Analyzing Ride-Sourcing Market Equilibrium and Its Transitions with Heterogeneous Users

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1. Introduction

Thanks to the development of smartphone applications and information and communication technologies, ride-sourcing service [1] has become a popular form of urban shared mobility in recent years [2]. In a ride-sourcing market, a ride-sourcing firm or transportation network company acts as an intermediary platform, utilizing its market power and technologies and user devices to provide matching services for passengers and drivers. Passengers place trip orders to the platform and fulfill their trips served by drivers, while drivers answer the ride-sourcing platform and provide passenger trip services using their own or leased vehicles.

Although sharing many similarities, there are several differences between the traditional taxi and ride-sourcing markets, including entry restriction and price regulation. Traditional taxi markets can be divided into three broad categories: stand, hailing, and dispatching markets [3]. Unlike the stand or hailing taxi market, the ride-sourcing market is a typical two-sided market [4–6], where passengers and drivers form two user groups and interact via the ride-sourcing platform. Another critical difference between stand/hailing and ride-sourcing markets is whether the matching and meeting between a passenger and a driver in a trip occur simultaneously. As shown in Figure 1, a passenger and a driver in a stand/hailing taxi trip match and meet simultaneously, while in a ride-sourcing trip, there is a deadlocked pick-up period after platform matching and before users’ meeting.

While dispatching taxi and ride-sourcing markets are more alike in ways that both are two-sided markets, service is demand-responsive, and users match and meet sequentially, in the dispatching taxi market, passengers telephone a dispatching center requesting an immediate or scheduled taxi service at a designated place. Compared with dispatching taxi services, smartphone apps in ride-sourcing...
services enable more instant, flexible, and personalized user experiences. Moreover, ride-sourcing service operation can be more advanced with various and automated user matching rules, where passengers and drivers may even choose each other in a bidirectional manner.

Modeling the matching and meeting process is critical in either the traditional taxi market or ride-sourcing market modeling. The hailing taxi market is often characterized by a general bilateral searching and meeting function [7]. Such a mathematical framework is rooted in the search and matching theory in economics (e.g., [8–12]), while in the ride-sourcing market modeling and management literature, two noteworthy approaches emerge to modeling the before-service period from the point shown in Figure 1. In the first approach, the matching function is directly transplanted from hailing taxi market modeling to ride-sourcing market modeling (e.g., [13–16]). The before-service period is treated unsegmented. The matching and meeting points are not differentiated. Users’ matching and meeting are comprehensively formulated by a matching function at a macroscopic level. The second approach segments the before-service period and notices that a passenger and a driver in a trip commonly experience the pick-up period (e.g., [17–19]).

Prior to this study, research using either approach did not notice the modeling difference and compare with each other. Both models have their advantages and disadvantages. On the one hand, the first approach is flexible to approximate users’ matching and meeting process with adjustable forms and parameter settings of the matching function (despite at a macroscopic level). Furthermore, the first approach can deal with various situations, e.g., equilibrium and nonequilibrium analyses and applications to aggregate and disaggregate markets. On the other hand, the second approach can yield more explicit solutions when general user heterogeneity is not considered and can analyze the WGC phenomenon (Nevertheless, some studies such as Ke et al. [19] prove that the WGC regime does not normally arise). Besides, the second approach requires a strong assumption that the platform uses the first-dispatch protocol and has perfect efficiency with no matching time delay.

With the justification of a comprehensive matching function in ride-sourcing market modeling, this study pushes forward this approach considering general user heterogeneity and further analyzes the ride-sourcing market equilibrium and its transitions with heterogeneous users. User heterogeneity has been successfully modeled for the transit market by Zhang et al. [20, 21]. Nevertheless, there is a major research gap to extend the model of general user heterogeneity from the transit market to the ride-sourcing market. Because the ride-sourcing market is two-sided, not only passengers’ heterogeneity but also drivers’ heterogeneity should be considered. Hence, heterogeneous user behavior and equilibrium in the ride-sourcing market are significantly different from those in the transit market.

We begin our analysis with a single origin-destination (O-D) market. Passenger characteristics vary in their value of time and willingness to pay, while driver characteristics vary in time and willingness to accept. Sufficient conditions that guarantee a unique ride-sourcing market equilibrium are derived. The concepts of the time frame of concern and the ride-sourcing factor emerge in these sufficient conditions. We assume that the ride-sourcing factor is positive within the time frame of concern. Under this assumption, we explore equilibrium platform operation for profit maximization. We find that when the corresponding price is optimized, the platform commission rate is closely related to the fare elasticity of passenger demand or the wage elasticity of driver supply. What is more, the sum of the two price elasticities is negative when both fare and wage are simultaneously optimized.

Equilibrium transitions are investigated. Comparative statics are conducted to analyze an equilibrium transition caused by the change in passenger fare, potential demand, driver wage, potential supply, and in-vehicle service time, respectively. Two useful propositions are presented for equilibrium operation under the minimization of passenger

Figure 1: A comparison between stand/hailing taxi and ride-sourcing trips.
waiting time or driver idle time. Nonequilibrium modeling to understand the transition path is also analyzed, revealing how the detailed transition evolves. The single O-D market model is further extended to aggregate and disaggregate ride-sourcing markets and incorporate travel time reliability. Numerical experiments, which cover an aggregate/single O-D base case, sensitivity analysis, equilibrium transitions, and a two-node disaggregate market, are presented to illustrate the theoretical model. The larger network of Sioux Falls is also tested as a demonstration of the disaggregate market equilibrium. A flowchart describing our main content organization and analysis procedure is shown in Figure 2.

The remainder of this study is organized as follows. Section 2 reviews the existing literature on taxi and ride-sourcing market modeling and user heterogeneity in ride-sourcing market modeling. Section 3 presents model settings and analyzes the equilibrium and operation of a single O-D ride-sourcing market. Section 4 explores equilibrium transitions due to exogenous variables and models the nonequilibrium transition paths. Section 5 applies the single O-D market model to aggregate and disaggregate ride-sourcing markets and briefly discusses travel time reliability. Section 6 presents several numerical experiments to illustrate the theoretical model. Section 7 discusses operation and policy implications, draws conclusions, and provides an outlook on future research.

2. Literature Review

2.1. Taxi and Ride-Sourcing Market Modeling. Taxi market modeling has received much research attention over the past half a century. Douglas [22] developed a demand-supply equilibrium model for a cruising taxicab market. Schroeter [23] presented a taxi service model in a regulated market where radio dispatch and airport cabstand are the primary modes of operation. Arnott [24] considered an illustrative structural model for dispatching taxi services and found that taxi travel should be subsidized in a first-best environment. Schaller [25] showed that entry controls in taxi regulation have quite different impacts between the dispatching taxi market and the stand/hailing taxi market. Yang and Yang [7] introduced a general meeting function and investigated the equilibrium properties of taxi markets with search frictions. Salanova et al. [26] presented a well-organized review of the different models developed for the taxicab problem.

Since 2012, the taxi business model was upended by transportation network companies such as Uber and Lyft [27]. In the wake of the industry trend of ride-sourcing services, numerous studies focus on the modeling and management of the ride-sourcing market. Recent comprehensive reviews can be found in [6]. For example, Zha et al. [13] provided a quantitative analysis on the market structure of ride-sourcing services and explored effective regulation policies. Wang et al. [14] examined the equilibrium model of a taxi-hailing platform and evaluated the impacts of the platform’s pricing strategies. Using a Cobb–Douglas type production function, Wang et al. [15] provided a customer order cancellation model for coupled ride-sourcing and taxi markets. Nourinejad and Ramezani [16] introduced a dynamic nonequilibrium ride-sourcing model and extended the bilateral meeting function to time-varying conditions. Castillo et al. [17] proposed a ride-sourcing equilibrium model without using the matching function. They showed that the supply curve is non-monotonic, and the market may enter a so-called wild goose chase (WGC) mode. Xu et al. [18] further studied the WGC phenomenon and the supply curves of ride-hailing systems under different market conditions. Ke et al. [19] adopted the assumption of Castillo et al. [17] and extended the model to on-demand ride-pooling markets. They found that the monopoly optimum, social optimum, and second-best solutions in both ride-pooling and non-pooling markets were always in a normal regime rather than the WGC regime.

2.2. User Heterogeneity in Ride-Sourcing Market Modeling. In the ride-sourcing market modeling literature, very limited studies consider heterogeneous users, but only consider restricted user heterogeneity. For instance, Wang et al. [28] examined the impacts of rider-driver cost-sharing strategies and equilibria in a ridesharing program by modeling the mode choices of a group of heterogeneous travelers with continuously distributed values of time, while drivers’ heterogeneity was not considered. Bai et al. [29] studied an on-demand service platform with heterogeneous passengers and drivers in their reservation prices, without considering user heterogeneity in value of time. We made our initial effort in [30] but focused on network effects in the ride-sourcing market.

3. Single O-D Market

Consider a single O-D ride-sourcing market. Passengers and drivers (labelled $U$ and $V$, respectively) are connected via smartphone apps operated by a monopolistic ride-sourcing company. Drivers provide trip services for passengers from one origin to one destination. For the sake of modeling simplicity, the two main assumptions used in this single O-D market are summarized as follows.

