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Research Article

The Effect of Key Indicators on the Operation Costs for Public Toll Roads

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At the end of the build-operate-transfer road concession period, an optimal model for the operation of public toll roads is created based on user heterogeneity regarding the values of time for different road users. The impact of user heterogeneity on operation costs for government and private firms is subsequently analyzed on the following critical variables: user values of time, road volume/capacity ratio, and road capacity. Concerning the values of time for different road users, the mean residual and failure functions are established to describe three optimization hypotheses: maximization of social welfare with operation by the government, two extreme cases with operation by a private firm, and a Pareto-optimal solution with operation by a private firm. It is concluded that the mean residual values of the time function are a linear function of the user values of time under a Pareto-optimal operation by the government. It is also determined that private profit is related to the demand-related operational cost of the government and private firm under a Pareto-optimal operation by a private firm. These conclusions suggest relevant recommendations for the government on policymaking for the operation of public toll roads.

1. Introduction

Many roads, especially highways, have been built according to the build-operate-transfer (BOT) model in the past forty years since it was established and applied in Turkey. Roads built based on the BOT model can be operated by a private firm for a specific time, usually lasting 30–50 years, depending on the contract. In some cases, the BOT concession period can be as long as 99 years, such as Highway 407 in Toronto, Canada [1].

At present, an increasing number of BOT roads are being transferred to the government with the expiration of the BOT contracts. After this transfer, the BOT road is normally expected to become public, which allows the drivers to use it for free, as described in Figure 1. The government then takes responsibility for operating the road using government funds. De Palma et al. [2] assumed that the required standard could not be maintained for the service provided to road users after the BOT concession period. However, as an essential means of transportation between cities, it is critical to ensure the necessary availability of the road to users after it has been transferred to the government.

Although the operation cost (OC), including but not limited to the maintenance cost, is not comparable to the initial construction cost of the road, it is still considered a considerable amount of government expenditure. The solution is to collect a toll for road use to compensate for the OC. This type of road is called a public toll road (PTR) [3], as shown in Figure 2. These roads are being charged for transportation use in China [4]. The Chinese government can whether operate the PTR road on its own by collecting certain amount of toll from the users or entrust a private firm with the same responsibility, which differs from the case of the BOT model.

Sufficient studies have focused on the construction and operation of BOT roads with heterogeneous users [1, 5–11]; however, few researchers have focused on the operation of PTR

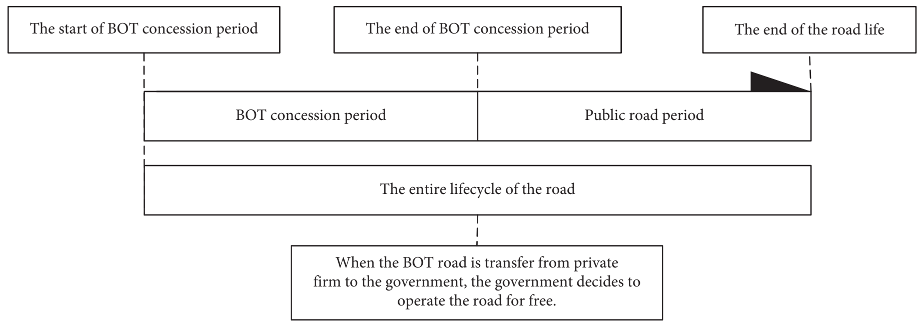

Figure 1: A BOT road becomes a public road.

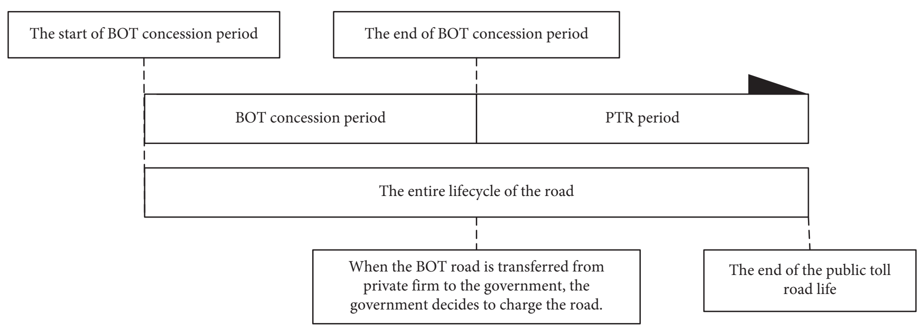

Figure 2: A BOT road becomes a public toll road (PTR).

with heterogeneous users. To the best of our knowledge, the function of OC during PTR has not been sufficiently studied.

In contrast to existing research, we explicitly examine the influence of heterogeneous users on the OC of PTR after the expiration of the BOT concession period by considering the benefits for government and private firms. Two options are available if the government continues to charge a toll for the road during the PTR period. The government can either operate the road independently or choose a private firm to operate it. Thus, the government must make decisions. In our model, by considering the OC in the objective function and introducing an additional constraint on OC, we are in a position to determine not only the effect of the value of time (VOT) on the OC of PTR but also the length of the PTR period as well as the level of toll to be charged from the users.

To analyze the impact of PTR on OC, we introduce a mean residual VOT function and a failure rate to characterize user heterogeneity. In addition, we examine the properties of OC with three basic variables: the length of the PTR period, the road capacity, and the price of the toll to maximize the social welfare of PTR with government operation. We further investigate two extreme cases of PTR with private firm operations and achieve the Pareto-optimality for PTR. Based on the studies mentioned above, this paper can provide insight, defining government policy on transportation.

The rest of the paper is organized as follows: Section 2 reviews the literature. Section 3 introduces the relevant definitions and assumptions. Section 4 presents the properties of OC by analyzing the operation by the government, two extreme cases with operation by a private firm, and Pareto-efficient PTR. Finally, Section 5 concludes the paper.

2. Literature Review

Most existing literature on BOT road studies focuses on capacity determination, toll charges, length of the concession period, private profit, and social welfare related to BOT roads. The analysis of social welfare can be traced back to Pigou [12]. Beckmann et al. [13] analyzed the social welfare of transportation (system with an economic model) and created a classic traffic (or transportation) model known as the Beckmann transformation, which had a significant impact on subsequent research. Subsequently, Walters [14] and Vickrey [15, 18] developed additional transportation models. A notable achievement of BOT studies is the self-financing theorem, deduced under the first-best condition on a one-way road with homogeneous users. The toll charge given by the self-financing theorem, which states that the total revenue of the toll road collected from users only covers its investment under certain conditions, is equal to the difference between the social and private marginal costs.

Further research was conducted on a general transportation network in both congestion pricing and the relationship between road investment and toll revenue [17, 18]. Niu and Zhang [19] examined the price, road capacity, and concession period decisions of Pareto-efficient BOT contracts under uncertain demand. Shi et al. [20] examined the optimal choice of BOT road capacity, toll, and subsidy underpaid minimum traffic guidance. Feng et al. [21] conducted studies on contracts renegotiated with a loss-averse private firm on the BOT road [21]. Based on whether the two periods, including construction period and private operation period, are defined together or not, Zhang et al. [25] made an analysis on the effects of concession period structures on contracts of BOT road.

Private profit and social welfare are the two main objective functions of the traditional transportation economic model. To consider these two objectives simultaneously, Daganzo [26], viewing variable tolls as a hybrid between pricing and rationing, designed a Pareto-improving model with the distribution of losses and gains of all road users via modifying the traditional social welfare approach. Guo and Yang [10] conducted a preliminary study of the concession period and examined unconstrained, profit-constrained, and social welfare-maximizing BOT contracts via three essential variables: road capacity, toll charge, and concession period with homogeneous users. Tan et al. [24] generalized the Pareto-optimal results from Guo and Yang [10]. They further examined the Pareto-optimal contract of the BOT by maximizing private profit and social welfare simultaneously with the assumption that both government and private firms are fully informed of the travel demand and toll charge. They also examined the impact of several regulatory regimes on the behavior of private firms. In reality, the road user value over time is heterogeneous.

