Research Article

Sufficient Conditions for Hamiltonicity of Graphs with Respect to Wiener Index, Hyper-Wiener Index, and Harary Index

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In this paper, with respect to the Wiener index, hyper-Wiener index, and Harary index, it gives some sufficient conditions for some graphs to be traceable, Hamiltonian, Hamilton-connected, or traceable for every vertex. Firstly, we discuss balanced bipartite graphs with $\delta(G) \geq t$, where $\delta(G)$ is the minimum degree of G , and gain some sufficient conditions for the graphs to be traceable or Hamiltonian, respectively. Secondly, we discuss nearly balanced bipartite graphs with $\delta(G) \geq t$ and present some sufficient conditions for the graphs to be traceable. Thirdly, we discuss graphs with $\delta(G) \geq t$ and obtain some conditions for the graphs to be traceable or Hamiltonian, respectively. Finally, we discuss t -connected graphs and provide some conditions for the graphs to be Hamilton-connected or traceable for every vertex, respectively.

1. Introduction

We consider only a simple graph. For a graph $G = (V(G), E(G))$, we use n to denote $|V(G)|$ and $e(G)$ to $|E(G)|$. We write $d_G(x, y)$ as the minimum length of the paths between x and y in G , which is called the distance between two vertices x and y of G . Denote $\delta(G) = \min\{d_G(v), v \in V(G)\}$. For a bipartite graph $G = (X, Y; E)$, if $|X| = |Y|$, it is called a *balanced bipartite graph*; if $|X| = |Y| + 1$, it is called a *nearly balanced bipartite graph*. For disjoint graphs G_1 and G_2 , the graph $G_1 + G_2$ is called the union of G_1 and G_2 , where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$; the graph $G_1 \vee G_2$ is called the join of G_1 and G_2 , where $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1 + G_2) \cup \{xy : \forall x \in V(G_1), \forall y \in V(G_2)\}$. For example, $K_{n,m} = O_n \vee O_m$. The graph $\bar{G} = (V(\bar{G}), E(\bar{G}))$ means the *complement* of G , where $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{xy : x, y \in V(G), xy \notin E(G)\}$. The graph $\bar{G} := (X, Y; E')$, $E' = \{xy : x \in X, y \in Y, xy \notin E\}$ is called the *quasicomplement* of bipartite $G = (X, Y; E)$.

In [1], Wiener introduced the *Wiener index*, $W(G) = \sum_{v_i, v_j \in V(G)} d_G(v_i, v_j)$, for a connected graph G in 1947.

We denote $D_i(G) = \sum_{v_j \in V(G)} d_G(v_i, v_j)$; then,

$$W(G) = \frac{1}{2} \sum_{i=1}^n D_i(G). \quad (1)$$

For a connected graph G , $WW(G) = (1/2) (\sum_{v_i, v_j \in V(G)} d_G(v_i, v_j) + \sum_{v_i, v_j \in V(G)} d_G^2(v_i, v_j))$ is called its *hyper-Wiener index*, which is introduced by Klein et al. [2] in 1995.

We denote $DD_i(G) = \sum_{v_j \in V(G)} d_G^2(v_i, v_j)$; then,

$$WW(G) = \frac{1}{4} \sum_{i=1}^n (D_i(G) + DD_i(G)). \quad (2)$$

In [3, 4], Plavšić et al. and Ivanciuc et al. gave the *Harary index*, $H(G) = \sum_{v_i, v_j \in V(G)} 1/(d_G(v_i, v_j))$ for a connected graph G , independently.

We denote $\bar{D}_G(v_i) = \sum_{v_j \in V(G)} 1/(d_G(v_i, v_j))$; then,

$$H(G) = \frac{1}{2} \sum_{i=1}^n \tilde{D}_G(v_i). \quad (3)$$

For a graph G , if it contains a path (cycle) containing all vertices of G , it is *traceable* (*Hamiltonian*); if there are paths containing all vertices of G between any two vertices in G , it is called to be *Hamilton-connected*; if there are paths containing all vertices of G from any vertex in G , it is called to be *traceable from every vertex*.

Topological indices have attracted much attention in the literature, and they are widely applied in various fields of science and technology. Recently, some interesting results have been obtained, see [5–11]. Especially, some topological indices are used to characterize the Hamiltonian property of graphs. We refer readers to see [2, 12–21]. Among them, Hua and Ning [15] gave Hamiltonian conditions for a balanced bipartite graph with respect to the Wiener index and Harary index. Cai et al. [13], by the hyper-Wiener index, gave conditions for a balanced bipartite graph to be traceable and Hamiltonian and a t -connected graph to be Hamiltonian, respectively. Li [18, 19] gave Hamiltonian conditions for a t -connected graph with respect to the Wiener index and Harary index, respectively. With the Wiener index, hyper-Wiener index, and Harary index, Yu et al. [20] gave conditions for a t -connected graph to be Hamilton-connected and then got conditions for it to be traceable from every vertex, as well as gave conditions for a nearly balance bipartite graph to be traceable. Especially, Liu et al. [16, 17] studied the Hamiltonian property of graphs with respect to the Wiener index and Harary index of complement of graph or quasicomplement of bipartite graph, respectively. As a continuance of these results, we also study the similar problems.