**Assumption 1.** Each driver carries only one passenger per trip.

**Assumption 2.** There is a sole and uncongested trip route between the origin and the destination.

Assumption 2 can be relaxed when discussing travel time reliability in Section 5.3. The in-vehicle service time shared by both passenger and driver in a trip is denoted by $T_i$. Without traffic congestion, $T_i$ is exogenously determined. The ride-sourcing platform charges trip fare $P^U$ from the passenger, collects trip commission $\chi$, and pays trip wage $P^V = P^U - \chi$ when the service is accomplished. The definitions of primary variables used in the single O-D market are listed in Table 1.
3.1. Demand and Supply. For a passenger at the origin who requests a trip service, his or her waiting time (including waiting time to be matched and waiting time for pick-up) is denoted by $T_{w}^{U}$. A passenger with a value of waiting time $\beta_{w}^{U}$ and value of in-vehicle time $\beta_{i}^{U}$ experiences generalized cost $C^{U} = P^{U} + \beta_{w}^{U} \cdot T_{w}^{U} + \beta_{i}^{U} \cdot T_{i}^{U}$. For ease of analysis, we assume only one dimension of passengers’ heterogeneity on value of time (VOT) $\beta_{w}^{U}$ and suppose a scaling factor $\delta_{U}^{*} = \beta_{i}^{U} / \beta_{w}^{U}$ that is identical among all passengers. $T_{U} = T_{w}^{U} + \delta_{U}^{*} \cdot T_{i}^{U}$ is denoted as the scaled passenger trip time, and then, a passenger’s generalized cost is as follows:

$$C^{U} = c^{U} \left( P^{U}, T_{w}^{U}, T_{i}^{U} \right),$$

$$= P^{U} + \beta_{w}^{U} \cdot T_{w}^{U},$$

$$= P^{U} + \beta_{w}^{U} \cdot \left( T_{w}^{U} + \delta_{U}^{*} \cdot T_{i}^{U} \right).$$

Passenger demand is derived in terms of passengers’ VOT and willingness to pay (WTP). WTP establishes the reserved maximum cost that an individual is willing to make a trip. A passenger’s WTP is denoted by $\omega^{U}$. A passenger chooses the ride-sourcing service if and only if his or her WTP is at least as great as the generalized cost, i.e., $\omega^{U} \geq C^{U}$.

Passengers differ in both their VOT and WTP. Let $h^{U} \left( \beta^{U}^{*}, \omega^{U} \right)$ denote the joint probability density function of $(\beta^{U}, \omega^{U}^{*})$, with the support of $\beta^{U}^{*} \in [0, \beta^{U}]$ and $\omega^{U} \in [0, \omega^{U}]$ (user distribution on an infinite support may be mathematically more rigorous but realistically unnecessary). Users’ value of time and willingness to pay cannot be arbitrarily large or negative in reality. In numerical experiments, we assume bivariate normal distributions with infinite supports, but still conduct approximate calculations on finite supports based on the empirical three-sigma rule). As shown in Appendix A, the actual or realized demand of passengers can be written as follows:
\[ Q^U = q^U \left( c^U, \Omega^U \right), \]
\[ = \tilde{q}^U \left( p^U, T^U, t, \Omega^U \right), \]
\[ = \Omega^U \cdot \int_{(\beta^U, \omega^U) \in \Omega^U} h^U(\beta^U, \omega^U) d\omega^U d\beta^U, \]

where \( \Omega^U \) is potential passenger demand and \( \Omega^U \) is the market segment of passengers.

Accordingly, a driver around the origin cruises (or waits) and proceeds en route pick-up before serving a passenger. We consider this as a whole driver idle period, as discussed in Section 1. The driver’s idle time \( T^U \) consists of the driver’s cruising (or waiting) time and pick-up time. The driver experiences the vehicle operation and fuel cost of \( C^V = c^V \left( T^U, T_1 \right) \), where \( \delta c^V / \partial T^U > 0 \) and \( \delta c^V / \partial T_1 > 0 \). We assume that drivers are generally heterogeneous, too. We also consider only one dimension of drivers’ heterogeneity on VOT. A driver with a VOT \( \beta^U \) gains a generalized benefit of:

\[ B^V = b^V \left( p^V, T^V, T_1 \right), \]
\[ = p^V - C^V - \beta^V \cdot T^V, \]
\[ = p^V - C^V - \beta^V \cdot \left( T^V + \delta^V \cdot T_1 \right), \]

where \( \delta^V \) is a scaling factor between the driver’s value of in-service time and value of idle time, and \( T^V = T^U + \delta^V \cdot T_1 \) is the scaled driver trip time.

Driver supply is derived in terms of drivers’ VOT and willingness to accept (WTA). WTA establishes the minimum gains that a driver is willing to provide trip service for a passenger. A driver’s WTA is denoted by \( \alpha^V \). A driver chooses to serve a passenger if and only if the generalized benefit is no less than his or her WTA, i.e., \( B^V \geq \alpha^V \). With both drivers’ VOT and WTA differ, let \( h^V(\beta^V, \omega^V) \) denote the joint probability density function of \( \beta^V, \omega^V \), with the support of \( \beta^V \in [0, \bar{\beta}^V] \) and \( \omega^V \in [0, \bar{\omega}^V] \). Let \( \Omega^V \) denote potential driver supply and \( Q^V \) denote the market segment of drivers. Driver supply affected by market entry and exit is implied in \( Q^V \). The actual or realized supply of drivers is as follows:

\[ Q^V = q^V \left( b^V, \Omega^V \right), \]
\[ = q^V \left( p^V, T^V, T_1, \Omega^V \right), \]
\[ = \Omega^V \cdot \int_{(\beta^V, \omega^V) \in \Omega^V} h^V(\beta^V, \omega^V) d\omega^V d\beta^V. \]

Other market variables, such as the mean VOTs of marginal users \( E_M^U(\beta^U) \) and \( E_M^V(\beta^V) \), and the characteristic times \( \lambda^U_1, \lambda^U_2, \lambda^V_1, \) and \( \lambda^V_2 \), are specified in Appendix A.

### 3.2. The Market Equilibrium

At the origin, let \( N^U \) denote the number of unserved passengers, \( N^U \) be the number of vacant drivers, \( M^U-V \) be the comprehensive platform matching rate between passengers and drivers. At market equilibrium, the market clearance condition yields the following:

\[ Q^V = M^U-V, \]
\[ = Q^V. \]

The ride-sourcing market variables are invariant, and we have \( N^U = T^U \cdot Q^U \) and \( N^V = T^V \cdot Q^V \). Following previous studies [7, 13], the matching rate is characterized by a continuous and differentiable matching function:

\[ M^U-V = m \left( N^U, N^V \right), \]
\[ = m \left( T^U \cdot Q^U, T^V \cdot Q^V \right). \]

with \( \partial m / \partial N^U > 0 \) and \( \partial m / \partial N^V > 0 \), and \( M^U-V \longrightarrow 0 \) as either \( N^U \longrightarrow 0 \) or \( N^V \longrightarrow 0 \). The elasticities of the matching rate with respect to the number of waiting passengers and the number of vacant vehicles are \( \alpha_1 = \frac{\partial M^U-V}{\partial N^U} \) and \( \alpha_2 = \frac{\partial M^U-V}{\partial N^V} \), respectively. It is generally expected that \( 0 < \alpha_1, \alpha_2 < 1 \). The matching function that is homogeneous of degree \( \alpha_1 + \alpha_2 \) is said to exhibit increasing, constant, or decreasing returns to scale if \( \alpha_1 + \alpha_2 > 1 \), \( \alpha_1 + \alpha_2 = 1 \), and \( \alpha_1 + \alpha_2 < 1 \), respectively [7]. Conditionally, \( T^U \) and \( T^V \) are endogenously determined by simultaneously solving equations (2), (4), and (5), and we write the following:

\[ \begin{cases} T^U_w = t_w^U \left( p^U, p^V, \Omega^U, \Omega^V, T_1 \right), \\ T^V_w = t_w^V \left( p^U, p^V, \Omega^U, \Omega^V, T_1 \right). \end{cases} \]

The existence and uniqueness conditions of the solution to (7) are provided in the following proposition. The proof is given in Appendix B.

**Proposition 1.** For given \( \Omega^U \), \( \Omega^V \), \( T_1 \), \( p^U \), \( p^V \), and \( \Omega^V \), the equilibrium solution of passenger waiting time \( T^U_w \) and driver idle time \( T^V_w \) exists and is unique within \( T^U_w \in (0, T^U_w) \) and \( T^V_w \in (0, T^V_w) \) if the following conditions hold:

(a) \( Q^U > Q^V \) when \( T^U_w \longrightarrow 0 \);

(b) There exists such \( T^U_w \) that \( Q^U - M^U-V < 0 \) at \( T^U_w = T^U_w \) and \( T^U_w = T^U_w \), where \( T^U_w \) is the solution of \( Q^U = Q^V \) at \( T^U_w = T^U_w \) (solution \( T^U_w \) is unique if condition (a) is satisfied); and

(c)

\[ \eta = 1 - \alpha_1 - \alpha_2 + \alpha_1 \frac{\lambda^U_1}{T^U_w} + \alpha_2 \frac{\lambda^V_2}{T^V_w} > 0, \quad \forall T^U_w \in \left( 0, T^U_w \right), \]

\[ T^V_w \in \left( 0, T^V_w \right). \]

Proposition 1 provides a sufficient condition with significant implications. Condition (a) conveys the message that driver supply should be adequate to guarantee an equilibrium solution. Conditions (b) and (c) define the time
frame of concern for a unique equilibrium solution. Other mathematically feasible solutions may exist beyond this time frame but probably lack practical significance. The term \( \eta \) in condition (c), henceforth called the ride-sourcing factor, should be positive within the time frame of concern. It synthetically reflects platform matching properties (characterized by elasticities of the matching function), user statistical attributes (characterized by passengers’ and drivers’ characteristic times), and matching service qualities (characterized by passenger waiting time and driver idle time) of the ride-sourcing market. Based on Proposition 1, we suppose that the following assumption is valid.

**Assumption 3.** The ride-sourcing factor is positive within the time frame of concern. (Note that if the matching function that is homogeneous of degree \( \alpha_1 + \alpha_2 \) exhibits decreasing returns to scale (i.e., \( \alpha_1 + \alpha_2 < 1 \)), then the ride-sourcing factor \( \eta \) must be positive.)

Price elasticities of equilibrium passenger demand and driver supply can be, respectively, derived from Appendixes C.1 and C.2:

\[
\begin{align*}
\epsilon_{pu}^v &= \frac{\partial Q^v}{\partial P^u}\Delta_{\Delta-P^u} \cdot \frac{P^u}{Q^v} \\
&= \frac{\alpha_1 \cdot P^u}{\eta \cdot T^v_w \cdot E^{M^v}(\beta^v)} \\
\epsilon_{pv}^u &= \frac{\partial Q^u}{\partial P^v}\Delta_{\Delta-P^v} \cdot \frac{P^v}{Q^u} \\
&= \frac{\alpha_2 \cdot P^v}{\eta \cdot T^v_w \cdot (\frac{\partial C^v}{\partial T^v_w} + E^{M^v}(\beta^v))},
\end{align*}
\]

where \( \Delta = \cdot \) means that the differential is calculated with respect to a particular variable. In other words, only this particular variable changes in a multivariate function. It is interesting to see that both equilibrium elasticities in (9) are connected with the ride-sourcing factor. Under Assumption 3, we have \( \epsilon_{pu}^v < 0 \) and \( \epsilon_{pv}^u > 0 \).