Therefore, user heterogeneity exists in road pricing. Mohring [25] generalized self-financing theorems for both homogeneous and heterogeneous users of their unique value of time. Using the basic bottleneck model, Arnott and de Palma [5] examined the effect of congestion tolls on social welfare using the inelastic demand of heterogeneous users. Two cases were conducted: one when the toll revenue is refunded to all road users as an equal lump-sum payment. Mayet and Hansen [6] analyzed second-best congestion pricing based on the continuously distributed VOT of road users. They designed different toll charge cases depending on what social welfare is measured and whether toll revenue is part of the benefit. Yang et al. [7] investigated how heterogeneous users affect the social welfare and profit generated from a toll road in a general transportation network. Verhoef and Small [8] explored the properties of various public and private toll charges in a congested road network with heterogeneous users under elastic demand. They found that the toll of revenue maximization is more inefficient than the pricing of welfare maximization. User heterogeneity can mitigate these differences. The model of public-private partnership built the road to gain broad support for road pricing. Rouhani et al. [26] introduced a new approach that stimulates public support for road pricing, followed by an in-depth analysis of social welfare for road pricing. Wang and Wang et al. [27] developed a Stackelberg game model to optimize the governments subsidies' effectiveness for the public-private partnerships performance mode.

To examine the effects of new roads, Xiao and Yang [9] analyzed the likely bias in a monopoly environment away from the social optimum under the more realistic assumption that each trip maker has a unique value of time. Using cumulative distribution, they also investigated the road capacity, toll set, and efficiency loss by a monopolist under different regulatory mechanisms. With the fixed demand of heterogeneous users, Guo and Yang [10] built a biobjective optimization model based on travel time and travel cost. Using the failure rate and mean residual function, Tan and Yang [1] examined the impact of heterogeneous users on toll road franchising by analyzing the properties of Pareto-optimal BOT contracts and regulation mechanisms. They used a new solution with VOT and volume/capacity ratio substituting demand and capacity to gain further insight. Under the condition of a continuous distribution of the value of time (VOT), Nie and Liu [11] examined the impact of VOT on a pricing-refunding scheme that is both Pareto-improving and self-financing in a static congestion pricing model with two modes.

3. The Model

In this section, we first introduce relevant definitions for further analysis. In the following sections, we define the decision variables of the government and private firms by subscripts g and s, respectively. If the variable does not have subscripts g and s in the paper, it will be given subscripts g and s when used in the analysis.

It is assumed that there are two roads between origin A and destination B in the general network. One is the toll-free road, and the other is the PTR connecting two nodes (A and B) directly. PTR is the BOT road transferred from a private firm to the government. The PTR is assumed to hold the original BOT traffic property, which can shorten the travel time. Owing to its large OC, the government tends to charge for use of the PTR. Based on these two options, the government operates the road independently or determines a private firm to operate it. Section 1 defines the road life after the BOT concession period as the PTR period. Tan and Yang [1] proved that private firms intend to operate the BOT road for their entire road life. The results of this study are also applicable to our paper; namely, by obtaining the PTR franchising for the OC concession period, the private firm is willing to operate the PTR during the entire PTR period; thus, the OC concession period equals the PTR period. Meanwhile, if the government decides to operate the PTR independently, it will also operate it during the entire PTR period. The relationship between OC concession and PTR periods is shown in Figure 3. Based on the above analysis, the variable T does not need to be analyzed in our study.

For the toll-free road, it is assumed that there is no congestion, the travel time t_e is fixed, and the existing capacity is y_1. The travel time of PTR follows a continuously differentiable function $t(q, y)$ where $y > y_1$ and $q \geq 0$ are the capacity and travel demand (traffic flow) of the PTR, respectively. Without loss of generality, the function $t(q, y)$ is of the following properties $\partial t/\partial q > 0, \partial^2 t/\partial^2 q > 0$ for any $y > y_1$ and $\partial t/\partial y < 0$ for any $q \geq 0$. It is also assumed that $t_e > t_0$, where t_0 is the travel time for the toll-free road with free flow and the travel time of PTR $t(q, y)$ is shorter than t_e all the time so that the PTR will be able to attract those travelers whose saved VOT is either equal to or exceeds the amount of the toll p which they pay.

Assuming the number of total users, Q, including the PTR and the toll-free road users, is fixed, the VOT of Q follows a continuously differentiable cumulative distribution $F(\beta)$, and $f(\beta)$ is the corresponding probability

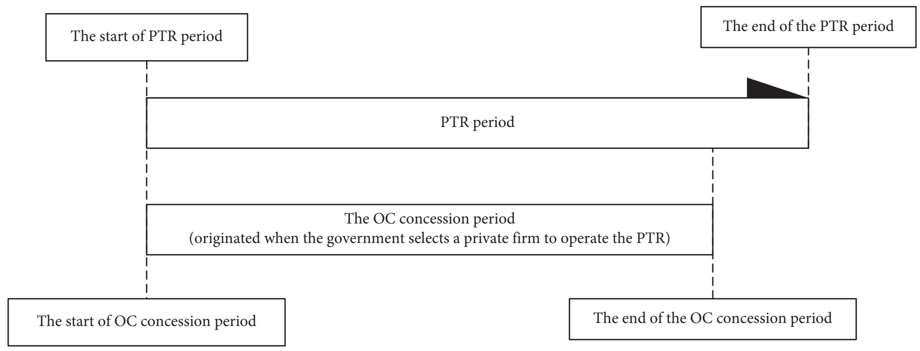

Figure 3: The relationship between OC concession period and PTR period.

density function that supports $\Theta = (\beta_0, \beta^0) \subset (0, +\infty)$. In this study, we further examine the case of the separating equilibrium [8].

In reality, the VOT varies with different PTR users. We assume that the distribution of all road users' VOT can be expressed by a continuously decreasing function $\beta(x)$. Here, $\beta(x)$ represents the VOT of the x^{th} user. Let $F(\beta)$ be the cumulative distribution function of VOT by $\beta(x) = F^{-1}(1 - x/Q)$ [6, 9].

β_c is defined as the VOT of the marginal user using PTR. It is assumed that users of PTR with a VOT higher than or equal to β_c decide to choose PTR for their travel, and the other users whose VOT is less than β_c decide to select the toll-free road. Users will choose a toll-free road if the saved VOT is less than the toll charge. Hereafter, we use β to substitute the cutoff value β_c.

Using the notation VOT and $F(\beta)$, we determine the relationship between the travel demand q of PTR and the cutoff VOT β as follows:

$$q = Q \cdot [1 - F(\beta)] = Q \cdot \overline{F}(\beta), \tag{1}$$

where $\overline{F}(\beta)$ is the probability of road users using PTR. With (1), demand q has a one-to-one correspondence with the cutoff VOT β. The users of PTR with more VOT tend to pay the toll to save time and generate more profit. It is assumed that the toll amount is a function of demand q and road capacity y:

$$p = \beta \cdot [t_e - t(q, y)]. \tag{2}$$

With (1) and (2), we can determine variables p and y by variables β and capacity y. Hereafter, we use β and y as independent variables substituting toll p and capacity y, respectively. In the following section of the paper, the toll p is assigned a subscript s when the private firm operation is analyzed and a subscript g when the government operation is analyzed.

