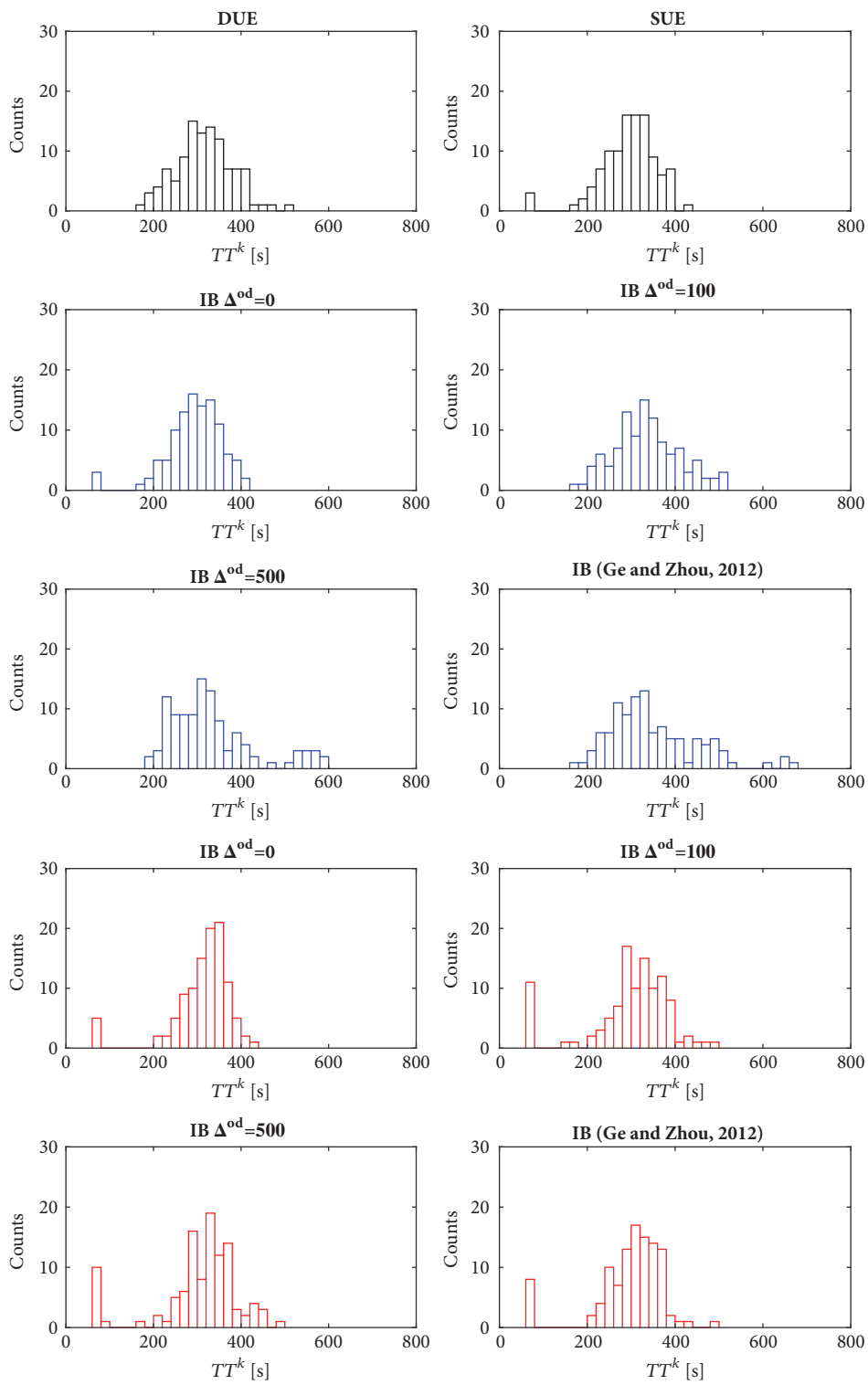

Figure 6: Average route travel time distributions for the DUE, SUE, and different settings of the indifference band.

search order and the different settings of the indifference band. While, for the indifferent preference search order, the network capacity decreases with an increase of the Δ^{od}. This is also evidenced in Figure 7 (ii) by the decrease of α_{TTD} as Δ^{od} increases. The average waiting time for the vehicles to enter the network also increases. The average waiting times per vehicle are 51 s for the DUE; 52 s for the SUE; 52 s for $\Delta^{od} = 0$; 54 s for $\Delta^{od} = 100$; 61 s for $\Delta^{od} = 500$; and 57 s for the setting of the indifference band defined by Ge and Zhou [20]. Note that these are the averaging waiting times for the indifferent