### 3.3. A Profit-Maximizing Platform

Based on the ride-sourcing market equilibrium, we look at optimal trip fare and wage settings. The operation cost of the ride-sourcing platform (such as the organization and technology cost) usually does not vary significantly with market share. Therefore, following some previous studies [29, 31], the profit of the platform has the form of:

\[
\pi = P^u \cdot Q^u - P^v \cdot Q^v.
\]

Necessary first-order conditions for maximizing platform profit with respect to passenger fare and driver wage are derived in Appendixes C.1 and C.2, respectively. Optimal passenger fare \( P^u_{\pi\pi} \) is determined as follows:

\[
\alpha_1 \cdot \chi_{P^u-P^u\pi} = \eta \cdot T^v_w \cdot E^{M^v}(\beta^v)|_{P^u-P^u\pi}. \tag{11}
\]

Optimal driver wage \( P^v_{\pi\pi} \) is determined as follows:

\[
\alpha_2 \cdot \chi_{P^v-P^v\pi} = \eta \cdot T^v_w \cdot (\frac{\partial C^v}{\partial T^v_w} + E^{M^v}(\beta^v))|_{P^v-P^v\pi}. \tag{12}
\]

The values of the commission rate under the equilibrium state are summarized in the following proposition.

**Proposition 2.** Under the equilibrium state, the commission rate set by the profit-maximizing platform is \( -\epsilon_{pu}^v \) if passenger fare is optimized and is \((1 + \epsilon_{pu}^v)^{-1}\) if driver wage is optimized. If fare and wage are simultaneously optimized, the sum of the two equilibrium elasticities is negative one; i.e., \( \epsilon_{pu}^v + \epsilon_{pv}^u = -1 \).

**Proof.** By equations (9) and (11), the corresponding commission rate under optimal passenger fare is as follows:

\[
\chi_{P^u-P^u\pi} = 1 - \frac{P^v}{P^u_{\pi\pi}},
\]

\[
= \frac{\eta \cdot T^v_w \cdot E^{M^v}(\beta^v)|_{P^u-P^u\pi}}{\alpha_1 \cdot P^u_{\pi\pi}} = -\epsilon_{pu}^v. \tag{13}
\]

By equations (9) and (12), the corresponding commission rate under optimal driver wage is as follows:

\[
\chi_{P^v-P^v\pi} = 1 - \frac{P^v}{P^v_{\pi\pi}},
\]

\[
= \frac{\eta \cdot T^v_w \cdot (\frac{\partial C^v}{\partial T^v_w} + E^{M^v}(\beta^v))|_{P^v-P^v\pi}}{\alpha_2 \cdot P^v_{\pi\pi}} = (1 + \epsilon_{pu}^v)^{-1}. \tag{14}
\]

If passenger fare and driver wage are simultaneously optimized, both equations (13) and (14) satisfy and yield \( \epsilon_{pu}^v + \epsilon_{pv}^u = -1 \).

Proposition 2 provides a measurable approach to judge whether trip fare or wage is at its optimal level.

### 4. Equilibrium Transitions

As shown in equation (7), the ride-sourcing market equilibrium depends on exogenous variables \( P^u, Q^u, P^v, Q^v \), and \( T^v \). Any disturbances of these variables probably cause imbalances between demand and supply, which result in a nonequilibrium state with either insufficient drivers (\( Q^u > Q^v \)) or a nonequilibrium state with insufficient passengers (\( Q^u < Q^v \)). If the nonequilibrium market can stabilize autonomously or with specific regulations, a new equilibrium state will eventually emerge where \( Q^u = M^{U-V} = Q^v \) holds again. In this section, we investigate such an equilibrium transition.
4.1. Comparative Statics. If a new equilibrium state is supposed to be attainable, comparative statics is used to compare the equilibrium states before and after the transition. The detailed comparative static analysis is provided in Appendix C. A numerical illustration will be demonstrated in Figure 3 in Section 6.3.

4.1.1. Transition Caused by the Change in Passenger Fare. If passenger fare $P^U$ is the only changing exogenous variable, we have (see Appendix C.1) the following:

$$
\begin{align*}
\left. \frac{dQ^U}{dP^U} \right|_{\Delta = P^U} &= \frac{\alpha_1 \cdot Q^U}{\eta \cdot T^U_w \cdot E_{M^U}(\beta^U)} \\ \left. \frac{dT^U_w}{dP^U} \right|_{\Delta = P^U} &= \frac{\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w}{\eta \cdot T^U_w \cdot E_{M^U}(\beta^U)} , \\
\left. \frac{d\lambda^U_1}{dP^U} \right|_{\Delta = P^U} &= \frac{\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w}{\eta \cdot T^U_w \cdot E_{M^U}(\beta^U)} .
\end{align*}
$$

Under Assumption 3, both equilibrium demand and supply decrease with passenger fare in equation (15); while equilibrium driver idle time increases with passenger fare, the direction of the change in equilibrium passenger waiting time depends on the sign of $\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w$. If $\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w < 0$, equilibrium passenger waiting time decreases in fare; if $\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w > 0$, passenger waiting time increases in fare.

4.1.2. Transition Caused by the Change in Potential Demand. If potential demand $\overline{Q}$ is the only changing exogenous variable, we have (see Appendix C.3) the following:

$$
\begin{align*}
\left. \frac{dQ^U}{d\overline{Q}} \right|_{\Delta = \overline{Q}} &= \frac{\eta \cdot T^U_w}{\alpha_1 \cdot \lambda^U_1 \cdot G_{\overline{Q}^U}} , \\
\left. \frac{dT^U_w}{d\overline{Q}} \right|_{\Delta = \overline{Q}} &= \frac{\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w}{\alpha_1} , \\
\left. \frac{d\lambda^U_1}{d\overline{Q}} \right|_{\Delta = \overline{Q}} &= \frac{\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w}{\alpha_1} ,
\end{align*}
$$

Under Assumption 3, equilibrium demand and supply increase with potential demand in equation (16); equilibrium driver idle time decreases when passenger demand increases; the direction of change in equilibrium passenger waiting time also depends on the sign of $\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w$. If $\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w < 0$, equilibrium passenger waiting time increases in equilibrium demand (and potential demand); if $\alpha_1 \cdot \lambda^U_1 - \eta \cdot T^U_w > 0$, equilibrium passenger waiting time decreases in demand. By the analyses of Sections 4.1.1 and 4.1.2, we have the following proposition.

**Proposition 3.** Under Assumption 3, if equilibrium passenger waiting time has a minimum when either passenger fare or potential passenger demand varies, then the minimum occurs at:

$$
\alpha_1 \cdot \lambda^U_1 \big|_{\Delta = P^U \text{ or } \overline{Q}} = \frac{\eta \cdot T^U_w}{\alpha_1} .
$$

Proposition 3 is a useful reference for the equilibrium operation of passenger waiting time minimization rather than platform profit maximization.

4.1.3. Transition Caused by the Change in Driver Wage. If driver wage $P^V$ is the only changing exogenous variable, we have (see Appendix C.2) the following:

$$
\begin{align*}
\left. \frac{dQ^V}{dP^V} \right|_{\Delta = P^V} &= \frac{\alpha_2 \cdot Q^V}{\eta \cdot T^V_w \cdot (\partial c^V / \partial T^V_w + E_{M^V}(\beta^V))} , \\
\left. \frac{dT^V_w}{dP^V} \right|_{\Delta = P^V} &= \frac{\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w}{\eta \cdot T^V_w \cdot (\partial c^V / \partial T^V_w + E_{M^V}(\beta^V))} , \\
\left. \frac{d\lambda^V_1}{dP^V} \right|_{\Delta = P^V} &= \frac{\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w}{\eta \cdot T^V_w \cdot (\partial c^V / \partial T^V_w + E_{M^V}(\beta^V))} .
\end{align*}
$$

Under Assumption 3, equilibrium passenger demand and driver supply increase in the driver wage, while equilibrium passenger waiting time decreases in the driver wage; the direction of the change in equilibrium driver idle time depends on the sign of $\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w$. If $\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w < 0$, equilibrium driver idle time increases with driver wage; if $\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w > 0$, it decreases in the driver wage.

4.1.4. Transition Caused by the Change in Potential Supply. If potential supply $\overline{Q}$ is the only changing exogenous variable, we have (see Appendix C.4) the following:

$$
\begin{align*}
\left. \frac{dQ^V}{d\overline{Q}} \right|_{\Delta = \overline{Q}} &= \frac{\eta \cdot T^V_w}{\alpha_2 \cdot \lambda^V_1 \cdot G_{\overline{Q}^V}} , \\
\left. \frac{dT^V_w}{d\overline{Q}} \right|_{\Delta = \overline{Q}} &= \frac{\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w}{\alpha_2} , \\
\left. \frac{d\lambda^V_1}{d\overline{Q}} \right|_{\Delta = \overline{Q}} &= \frac{\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w}{\alpha_2} .
\end{align*}
$$

Under Assumption 3, equilibrium demand and supply increase in potential supply; equilibrium passenger waiting time decreases when equilibrium passenger demand increases; the direction of change in driver idle time depends on the sign of $\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w$, too. If $\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w < 0$, equilibrium driver idle time increases in equilibrium demand; if $\alpha_2 \cdot \lambda^V_1 - \eta \cdot T^V_w > 0$, it decreases as equilibrium demand increases. Analogous to Proposition 3, we have a pairwise proposition for the equilibrium operation of driver idle time minimization.
Proposition 4. Under Assumption 3, if equilibrium driver idle time has a minimum when either driver wage or potential driver supply varies, then the minimum occurs at:

\[ \alpha_2 \cdot \lambda^\gamma_{\Delta = P^V \text{ or } Q^V} = \eta \cdot T^{V}_{w|\Delta = P^V \text{ or } Q^V}. \]  

(20)
4.1.5. Transition Caused by the Change in In-Vehicle Time.