To examine the case with a tractable analytical framework, two essential concepts—the mean residual VOT function and failure rate function—are proposed for reliability. If q users with a cutoff VOT β choose PTR, the probability of the following user not choosing the road is defined as the failure rate. Mathematically,

$$h_\beta \equiv \frac{f(\beta)}{\overline{F}(\beta)}. \tag{3}$$

If we assume that users choosing PTR have a VOT higher than the cutoff VOT β, the mean residual VOT will equal the result of the average excess VOT for the users subtracting the cutoff VOT β; namely,

$$l(\beta) = [v \mid v \geq \beta] - \beta, \tag{4}$$

where $E[\cdot]$ denotes the expectation value. It is well known that among the distribution functions $F(\beta)$, mean residual VOT function $l(\beta)$, or failure rate function $h(\beta)$, any one can be determined by the other two functions [28]. Bryson and Siddiqui [29] established the relationship between the mean residual VOT and failure rate as follows:

$$h(\beta) \cdot l(\beta) = l'(\beta) + 1. \tag{5}$$

With the mean residual VOT function and survival probability, the toll revenue per unit time $R(\beta, y)$ can be calculated as follows:

$$R(\beta, y) = q \cdot p = Q \cdot \overline{F}(\beta) \cdot p = Q \cdot \overline{F}(\beta) \cdot \beta \cdot [t_e - t(q, y)]. \tag{6}$$

In the second paragraph of this section, we point out that the PTR period T is not considered in this study. Thus, the total social surplus $S(\beta, y)$ is generated by the saved time, the value of which is measured in monetary terms. Hence, based on (4),

$$\begin{aligned} S(\beta, y) &= Q \cdot \overline{F}(\beta) \cdot [t_e - t(q, y)] \cdot E\left(\frac{v}{v} \geq \beta\right) - R(\beta, y) \\ &= Q \cdot \overline{F}(\beta) \cdot [t_e - t(q, y)] \cdot [l(\beta) + \beta] - R(\beta, y). \end{aligned} \tag{7}$$

In line with Shi et al. [30], we assume that the OC cost is composed of two parts: demand-related OC and capacity-related OC. Demand-related OC is related to road traffic demand, and capacity-related OC is related to road capacity. Let $m_s q, m_g q, I(y)$ be the perfect information for private firms and the government. m_s, m_g are the demand-related OC of private firms and government per unit travel time, respectively. Let $M_s(q)$ and $M_g(q)$ be private and governmental OC during the PTR period, respectively. Note that $I(y)$ is a function of PTR capacity, so we substitute "capacity-related OC" with "the sum of maintenance and construction costs" when using $I(y)$ in the following section for convenience.

There would be three objective functions with heterogeneous users with the above notations: private profit objective function, social welfare objective functions with government operations, and social welfare objective function with private firm operations. To obtain satisfactory results, we can choose a combination of (β, y) to maximize the above objective functions or optimize private profit and social welfare simultaneously (Pareto-optimal). These functions are expressed as follows:

Supposing that the government tends to earn a profit, the profit equals the result of the total revenue from which the sum of demand-related OC and capacity-related OC during the PTR period is subtracted:

$$P_g(\beta, y) = R(\beta, y) - M_g(q) - I(y). \tag{8}$$

The private profit equals the total revenue from which the sum of demand-related OC and capacity-related OC during the PTR period is subtracted:

$$P_s(\beta, y) = R(\beta, y) - M_s(q) - I(y). \tag{9}$$

The total social welfare of PTR with government operation, $W_g(\beta, y)$, equals the sum of the total consumer surplus and the government profit during the PTR period:

$$W_g(\beta, y) = S(\beta, y) + P_g(\beta, y) = Q \cdot \overline{F}(\beta) \cdot \left[t_e - t(q, y)\right] \cdot [l(\beta) + \beta] - M_g(q) - I(y). \tag{10}$$

The total social welfare of PTR with private firm operation, $W_s(\beta, y)$, equals the sum of the total consumer surplus and the private firm profit during the PTR period:

$$W_s(\beta, y) = S(\beta, y) + P_s(\beta, y) = Q \cdot \overline{F}(\beta) \cdot \left[t_e - t(q, y)\right] \cdot [l(\beta) + \beta] - M_s(q) - I(y). \tag{11}$$

When a private firm operates the PTR, the Pareto-optimal OC objective function for heterogeneous users can be determined by capacity y and the cutoff VOT β to meet the requirements of the government and private firms simultaneously. We then formulate a biobjective programming function as follows:

$$\max_{(T,\beta,r)\in\Omega} \begin{pmatrix} W_s(\beta, y) \\ P_s(\beta, y) \end{pmatrix}, \tag{12}$$

where $\Omega = \{(\beta, r)\colon\ y > y_1, \beta \in \Omega\}$ and capacity y are higher than y_1, and y_1 is the road capacity when the road is transferred from a private firm. Private profit $P_s(\beta, y)$ and social welfare $W_s(\beta, y)$ are defined in (9) and (11), respectively. To gain further insight, we define the following Pareto-optimal OC contract.

Definition 1 (a Pareto-optimal OC contract). An OC combination $(\beta^*, y^*) \in \Omega$ is called a Pareto-optimal contract if there is no other triple combination $(\beta, y) \in \Omega$ such that $P(\beta, y) \geq P(\beta^*, y^*)$ and $W(\beta, y) \geq W(\beta^*, y^*)$ have at least one strict inequality.

With the Pareto concept and above Pareto-optimal contract definition, we know that the improvement of the profit of one participant would make the other worse off; namely, neither participant can improve their profit simultaneously. In the following sections, we examine the impact of heterogeneous users on social welfare with governmental operations and investigate two extreme cases with private firm operations, the Pareto-optimal contract, and various regulatory regimes.

Note that although the discounting rate has an important effect on the optimal contract for private profit and social welfare, it does not affect the above results because both social welfare and private profit are invariant with calendar time in this research (for more details, see [1, 10]).

4. Analysis of Optimal OC Contracts

In this section, we first examine the properties of governmental OC using (10) and then investigate how to maximize social welfare and private profit under the PTR period with the first-order conditions of the two extreme cases. We then examine the properties of Pareto-efficient contracts of the OC problem in (12). To analyze the problem reasonably, we introduce three common assumptions, which form the analytical basis for the following subsections.

Assumption 1. Term $q\beta(q)$ is the concave function of q; $\beta h(\beta)$ increases with β.

Assumption 2. The travel time function $t(q, y)$ is homogeneous of degree zero in flow q and road capacity y; that is, $t(\alpha q, \alpha y) = t(q, y)$ for any $\alpha > 0$.

We set $r = q/y$ as the volume-to-capacity ratio in the following subsections to analyze the OC problem thoroughly. Wang et al. [31] studied the fundamental properties of the volume-to-capacity ratio of BOT roads in general networks. To simplify the case for better understanding,

$t(q, y)$ can be written as $t(r)$, which is an increasing and strictly convex continuous function of v/c ratio r.

Assumption 3. The return to scale in demand-related OC of private firms and governments is constant; namely, $M_s(q) = m_s q$ and $M_g(q) = m_g q$; capacity-related OC is also a constant return to scale; namely $I(y) = ky$, where m_s, m_g represent the constant costs per demand-related unit, and k represents the constant cost per capacity-related unit.

Based on the assumptions mentioned above, equations (9)–(11) can be converted into

$$P(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{\beta \cdot \left[t_e - t(r)\right] - m_s - \frac{k}{r}\right\}, \tag{13}$$

$$W_g(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{\left[l(\beta) + \beta\right] \cdot \left[t_e - t(r)\right] - m_g - \frac{k}{r}\right\}, \tag{14}$$

$$W_s(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{\left[l(\beta) + \beta\right] \cdot \left[t_e - t(r)\right] - m_s - \frac{k}{r}\right\}. \tag{15}$$

Note that with the definition of $r = q/y$, it is clear that a feasible solution (β, r) of (13)–(15) correspond one-to-one to (β, y).

4.1. Toll and Capacity of the PTR with Governmental Operation. If m_g is less than m_s, the government tends to operate the road independently, and for the PTR period, social welfare is maximized with a determined combination of optimal price and capacity [3]. Equation (14) is the objective function in this case.

Let $(\widetilde{\beta}_g, \widetilde{r}_g)$ be the solution to maximize social welfare $W_g(\beta, r)$ when the government operates the PTR. With the incorporation of $(\widetilde{\beta}_g, \widetilde{r}_g)$ as the corresponding solution of $W_g(\beta, r)$, the demand $\widetilde{q}_g$ corresponds to the cutoff VOT $\widetilde{\beta}_g$ with (1). We then deduce the following first-order optimal condition with $(\widetilde{\beta}_g, \widetilde{r}_g)$,

$$\widetilde{\beta}_g = \frac{\widetilde{r}_g \cdot m_g + k}{\widetilde{r}_g \cdot \left[t_e - t\left(\widetilde{r}_g\right)\right]}, \tag{16}$$

$$l\left(\widetilde{\beta}_g\right) + \widetilde{\beta}_g = \frac{k}{\widetilde{r}_g^2 t'\left(\widetilde{r}_g\right)}, \tag{17}$$

where $\widetilde{r}_g$ denotes the social optimal v/c ratio (or service quality) when the government operates the road; namely $\widetilde{r}_g = \widetilde{q}_g/\widetilde{y}_g$.