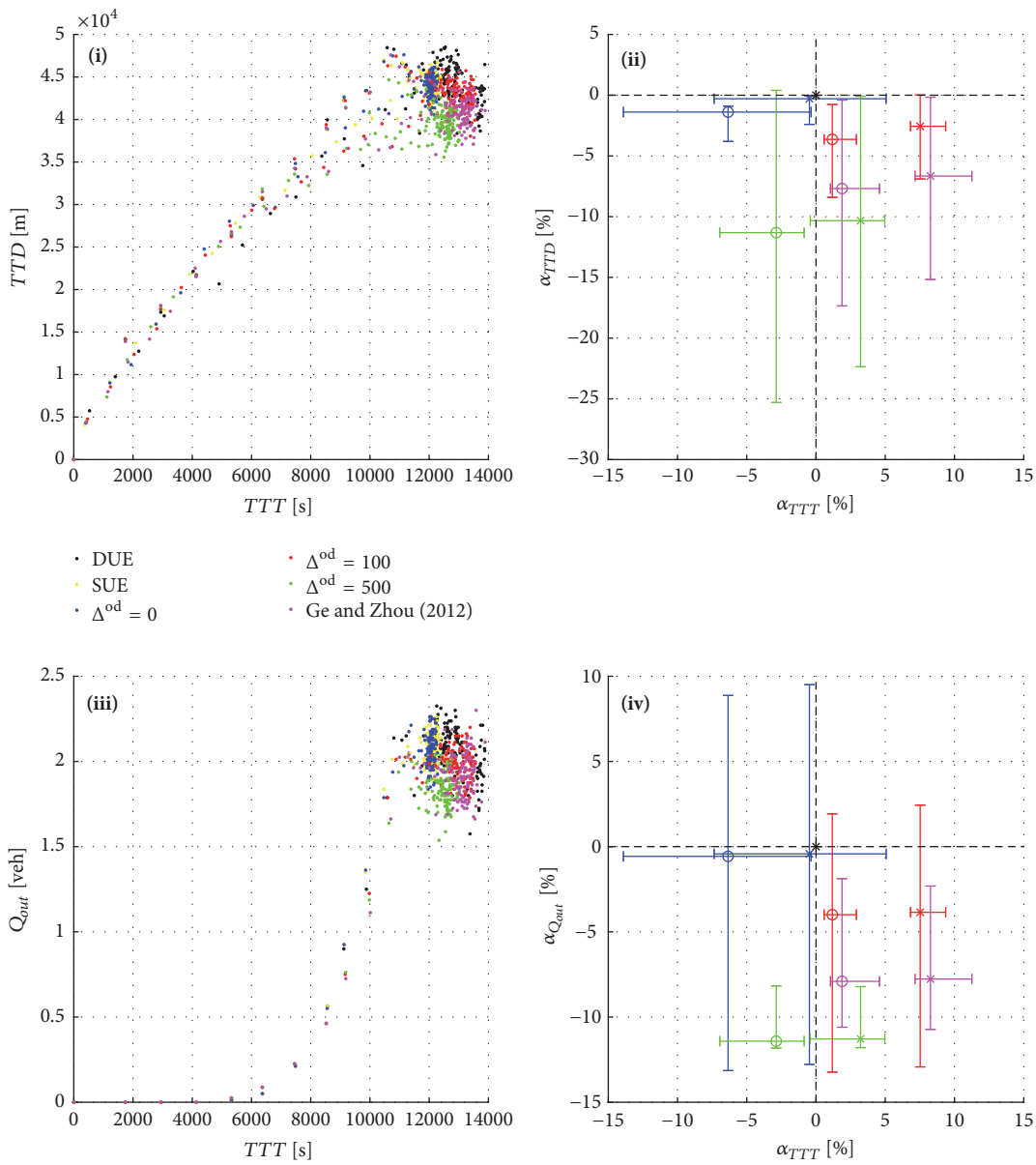

FIGURE 7: *(i)* Total travel distance (TTD) [in m] as function of the total travel time (TTT) [in s]. *(ii)* α_{TTD} versus α_{TTT} for the four settings of the indifference band. *(iii)* Vehicles outflow (Q_{out}) as function of the total travel time (TTT). *(iv)* $\alpha_{Q_{out}}$ versus α_{TTT} for the four settings of the indifference band. In subplots *(i)* and *(iii)*, the results are shown for the DUE, SUE, and the four settings of the indifference band. In subplots *(ii)* and *(iv)*, the circle dots represent the relative differences between the four settings of the indifference band and the DUE. The cross dots represent the relative differences between the four settings of the indifference band and the SUE. These results are for the indifferent preferences search order.