\[
\begin{align*}
\frac{dT_i}{dt} &= \frac{-\eta \cdot \lambda_2 \cdot \nabla \cdot T_w \cdot T_w \cdot T_w \cdot T_w \cdot T_w \cdot T_w}{\alpha_1 \cdot \lambda_1 \cdot \lambda_2 \cdot T_w \cdot \alpha_2 \cdot \lambda_2 \cdot \lambda_1 \cdot T_w} \left( \frac{dQ^U}{Q^U} \right) \Delta = T_i, \\
\frac{dT_w^U}{dt} &= \left( \frac{\eta \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_1 \cdot T_w}{\alpha_1 \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_2 \cdot \lambda_1 \cdot T_w} - \lambda_1 \right) \left( \frac{dQ^U}{Q^U} \right) \Delta = T_i, \\
\frac{dT_w^V}{dt} &= \left( \frac{\eta \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_1 \cdot T_w}{\alpha_1 \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_2 \cdot \lambda_1 \cdot T_w} \right) \left( \frac{dQ^V}{Q^V} \right) \Delta = T_i.
\end{align*}
\]

Under Assumption 3, equilibrium demand and supply decrease in the in-vehicle service time; how equilibrium passenger waiting time or driver idle time change with respect to in-vehicle time depend on the relative magnitude between \( \alpha_1 \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_2 \cdot \lambda_1 \cdot T_w \) and \( \eta \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_1 \cdot T_w \) or \( \eta \cdot \lambda_1 \cdot \lambda_2 \cdot \alpha_1 \cdot \lambda_2 \cdot \alpha_2 \cdot \lambda_1 \cdot T_w \), respectively.

4.2. Transition Path. The path of equilibrium transition (i.e., the detailed process of how transition evolves) involves nonequilibrium modeling. Nourinejad and Ramezani [16] introduced a nonequilibrium ride-sourcing model. Here, we propose a similar mechanism for nonequilibrium modeling:

\[
\begin{align*}
\theta^U \cdot \frac{dN^U}{dt} &= Q^U - M^U-V, \quad T_w = N^U \quad \text{and} \quad M^U-V, \\
\theta^V \cdot \frac{dN^V}{dt} &= Q^V - M^U-V, \quad T_w = N^V \quad \text{and} \quad M^U-V,
\end{align*}
\]

where \( \theta^U \) and \( \theta^V \) are positive parameters that reflect system efficiencies during the before-service period. Since we are using the approach based on a comprehensive matching function that does not differentiate users’ matching and meeting points, the values of \( \theta^U \) and \( \theta^V \) may not be 1 as indicated by a direct queuing model. Assumptions in equation (22) are intuitive that the number of waiting passengers/idle drivers changes proportionally to the difference between passenger demand/driver supply and the matching rate. The assumptions of steady-state approximations \( N^U = T_w \cdot M^U-V \) and \( N^V = T_w \cdot M^U-V \) were explained by Nourinejad and Ramezani [16]. To describe this nonequilibrium system with standard forms of differential equations, we need to calculate the total derivatives for \( N^U = T_w \cdot M^U-V \) and \( N^V = T_w \cdot M^U-V \):

\[
\begin{align*}
\frac{dN^U}{dt} &= T_w \cdot \frac{dM^U-V}{dt} + M^U-V \cdot \frac{dT_w}{dt}, \\
\frac{dN^V}{dt} &= T_w \cdot \frac{dM^U-V}{dt} + M^U-V \cdot \frac{dT_w}{dt}.
\end{align*}
\]

Because (6) still hold under steady-state approximations, we have the following:

\[
\begin{align*}
\frac{dM^U-V}{dt} &= \frac{\partial m}{\partial N^U} \cdot \frac{dN^U}{dt} + \frac{\partial m}{\partial N^V} \cdot \frac{dN^V}{dt} \\
&= \frac{\alpha_1 \cdot \frac{dN^U}{dt} + \alpha_2 \cdot \frac{dN^V}{dt}}{T_w}.
\end{align*}
\]

Simultaneously solving equations (22)–(24), the nonequilibrium system (22) can be rewritten without the terms of \( N^U \) and \( N^V \):

\[
\begin{align*}
\frac{dT_w^U}{dt} &= \frac{1 - \alpha_1}{\theta^U} \left( \frac{Q^U}{M^U-V} - 1 \right) - \alpha_2 \cdot \frac{T_w^U}{T_w} \left( \frac{Q^V}{M^U-V} - 1 \right), \\
\frac{dT_w^V}{dt} &= \frac{1 - \alpha_2}{\theta^V} \left( \frac{Q^V}{M^U-V} - 1 \right) - \alpha_1 \cdot \frac{T_w^V}{T_w} \left( \frac{Q^U}{M^U-V} - 1 \right).
\end{align*}
\]

How passenger waiting time and driver idle time evolve is fully described by an autonomous nonlinear first-order differential equation (25). Clearly, the equilibrium state defined in Section 3 is a fixed-point solution to equation (22). Numerical illustrations of the transition paths are shown in Section 6.3. The stability of the nonequilibrium system can be theoretically analyzed using either the first method (linearization around the equilibrium point) or the second method (construction of a Lyapunov candidate function) of the Lyapunov stability theory [32].

5. Model Extensions

5.1. The Aggregate Market. We can apply the single O-D market model to the aggregate ride-sourcing market. The whole study area is treated as the sole origin and destination, and all trips are analyzed at their average level. In the supply settings of the single O-D market model, potential supply is exogenous but may vary. It means that drivers may potentially move in or leave the origin region. When applied to
the aggregate market model, drivers may enter or leave the ride-sourcing market during the study period.

The single O-D market model can also be applied to the conserved aggregate market, where the number of active drivers is fixed. The condition of fleet size conservation is then added:

$$N^V = Q^V \cdot T^V_w + Q^V \cdot T^V_i,$$

where $N^V$ is the fixed driver fleet size. Potential supply $Q^V$ becomes an endogenous variable, and the equilibrium solution to (7) is revised as follows:

$$
\begin{align*}
T^U_w &= t^U_w(p^U, p^V, T^V_i, N^V), \\
T^V_w &= t^V_w(p^U, p^V, T^V_i, N^V), \\
Q^V &= q^V(p^V, T^V_i, N^V).
\end{align*}
$$

5.2. The Disaggregate Market. The spatial dimension of different origins and destinations is also considered in the disaggregate ride-sourcing market. The study area is divided into subregions, denoted by the set $S$. Trips from one subregion to another may vary in their origins, destinations, and trip time. Since we are not tracing a passenger’s trip trajectory, trips of a user with an intermediate stop are treated as two different trips in the disaggregate ride-sourcing market. To discuss the equilibrium and operation of the disaggregate market will be even more complex. As analyzed in Section 3, the single O-D market equilibrium has to be satisfied for each O-D if a whole disaggregate market equilibrium exists.

For trips from origin $o \in S$ to destination $d \in S$, let $Q^U_{o,d}$, $Q^V_{o,d}$, and $M^U_{o,d}$ be the single O-D passenger demand, driver supply, and matching rate, respectively; let $T^V_{wod}$ and $T^V_{iod}$ be the corresponding driver idle time and in-vehicle service time, respectively. We suppose that the market is conserved, where drivers cannot enter or exit the market during the study period. Then, the whole disaggregate market equilibrium, if it exists, has to satisfy the market clearance condition between each O-D:

$$Q^U_{o,d} = M^U_{o,d},$$

and the fleet size conservation is as follows:

$$\sum_{o \in S} \sum_{d \in S} Q^V_{o,d} \cdot (T^V_{wod} + T^V_{iod}) = N^V,$$

and the driver flow conservation at each subregion is as follows:

$$\sum_{o \in S} Q^V_{o,k} = \sum_{d \in S} Q^V_{d,k} s, \quad \forall k \in S.$$

In Section 6.4, we present two numerical examples of different network scales to illustrate the disaggregate market equilibrium.

5.3. Travel Time Reliability. Our model also allows the consideration of travel time reliability [33, 34]. For example, if passengers’ in-vehicle travel time uncertainties (e.g., due to congestion) are considered in the single O-D market, equation (1) can be revised as follows:

$$
C^U = p^U + \beta^U \cdot \left( T^U_w + \delta^U \cdot (T^V_i + \epsilon^U) \right),
$$

where $\epsilon^U$ is a random variable that characterizes the in-vehicle travel time reliability of passengers. We may suppose that $\epsilon^U$ follows a particular distribution with support of $[\epsilon^U, \epsilon^U]$ and zero mean and is independent of passengers’ VOT and WTP. This problem is then discussed with three scenarios. In the first scenario, if heterogeneity in passengers’ VOT is more significant than the uncertainty of in-vehicle travel time and dominates the variations of passengers’ generalized cost, then (31) can be approximated to zero. Passenger demand is the same as shown by equation (2). In the second scenario, if the uncertainty of in-vehicle travel time dominates, then passengers’ VOT $\beta^U$ is treated as fixed in equation (31). Similar to Appendix A, passenger demand should be calculated in the two-dimensional space of $(\epsilon^V, \omega^V)$, where the market boundary remains a plane. In the third scenario, if both VOT and in-vehicle travel time variabilities are significant and cannot be omitted, then passenger demand is integrated based on the joint distribution in the three-dimensional space of $(\epsilon^V, \beta^V, \omega^V)$. Zhang et al. [21] discussed demand calculation for congested transit services in the three-dimensional space where the market boundary is a plane. In this third scenario, demand calculation will be more complex because the market boundary, as indicated by equation (31), is now a hyperbolic paraboloid. In the latter two scenarios, the consideration of travel time variability alters the market boundary, changes the shape and size of the market segment, and affects the corresponding passenger decision and market equilibrium.

Drivers’ in-vehicle travel time reliability can be similarly analyzed with the revision of equation (3). When both demand and supply are derived, the process of analyzing market equilibrium and equilibrium transitions is the same. For the disaggregate market, the discussion of in-vehicle travel time reliability should be O-D-specific, too. More dimensions on travel time reliability can be further considered, including passenger waiting time and driver idle time uncertainties. The introduction of more random variables in the passenger cost or driver benefit function will probably lead to users’ market segment in hyperspace and increase the computation complexity of demand or supply. An independent and thorough study that expands these analyses on travel time reliability is anticipated.