In this paper, we discuss the hamiltonicity of graph by the Wiener index, hyper-Wiener index, and Harary index for quasicomplement or complement. In Section 2, we present some notations and some lemmas needed in the following. In Sections 3 and 4, we present some conditions for a balanced bipartite graph G with $\delta(G) \geq t$ to be traceable and Hamilton, respectively. In Section 5, we give sufficient conditions for a nearly balanced bipartite graph G with $\delta(G) \geq t$ to be traceable. In Sections 6 and 7, we present some conditions for a graph G with $\delta(G) \geq t$ to be traceable and Hamiltonian, respectively. In Sections 8 and 9, we provide some conditions for a t -connected graph to be Hamilton-connected and traceable from every vertex, respectively.

2. Preliminaries

Especially, in the following, for bipartite graphs, we always fix their partite sets. For instance, $O_{s,t}$ and $O_{t,s}$ are seen as different bipartite graphs unless $s = t$.

A graph $G_1 \sqcup G_2$ is the graph obtained from $G_1 \vee G_2$ by deleting all possible edges between X_1 and X_2 and possible edges between Y_1 and Y_2 , where $G_i(X_i, Y_i, E_i)$, ($i = 1, 2$) be bipartite graphs. Next, we denote some special classes of graphs:

$$\begin{aligned} B_n^t &= O_{t,n-t} \sqcup K_{n-t,t} \left(1 \leq t \leq \frac{n}{2} \right), \\ C_n^t &= O_{t,n-t} \sqcup K_{n-t-1,t} \left(1 \leq t \leq \frac{n}{2} \right), \\ R_n^t &= K_{t,t} + K_{n-t,n-t} \left(1 \leq t \leq \frac{n}{2} \right), \\ Q_n^t &= O_{t+1,n-t} \sqcup K_{n-t-1,t} \left(1 \leq t \leq \frac{n}{2} \right), \\ L_n^t &= K_1 \vee (K_t + K_{n-t-1}), \left(1 \leq t \leq \frac{(n-1)}{2} \right), \\ N_n^t &= K_t \vee (K_{n-2t} + tK_1), \left(1 \leq t \leq \frac{(n-1)}{2} \right), \\ \underline{L}_n^t &= K_{t+1} + K_{n-t-1}, \left(1 \leq t \leq \frac{(n-1)}{2} \right), \\ \underline{N}_n^t &= K_t \vee (K_{n-2t-1} + (t+1)K_1), \left(1 \leq t \leq \frac{(n-1)}{2} \right). \end{aligned} \quad (4)$$

Note that $e(C_n^t) = n(n-t-1) + t^2$, and C_n^t is not traceable.

Lemma 1. Denote $\hat{G}$ as a quasicomplement of bipartite G with $2n$ vertices. If $\hat{G}$ is a connected balanced bipartite graph, then

$$W(\hat{G}) \leq 2n^3 - 3n^2 + 2n + 2(n-1)e(G). \quad (5)$$

Proof. As $\hat{G}$ is a quasicomplement,

$$\begin{aligned} W(\hat{G}) &= \sum_{v_i, v_j \in V(\hat{G})} d_{\hat{G}}(v_i, v_j), \\ &= \frac{1}{2} \sum_{j=1}^{2n} D_j(\hat{G}), \\ &\leq \frac{1}{2} \left(\sum_{j=1}^{2n} (d_{\hat{G}}(x_j) + (2n-1)(n-d_{\hat{G}}(x_j))) + 2n^2 \right. \\ &\quad \left. - 4n + 2 \right), \\ &= 4n^3 - 5n^2 + 2n + (1-n) \sum_{j=1}^{2n} d_{\hat{G}}(x_j), \\ &= 4n^3 - 5n^2 + 2n + (1-n) \sum_{j=1}^{2n} (n - d_G(x_j)), \\ &= 2n^3 - 3n^2 + 2n + (n-1) \sum_{j=1}^{2n} d_G(x_j), \\ &= 2n^3 - 3n^2 + 2n + 2(n-1)e(G). \end{aligned}$$

(6)
□

Lemma 2. Let $\widehat{G}$ be a quasicomplement of bipartite G with $2n$ vertices. If $\widehat{G}$ is a connected balanced bipartite graph, then

$$WW(\widehat{G}) \leq 2n^4 - 5n^3 + 5n^2 - n + (2n^2 - n - 1)e(G). \quad (7)$$

Proof. As $\widehat{G}$ is a quasicomplement,

$$\begin{aligned} WW(\widehat{G}) &= \frac{1}{2} \sum_{v_i, v_j \in V(\widehat{G})} (d_{\widehat{G}}(v_i, v_j) + d_{\widehat{G}}^2(v_i, v_j)), \\ &= \frac{1}{4} \sum_{j=1}^{2n} (D_j(\widehat{G}) + DD_j(\widehat{G})), \\ &\leq \frac{1}{4} \sum_{j=1}^{2n} (d_{\widehat{G}}(x_j) + (2n-1)(n - d_{\widehat{G}}(x_j)) \\ &\quad + 2n^2 - 4n + 2) \\ &\quad + \frac{1}{4} \sum_{j=1}^{2n} (d_{\widehat{G}}(x_j) + (4n^2 - 4n + 1)(n - d_{\widehat{G}}(x_j)) \\ &\quad + 4(n-1)^3), \\ &= 4n^4 - 6n^3 + 4n^2 - n \\ &\quad - (n^2 - 0.5n - 0.5) \sum_{j=1}^{2n} d_{\widehat{G}}(x_j), \\ &= 4n^4 - 6n^3 + 4n^2 - n \\ &\quad - (n^2 - 0.5n - 0.5) \sum_{j=1}^{2n} (n - d_G(x_j)), \\ &= 2n^4 - 5n^3 + 5n^2 - n + (2n^2 - n - 1)e(G). \end{aligned} \quad (8)$$