With equations (2) and (16), the social optimal toll charge $\widetilde{p}_g$ when the government operates the road can be calculated as follows:

$$\widetilde{p}_g = \widetilde{\beta}_g \cdot \left[t_e - t\left(\widetilde{r}_g\right)\right] = m_g + \frac{k}{\widetilde{r}_g} = m_g + \frac{k \cdot \widetilde{y}_g}{\widetilde{q}_g}. \tag{18}$$

The government can operate the PTR with the toll charge in (18) to achieve optimal social welfare: (18) indicates that the social optimal toll charge is precisely equal to the sum of the OC per demand-related unit and the capacity-related OC for each trip. According to (18), the government does not profit from the PTR if the toll is charged. The total revenue generated from charging users' tolls equals the sum of demand-related OC and capacity-related OC, obeying the self-financing theory [32–34]. Under the first-order condition (17), the social optimal toll charge can also be expressed as

$$\widetilde{p}_g = m_g + \frac{k}{\widetilde{r}_g} = m_g + \left[l\left(\widetilde{\beta}_g\right) + \widetilde{\beta}_g\right] \cdot \widetilde{r}_g \cdot t'\left(\widetilde{r}_g\right) = m_g + E\left[x | x \geq \widetilde{\beta}_g\right] \cdot \widetilde{r}_g \cdot t'\left(\widetilde{r}_g\right), \tag{19}$$

where $E[x|x \geq \widetilde{\beta}_g]$ is the mathematical expectation of VOT when it is higher than $\widetilde{\beta}_g$, which is also referred to as the average VOT of the actual PTR users. Equations (18) and (19) follow the first-best road capacity and pricing rules. Namely, the social optimal toll charge equals the sum of the congestion externality and demand-related OC [34].

The government would like the revenue $R(\beta, r)$ collected from road users to cover the sum of demand-related OC $M_g(q)$ and capacity-related cost $I(y)$, so it is necessary to analyze maximized social welfare without operational profit. Therefore, the issue can be described as follows:

$$W_g(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{\left[l(\beta) + \beta\right] \cdot \left[t_e - t(r)\right] - m_g - \frac{k}{r}\right\}, \tag{20}$$

subject to

$$P_g(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{\beta \cdot \left[t_e - t(r)\right] - m_g - \frac{k}{r}\right\} = 0, \tag{21}$$

where the profit $P_g(\beta, r)$ made by the government is zero. Referring to the above optimization, we notice that the optimization approach suggested for the private firm when operating the BOT road is suitable for government operation during the PTR period as well. However, the sufficient conditions of this optimization issue are changed.

We assume that (β_g^*, y_g^*) is a zero-profit optimal solution to the maximization function (14) and that (β_g^*, y_g^*) denotes the corresponding zero-profit solution to (21). Using the Lagrange method, the constraint problem above can be transformed into the following Lagrange problem:

$$L_g(\beta, r, \eta) = W_g(\beta, r) + \eta_g P_g(\beta, r), \tag{22}$$

where $\eta_g \geq 0$ denotes the Lagrange multiplier. Therefore, we obtain the following first-order condition:

$$\begin{aligned}\frac{\partial L_g}{\partial \beta}\Big|_{\beta_g^*, r_g^*} &= Q \cdot \overline{F}\left(\beta_g^*\right)\left[-h\left(\beta_g^*\right)\left(\left(l\left(\beta_g^*\right)+\beta_g^*\right)\left(t_e - t\left(r_g^*\right)\right) - m_g - \frac{k}{r_g^*}\right) + \left(l\prime\left(\beta_g^*\right)+1\right)\left(t_e - t\left(r_g^*\right)\right)\right] \\ &\quad + \eta_g \cdot Q\overline{F}\left(\beta_g^*\right)\left[-h\left(\beta_g^*\right)\left(\beta_g^*\left(t_e - t\left(r_g^*\right)\right) - m_g - \frac{k}{r_g^*}\right) + \left(t_e - t\left(r_g^*\right)\right)\right],\end{aligned} \tag{23}$$

$$\frac{\partial L_g}{\partial r}\Big|_{\beta_g^*, r_g^*} = Q \cdot \overline{F}\left(\beta_g^*\right)\left\{\left[l\left(\beta_g^*\right)+\beta_g^*\right]\left[-t'\left(r_g^*\right)\right] + \frac{k}{\left(r_g^*\right)^2}\right\} + \eta_g \cdot Q\overline{F}\left(\beta_g^*\right)\left\{\beta_g^*\left[-t'\left(r_g^*\right)\right] + \frac{k}{\left(r_g^*\right)^2}\right\} = 0, \tag{24}$$

$$\frac{\partial L_g}{\partial \eta_g}\Big|_{\beta_g^*, r_g^*} = P_g\left(\beta_g^*, r_g^*\right) = Q \cdot \overline{F}\left(\beta_g^*\right) \cdot \left\{\beta_g^* \cdot \left[t_e - t\left(r_g^*\right)\right] - m_g - \frac{k}{r_g^*}\right\} = 0. \tag{25}$$

From equation (5), equations (24) and (25) can be rewritten as

$$\beta_g^* \cdot \left[1 - \frac{\eta_g}{1+\eta_g} \cdot \frac{1}{\beta_g^* h\left(\beta_g^*\right)}\right] = \frac{m_g \cdot r_g^* + k}{r_g^*\left[t_e - t\left(r_g^*\right)\right]}, \tag{26}$$

$$\beta_g^* + \frac{l\left(\beta_g^*\right)}{1+\eta_g} = \frac{k}{\left(r_g^*\right)^2 t'\left(r_g^*\right)}. \tag{27}$$

Eliminating Lagrange multiplier η_g from (26) and (27), we obtain

$$\frac{1}{h\left(\beta_g^*\right) \cdot l\left(\beta_g^*\right)} \cdot \frac{k}{r_g^* t'\left(r_g^*\right)} - \frac{m_g \cdot r_g^* + k}{t_e - t\left(r_g^*\right)} = r_g^* \beta_g^*\left[\frac{1}{\beta_g^* h\left(\beta_g^*\right)} + \frac{1}{h\left(\beta_g^*\right) \cdot l\left(\beta_g^*\right)} - 1\right], \tag{28}$$

which determines the relationship between the optimal v/c ratio r_g^* and the cutoff VOT β_g^* from the zero-profit solution (β_g^*, r_g^*). We regard r_g^* as a function of β_g^*, and the function $r_g^* = r_g^*(\beta_g^*)$ is continuous and differentiable. Applying the derivation rule of the implicit function, we can derive r_g^* with respect to β_g^*:

$$\begin{aligned}&\left\{\frac{d}{dr}\left(\frac{k}{r_g^* t'\left(r_g^*\right)}\right) \cdot \frac{1}{h\left(\beta_g^*\right) l\left(\beta_g^*\right)} + \beta_g^* - \frac{d}{dr}\left(\frac{m_s \cdot r_g^* + k}{t_e - t\left(r_g^*\right)}\right) - \left[\frac{\beta_g^*}{h\left(\beta_g^*\right) l\left(\beta_g^*\right)} + \frac{1}{h\left(\beta_g^*\right)}\right]\right\} \frac{\mathrm{d}r}{\mathrm{d}\beta} \\ &= -r_g^* \cdot \frac{\eta_g}{1+\eta_g} \cdot \frac{l''\left(\beta_g^*\right)}{h^2\left(\beta_g^*\right) l\left(\beta_g^*\right)}.\end{aligned} \tag{29}$$

With (26) and (27), we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{k}{r_g^* t'\left(r_g^*\right)}\right) \cdot \frac{1}{h\left(\beta_g^*\right) l\left(\beta_g^*\right)} + \beta_g^* - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{m_s \cdot r_g^* + k}{t_e - t\left(r_g^*\right)}\right) - \left[\frac{\beta_g^*}{h\left(\beta_g^*\right) l\left(\beta_g^*\right)} + \frac{1}{h\left(\beta_g^*\right)}\right] = 0. \tag{30}$$

Then, if $r_g^* \eta_g / [(1+\eta_g) \cdot h^2(\beta_g^*) m(\beta_g^*)] > 0$, we have $l''(\beta_g^*) = 0$. Based on the analysis above, we posit Proposition 1.