6. Numerical Experiments

6.1. Base Case: Aggregate/Single O-D Market. Consider an aggregate market where drivers are free to enter or leave. As explained, it can be treated as a single O-D market where trips are analyzed at their average level. The following
Numerical settings are based on real-world data in the city of Ningbo, China. The in-vehicle time is $T_i = 13.40 \text{min} = 0.2233 \text{hr}$. Potential passenger demand is $Q^* = 4.5 \times 10^4 \text{hr}$. Passengers' scaling factor for VOT is $\delta^U = 0.5$. Characteristics of passengers’ VOT and WTP follow a bivariate normal distribution (Empirical estimation of user preferences is a priority but may be challenging [21]). The mean VOT and WTP are selected based on users’ income level and sample ride-sourcing trip order data in Ningbo, China, so are the mean VOT and WTA in equation (33). The numerical estimation of the joint distribution of user characteristics does not exactly reflect reality, but we try to make sure that the order of magnitude is accurate. For readers’ information, the 2019 Ningbo statistical data show that the average wage of working staff and workers in urban collective-owned units and above is 106,883 CNY/yr ($15,490/yr) and the annual disposable income of urban residents is 64,886 CNY/yr ($9,404/yr). A sample 2019 monthly data of the ridesourcing market in Ningbo show that drivers’ average monthly salary is 8,762 CNY ($1,270).

$$\left( \beta^U, w^U \right) \sim A \left( \mu^U, \Sigma^U \right),$$

$$\mu^U = (40,36)^T,$$

$$\Sigma^U = \begin{pmatrix} 1600 & 9 \\ 9 & 0 \end{pmatrix},$$

with the total factor productivity $A = 0.1$ and matching elasticities $\alpha_1 = \alpha_2 = 0.75$. The default matching function exhibits increasing returns to scale.

The existence and uniqueness of the equilibrium can be checked using Proposition 1. For example, for $P^U = 25 \text{ CNY} ($3.62$) and $P^W = 20 \text{ CNY} ($2.90$), we first have $q(P^U, 0, T^*, Q^*) = 59159/\text{hr} > Q^*$, and condition (a) of Proposition 1 is satisfied. Then, we consider $T^* = 0.8 \text{ hr}$. Solving $Q^* = Q^W$, we have $T^* = 0.4 \text{ hr}$, and $Q^* - M^U - V = -3979/hr < 0$ at $T^U = 0.8 \text{ hr}$ and $T^V = 0.4 \text{ hr}$. Condition (b) is satisfied. The time frame of concern is $T^U = (0, 0.8)$ and $T^V = (0, 0.4)$. Third, it can be verified that the ride-sourcing factor $\eta$ is always positive within the time frame of concern. Therefore, condition (c) is satisfied as well. Consequently, the unique equilibrium exists within the time frame of concern. In fact, the equilibrium can be solved via the bisection method, which yields $T^U = 0.1441 \text{ hr} = 8.656 \text{ min}$, $T^V = 0.1823 \text{ hr} = 10.94 \text{ min}$, and $Q^* = Q^W = M^U - V = 23485/\text{hr}$.

The optimal settings of trip fare and wage by a profit-maximizing platform are solved using a bi-level program. We compute the equilibrium solutions of passenger waiting time and driver idle time at the lower level for each given passenger fare and driver wage. At the upper level, we solve (11) and (12) based on the market equilibrium obtained at the lower level. Equilibrium market variables under profit maximization are listed in Table 2. We have $e_{pu} = (1 + e_{pu})^{-1} = 35.8\%$ and $e_{pw} + e_{pw} = -1$. Proposition 2 is verified.

6.2 Sensitivity Analysis. Based on base case equilibrium under profit maximization, we do several sensitivity analyses to see whether and to what extent the equilibrium state is sensitive to different input parameters, and how different stakeholders (i.e., passengers, drivers, and the platform) react in different situations. The results are arranged in two panels in a figure (see Figures 4–9). In every figure, the left panel shows how equilibrium passenger waiting time $T^U$, equilibrium driver idle time $T^V$, identical equilibrium passenger supply $Q^*$, equilibrium driver supply $Q^W$, and platform matching rate $M^U - V$ change with respect to the corresponding varying parameter, and the right panel shows how optimal passenger fare $P^U$, optimal driver wage $P^W$, and maximum platform profit $\pi$ change with respect to the corresponding varying parameter.

6.2.1 Sensitivity Analysis with Respect to Passengers’ Correlation Coefficient. The correlation coefficient between passengers’ VOT and WTP, denoted by $\rho$, varies from $-0.7$ to $0.7$ with a step size of 0.1. The results are shown in Figure 4. As passengers’ correlation coefficient increases, optimal passenger fare decreases. This is because fewer passengers in the passengers’ market segment have a low VOT and a high WTP, whereas more passengers lie along the market boundary. Passenger demand becomes more elastic, and the platform lowers passenger fare to attract more passengers and reduce profit loss. As a result, equilibrium passenger demand and supply and the platform matching
rate increase. Equilibrium passenger waiting time decreases mainly because more drivers are available for services, while waiting passengers are still not congested. The driver side is less sensitive with respect to passengers’ correlation coefficient. As equilibrium driver supply increases, both driver wage and equilibrium driver idle time almost stay the same.
6.2.2. Sensitivity Analysis with Respect to Drivers’ Correlation Coefficient. The correlation coefficient between drivers’ VOT and WTA, denoted by $\sigma$, also varies from $-0.7$ to $0.7$ with a step size of $0.1$. The results are shown in Figure 5. As drivers’ correlation coefficient increases, the optimal driver wage decreases. This is because fewer drivers lie along the drivers’ market boundary, and more drivers in the market segment have a low VOT and a low WTA. Driver supply becomes less elastic, and the platform is able to lower the driver wage to make more profit. Accordingly, equilibrium passenger waiting time increases. On the passenger side, equilibrium passenger waiting time and optimal passenger fare almost keep unchanged when drivers’ correlation coefficient varies.

6.2.3. Sensitivity Analysis with Respect to Passengers’ Scaling Factor. The scaling factor of passengers $\delta_U$ varies from $0$ to $1$, with a step size of $0.1$. The results are shown in Figure 6. As passengers’ scaling factor increases, passengers value more on in-vehicle time than waiting time. With the fixed in-vehicle service time, equilibrium passenger waiting time increases in balance with the value loss on in-vehicle time. Passengers cost more, and equilibrium passenger demand decreases. The platform lowers passenger fare to reduce passenger loss, and the platform profit decreases. Equilibrium driver idle time increases mainly because both passenger demand and driver supply become scarce. The platform chooses to keep the optimal driver wage almost the same.
6.2.4. Sensitivity Analysis with Respect to Drivers’ Scaling Factor. The scaling factor of drivers $\delta^V$ varies from 0 to 1, with a step size of 0.1, too. The results are shown in Figure 7. As drivers’ scaling factor increases, equilibrium driver idle time increases. This is because drivers value more on in-vehicle service time than idle time when the scaling factor of drivers increases. Given that the in-vehicle service time keeps constant, equilibrium driver idle time increases in balance with the value loss on in-vehicle time. As a result, drivers’ market segment shrinks, and equilibrium driver supply decreases. With fewer drivers in the market, equilibrium passenger waiting time increases, and equilibrium passenger demand decreases. The platform chooses to increase driver wage to reduce driver loss and increase passenger fare to reduce more profit loss.

6.2.5. Sensitivity Analysis with Respect to Platform Matching Elasticities. For the Cobb–Douglas platform matching function (34) we set matching elasticities $\alpha_1 = \alpha_2$ for ease of sensitivity analysis. The elasticities are varied from 0.7 (There is no equilibrium solution given these numerical settings if the elasticities are too small (e.g., the matching function exhibits decreasing returns of scale).) to 1 with a step size of 0.05. The results are shown in Figure 8. As the matching elasticities increase, the platform capacity of matching is enhanced exponentially. Consequently, both equilibrium passenger waiting time and driver idle time shorten sharply. Both equilibrium passenger demand and driver supply increase rapidly. The platform chooses to reduce driver wage but keep passenger fare almost the same to gain more profit.
6.2.6. Sensitivity Analysis with Respect to Total Factor Productivity. Furthermore, we do sensitivity analysis with respect to the total factor productivity \( A \) in the Cobb–Douglas matching function (34) The total factor productivity varies from 0.1 to 1 with a step size of 0.1. The results are shown in Figure 9. As the total factor productivity increases, the platform capacity of matching improves linearly. As a result, both equilibrium passenger waiting time and driver idle time decrease relatively fast, and equilibrium passenger demand and driver supply increase. The platform lowers driver wage for more profit returns.

6.3. Equilibrium Transitions. Figure 3 illustrates how the equilibrium transition occurs due to different variables \( P^U, P^V, Q^U, Q^V, \) and \( T_V \), respectively. The numerical settings are the same as the base case in Section 6.1, especially that we set passenger fare \( P^U = 25 \text{ CNY} (\$3.62) \) and \( P^V = 20 \text{ CNY} (\$2.90) \) when they are not varying.

In Figure 3(a), the minimum equilibrium passenger waiting time \( T^U = 0.1437/\text{hr} = 8.62 \text{ min} \), which occurs around \( P^U = 24 \text{ CNY} (\$3.48) \), while in Figure 3(b), the minimum \( T^U = 0.1437/\text{hr} = 8.62 \text{ min} \), which occurs at \( Q^V = 4.94 \times 10^4/\text{hr} \). The corresponding market variable at which \( T^U \) is the minimum in either Figure 3(a) or Figure 3(b) satisfies equation (17). Therefore, Proposition 3 is verified. Similarly, Proposition 4 can be verified in Figures 3(c) and 3(d).

For nonequilibrium transition paths, we change each exogenous variable to a new value, compare two equilibria, and see whether the equilibrium transition can be successful. Same as the base case in Section 6.1, we set default passenger fare \( P^U = 25 \text{ CNY} (\$3.62) \) and \( P^V = 20 \text{ CNY} (\$2.90) \). The default equilibrium \( e_0 \) is \( (T^U_{w_0}, T^V_{w_0}) = (0.1441, 0.1823) \). We consider different equilibria to explain more about equilibrium transitions.