□

Lemma 3. Let $\widehat{G}$ be a quasicomplement of bipartite G of order $2n$. If $\widehat{G}$ is a connected balanced bipartite graph, then

$$H(\widehat{G}) \geq \frac{4n^3 - n}{4n - 2} - \frac{2n - 2}{2n - 1} e(G). \quad (9)$$

Proof. As $\widehat{G}$ is a quasicomplement,

$$\begin{aligned} H(\widehat{G}) &= \sum_{v_i, v_j \in V(\widehat{G})} \frac{1}{d_{\widehat{G}}(v_i, v_j)}, \\ &= \frac{1}{2} \sum_{j=1}^{2n} \bar{D}_{\widehat{G}}(v_j), \\ &\geq \frac{1}{2} \sum_{j=1}^{2n} \left(d_{\widehat{G}}(x_j) + \frac{1}{2n-1} (n - d_{\widehat{G}}(x_j)) + \frac{1}{2} \right), \\ &= \frac{n(4n-1)}{2(2n-1)} + \frac{n-1}{2n-1} \sum_{j=1}^{2n} d_{\widehat{G}}(x_j), \\ &= \frac{n(4n-1)}{2(2n-1)} + \frac{n-1}{2n-1} \sum_{j=1}^{2n} (n - d_G(x_j)), \\ &= \frac{4n^3 - n}{4n-2} - \frac{n-1}{2n-1} \sum_{j=1}^{2n} d_G(x_j), \\ &= \frac{4n^3 - n}{4n-2} - \frac{2n-2}{2n-1} e(G). \end{aligned} \quad (10)$$

□

Lemma 4. Let $\overline{G}$ be a complement of G with n vertices. If $\overline{G}$ is a connected graph, then

$$W(\overline{G}) \leq \frac{1}{2} n(n-1) + (n-2)e(G). \quad (11)$$

Proof.

$$\begin{aligned} W(\overline{G}) &= \sum_{v_i, v_j \in V(\overline{G})} d_{\overline{G}}(v_i, v_j), \\ &= \frac{1}{2} \sum_{j=1}^n D_j(\overline{G}), \\ &\leq \frac{1}{2} \sum_{j=1}^n (d_{\overline{G}}(x_j) + (n-1)(n-1 - d_{\overline{G}}(x_j))), \\ &= \frac{n}{2} (n^2 - 2n + 1) + \frac{n}{2} (2-n) \sum_{j=1}^n d_{\overline{G}}(x_j), \\ &= \frac{n}{2} (n^2 - 2n + 1) - \frac{n}{2} (n-2) \sum_{j=1}^n (n-1 - d_G(x_j)), \\ &= \frac{n}{2} (n-1) + \frac{(n-2)}{2} \sum_{j=1}^n d_G(x_j), \\ &= \frac{1}{2} n(n-1) + (n-2)e(G). \end{aligned} \quad (12)$$

□

Lemma 5. Let $\overline{G}$ be a complement of G with n vertices. If $\overline{G}$ is a connected graph, then

$$WW(\overline{G}) \leq \frac{1}{2}n(n-1) + \frac{1}{2}(n^2 - n - 2)e(G). \quad (13)$$

Proof

$$\begin{aligned} WW(\overline{G}) &= \frac{1}{2} \sum_{v_i, v_j \in V(\overline{G})} (d_{\overline{G}}(v_i, v_j) + d_{\overline{G}}^2(v_i, v_j)), \\ &= \frac{1}{4} \sum_{j=1}^n (D_j(\overline{G}) + DD_j(\overline{G})), \\ &\leq \frac{1}{4} \sum_{j=1}^n (d_{\overline{G}}(x_j) + (n-1)(n-1-d_{\overline{G}}(x_j))), \\ &\quad + \frac{1}{4} \sum_{j=1}^n (d_{\overline{G}}(x_j) + (n-1)^2(n-1-d_{\overline{G}}(x_j))) \\ &= \frac{n^2}{4}(n^2 - 2n + 1) + \frac{(2+n-n^2)}{4} \sum_{j=1}^n d_{\overline{G}}(x_j), \\ &= \frac{n^2}{4}(n^2 - 2n + 1) + \frac{(2+n-n^2)}{4} \sum_{j=1}^n (n-1-d_G(x_j)), \\ &= \frac{n}{2}(n-1) + \frac{(n^2-n-2)}{4} \sum_{j=1}^n d_G(x_j), \\ &= \frac{1}{2}n(n-1) + \frac{1}{2}(n^2-n-2)e(G). \end{aligned} \quad (14)$$