Proposition 1. *With Assumptions 1–3, for any optimal solution to social welfare maximization (14) under government operation, the mean residual VOT function $l(\beta)$ is a linear function of β that is unrelated to r_g and β_g.*

Proposition 1 shows that regardless of the relationship between r_g and β_g, $l(\beta)$ is always a linear function of β_g with the optimal solution.

4.2. Two Extreme Cases under Private Firm Operation. In this section, we analyze the case in which the road is operated by a private firm and two important extreme cases, either the optimization of social welfare (SO) or the optimization of the private profit made by a private firm (OP). Let $(\tilde{\beta}, \tilde{r})$ and $(\bar{\beta}, \bar{r})$ be the solutions to SO and OP, respectively, which maximize social welfare $W_s(\beta, r)$ and private profit $P_s(\beta, r)$. The demand $\tilde{q}$ corresponds to the $\tilde{\beta}$ in (1), and the demand $\bar{q}$ also corresponds to $\bar{\beta}$ with (1). Note that from the notation in Section 3 and Assumptions 1–3, the solutions to SO and MO, that is, $(\tilde{\beta}, \tilde{r})$ and $(\bar{\beta}, \bar{r})$, are globally unique extreme values of function W and profit function P. We derive the following first-order conditions: $(\tilde{\beta}, \tilde{r})$ and $(\bar{\beta}, \bar{r})$.

$$\tilde{\beta} = \frac{\tilde{r} \cdot m_s + k}{\tilde{r} \cdot [t_e - t(\tilde{r})]}, \tag{31}$$

$$l(\tilde{\beta}) + \tilde{\beta} = \frac{k}{\tilde{r}^2 t'(\tilde{r})}, \tag{32}$$

$$\bar{\beta} - \frac{1}{h(\bar{\beta})} = \frac{\bar{r} \cdot m_s + k}{\bar{r} \cdot [t_e - t(\bar{r})]}, \tag{33}$$

$$\bar{\beta} = \frac{\theta}{\bar{r}^2 t'(\bar{r})}. \tag{34}$$

Based on the first-order condition in (31), the toll charge of the PTR can be converted to

$$\begin{aligned} \tilde{p} &= \tilde{\beta} \cdot [t_e - t(\tilde{r})] \\ &= m_s + \frac{k}{\tilde{r}} \\ &= m_s + \frac{k \cdot \tilde{y}}{\tilde{q}}. \end{aligned} \tag{35}$$

Equation (35) implies that the toll charge of the SO is equal to the OC per demand-related unit and capacity-related OC for each trip. The private firm does not profit from its operation. The total revenue achieved is exactly equal to the sum of OC, which obeys the self-financing theory [33]. With the first-order condition in (32), the toll charge of the SO can be expressed as

$$\begin{aligned} \tilde{p} &= m_s + \frac{k}{\tilde{r}} \\ &= m_s + [l(\tilde{\beta}) + \tilde{\beta}] \cdot \tilde{r} \cdot t'(\tilde{r}) \\ &= m_s + E[x | x \geq \tilde{\beta}] \cdot \tilde{r} \cdot t'(\tilde{r}), \end{aligned} \tag{36}$$

where $E[x | x \geq \tilde{\beta}]$, the average VOT of the actual PTR users, is the mathematical expectation of VOT when it is greater than $\tilde{\beta}$. Equations (35) and (36), similar to those in Section 4.1, also obey the first-best road capacity and pricing rules.

With first-order conditions (33) and (34), the toll charge of MO can be formulated as

$$\bar{p} = m_s + \frac{k}{\bar{r}} + \frac{[t_e - t(\bar{r})]}{h(\bar{\beta})} = m_s + \bar{\beta} \bar{r} t'(\bar{r}) + \frac{[t_e - t(\bar{r})]}{h(\bar{\beta})}. \tag{37}$$

The toll charge of the MO comprises three parts. The first term on the left-hand side of (37) is the demand-related OC. The second term represents the congestion charge, which equals the marginal external cost.

Although there is a difference between the solutions of MO and SO, based on Assumptions 1–3, we can also deduce a close relationship between them.

Proposition 2. *Under Assumptions 1–3, $\tilde{y} \geq \bar{y} \geq y_1$, $\tilde{p} \leq \bar{p}$ and $\tilde{\beta} \leq \bar{\beta}$, $\tilde{q} \geq \bar{q}$.*

Proof. $\tilde{p} \leq \bar{p}$, $\tilde{\beta} \leq \bar{\beta}$, and $\tilde{q} \geq \bar{q}$ are demonstrated by Tan and Yang [1] and applied in our model.

We only provide the proofing of $\tilde{y} \geq \bar{y} \geq y_1$.

$$W(\tilde{\beta}, \tilde{r}) = k\tilde{y} \cdot \left[\frac{t_e - t(\tilde{r})}{\tilde{r} t'(\tilde{r})} - \frac{m_s}{k} \cdot \tilde{r} - 1 \right], \tag{38}$$

$$W(\bar{\beta}, \bar{r}) = k\bar{y} \cdot \left\{ \frac{t_e - t(\bar{r})}{\bar{r} t'(\bar{r})} \left[\frac{l(\bar{\beta})}{\bar{\beta}} + 1 \right] - \frac{m_s}{k} \cdot \bar{r} - 1 \right\}. \tag{39}$$

The term $[t_e - t(r)]/rt'(r)$ is a decreasing function of r for $r \leq \hat{r}$ and $l(\bar{\beta})/\bar{\beta} \geq 0$. Then,

$$\frac{t_e - t(\tilde{r})}{\tilde{r} t'(\tilde{r})} - \frac{m_s}{k} \cdot \tilde{r} - 1 \leq \frac{t_e - t(\bar{r})}{\bar{r} t'(\bar{r})} - \frac{m_s}{k} \cdot \bar{r} - 1 \leq \frac{t_e - t(\bar{r})}{\bar{r} t'(\bar{r})} \left[\frac{l(\bar{\beta})}{\bar{\beta}} + 1 \right] - \frac{m_s}{k} \cdot \bar{r} - 1. \tag{40}$$

This implies that the capacity $\tilde{y}$ of SO is higher than the capacity $\overline{y}$ of MO if $W(\tilde{\beta},\tilde{r}) \geq W(\overline{\beta},\overline{r})$. The government must maintain the existing road capacity not less than y_1, which is also required for private firms. This completes this proof.

Proposition 2 implies that private firms tend to determine a higher toll charge, lower road capacity, and not offer traffic services to unnecessarily more road users (users with a higher cutoff VOT β), which results in road users' demand for traffic being lower. □

4.3. Properties of Pareto-Efficient OC Contracts. According to (13) and (15), the Pareto-optimal problem can be defined as

$$\max_{(\beta,r)\in\Omega} \begin{pmatrix} W_s(\beta,r) \\ P(\beta,r) \end{pmatrix} = \begin{pmatrix} Q\cdot\overline{F}(\beta)\cdot\left\{[l(\beta)+\beta]\cdot[t_e - t(r)] - m_s - \dfrac{k}{r}\right\} \\ Q\cdot\overline{F}(\beta)\cdot\left\{\beta\cdot[t_e - t(r)] - m_s - \dfrac{k}{r}\right\} \end{pmatrix}, \tag{41}$$

where $\Omega = \{(\beta,r)\colon \beta\in\Omega, r>0\}$. The relationship between the solution and (41), and the combination of the VOT and road capacity (β, y) is a one-to-one correspondence. With the definition of the Pareto-optimal OC contract in Section 3, we know that if a solution pair $(\beta,r)\in\widetilde{\Omega}$, such as $P(\beta,r)\geq P(\beta^*,r^*)$ and $W(\beta,r)\geq W(\beta^*,r^*)$ with at least one strict inequality, is not found, a pair $(\beta^*,r^*)\in\widetilde{\Omega}$ of OC function (41) can be called a Pareto-optimal solution.