Figure 10 demonstrates the equilibrium transition when driver wage \( P^V \) decreases to 18 CNY (\$2.61). By equation (18), a lower wage increases passengers’ waiting time \( T^V \) to 0.2018 hr, while drivers’ idle time decreases to 0.1439 hr. Figure 11 shows the transition when in-vehicle service time \( T^V \) decreases to 10 min, and the new equilibrium \( e_2 \) is (0.1107, 0.2159). The decrease in \( T^U \) and increase in \( T^V \) can be verified according to equation (21). Figure 12 illustrates the transition when \( Q^V \) changes to 5.0 \times 10^4/\text{hr} \). The results indicate that whether or not the nonequilibrium system evolves to the new equilibrium depends on and is very sensitive to the parameters of \( \theta^U \) and \( \theta^V \).

6.4. Disaggregate Market Equilibrium

6.4.1. Two-Node Disaggregate Market. We further consider a two-node disaggregate market, as shown in Figure 13. There are bilateral ride-sourcing trips between subregions \( A \) and \( B \). The market is conserved with the number of active drivers \( N^V = 2.0 \times 10^3 \).

For trips from origin \( A \) to destination \( B \), the in-vehicle service time is \( T^V_{w_1} = 12 \text{ min} = 0.2 \text{ hr} \); potential demand is \( Q^U_{w_1} = 4.0 \times 10^5/\text{hr} \) and realized demand is \( Q^V_1 \); potential and realized driver supply is denoted by \( Q^V_{w_1} \) and \( Q^V_{w_2} \), respectively; passenger fare is \( P^U_{w_1} = 25 \text{ CNY} (\$3.62) \) and driver wage is \( P^V_{w_1} = 20 \text{ CNY} (\$2.90) \); and passenger waiting time, driver idle time, the number of unserved passengers, and the number of vacant drivers are denoted by \( T^U_{w_1}, T^V_{w_1}, N^U_{w_1}, \) and \( N^V_{w_1} \), respectively.

For trips from origin \( B \) to destination \( A \), it takes a bit longer time and in-vehicle service time is \( T^V_{w_1} = 15 \text{ min} = 0.25 \text{ hr} \); potential demand is \( Q^U_{w_2} = 6.0 \times 10^5/\text{hr} \) and realized demand is \( Q^V_{w_2} \); potential and realized driver supply is denoted by \( Q^V_{w_1} \) and \( Q^V_{w_2} \), respectively; passenger fare and driver wage also, respectively, increase to \( P^U_{w_2} = 28 \text{ CNY} (\$4.06) \) and \( P^V_{w_2} = 22 \text{ CNY} (\$3.19) \) due to longer travel time; passenger waiting time, driver idle time, the number of unserved passengers, and the number of vacant drivers are denoted by \( T^U_{w_2}, T^V_{w_2}, N^U_{w_2}, \) and \( N^V_{w_2} \), respectively.

The two matching functions are \( M^U_{w_1} \) at origin \( A \) and \( M^U_{w_2} \) at origin \( B \). For simplicity, both are assumed to have the same Cobb–Douglas type functional form and parameters:

\[
M^U_{w_1} = A' \cdot (N^U_{w_1})^{\alpha_1} \cdot (N^V_{w_1})^{\alpha_2},
\]

\[
M^U_{w_2} = A' \cdot (N^U_{w_2})^{\alpha_1} \cdot (N^V_{w_2})^{\alpha_2},
\]

where \( A', \alpha_1, \) and \( \alpha_2 = 0.75 \). Equation (35) has a different setting on the total factor productivity compared with (34), because origins \( A \) and \( B \) have a smaller spatial scale than the aggregate market in the base case, and demand and supply are much fewer. Other settings, such as passengers’ scaling factor and characteristics, drivers’ scaling factor and characteristics, and drivers’ operation and fuel cost function, are the same as the base case in Section 6.1.

Based on equations (28)–(30) in Section 5.2, at market equilibrium we have the following:

\[
\begin{aligned}
Q^U_{w_1} &= Q^V_{w_1} = M^U_{w_1}, \\
Q^U_{w_2} &= Q^V_{w_2} = M^U_{w_2}, \\
Q^V_{w_1} &= Q^V_{w_2}, \\
Q^V_{w_1} \cdot (T^V_{w_1} + T^V_{w_2}) + Q^V_{w_2} \cdot (T^V_{w_2} + T^V_{w_1}) &= N^V.
\end{aligned}
\]

Solving equation (36) numerically within the time frame of concern, we have a unique two-node disaggregate market equilibrium given in Table 3. Compared with origin \( B \), passenger waiting time at origin \( A \) is shorter, while driver idle time at origin \( A \) is longer. This is because potential demand is higher at origin \( B \), and passengers tend to wait longer for trip service; on the other hand, drivers at origin \( B \) are easier to be matched with a passenger. As a result, the number of unserved passengers is fewer at origin \( A \); the number of vacant drivers is greater at origin \( A \). The potential driver supply at origin \( A \) is relatively higher to maintain the same amount of realized driver supply between \( A \) and \( B \).
Figure 10: Equilibrium transition when $P^V = 18$ CNY ($\approx 2.61$) ($\theta^U = 1, \theta^V = (1/7.2564)$). (a) Transition path. (b) Evolution of $T^U_w$. (c) Evolution of $T^V_w$.

Figure 11: Equilibrium transition when $T_i = 10$ min ($\theta^U = 1, \theta^V = (1/18.8222)$). (a) Transition path. (b) Evolution of $T^U_w$. (c) Evolution of $T^V_w$.

Figure 12: Continued.
The potential demand rate of each O-D pair is referred to [37], which considers all the 552 O-D pairs among 24 nodes and 76 links. The total number of trips is 360,600. The number of active drivers is set as 4,000, while the total number of equations given by equations (28)–(30) is 4m + n. Therefore, there is a higher probability of multiple equilibrium solutions in more complicated networks. We test the Sioux Falls network to show that the model application to the disaggregate market can be extended to a larger network. Note that the Sioux Falls network used here is not a real road network anymore. The nodes in the Sioux Falls network now represent a region rather than physical streets. We solve the equilibrium problem by adopting an iterative approach [35, 36]. One practically meaningful solution is obtained using the following settings. It costs 788 seconds for a single core (AMD Ryzen 7 4800HS, 4.0-4.1 GHz) to converge with the stop criterion given as follows:

$$\sum_{o \in S} \sum_{d \in S} \left( Q^U_{od}(i) - Q^U_{od}(i - 1) \right)^2 + \left( Q^{VU}_{od}(i) - Q^{VU}_{od}(i - 1) \right)^2 < 10^{-3},$$

where $i$ represents the number of iterations.

The potential demand rate of each O-D pair is referred to [37], which considers all the 552 O-D pairs among 24 nodes and 76 links. The total number of trips is 360,600. The number of active drivers is set as $N^V = 1.0 \times 10^5$. We also adopt the same Cobb–Douglas type matching function as (35), with $\alpha = 5$ and $\lambda_1 = \lambda_2 = 0.6$. To avoid computational complexity, we assume that all the users choose the shortest paths and do not consider the congestion effect on each link. The in-vehicle service time is calculated based on the length and vehicle speed on each link, where the length of each link is referred to the GitHub (https://github.com/bstabler/TransportationNetworks/tree/master/SiouxFalls) and vehicle speed on each link is randomly generated within [8, 24] (km/hr). Passenger fares in different O-D pairs are randomly generated within [37, 42] CNY ([$5.36, $6.09]), and driver wages are randomly generated within [30, 35] CNY ([$4.35, $5.07]).

Figures 14 and 15 illustrate the market equilibrium of the Sioux Falls network. As realized demand and supply are identical to the O-D ride-sourcing trip flow under equilibrium, the ride-sourcing trip flows on each link and between each O-D pair are demonstrated. The two figures
show that most trip orders are achieved around the upper-right CBD area. Fewer trips are achieved on long-distance O-D pairs due to the higher in-vehicle time cost. The mean driver idle time on all O-D pairs is 0.790 hr. 37.4% of the drivers experienced driver idle time less than 10 min, 24.1% of them experience driver idle time between 10 and 30 min, and the remaining 38.5% experience more than 30 min before service. The mean passenger waiting time on all O-D pairs is 0.057 hr. Only 44.3% of the passengers wait for more than 3 min, and only 0.7% of them wait for more than 10 min.

7. Discussions and Conclusions

7.1. Operation and Policy Implications. Our robust mathematical model for analyzing ride-sourcing markets with heterogeneous users has several operation and policy implications:

(i) Sensible ride-sourcing market evaluation. Market variables are correlated, including user prices, demand and supply, and in-service and before-service times. One can evaluate the ride-sourcing market state based on these variables, such as judging whether the market is at equilibrium or not, inferring passenger waiting time and driver idle time at equilibrium, and calculating realized or potential demand and supply from other variables. After such evaluation, the market can be adjusted to another state if necessary with appropriate measures as indicated in the analyses in equilibrium transitions.

(ii) More realistic ride-sourcing market decision-making. Incorporating user heterogeneity of both passengers and drivers into ride-sourcing market modeling can benefit ride-sourcing operators strategically and improve their business in the real world. User prices and the commission rate are set more reasonably. Different operation goals that include profit maximization, passenger waiting time minimization, or driver idle time minimization can be considered. Simulations based on the disaggregate market modeling will help with practical ride-sourcing market operation in a real transportation network.

(iii) Better ride-sourcing trip experience. The improved market operation also enhances user experiences. Passengers are charged with reasonable fares, and drivers are paid appropriate wages. Both users can enjoy more convenient ride-sourcing trips with optimized passenger waiting time and driver idle time at equilibrium. What is more, the model extension on travel time reliability will consider trip uncertainties.

(iv) Multiple ride-sourcing market regulations. Our model implies a feasible policy toolkit to regulate the market. Price regulation, supply regulation, or other policies can be established based on comparative static analysis under a particular equilibrium transition regime. Sensitivity analyses show how relevant market stakeholders respond with respect to different parameters of passengers, drivers, and the platform. A policymaker can utilize this to improve sustainable mobility in the region. For
example, a policymaker may impose regulations on platform matching rules via a restrictive matching function in case of need.