Lemma 6. Let $\overline{G}$ be a complement of G with n vertices. If $\overline{G}$ is a connected graph, then

$$H(\overline{G}) \geq \frac{1}{2}(n^2 - n) - \frac{n-2}{n-1}e(G). \quad (15)$$

Proof

$$\begin{aligned} H(\overline{G}) &= \sum_{v_i, v_j \in V(\overline{G})} \frac{1}{d_{\overline{G}}(v_i, v_j)}, \\ &= \frac{1}{2} \sum_{j=1}^n \overline{D}_{\overline{G}}(v_j), \\ &\geq \frac{1}{2} \sum_{j=1}^n \left(d_{\overline{G}}(x_j) + \frac{1}{n-1}(n-1-d_{\overline{G}}(x_j)) \right), \\ &= \frac{n}{2} + \frac{n-2}{2(n-1)} \sum_{j=1}^n d_{\overline{G}}(x_j), \\ &= \frac{n^2-n}{2} - \frac{n-2}{2(n-1)} \sum_{j=1}^n d_G(x_j), \\ &= \frac{1}{2}(n^2 - n) - \frac{n-2}{n-1}e(G). \end{aligned} \quad (16)$$

3. Traceable of Balanced Bipartite Graphs

Lemma 7 (see [22]). Let G be a balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$e(G) > n^2 - nt - 2n + t^2 + 4t + 4, \quad (17)$$

for some integer t , then G is traceable or $G \subseteq Q_n^t$ or $t = 1$, $G \subseteq R_n^1$.

Theorem 1. Let $\widehat{G}$ be a connected balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$W(\widehat{G}) > 4n^3 - (2t+9)n^2 + (2t^2 + 10t + 14)n - 2(t+2)^2, \quad (18)$$

for some integer t , then G is traceable or $t = 1$ and $G \subseteq R_n^1$.

Proof. Since $W(\widehat{G}) > 4n^3 - (2t+9)n^2 + (2t^2 + 10t + 14)n - 2(t+2)^2$, by Lemma 1, we get $e(G) > n^2 - nt - 2n + t^2 + 4t + 4$. By Lemma 7, we obtain that G is traceable or $G \subseteq Q_n^t$ or $t = 1$, $G \subseteq R_n^1$.

If $G \subseteq Q_n^t$: because $\widehat{G}$ is connected and $\delta(G) \geq t \geq 1$, we get $e(G) \leq n^2 - nt - n + t^2 + t - (n-t-1) - t < n^2 - nt - 2n + t^2 + 4t + 4$. It is in contradiction with the above.

If $t = 1$: because $\widehat{G}$ is connected, $G \subseteq R_n^1$. $\square$

Theorem 2. Let $\widehat{G}$ be a connected balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$WW(\widehat{G}) > 4n^4 - (2t+10)n^3 + (2t^2 + 9t + 14)n^2 - (t^2 + 3t + 3)n - (t+2)^2, \quad (19)$$

for some integer t , then G is traceable or $t = 1$, $G \subseteq R_n^1$.

Proof. Since $WW(\widehat{G}) > 4n^4 - (2t+10)n^3 + (2t^2 + 9t + 14)n^2 - (t^2 + 3t + 3)n - (t+2)^2$, by Lemma 2, $e(G) > n^2 - nt - 2n + t^2 + 4t + 4$. By Lemma 7, G is traceable or $G \subseteq Q_n^t$ or $t = 1$, $G \subseteq R_n^1$. By the same discussion as the proof of Theorem 1, the conclusion is established. $\square$

Theorem 3. Let G be a connected balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$H(\widehat{G}) < \frac{(4t+12)n^2 - (4t^2 + 20t - 25)n + 4t^2 + 16t + 16}{4n-2}, \quad (20)$$

for some integer t , then G is traceable or $t = 1$, $G \subseteq R_n^1$.

Proof. Since $H(\widehat{G}) < ((4t+12)n^2 - (4t^2 + 20t - 25)n + 4t^2 + 16t + 16)/(4n-2)$, by Lemma 3, we get $e(G) > n^2 - nt - 2n + t^2 + 4t + 4$. By Lemma 7, we obtain that G is traceable or $G \subseteq Q_n^t$ or $t = 1$, $G \subseteq R_n^1$. By the same discussion as the proof of Theorem 1, the conclusion is established. $\square$

4. Hamiltonian of Balanced Bipartite Graphs

Lemma 8 (see [23]). Let G be a balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$e(G) > n^2 - nt - n + t^2 + 2t + 1, \quad (21)$$

for some integer t , then G is Hamiltonian or $G \subseteq B_n^t$.

Theorem 4. Let $\widehat{G}$ be a connected balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$W(\widehat{G}) > 4n^3 - (2t + 7)n^2 + (2t + 2(t + 1)^2 + 4)n - 2(t + 1)^2, \quad (22)$$

for some integer t , then G is Hamiltonian.

Proof. Since $W(\widehat{G}) > 4n^3 - (2t + 7)n^2 + (2t + 2(t + 1)^2 + 4)n - 2(t + 1)^2$, by Lemma 1, we get $e(G) > n^2 - nt - n + t^2 + 2t + 1$. By Lemma 8, G is Hamiltonian or $G \subseteq B_n^t$.