Under Assumptions 1–3, we obtain the upper bound of the *v*/*c* ratio in the Pareto-optimal function in (41) using the same terms in $r\cdot[t_e - t(r)]$, as described by Tan and Yang [1] for a similar reason (see Appendix A for further details).

Proposition 3. *Under Assumptions 1–3, if (β^*,r^*) is a Pareto-optimal solution to (41), then $r^*\leq\hat{r}$, where*

$$\hat{r} = \arg\max_{r\geq 0}\{r\cdot[t_e - t(r)]\}. \tag{42}$$

From the optimal condition (42), we have

$$t_e - t(\hat{r}) = \hat{r}t'(\hat{r}). \tag{43}$$

The term $t_e - t(\hat{r})$ of (43) is the travel time saved if travelers choose to use the PTR. The term $\hat{r}t'(\hat{r})$ of (43), which can be written as $q\cdot\partial t(q,y)/\partial q$, is the congestion externality. Proposition 3 implies that none of the Pareto-optimal *v*/*c* ratios r can exceed a critical ratio $\hat{r}$, in which the saved travel time is precisely equal to its congestion externality per unit time.

Additionally, the same term $Q\cdot\overline{F}(\beta)$ of social welfare $W_s(\beta,r)$ and private profit $P_s(\beta,r)$ in (41) denotes the total number of cars using the PTR during its whole life. The term $\beta\cdot[t_e - t(r)]$ of private profit at $r=\hat{r}$ is the average profit made from every single trip of a car.

With (43), we have

$$\beta[t_e - t(\hat{r})] - \left(m_s + \frac{k}{r}\right) = \beta q\frac{\partial t(q,y)}{\partial q} - \left(m_s + \frac{ky}{q}\right), \tag{44}$$

where the last term $m_s + k\cdot y/q = (m_s q + ky)/q$ is the sum of the demand-related OC and capacity-related OC of the private firm for each car trip. The term $(l(\beta)+\beta)\cdot(t_e - t(r))$ in (41) at $r=\hat{r}$ is the average social surplus on each trip, which is equal to the sum of the average private firm surplus and the average consumer surplus. Then, the average social surplus on each trip can be written as

$$[l(\beta)+\beta][t_e - t(\hat{r})] - \left(m_s + \frac{k}{r}\right) = [l(\beta)+\beta]q\frac{\partial t(q,y)}{\partial q} - \left(m_s + \frac{k\cdot y}{q}\right). \tag{45}$$

Therefore, (45) and (44) imply that the average social surplus equals the difference between the cost-based congestion externality and the OC per trip. From the proof of Proposition 3 in Appendix A, we know that the private firm obtains a negative profit, while profit and social welfare strictly decrease with r and in the domain $(\hat{r},+\infty)$ for any cutoff VOT β. Therefore, we restrict our research to the domain of $(\hat{r},+\infty)$.

Under Assumptions 1–3, especially if $qB(q)$ is concave, the revenue function $R(q,y)$ is also concave, social welfare W is a strictly concave function of demand q and capacity y, and private profit P is a unimodal function of demand q for any given y and strictly concave function of y. We also assume that for the Pareto-optimal problem (41), the Pareto-optimal outcome corresponds one-to-one to the Pareto-optimal solution. We then apply the ε-constraint method [35] to solve (41). Miettinen [35] pointed out that one of the objective functions can be selected for optimization, and the other can be transformed into a constraint condition by setting a lower bound. Using this method, we transform Pareto-optimal problem (41) into the following problem:

$$\max_{(\beta,r)} W(\beta,r), \tag{46}$$

subject to

$$P_s(\beta,r) = Q\cdot\overline{F}(\beta)\cdot\left\{\beta\cdot[t_e - t(r)] - m_s - \frac{k}{r}\right\} \geq P^*, \tag{47}$$

$$Q\cdot\overline{F}(\beta)\cdot\left[p(\beta,r) - m_s - \frac{k}{r}\right] \leq Q\cdot\overline{F}(\beta)\cdot\left(m_g - m_s\right), \tag{48}$$

$$m_g \geq m_s, q \geq 0, y \geq y_1, \tag{49}$$

where $P^* \geq 0$ is the lower bound of the profit that the private firm can accept. Profit P^* is associated with the Pareto-efficient optimal solution (β^*, r^*). Constraint condition (47) means private profit will not be higher than the OC difference between government and private firms. If private profit exceeds the difference, the government decides to operate the road itself.

Condition (48) can be written as

$$Q\overline{F}(\beta)\cdot\left[p(\beta,r) - m_g - \frac{k}{r}\right] \leq 0. \tag{50}$$

Equation (50) implies that government profit is negative or equal to zero if the government operates the road itself. In this case, a private firm will be offered a contract to operate the road, while the government will avoid the loss. This constraint is identical to the condition in $m_g > m_s$. With the constraint condition of (50), our model differs from the traditional BOT model.

The above-transformed problem is generally solved using the Kuhn–Tucker method [32]. We set η_1 and η_2 as the Lagrange multipliers of the constraints in (47) and (48), respectively. To gain further insights into the impact of heterogeneous users on Pareto-efficient OC contracts with OC, we assume the Lagrange multipliers η_1 and η_2 as follows.

Assumption 4. The Lagrange multipliers $\eta_1 \geq \eta_2$.

The Lagrange multiplier is usually defined as a sensitivity coefficient (or shallow price) in economic explanation, and it eliminates marginal profit for some additional amount of resources. So, the condition $\eta_1 \geq \eta_2$ implies that the constraint of private profit (47) is more critical than the constraint of (48); namely, the private firm is more sensitive to its profit.

We obtain first-order conditions with Pareto-efficient optimal solution (β^*, r^*) using the Kuhn–Tucker method (see Appendix B for details).

$$\beta^*\left[1 - \frac{\eta_1 - \eta_2}{1+\eta_1-\eta_2}\cdot\frac{1}{\beta^* h(\beta^*)}\right] = \frac{1}{[t_e - t(r^*)]}\left[\left(\frac{m_s + k}{r^*}\right) - \frac{\eta_2\left(m_g - m_s\right)}{1+\eta_1-\eta_2}\right], \tag{51}$$

$$\beta^* + \frac{l(\beta^*)}{1+\eta_1-\eta_2} = \frac{k}{(r^*)^2 t'(r^*)}. \tag{52}$$

From the Kuhn–Tucker condition, the Lagrange multipliers are $\eta_1 \geq 0$ and $\eta_2 \geq 0$. From the previous section, we know that constraints (47) and (48) must be satisfied. If the Lagrange multipliers are $\eta_1 = 0$ or $\eta_2 = 0$, constraints (47) and (48) will become inactive, the Lagrange multipliers η_1 and η_2 will not be equal to zero, and then $\eta_1 > 0$ and $\eta_2 > 0$. If $\eta_1 > 0$ and $\eta_2 > 0$, from the first-order conditions (B.4) to (B.7), we obtain the following equations:

$$P(\beta,r) - P^*(\beta^*,r^*) = qp - m_s q - ky - P^*(\beta^*,r^*) = 0, \tag{53}$$

$$Q\overline{F}(\beta)\cdot p(\beta,r) - m_g q - ky = 0. \tag{54}$$

From (53) and (54), we have

$$P^* = \left(m_g - m_s\right)q. \tag{55}$$

Proposition 4. *Under Assumptions 1–3, for any Pareto-optimal solution to (41), the private profit is determined by the demand-related OC of the government and private firm.*

Proposition 4 shows that for any Pareto-optimal solution to (41), the lower bound of the private firm profit P^* is equal to $(m_g - m_s)q$. The Pareto-optimal private profit is a linear function of travel demand. Private profit equals the difference between the demand-related OC of the government and private firms during the PTR period.