7.2. Summary. With the justification of a comprehensive matching function approach, this study models and analyzes the ride-sourcing market with heterogeneous users. A single O-D market model is first developed. Drivers provide trip services for passengers via a monopolistic ride-sourcing platform. Passenger demand and driver supply are derived with a monetary-metric utility approach, where users differ in their characteristics. Passengers and drivers are matched by a comprehensive bilateral matching function.

Under the single O-D market equilibrium, passenger waiting time and driver idle time are endogenously determined. We derive sufficient conditions that guarantee a unique ride-sourcing market equilibrium. The concepts of the time frame of concern and the ride-sourcing factor are defined based on these sufficient conditions. Equilibrium market operation with a profit-maximizing platform is explored. The commission rate can be expressed in the forms of fare elasticity of demand or the wage elasticity of supply when the corresponding price is optimized; the sum of the two price elasticities is negative if fare and wage are simultaneously optimized.

Equilibrium transitions for the single O-D market are investigated. Comparative static analyses are conducted for equilibrium transitions with respect to the changes in different exogenous market variables. Two propositions are presented as useful references for the equilibrium operation of passenger waiting time minimization and driver idle time minimization, respectively. Nonequilibrium modeling to understand the transition paths is analyzed. The single O-D market model is then extended to aggregate and disaggregate markets and incorporation of travel time reliability. Numerical experiments are presented to illustrate the theoretical model. Discussions are made regarding operation and policy implications.

7.3. Limitations and Future Work. The current exposition of the study has some limitations and thus paves several interesting avenues for future studies:

(i) As explained in Section 1, the two approaches to model the matching and meeting process have advantages and disadvantages, while our paper uses the first approach. A parallel ride-sharing market model with heterogeneous users is preferred using the second approach with the segmented before-service period. More comparisons can be made between the two approaches during further studies.

(ii) More complex but realistic model settings can be considered, such as two or more passengers sharing the same trip, ride-splitting with heterogeneous users, trip route alternatives, the values of reliability for in-service and before-service times, and equilibrium stability nonequilibrium analysis. For example, when adapting our model to the ride-pooling market with multi-passenger services, one method is to revise the passenger cost and driver benefit functions by incorporating a pairing rate and detour time [19].

(iii) Discussions on the disaggregate market are limited. Therefore, future explorations on disaggregate market modeling will be worthy of investigation, e.g., equilibrium properties, multiple equilibria, routing choices, and price discrimination. Simulations based on more real transport networks can be conducted.

(iv) The non-monopolistic environment will be interesting but challenging. For the transit market, the oligopoly environment and service differentiation are explored in [38]. Under the current deterministic assumptions, services are probably differentiated between different non-monopolistic transportation network companies, or stochastic choice assumptions such as a logit model may be introduced and combined with our model.

(v) More empirical studies can be done using the ride-sourcing market data from different cities worldwide. Both model assumptions and analytical results will be tested in such studies, which is helpful to better comprehend the ride-sourcing market.

Appendix

A. Heterogeneous User Characteristics

This ride-sourcing market with heterogeneous users was modeled in Zhang et al. [30], which studies the same-side and cross-side network effects for passengers and drivers. We inherit this framework and use consistent mathematical symbols. As shown in Figure 16, the general user heterogeneity of passengers and drivers is characterized by two dimension spaces, respectively.

From the left panel of Figure 16, the market segment of passengers is as follows:

\[ \Omega^U = \{(\beta^U, \omega^U) | \beta^U \cdot T^U \leq \omega^U \leq \omega^U, \beta^U \geq 0 \}. \]  \hspace{1cm} (A.1)

The market boundary of passengers is as follows:

\[ M^U = \{(\beta^U, \omega^U) | \omega^U = \beta^U \cdot T^U, \beta^U \leq \omega^U \leq \omega^U \}. \]  \hspace{1cm} (A.2)

Let \( \Omega^U \) denote potential passenger demand. The actual or realized demand of passengers is as follows:

\[ Q^U = \bar{Q}^U \left( \vec{C}^U, \vec{Q}^U \right), \]
\[ = \bar{Q}^U \left( \beta^U, T^U, \omega^U, \vec{Q}^U \right), \]
\[ = \bar{Q}^U \cdot \int (\beta^U, \omega^U) \in \Omega^U h^U(\beta^U, \omega^U) d\omega^U d\beta^U. \]  \hspace{1cm} (A.3)

Expressed as a fraction of potential demand, the market segment of all passengers is as follows:
Willingness to pay $\omega_U$ follows:

$$G_{\Omega^U} = \frac{Q^U}{Q^V},$$

$$= \int_{\omega_U - P^U}^{\omega_U - P^U + \beta^U \cdot T^U} d\omega_U. \tag{A.4}$$

The probability density of marginal passengers on the market boundary $M^U$ is as follows:

$$g_{MU} = \frac{\omega_U - P^U}{\omega_U - P^U + \beta^U \cdot T^U} h_U^U(\beta^U, P^U, \omega_U) d\beta_U. \tag{A.5}$$

The mean VOT for marginal passenger on the market boundary $M^U$ is as follows:

$$E_{MU}(\beta^U) = \int_{(\beta^U, \omega^U) \in M^U} \omega^U \cdot h_U^U(\beta^U, \omega^U) d\omega^U d\beta^U. \tag{A.6}$$

The characteristic times of passengers are defined as follows:

$$\lambda_1^U = \frac{G_{\Omega^U}}{(g_{MU} \cdot E_{MU}(\beta^U))}, \tag{A.7}$$

$$\lambda_2^U = \frac{G_{\Omega^U}}{(\beta^U \cdot g_{MU} \cdot E_{MU}(\beta^U))}. \tag{A.8}$$

From the right panel of Figure 16, the market segment of drivers is as follows:

$$\Omega^V = \{(\beta^V, \omega^V) | 0 \leq \omega^V \leq P^V - C^V - \beta^V \cdot T^V, \beta^V \geq 0\}. \tag{A.9}$$

The market boundary of drivers is as follows:

$$M^V = \{(\beta^V, \omega^V) | \omega^V = P^V - C^V - \beta^V \cdot T^V, 0 \leq \omega^V \leq P^V - C^V\}. \tag{A.10}$$

Let $Q^V$ denote potential supply. The actual or realized supply of drivers is as follows:

$$Q^V = q^V \left( B^V, Q^V \right),$$

$$= q^V \left( P^V, T^V, T_i, Q^V \right), \tag{A.11}$$

$$= Q^V \cdot \int_{(\beta^V, \omega^V) \in \Omega^V} h_V^V(\beta^V, \omega^V) d\omega^V d\beta^V.$$

Expressed as a fraction of potential supply, the market segment of all drivers is as follows:

$$G_{\Omega^V} = \frac{Q^V}{Q^V},$$

$$= \int_{\omega - P^V}^{\omega - P^V + \beta \cdot T^V} d\omega_U. \tag{A.12}$$

The probability density of marginal drivers on the market boundary $M^V$ is as follows:

$$g_{MV} = \frac{\omega - P^V}{\omega - P^V + \beta \cdot T^V} h_V^V(\beta^V, P^V - C^V - \beta^V \cdot T^V) d\omega^V. \tag{A.13}$$

The mean VOT for marginal drivers on the market boundary $M^V$ is as follows:

$$E_{MV}(\beta^V) = \int_{(\beta^V, \omega^V) \in M^V} \omega^V \cdot h_V^V(\beta^V, \omega^V) d\omega^V d\beta^V. \tag{A.14}$$

The characteristic times of drivers are defined as follows:

$$\lambda_1^V = \frac{G_{\Omega^V}}{(g_{MV} \cdot \partial \omega^V / \partial T_i + E_{MV}(\beta^V))},$$

$$\lambda_2^V = \frac{G_{\Omega^V}}{(g_{MV} \cdot \partial \omega^V / \partial T_i + \delta^V \cdot E_{MV}(\beta^V))}. \tag{A.15}$$
**B. Proof of Proposition 1**

Note that $Q^U$, $Q^V$, $T_U$, $P^U$, and $P^V$ are given. First, for any given $T^U_w > 0$, let:

$$f(T^V_w) = Q^U - Q^V,$$

$$= \tilde{q}^U(P^U, t^U_w, T_U, Q^U) - \tilde{q}^V(P^V, t^V_w, T^V_w, Q^V). \tag{B.1}$$

Since $Q^V$ monotonically decreases in $T^V_w$, function $f(T^V_w)$ increases in $T^V_w$. When $T^V_w \to \infty$, $Q^V \to 0$, and $f(T^V_w) > 0$; when $T^V_w \to 0$, $f(T^V_w) = Q^V - Q^V < Q^U - Q^V < 0$ by condition (a). By the intermediate value theorem, $f(T^V_w) = Q^V - Q^V = 0$ exists and is unique for any given $T^U_w > 0$. This means that for any given $T^U_w > 0$, there is a unique value of $T^V_w$ that satisfies $Q^U = Q^V$, and we write $T^V_w = t(T^U_w)$.

Second, let

$$g(T^U_w) = Q^U - M^U - V,$$

$$= \tilde{q}^U(P^U, t^U_w, T_U, Q^U) - m(T^U_w, Q^U, t(T^U_w) \cdot Q^V). \tag{B.2}$$

Given the conditions that $Q^U = Q^V$ and $T^V_w = t(T^U_w)$, the entire problem reduces to the existence and uniqueness of a univariate problem $g(T^U_w) = 0$, where $T^U_w$ is the only changing variable. When $T^U_w \to 0$, $M^U \to 0$, and $g(T^U_w) > 0$; when $T^U_w = T^U_w$, $g(T^U_w) < 0$ by condition (b). To check the monotonicity of function $g(T^U_w)$, we need to calculate the corresponding differentials of demand function (equation (2)) and supply function (equation (4)). Using Leibniz’s rule for differentiation under the integral sign, we have the following:

$$\frac{dQ^U}{dT^U_w} = Q^U \cdot \frac{g_{M^U} \cdot E_{M^U}(\beta^U)}{\lambda^1} \cdot \frac{dT^U_w}{\lambda^1}.$$

(B.3)

where $\frac{dQ^U}{dT^U_w}$ yields $(-Q^U/\lambda^U)$ and $\frac{dQ^V}{dT^U_w} = \frac{dQ^U}{dT^U_w}$ under $Q^U = 0$. $t(T^U_w)$ is increasing in $T^U_w$.