If $G \subseteq B_n^t$, because $\widehat{G}$ is connected and $\delta(G) \geq t \geq 1$, we get $e(G) \leq (n - t)^2 + nt - (n - t) - t < n^2 - nt - n + t^2 + 2t + 1$. It is in contradiction with the above. $\square$

Theorem 5. Let $\widehat{G}$ be a connected balanced bipartite graph with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$WW(\widehat{G}) > 4n^4 - (2t + 8)n^3 + (2t^2 + 5t + 7)n^2 - (t^2 + t + 1)n - (t + 1)^2, \quad (23)$$

for some integer t , then G is Hamiltonian.

Proof. Since $WW(\widehat{G}) > 4n^4 - (2t + 8)n^3 + (2t^2 + 5t + 7)n^2 - (t^2 + t + 1)n - (t + 1)^2$, by Lemma 2, we get $e(G) > n^2 - nt - n + t^2 + 2t + 1$. By Lemma 8, G is Hamiltonian or $G \subseteq B_n^t$. By the same discussion as the proof of Theorem 4, the conclusion is established. $\square$

Theorem 6. Let $\widehat{G}$ be a connected balanced bipartite graph of with $2n$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 3$, and

$$H(\widehat{G}) < \frac{(4t + 8)n^2 - (4t^2 + 12t + 9)n + 4t^2 + 8t + 4}{4n - 2}, \quad (24)$$

for some integer t , then G is Hamiltonian.

Proof. Since $H(\widehat{G}) < ((4t + 8)n^2 - (4t^2 + 12t + 9)n + 4t^2 + 8t + 4)/(4n - 2)$, by Lemma 3, $e(G) > n^2 - nt - n + t^2 + 2t + 1$. By Lemma 8, G is Hamiltonian or $G \subseteq B_n^t$. By the same discussion as the proof of Theorem 4, the conclusion is established. $\square$

5. Traceable of Nearly Balanced Bipartite Graphs

Lemma 9 (Yu, Fang, and Fan [24]). Let G be a nearly balanced bipartite graph with $2n - 1$ vertices. If $\delta(G) \geq t \geq 1$, $n \geq 2t + 1$, and

$$e(G) > n^2 - nt - 2n + t^2 + 2t + 1, \quad (25)$$

for some integer t , then G is traceable or $G \subseteq C_n^t$.

Theorem 7. Let $\widehat{G}$ be a connected nearly balanced bipartite graph with $2n - 1$ vertices. If $n \geq 2t + 1$, $\delta(G) \geq t \geq 1$, and

$$W(\widehat{G}) > 4n^3 - (2t + 14)n^2 + (4t + 2(t + 1)^2 + 16)n - 4(t + 1)^2 - 4, \quad (26)$$

for some integer t , then G is traceable.

Proof. Let $G = G(X, Y; E)$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}\}$, then

$$\begin{aligned} W(\widehat{G}) &= \sum_{v_i, v_j \in V(\widehat{G})} d_{\widehat{G}}(v_i, v_j) \\ &= \frac{1}{2} \sum_{j=1}^{2n-1} D_j(\widehat{G}) \\ &\leq \frac{1}{2} \left(\sum_{j=1}^n (d_{\widehat{G}}(x_j) + (2n - 3)(n - 1 - d_{\widehat{G}}(x_j)) + 2(n^2 - 2n + 1)) \right) \\ &\quad + \frac{1}{2} \left(\sum_{j=1}^{n-1} (d_{\widehat{G}}(y_j) + (2n - 3)(n - d_{\widehat{G}}(y_j)) + 2(n^2 - 4n + 4)) \right) \\ &= 4n^3 - 12n^2 + 12n - 4 - (n - 2) \\ &\quad \cdot \left(\sum_{j=1}^n d_{\widehat{G}}(x_j) + \sum_{j=1}^{n-1} d_{\widehat{G}}(y_j) \right) \\ &= 2n^3 - 6n^2 + 8n - 4 + (n - 2) \\ &\quad \cdot \left(\sum_{j=1}^n d_G(x_j) + \sum_{j=1}^{n-1} d_G(y_j) \right) \\ &= 2n^3 - 6n^2 + 8n - 4 + 2(n - 2)e(G), \end{aligned} \quad (27)$$

where $d_{\widehat{G}}(x_j) = n - 1 - d_G(x_j)$ and $d_{\widehat{G}}(y_j) = n - d_G(y_j)$. Because $W(\widehat{G}) > 4n^3 - (2t + 14)n^2 + (4t + 2(t + 1)^2 + 16)n - 4(t + 1)^2 - 4$, we get $e(G) > n^2 - nt - 2n + t^2 + 2t + 1$. By Lemma 9, we get G is traceable or $G \subseteq C_n^t$.

If $G \subseteq C_n^t$, because $\widehat{G}$ is connected and $\delta(G) \geq t \geq 1$, we get $e(G) \leq t^2 + n(n-t-1) - (n-t-1) - t < n^2 - nt - 2n + t^2 + 2t + 1$. It is in contradiction with the above. $\square$

Theorem 8. Let $\widehat{G}$ be a connected nearly balanced bipartite graph with $2n-1$ vertices. If $n \geq 2t+1$, $\delta(G) \geq t \geq 1$, and

$$WW(\widehat{G}) > 4n^4 - (2t+18)n^3 + \left(2t^2 + 9t + \frac{65}{2}\right)n^2 - \left(5t^2 + 12t + \frac{53}{2}\right)n + 2t^2 + 4t + 8, \quad (28)$$

for some integer t , then G is traceable.