5. Conclusions and Policy Implications

5.1. Conclusions. This study's results reveal that the OC exercises a vital function in PTR projects, while this part of the cost is usually neglected during the BOT concession period. We further examined the effect of different VOTs of heterogeneous users on OC for the PTR, characterized by the mean residual function and the failure rate function. We classified and analyzed the optimal solutions for both contracts concerning OC for government and private firms. We also studied Pareto-optimal solutions with private operations using biobjective programming. This study defined the optimal toll price under the maximization of social welfare with government operation. We proved that the mean residual function is a linear function of value-of-time of heterogeneous users under Pareto-optimum under the Pareto-optimum with government operation. In contrast, private profit equals the difference between OC of government and private firm under Pareto-optimum with private firm operation. We also compared the PTR variables, including road capacity, toll for users, and traffic demand, in two extreme cases under private firm operation: the optimization of social welfare and the optimization of private profit. The private firm operates on the road during the entire PTR period. The lower-bound profit of the private firm on the Pareto-optimum is a linear function of demand.

5.2. Policy Implications. Based on the discussion of the above models, some policy implications can be drawn for PTR road operations. According to the regulations, road tolls are divided into debt repayment period tolls and maintenance period tolls. Road maintenance period tolls are distinguished from debt repayment period tolls because the latter do not consider construction funding. The first implication pertains to the management of a road when it becomes a PTR. In the operation and management of PTR, we assume that the government can choose to operate the road by itself or choose a private firm to continue the operation. At this time, the road tolls are the maintenance period tolls (i.e., the government does not need to consider construction funding), and the government's decision-making department mainly considers management efficiency and operational supervision, rather than the issue of construction funding, so the government will have more freedom to operate the road by itself or choose a private firm to operate it.

Second, if the government decides to operate the road independently, it can consider two goals: maximizing social welfare and self-financing. Governments generally charge road traffic to maximize social welfare. The discussion in this study provides a particular selectivity for the government to formulate relevant road toll policies. Owing to the different traffic flows of different PTR, which leads to obvious differences in the total tolls of each PTR in a certain period, the government can consider the operation of PTR from the perspective of the entire region by considering the maximization of social welfare and the tolling of PTR from a more extensive scope.

Third, it has been stated at the beginning of Section 5.2 that the government would have a greater say in choosing a private firm to continue operating the road, regardless of construction funding. The government has to consider two issues: selecting a private firm based on management efficiency [3] and supervision management. In Part 4, the Pareto model is used to analyze the process of contracting between government and private firms. The government can supervise and regulate the operation management of the private firm by formulating relevant clauses in the contract, including the price of tolls and flexible operation periods based on traffic volumes. For example, when traffic volumes increase to a certain level, the government recovers operation rights to maximize social welfare.

Fourth, according to the relationship between the heterogeneous traffic volumes and the toll prices in the model, the government can adopt differentiated toll policies based on traffic volumes and the vehicle types. For example, it may adopt different toll prices or adjust the toll prices of different vehicles during different periods.

This paper suggests practical applications that the government can consider during its decision-making process. These suggestions also can be referenced by researchers for future studies in the field of PTR operation. Our conclusions can be further applied to other construction projects to be developed in the BOT model.

The paper has some limitations. First, the charging mechanism of PTR is the focus in further research, such as how to adjust the charging mechanism in different PTR sections and different time. In addition, it is necessary to further study the operation of PTR network, such as the operation of PTR network under the maximum social welfare. We believe that further research into these issues will yield more results in the area of PTR.

Appendix

A. Proof of Proposition 3

We use the method of reduction to absurdity to prove Proposition 3; namely, if any feasible solution (β, r) with $r > \widehat{r}$ is not a Pareto-optimal solution to (41), Proposition 3 is true. From Section 4.2, we know that a social optimal solution $(\widetilde{\beta}, \widetilde{r})$ maximizes social welfare with zero profit $P(\widetilde{\beta}, \widetilde{r}) = 0$ under the government operation. Therefore, if a feasible solution (β, r) with $r > \widehat{r}$ gives rise to a negative profit, the social welfare optimal solution $(\widetilde{\beta}, \widetilde{r})$ is dominating compared to the pair (β, r) since the social welfare $W(\widetilde{\beta}, \widetilde{r}) \geq W(\beta, r)$ and the profit $P(\widetilde{\beta}, \widetilde{r}) > P(\beta, r)$. On the other hand, if a feasible solution (β, r) with $r > \widehat{r}$ gives rise to a nonnegative profit, we can also get nonnegative social welfare. We rewrite the social welfare and profit of (41) as follows.

$$W_s(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{ \frac{[m(\beta) + \beta] \cdot r \cdot [t_e - t(r)] - k}{r} - m_s \right\}, \tag{A.1}$$

$$P(\beta, r) = Q \cdot \overline{F}(\beta) \cdot \left\{ \frac{\beta \cdot r \cdot [t_e - t(r)] - k}{r} - m_s \right\}. \tag{A.2}$$

Under the condition that the single travel time function $t(r)$ is increasing and strictly convex of v/c ratio r, the same term $r \cdot [t_e - t(r)]$ of equations (A.1) and (A.2) is strictly decreasing function of r in the domain of $r > \widehat{r}$, where $\widehat{r}$ is defined in (44). Based on the above analysis, we know that social welfare $W_s(\beta, r)$ and profit $P(\beta, r)$ become strictly decreasing function of r and nonnegative in the domain of $(\widehat{r}, t + n\infty)$; namely, the value of $W_s(\beta, r)$ and $P(\beta, r)$ with $r = \widehat{r}$ dominates, compared to $W_s(\beta, r)$ and $P(\beta, r)$ with $r > \widehat{r}$. This completes this proof.

B. Proof of Properties of Pareto-Efficient Solution

It is assumed that (β^*, r^*) is any Pareto-optimal solution to the OC problem (12), and (β^*, r^*) is the Pareto-optimal solution to the OC problem (41). From the constrained programming problem (46)–(49), the following Lagrange-like function can be defined as

$$L = W(\beta, r) + \eta_1 [P(\beta, r) - P^* (\beta^*, r^*)] + \eta_2 \left[-Q\overline{F}(\beta) \cdot p(\beta, r) + M_g 0(q) + I(y)\right], \tag{B.1}$$

where $\eta_1 \geq 0$ and $\eta_2 \geq 0$ are Lagrange multipliers associated with the constraints in (47) and (48), respectively. Using (3), the following first-order conditions of Kuhn–Tucker for the optimality of the above problem hold [32].

$$\begin{aligned} \frac{\partial L}{\partial \beta}\Big|_{(\beta^*, r^*)} &= Q \cdot \overline{F}(\beta^*) \left[-h(\beta^*) \left((m(\beta^*) + \beta^*)(t_e - t(r^*)) - m_s - \frac{k}{r^*} \right) + (m'(\beta^*) + 1)(t_e - t(r^*)) \right] \\ &+ \eta_1 \cdot Q \cdot \overline{F}(\beta^*) \left[-h(\beta^*) \left(\beta^* (t_e - t(r^*)) - m_s - \frac{k}{r^*} \right) + (t_e - t(r^*)) \right] \\ &+ \eta_2 \cdot Q \cdot \overline{F}(\beta^*) \left[-h(\beta^*) \left(-\beta^* (t_e - t(r^*)) + m_g + \frac{k}{r^*} \right) - (t_e - t(r^*)) \right] = 0, \end{aligned} \tag{B.2}$$

$$\begin{aligned} \frac{\partial L}{\partial r}\Big|_{(\beta^*, r^*)} &= Q \cdot \overline{F}(\beta^*) \left\{ [m(\beta^*) + \beta^*][-t'(r^*)] + \frac{k}{(r^*)^2} \right\} \\ &+ \eta_1 \cdot Q \cdot \overline{F}(\beta^*) \left\{ \beta^* [-t'(r^*)] + \frac{k}{(r^*)^2} \right\} + \eta_2 \cdot Q \cdot \overline{F}(\beta^*) \left\{ \beta^* t'(r^*) - \frac{k}{(r^*)^2} \right\} = 0, \end{aligned} \tag{B.3}$$

$$P(\beta^*, r^*) - P^* \geq 0, \tag{B.4}$$

$$M_g(q^*) + I(y) - Q\overline{F}(\beta^*) \cdot p(\beta^*, r^*) \geq 0, \tag{B.5}$$

$$\eta_1 \cdot \frac{\partial L}{\partial \eta_1} = \eta_1 [P(\beta, r) - P^* (\beta^*, r^*)] = 0, \tag{B.6}$$

$$\eta_2 \cdot \frac{\partial L}{\partial \eta_2} = \eta_2 \left[M_g(q) + I(y) - Q\overline{F}(\beta^*) \cdot p(\beta^*, r^*)\right] = 0, \tag{B.7}$$

$$\eta_1 \geq 0, \tag{B.8}$$

$$\eta_2 \geq 0. \tag{B.9}$$

With (5), the first-order conditions (B.2) and (B.3) can be written as (51) and (52). With (53) and (54), the above equation can be transformed into the following:

$$L = W(\beta, r) + \eta_1 [P(\beta, r) - P^* (\beta^*, r^*)] + \eta_2 \left[-Q\overline{F}(\beta^*) \cdot p(\beta^*, r^*) + M_g(q) + I(y)\right], \tag{B.10}$$

s.t. equations (53) and (54).