preference search order. On the other hand, since the network capacity is approximately similar for the strict preference search order and the different settings of Δ^{od}, the average waiting times per vehicle to enter the network are similar. The average waiting times for the strict preference search order are 74 s for $\Delta^{od} = 0$; 73 s for $\Delta^{od} = 100$; 75 s for $\Delta^{od} = 500$; and 74 s for the setting of the indifference band defined by Ge and Zhou [20]. From Figures 7 and 8, we can also observe a clear impact of the users search order on the total travel time spent on the network. For example, for $\Delta^{od} = 500$, the α_{TTT} is larger for the strict preference compared to the indifferent preference search order. This induces a lower vehicles mean speed and a lower internal network performance. Also, in the case of the indifferent preference search order, users will tend to choose routes with higher travel times as Δ^{od} increases. This leads to an increase of the accumulation of vehicles inside the network and consequently users spend more time to complete their trips. Also, the outflow Q_{out} of vehicles decreases as Δ^{od} increases (see Figures 7 and 8). Note that a lower outflow Q_{out} means lower system efficiency.

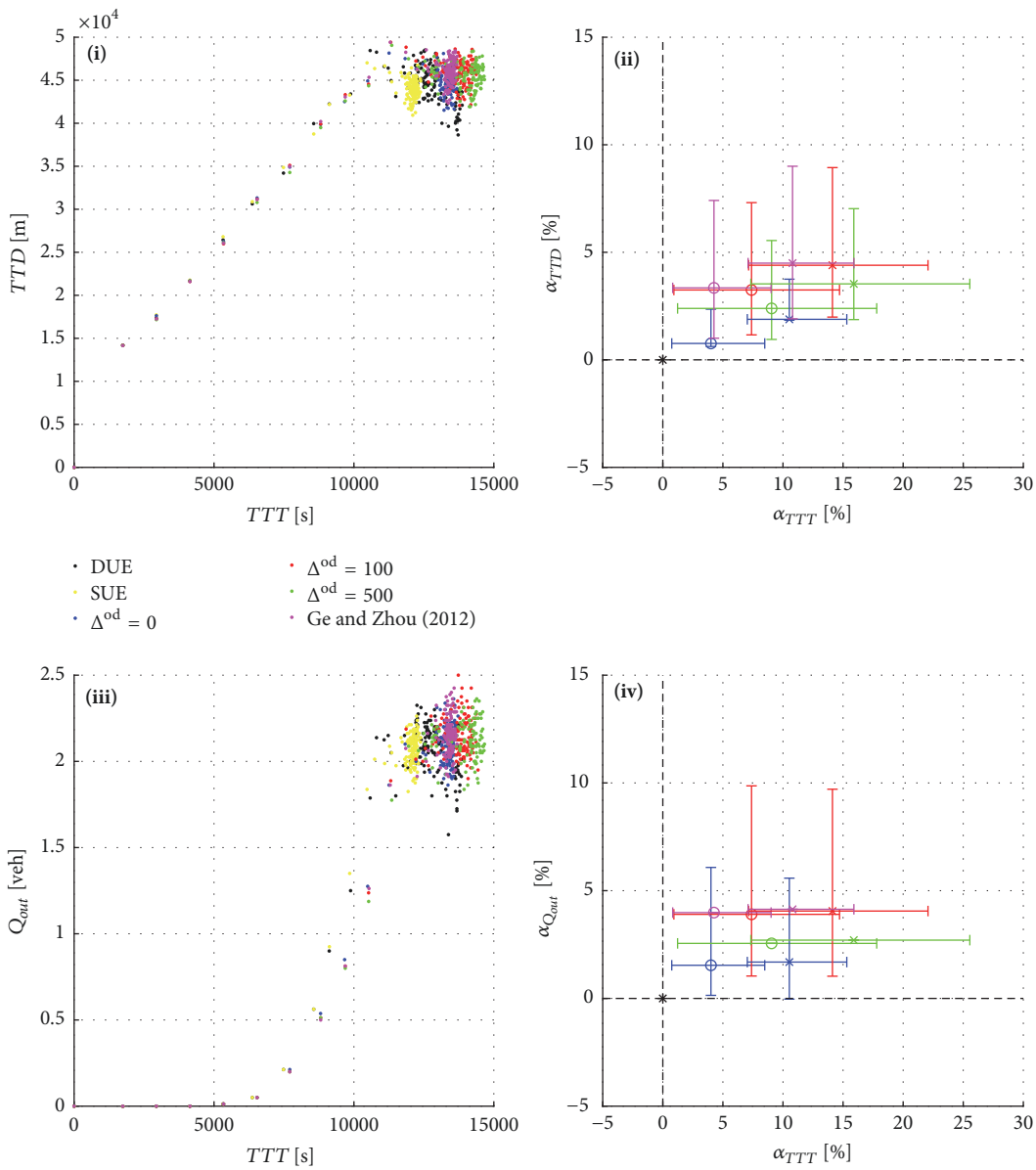

Figure 8: Same as in Figure 7, but for the strict preferences search order.