Proof. Let $G = G(X, Y : E)$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}\}$, then

$$\begin{aligned} WW(\widehat{G}) &= \frac{1}{2} \sum_{v_i, v_j \in V(\widehat{G})} (d_{\widehat{G}}(v_i, v_j) + d_G^2(v_i, v_j)) \\ &= \frac{1}{4} \sum_{j=1}^{2n-1} (D_j(\widehat{G}) + DD_j(\widehat{G})) \\ &\leq \frac{1}{4} \sum_{j=1}^n (d_{\widehat{G}}(x_j) + (2n-3)(n-1-d_{\widehat{G}}(x_j)) \\ &\quad + 2(n^2 - 2n + 1)) \\ &\quad + \frac{1}{4} \sum_{j=1}^n (d_{\widehat{G}}(x_j) + (2n-3)^2(n-1-d_{\widehat{G}}(x_j)) \\ &\quad + 4(n-1)^2(n-1)) \\ &\quad + \frac{1}{4} \sum_{j=1}^{n-1} (d_{\widehat{G}}(y_j) + (2n-3)(n-d_{\widehat{G}}(y_j)) \\ &\quad + (2n-4)(n-2)) \\ &\quad + \frac{1}{4} \sum_{j=1}^{n-1} (d_{\widehat{G}}(y_j) + (2n-3)^2(n-d_{\widehat{G}}(y_j)) \\ &\quad + (2n-4)^2(n-2)) \\ &= 4n^4 - 16n^3 + \frac{51}{2}n^2 - \frac{39}{2}n + 6 + \frac{1}{4}(-4n^2 + 10n - 4) \\ &\quad \cdot \left(\sum_{j=1}^n d_{\widehat{G}}(x_j) + \sum_{j=1}^{n-1} d_{\widehat{G}}(y_j) \right) \\ &= 2n^4 - 9n^3 + \frac{37}{2}n^2 - \frac{35}{2}n + 6 + \frac{1}{4}(4n^2 - 10n + 4) \\ &\quad \cdot \left(\sum_{j=1}^n d_G(x_j) + \sum_{j=1}^{n-1} d_G(y_j) \right) \\ &= 2n^4 - 9n^3 + \frac{37}{2}n^2 - \frac{35}{2}n + 6 + (2n^2 - 5n + 2)e(G), \end{aligned} \quad (29)$$

where $d_{\widehat{G}}(x_j) = n-1-d_G(x_j)$ and $d_{\widehat{G}}(y_j) = n-d_G(y_j)$. Because $WW(\widehat{G}) > 4n^4 - (2t+18)n^3 + (2t^2 + 9t + (65/2))n^2 - (5t^2 + 12t + (53/2))n + 2t^2 + 4t + 8$, we get $e(G) > n^2 - nt - 2n + t^2 + 2t + 1$. By Lemma 9, we obtain that G is traceable or $G \subseteq C_n^t$. By the same discussion as the proof of Theorem 7, the conclusion is established. $\square$

Theorem 9. Let $\widehat{G}$ be a connected nearly balanced bipartite graph with $2n-1$ vertices. If $n \geq 2t+1$, $\delta(G) \geq t \geq 1$, and

$$H(\widehat{G}) \leq \frac{(4t+8)n^2 - (4t^2 + 16t + 17)n + 8t^2 + 16t + 8}{4n-6}, \quad (30)$$

for some integer t , then G is traceable or $G \subseteq C_n^t$ ($t \leq 6$).

Proof. Let $G = G(X, Y : E)$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}\}$, then

$$\begin{aligned} H(\widehat{G}) &= \sum_{v_i, v_j \in V(G)} \frac{1}{d_{\widehat{G}}(v_i, v_j)} \\ &= \frac{1}{2} \sum_{j=1}^n \bar{D}_{\widehat{G}}(v_j) \\ &\geq \frac{1}{2} \sum_{j=1}^n \left(d_{\widehat{G}}(x_j) + \frac{1}{2n-3}(n-1-d_{\widehat{G}}(x_j)) + \frac{1}{2} \right) \\ &\quad + \frac{1}{2} \sum_{j=1}^{n-1} \left(d_{\widehat{G}}(y_j) + \frac{1}{2n-3}(n-d_{\widehat{G}}(y_j)) + \frac{1}{2} \right) \\ &= \frac{4n^2 - 5n}{4n-6} + \frac{n-2}{2n-3} \left(\sum_{j=1}^n d_{\widehat{G}}(x_j) + \sum_{j=1}^{n-1} d_{\widehat{G}}(y_j) \right) \\ &= \frac{4n^3 - 8n^2 + 3n}{4n-6} - \frac{n-2}{2n-3} \left(\sum_{j=1}^n d_G(x_j) + \sum_{j=1}^{n-1} d_G(y_j) \right) \\ &= \frac{4n^3 - 8n^2 + 3n}{4n-6} - \frac{2(n-2)}{2n-3} e(G), \end{aligned} \quad (31)$$

where $d_{\widehat{G}}(x_j) = n-1-d_G(x_j)$ and $d_{\widehat{G}}(y_j) = n-d_G(y_j)$. Because

$$H(\widehat{G}) \leq \frac{(4t+8)n^2 - (4t^2 + 16t + 17)n + 8t^2 + 16t + 8}{4n-6}, \quad (32)$$

we get $e(G) > n^2 - nt - 2n + t^2 + 2t + 1$. By Lemma 9, we obtain that G is traceable or $G \subseteq C_n^t$. Note that $t \geq 7$ and $H(\widehat{C}_n^t) = (1/4)(n^2 + 2tn - n - 2t^2) > ((4t+8)n^3 - (4t^2 + 24t + 33)n^2 + (16t^2 + 48t + 44)n - 16t^2 - 32t - 19)/(4n^2 - 14n + 12)$. If $G \subseteq C_n^t$, then $H(\widehat{G}) > H(\widehat{C}_n^t)$. The conclusion is established. $\square$