Then:

$$\frac{d\alpha(T^U_w)}{dT^U_w} = \frac{\partial Q^U}{\partial T^U_w} - \left( \frac{\partial m}{\partial N^w} \cdot \frac{\partial (T^U_w \cdot Q^U)}{\partial T^U_w} + \frac{\partial m}{\partial N^w} \cdot \frac{\partial t(T^U_w \cdot Q^U)}{\partial T^U_w} \right)$$

(B.5)

which is negative for $T^U_w \in (0, T^U_w)$ by condition (c). Therefore, the function $g(T^U_w)$ is decreasing in $T^U_w \in (0, T^U_w)$.

Again by the intermediate value theorem, $g(T^U_w) = Q^U - M^U - V = 0$, exists and is unique within the range of $T^U_w \in (0, T^U_w)$. This completes the whole proof.

$$\frac{dQ^U}{dT^U_w} = \frac{dQ^U}{T^U_w} \cdot T^U_w + \frac{dQ^U}{T^U_w} \cdot T^U_w \tag{C.1}$$

C. Comparative Static Analysis

The total derivative of market clearance condition (equation (5)) yields the following:

$$\frac{dQ^U}{d\lambda^U} = \frac{Q^U}{\lambda^U} \cdot \left( \frac{dP^U}{E_{M^U}(\beta^U)} + \frac{dT^U_w}{dT^U_w} \cdot \frac{dT^U_w}{\lambda^U} \right) \tag{C.2}$$

C.1. Differentials with Respect to Passenger Fare. When passenger fare $P^U$ is the only changing variable (\Delta = P^U), we calculate the corresponding differentials using Leibniz’s rule for differentiation under the integral sign.

For passenger demand and driver supply, we have the following:

$$\frac{dQ^U}{d\lambda^U} = \frac{Q^U}{\lambda^U} \cdot dT^U_w \cdot \frac{dT^U_w}{\lambda^U}.$$
The ride-sourcing factor $\eta = 1 - \alpha_1 - \alpha_2 + \alpha_1 \cdot \lambda_1^U / T_w^U + \alpha_2 \cdot \lambda_1^V / T_w^V$ is recalled. Substituting equation (C.2) into equation (C.1), we have the following:

$$dQ^U_{\Delta = p^U} = dQ^U_{\Delta = p^V},$$

$$= \frac{\alpha_1 \cdot Q^U}{\eta \cdot T_w^U \cdot E_M^U (\beta^U)} \cdot dP^V. \tag{C.3}$$

Substituting equation (C.3) into equation (C.2), we have the following:

$$dT^U_{\Delta = p^U} = \frac{\alpha_1 \cdot \lambda_1^U - \eta \cdot T_w^U}{\eta \cdot T_w^U \cdot E_M^U (\beta^U)} \cdot dP^U, \tag{C.4}$$

$$dT^V_{\Delta = p^U} = \frac{\alpha_1 \cdot \lambda_1^V}{\eta \cdot T_w^U \cdot E_M^U (\beta^U)} \cdot dP^U.$$

Based on equation (C.3), the differential of platform profit (equation (10)) with respect to $P^U$ is as follows:

$$d\pi|_{\Delta = p^U} = p^U \cdot dQ^U_{\Delta = p^V} + Q^U \cdot dP^U - p^V \cdot dQ^V_{\Delta = p^U},$$

$$= \left(1 - \frac{\alpha_1 \cdot \lambda_1}{\eta \cdot T_w^U \cdot E_M^U (\beta^U)}\right) \cdot Q^U \cdot dP^U. \tag{C.5}$$

C.2. Differentials with Respect to Driver Wage. When driver wage $P^V$ is the only changing variable $\Delta = Q^V$, similarly, for passenger demand and driver supply, we have the following:

$$dQ^U_{\Delta = p^V} = \frac{Q^U}{\lambda_1^V} \cdot dT^U_{\Delta = p^V},$$

$$dQ^V_{\Delta = p^V} = \frac{Q^V}{\lambda_1^V} \cdot \left(\frac{dP^V}{(\partial c^U / \partial T_w^V) + E_M^V (\beta^V)} - dT^V_{\Delta = p^V}\right). \tag{C.6}$$

Substituting equation (C.6) into equation (C.1), we have the following:

$$dQ^U_{\Delta = p^V} = dQ^V_{\Delta = p^V},$$

$$= \frac{\alpha_2 \cdot \lambda_1^V}{\eta \cdot T_w^V \cdot ((\partial c^V / \partial T_w^V) + E_M^V (\beta^V))} \cdot dP^V. \tag{C.7}$$

Substituting equation (C.7) into equation (C.6), we have the following:

$$dT^U_{\Delta = p^V} = \frac{\alpha_2 \cdot \lambda_1^U - \eta \cdot T_w^U}{\eta \cdot T_w^U \cdot ((\partial c^V / \partial T_w^V) + E_M^V (\beta^V))} \cdot dP^V,$$

$$dT^V_{\Delta = p^V} = \frac{\alpha_2 \cdot \lambda_1^V - \eta \cdot T_w^V}{\eta \cdot T_w^V \cdot ((\partial c^V / \partial T_w^V) + E_M^V (\beta^V))} \cdot dP^V. \tag{C.8}$$

Based on equation (C.7), the differential of platform profit with respect to $P^V$ is as follows:

$$d\pi|_{\Delta = p^V} = p^U \cdot dQ^U_{\Delta = p^V} - p^V \cdot dQ^V_{\Delta = p^V} - Q^V \cdot dP^V,$$

$$= \left(1 - \frac{\alpha_2 \cdot \lambda_1}{\eta \cdot T_w^V \cdot ((\partial c^V / \partial T_w^V) + E_M^V (\beta^V))}\right) \cdot Q^U \cdot dP^V. \tag{C.9}$$

Equations (C.5) and (C.9) lead to first-order condition equations (11) and (12).

C.3. Differentials with Respect to Potential Demand. When potential demand $Q^U$ is the only changing variable $\Delta = Q^V$, the differential of driver supply yields the following:

$$dT^U_{\Delta = Q^V} = \frac{dQ^U}{Q^U}|_{\Delta = Q^V},$$

$$= \frac{dQ^U}{Q^U}|_{\Delta = Q^V}. \tag{C.10}$$

With the ride-sourcing factor $\eta = 1 - \alpha_1 - \alpha_2 + \alpha_1 \cdot (\lambda_1^U / T_w^U) + \alpha_2 \cdot (\lambda_1^V / T_w^V)$, substituting equation (C.10) into equation (C.1) yields the following:

$$dT^U_{\Delta = Q^V} = \frac{\eta \cdot T_w^U - \alpha_1 \cdot \lambda_1^U}{\alpha_1} \cdot \frac{dQ^U}{Q^U}|_{\Delta = Q^V}. \tag{C.11}$$

Since $dQ^U|_{\Delta = Q^V} = G_{Q^U} \cdot dQ^U - (Q^U/\lambda_1^U) \cdot dT^U_{\Delta = Q^V}$, we have the following:

$$dQ^U = \frac{\eta \cdot T_w^U}{\alpha_1 \cdot \lambda_1^U \cdot G_{Q^U}} \cdot dQ^U|_{\Delta = Q^V}. \tag{C.12}$$

C.4. Differentials with Respect to Potential Supply. When potential supply $Q^V$ is the only changing variable $\Delta = Q^U$, the differential of passenger demand yields the following:
\[ \frac{dT^U_w}{\lambda_1|_{\Delta=\xi_i^V}} = \frac{dQ^V}{Q^V|_{\Delta=\xi_i^V}} = \frac{dQ^V}{\xi_i^V|_{\Delta=\xi_i^V}}. \] (C.13)

Substituting equation (C.13) into equation (C.1) yields the following:
\[ dT^U_w|_{\Delta=\xi_i^V} = \eta \cdot T^V_w - \frac{\alpha_2}{\alpha_1} \cdot dQ^V|_{\Delta=\xi_i^V}. \] (C.14)

Since \( dQ^V|_{\Delta=\xi_i^V} = dQ^V|_{\Delta=\xi_i^V} = G_{\alpha_1} \cdot dQ^V - (\alpha_1^V - \alpha_1)^V \cdot dT^U_w|_{\Delta=\xi_i^V} \), we have the following:
\[ dQ^V = \frac{\eta \cdot T^V_w}{\alpha_2} \cdot G_{\alpha_1} \cdot dQ^V|_{\Delta=\xi_i^V}. \] (C.15)

C.5. Differentials with Respect to In-Vehicle Time. When in-vehicle time \( T_i \) is the only changing variable (\( \Delta = T_i \)), total derivatives of demand function (equation (2)) and supply function (equation (4)) yield the following:
\[
\begin{align*}
\left\{ \begin{align*}
\frac{dQ^U}{dT^U_w} \cdot dT^U_w|_{\Delta=\Delta_i} + \frac{\partial Q^U}{\partial T_i} \cdot dT_i &= \frac{Q^U}{\lambda_1} \cdot dT^U_w|_{\Delta=\Delta_i} - \frac{Q^U}{\lambda_2} \cdot dT_i, \\
\frac{dQ^V}{dT^U_w} \cdot dT^V_w|_{\Delta=\Delta_i} + \frac{\partial Q^V}{\partial T_i} \cdot dT_i &= \frac{Q^V}{\lambda_1} \cdot dT^V_w|_{\Delta=\Delta_i} - \frac{Q^V}{\lambda_2} \cdot dT_i.
\end{align*} \right.
\end{align*}
\] (C.16)

Data Availability

The data used to support the findings of this study are included within the article. In particular, the Sioux Falls network data are replicated within the supplementary information file.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

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Supplementary Materials

The input data for the disaggregate market of the Sioux Falls network are replicated in this supplementary material file, which includes potential demand, randomly generated passenger fare, randomly generated driver wage, and in-vehicle time between each O-D of the Sioux Falls network (calculated based on the randomly generated vehicle speed and length of each link referred to https://github.com/bstabler/TransportationNetworks/tree/master/SiouxFalls). (Supplementary Materials)

References