In summary, we show that different types of bounded user rationality have different impacts on the network performance. Considering the indifferent preference search order where users present an indifference behavior for all of the *satisficing* routes, as Δ^{od} increases, the internal and outflow capacities of the network decrease. However, when considering the strict user preference order, both the internal and outflow capacities of the network are approximately similar as Δ^{od} increases.

5. Conclusions

Users route choices determine the level of congestion on a road network. Thus understanding the effects of users behavior is important for transportation network planning policies. In this paper, we investigated the influence of two types of bounded rational behavior, considering users preferences for the search order (i.e., indifferent and strict preferences), on individual route flows and network performance. To do this, we considered a dynamic implementation of a bounded rational framework, using a mesoscopic LWR traffic simulator [21]. The route costs were time-dependent and accounted for congestion, shock-waves, and spillback effects. To model the bounded rationality behavior, we relaxed the definition of the search order of the DUE and SUE frameworks [4]. Thus, instead of using an all-or-nothing procedure to assign the users to the route(s) with the minimum travel time(s), they were assigned according to a more flexible definition of the search order according to user preferences. We also considered both definitions of the indifference band ((2) and

(3)) for AL^{od}. To account for the distributions of travel times, we used Monte Carlo simulations [4] and algorithm based on the Method of Successive Averages was presented to solve the network equilibria.

This work extended the framework of bounded rationality applied to dynamic traffic assignment modeling in some directions. First, it incorporates the stochasticity of route travel times that are treated through Monte Carlo simulations. Second, this framework accounts for indifferent and strict users' preferences for their route choices. Our framework is reduced to the DUE or SUE, if the search order is defined for users that are utility minimizers. Third, this framework extends the work of Zhao and Huang [19] to a dynamic context considering the setting of the indifference band. In our framework, we do not need to solve suboptimization problems to calibrate the aspiration levels of the sub-preferred routes.

To first assess and gain insight into the changes of route flows at equilibrium, AL^{od}, and for both user search orders, we considered a static implementation on the toy Braess network. The results obtained with the indifferent preference search order revealed that (i) the bounded rational model framework is able to reproduce both DUE and SUE; (ii) when AL^{od} is sufficiently large, the route flows converge to 1/3, showing the user indifference for the route choice; (iii) the algorithm discussed converges. Also, based on this simple numerical test, we showed that we converge towards the same solution of the BR-SUE calculated, based on averaging over all local BR-DUE problems. In the second test, considering the strict user preference order, we showed that the route flows calculated between our model and the model of Zhao and Huang [19] reach good agreement. This validated our methodology applied to determine the search order in a dynamic context.

We also investigated the influence of the two settings of the users search order on the individual route flows and network performance, considering both definitions of AL^{od} as defined in (2) and (3). These tests were performed in a dynamic context, using the mesoscopic LWR traffic simulator [21]. We first showed that we were able to reproduce the SUE, by setting $\Delta^{od} = 0$ for the indifferent preference search order. For the strict preference search order, we did not obtain route flows similar to the SUE when setting $\Delta^{od} = 0$. This is due to the fact that routes with the most reliable travel times did not necessarily have the minimal travel time per each od pair. We also showed that for the indifferent preference search order, the route flows also tended to 1/3 as we increased Δ^{od}.

We then showed that bounded user rationality had a significant impact on network performance. For the indifferent preference search orders, the network inflow capacity decreased as Δ^{od} increased; and the network performance decreased as Δ^{od} increased. The outflow Q_{out} also decreased as Δ^{od} increased. For the strict preference search order, the network capacity was approximately similar for the different settings of the indifference band. However, since users were allowed to choose routes with longer travel times as Δ^{od} increased, the TTT increased and the internal performance of the network decreased. In brief, we showed that different types of bounded rationality have clearly different influences on network performance. This is very important when guiding policy makers to decide the best measures to implement in order to increase network performance.

As future work, we can extend this work in many directions. We first plan to extend this work to the macroscopic fundamental diagram (MFD) simulation. We also plan to extend this framework to heterogeneous classes of users. The heterogeneity can either be included in the search order or on the setting of the AL^{od}. And, we also plan to extend this model to a day-to-day assignment, by considering a learning process (e.g., based on reinforcement learning models) either on the AL^{od} or the users search order definition. We also emphasize that further research in the setting of the AL^{od} is required. The setting of the preference orders in the search process allows considering heterogeneous classes of users, with a preference for transportation mode, for example. However, we note that this users heterogeneity can also be included in the setting of the AL instead of the search order. In this case, the total demand Q of the od pair should be split into homogeneous groups of users with the same preference. The AL should then be defined per od and class of users. For each class of users, they are assigned based on an all-or-nothing assignment to one of the routes listed in ω^{od}.

Data Availability

This is a simulation study not a data-driven study. No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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