6. Traceable of Graphs

Lemma 10 (see [23]). Let G be a graph with $n(\geq 6t + 10)$ vertices, where t is an integer. If $\delta(G) \geq t$ and $e(G) > \binom{n-t-2}{2} + t^2 + 3t + 2$, then G is traceable unless $G \subseteq \underline{L}_n^t$ or $\underline{N}_n^t$.

Theorem 10. Let $\overline{G}$ be a connected graph with $n(\geq 6t + 10)$ vertices, where t is an integer. If $\delta(G) \geq t$ and

$$W(\overline{G}) > \frac{1}{2}(n^3 - (2t + 6)n^2 + (3t^2 + 15t + 19)n - 6t^2 - 22t - 20), \quad (33)$$

then G is traceable.

Proof. Since $W(\overline{G}) > (1/2)(n^3 - (2t + 6)n^2 + (3t^2 + 15t + 19)n - 6t^2 - 22t - 20)$, by Lemma 4, we get $e(G) > \binom{n-t-2}{2} + t^2 + 3t + 2$. By Lemma 10, we obtain that G is traceable unless $G \subseteq \underline{L}_n^t$ or $\underline{N}_n^t$.

If $G \subseteq \underline{L}_n^t$: note that $W(\underline{L}_n^t) = n^2 - 2n - tn + t^2 + 2t + 1$. Then, if $G \subseteq \underline{L}_n^t$, we have $W(\overline{G}) \leq W(\underline{L}_n^t) < (1/2)(n^3 - (2t + 6)n^2 + (3t^2 + 15t + 19)n - 6t^2 - 22t - 20)$, a contradiction.

If $G \subseteq \underline{N}_n^t$: note that $W(\underline{N}_n^t) = (1/2)(2n^2 - 4n - 6tn + 5t^2 + 7t + 2)$. Then, if $G \subseteq \underline{N}_n^t$, we have $W(\overline{G}) \leq W(\underline{N}_n^t) < (1/2)(n^3 - (2t + 6)n^2 + (3t^2 + 15t + 19)n - 6t^2 - 22t - 20)$, a contradiction. $\square$

Theorem 11. Let $\overline{G}$ be a connected graph with $n(\geq 6t + 10)$ vertices, where t is an integer. If $\delta(G) \geq t$ and

$$WW(\overline{G}) > \frac{1}{4}n^4 - \left(\frac{1}{2}t + \frac{3}{2}\right)n^3 + \left(\frac{3}{4}t^2 + \frac{13}{4}t + \frac{15}{4}\right)n^2 - \left(\frac{3}{4}t^2 + \frac{7}{4}t + \frac{1}{2}\right)n - \frac{3}{2}t^2 - \frac{11}{2}t - 5, \quad (34)$$

then G is traceable.

Proof. Since $WW(\overline{G}) > (1/4)n^4 - ((1/2)t + (3/2))n^3 + ((3/4)t^2 + (13/4)t + (15/4))n^2 - ((3/4)t^2 + (7/4)t + (1/2))n - (3/2)t^2 - (11/2)t - 5$, by Lemma 5, we get $e(G) > \binom{n-t-2}{2} + t^2 + 3t + 2$. By Lemma 10, we obtain that G is traceable or $G \subseteq \underline{L}_n^t$ or $\underline{N}_n^t$.

If $G \subseteq \underline{L}_n^t$: note that $WW(\underline{L}_n^t) = (1/2)(3n^2 - 7n - 4tn + 4t^2 + 8t + 4)$. Then, if $G \subseteq \underline{L}_n^t$, we have $WW(\overline{G}) \leq WW(\underline{L}_n^t) < (1/4)n^4 - ((1/2)t + (3/2))n^3 + ((3/4)t^2 + (13/4)t + (15/4))n^2 - ((3/4)t^2 + (7/4)t + (1/2))n - (3/2)t^2 - (11/2)t - 5$, a contradiction.

If $G \subseteq \underline{N}_n^t$: note that $WW(\underline{N}_n^t) = 3n^2 - 7n - 10tn + 9t^2 + 13t + 4$. Then, if $G \subseteq \underline{N}_n^t$, we have $WW(\overline{G}) \leq WW(\underline{N}_n^t) < (1/4)n^4 - ((1/2)t + (3/2))n^3 + ((3/4)t^2 + (13/4)t + (15/4))n^2 -$

$((3/4)t^2 + (7/4)t + (1/2))n - (3/2)t^2 - (11/2)t - 5$, a contradiction. $\square$

Theorem 12. Let $\overline{G}$ be a connected graph with $n(\geq 6t + 10)$ vertices, where t is an integer. If $\delta(G) \geq t$ and

$$H(\overline{G}) < \frac{(2t + 4)n^2 - (3t^2 + 15t + 18)n + 6t^2 + 22t + 20}{2n - 2}, \quad (35)$$

then G is traceable.