Further analysis is detailed in Section 4.3. This completes the proof.

Data Availability

All data, models, and codes that support the findings of this study are available from the authors upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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References

[1] Z. Tan and H. Yang, "The impact of user heterogeneity on road franchising," *Transportation Research Part E: Logistics and Transportation Review*, vol. 48, no. 5, pp. 958–975, 2012.

[2] A. De Palma, M. Kilani, and R. Lindsey, "Maintenance, service quality and congestion pricing with competing roads," *Transportation Research Part B: Methodological*, vol. 41, no. 5, pp. 573–591, 2007.

[3] B. Shang, Q. Sun, H. Feng, and J. Chang, "Determining the operator for the public toll road," *Journal of Advanced Transportation*, vol. 2021, Article ID 6652278, 14 pages, 2021.

[4] Ministry of Transport of the People's Republic of China, "Toll Road Management Regulations (Revised Draft), Ministry of Transport of the People's Republic of China," Beijing, China, 2018, http://www.gov.cn/xinwen/2018-12/21/content_5350738. htm.

[5] R. Arnott and A. de Palma, "The welfare effects of congestion tolls with heterogeneous commuters," *Journal of Transport Economics and Policy*, vol. 28, no. 2, pp. 139–161, 1994.

[6] J. Mayet and M. Hansen, "Congestion pricing with continuously distributed values of time," *Journal of Transport Economics and Policy*, vol. 34, no. 3, pp. 359–370, 2000.

[7] H. Yang, W. H. Tang, W. Man Cheung, and Q. Meng, "Profitability and welfare gain of private toll roads in a network with heterogeneous users," *Transportation Research Part A: Policy and Practice*, vol. 36, no. 6, pp. 537–554, 2002.

[8] E. T. Verhoef and K. A. Small, "Product differentiation on roads: constrained congestion pricing with heterogeneous users," *Journal of Transport Economics and Policy*, vol. 38, no. 1, pp. 127–156, 2004.

[9] F. Xiao and H. Yang, "Efficiency loss of private road with continuously distributed value-of-time," *Transportmetrica*, vol. 4, no. 1, pp. 19–32, 2008.

[10] X. Guo and H. Yang, "User heterogeneity and bi-criteria system optimum," *Transportation Research Part B: Methodological*, vol. 43, no. 4, pp. 379–390, 2009.

[11] Y. Nie and Y. Liu, "Existence of self-financing and Pareto-improving congestion pricing: impact of value of time distribution," *Transportation Research Part A: Policy and Practice*, vol. 44, no. 1, pp. 39–51, 2010.

[12] A. C. Pigou, *The Economics of Welfare*, Macmillan and Company, London, UK, 1920.

[13] M. Beckmann, C. B. Mcguire, and C. B. Winsten, *Studies in the Economics of Transportation*, Yale University Press, New Haven, CT, USA, 1956.

[14] A. A. Walters, "The theory and measurement of private and social cost of highway congestion," *Econometrica*, vol. 29, no. 4, pp. 676–699, 1961.

[15] W. Vickrey, "Pricing in urban and suburban transport," *The American Economic Review*, vol. 53, no. 2, pp. 452–465, 1963.

[16] W. Vickrey, "Congestion theory and transport investment," *The American Economic Review*, vol. 59, no. 2, pp. 251–261, 1969.

[17] H. Yang and H. J. Huang, *Mathematical and Economic Theory of Road Pricing*, Elsevier, Oxford, UK, 2005.

[18] K. A. Small and E. T. Verhoef, *The Economic of Urban Transportation (Version 2)*Routledge, London, UK, 2007.

[19] B. Niu and J. Zhang, "Price, capacity and concession period decisions of Pareto-efficient BOT contracts with demand uncertainty," *Transportation Research Part E: Logistics and Transportation Review*, vol. 53, pp. 1–14, 2013.

[20] S. Shi, Q. An, and K. Chen, "Optimal choice of capacity, toll, and subsidy for build-operate-transfer roads with a paid minimum traffic guarantee," *Transportation Research Part A: Policy and Practice*, vol. 139, pp. 228–254, 2020.

[21] Z. Feng, Y. Zhang, S. Zhang, and J. Song, "Contracting and renegotiating with a loss-averse private firm in BOT road projects," *Transportation Research Part B: Methodological*, vol. 112, pp. 40–72, 2018.

[22] Y. Zhang, Z. Feng, and S. Zhang, "The effects of concession period structures on BOT road contracts," *Transportation Research Part A: Policy and Practice*, vol. 107, pp. 106–125, 2018.

[23] C. F. Daganzo, "A Pareto optimum congestion reduction scheme," *Transportation Research Part B: Methodological*, vol. 29, no. 2, pp. 139–154, 1995.

[24] Z. J. Tan, H. Yang, and X. L. Guo, "Properties of Pareto-efficient contracts and regulations for road franchising," *Transportation Research Part B: Methodological*, vol. 44, no. 4, pp. 415–433, 2010.

[25] H. Mohring, "The peak load problem with increasing returns and pricing constraints," *The American Economic Review*, vol. 60, no. 4, pp. 693–705, 1970.

[26] O. M. Rouhani, R. R. Geddes, H. O. Gao, and G. Bel, "Social welfare analysis of investment public-private partnership approaches for transportation projects," *Transportation Research Part A: Policy and Practice*, vol. 88, pp. 86–103, 2016.

[27] D. Wang, X. Wang, L. Wang, H. Liu, M. Sing, and B. Liu, "Optimisation of government subsidies in infrastructure public-private partnerships," *Journal of Engineering, Design and Technology*, 2021, In press.

[28] C. Lai and M. Xie, *Stochastic Ageing and Dependence for Reliability*, Springer, New York, USA, 2006.

[29] M. C. Bryson and M. M. Siddiqui, "Some criteria for aging," *Journal of the American Statistical Association*, vol. 64, no. 328, pp. 1472–1483, 1969.

[30] S. Shi, Y. Yin, and X. Guo, "Optimal choice of capacity, toll and government guarantee for build-operate-transfer roads under asymmetric cost information," *Transportation Research Part B: Methodological*, vol. 85, pp. 56–69, 2016.

[31] S. Wang, Q. Meng, and Z. Liu, "Fundamental properties of volume-capacity ratio of a private toll road in general networks," *Transportation Research Part B: Methodological*, vol. 47, pp. 77–86, 2013.

[32] H. Mohring and M. Harwitz, *Highway Benefits: An Analytical Framework*, Northwestern University Press, Evanston, IL, USA, 1962.
[33] G. Roth, *Roads In a Market Economy, Avebury Technical*, Ashgate Publishing Limited, Aldershot, Hants, UK, 1996.
[34] D. M. Newbery, "Cost recovery from optimally designed roads," *Economica*, vol. 56, no. 222, pp. 165–185, 1989.
[35] K. M. Mietteinen, *Nonlinear Multi-Objective Optimization*, Kluwer Academic Publishers, Norwell, MA, USA, 1999.
[36] D. A. Wismer and R. Chattergy, *Introduction to Nonlinear Optimization: A Problem Solving Approach*, North-holland Publishing Company, New York, USA, 1978.