Proof. Since $H(\overline{G}) < ((2t + 4)n^2 - (3t^2 + 15t + 18)n + 6t^2 + 22t + 20)/(2n - 2)$, by Lemma 6, we get $e(G) > \binom{n-t-2}{2} + t^2 + 3t + 2$. By Lemma 10, we obtain that G is traceable unless $G \subseteq \underline{L}_n^t$ or $\underline{N}_n^t$.

If $G \subseteq \underline{L}_n^t$: note that $H(\underline{L}_n^t) = (1/4)(n^2 + n + 2tn - 2t^2 - 2)$. Then, if $G \subseteq \underline{L}_n^t$, we have $H(\overline{G}) \geq H(\underline{L}_n^t) > ((2t + 4)n^2 - (3t^2 + 15t + 18)n + 6t^2 + 22t + 20)/(2n - 2)$, a contradiction.

If $G \subseteq \underline{N}_n^t$: note that $H(\underline{N}_n^t) = (1/4)(n^2 + n + 2tn - 2t^2 - 6t - 2)$. Then, if $G \subseteq \underline{N}_n^t$, we have $H(\overline{G}) \geq H(\underline{N}_n^t) > ((2t + 4)n^2 - (3t^2 + 15t + 18)n + 6t^2 + 22t + 20)/(2n - 2)$, a contradiction. $\square$

7. Hamiltonian of Graphs

Lemma 11 (see [23]). Let G be a graph with $n(\geq 6t + 5)$ vertices, where t is an integer. If $\delta(G) \geq t$ and

$$e(G) > \binom{n-t-1}{2} + t^2 + 2t + 1, \quad (36)$$

then G is Hamiltonian or $G \subseteq \underline{L}_n^t$ or $\underline{N}_n^t$.

Theorem 13. Let $\overline{G}$ be a connected graph with $n(\geq 6t + 5)$ vertices, where t is an integer. If $\delta(G) \geq t$ and

$$W(\overline{G}) > \frac{1}{2}(n^3 - (2t + 4)n^2 + (3t^2 + 11t + 19)n - 6t^2 - 14t - 8), \quad (37)$$

then G is Hamiltonian.

Proof. Since $W(\overline{G}) > (1/2)(n^3 - (2t + 4)n^2 + (3t^2 + 11t + 19)n - 6t^2 - 14t - 8)$, by Lemma 4, we get $e(G) > \binom{n-t-1}{2} + t^2 + 2t + 1$. By Lemma 11, we obtain that G is Hamiltonian or $G \subseteq \underline{L}_n^t$ or $\underline{N}_n^t$.

If $G \subseteq \underline{L}_n^t$: note that $W(\underline{L}_n^t) = n^2 - tn - 3n + t^2 + t + 2$. Then, if $G \subseteq \underline{L}_n^t$, we have $W(\overline{G}) \leq W(\underline{L}_n^t) < (1/2)(n^3 - (2t + 4)n^2 + (3t^2 + 11t + 19)n - 6t^2 - 14t - 8)$, a contradiction.

If $G \subseteq \underline{N}_n^t$: note that $W(\underline{N}_n^t) = n^2 - 3tn - n + (5/2)t^2 + (3/2)t$. Then, if $G \subseteq \underline{N}_n^t$, we have $W(\overline{G}) \leq W(\underline{N}_n^t) <$

$(1/2)(n^3 - (2t+4)n^2 + (3t^2 + 11t + 19)n - 6t^2 - 14t - 8)$, a contradiction. $\square$

Theorem 14. Let $\delta(G) \geq t$ and $\overline{G}$ be a connected graph with $n(\geq 6t+5)$ vertices for some integers t satisfying the following condition:

$$WW(\overline{G}) > \frac{1}{4}n^4 - \left(\frac{1}{2}t + 1\right)n^3 + \left(\frac{3}{4}t^2 + \frac{9}{4}t + \frac{3}{2}\right)n^2 - \left(\frac{3}{4}t^2 + \frac{3}{4}t + \frac{1}{4}\right)n - \left(\frac{3}{2}t^2 + \frac{7}{2}t + 2\right). \quad (38)$$

Therefore, G is Hamiltonian.

Proof. Since $WW(\overline{G}) > (1/4)n^4 - ((1/2)t + 1)n^3 + ((3/4)t^2 + (9/4)t + (3/2))n^2 - ((3/4)t^2 + (3/4)t + (1/4))n - ((3/2)t^2 + (7/2)t + 2)$, by Lemma 5, we get $e(G) > \binom{n-t-1}{2} + t^2 + 2t + 1$. By Lemma 11, we obtain that G is Hamiltonian or $G \subseteq L_n^t$ or N_n^t .

If $G \subseteq L_n^t$: note that $WW(\overline{L}_n^t) = 3n^2 - 4tn - 9n + 4t^2 + 4t + 6$. Then, if $G \subseteq L_n^t$, we have $WW(\overline{G}) \leq WW(\overline{L}_n^t) < (1/4)n^4 - ((1/2)t + 1)n^3 + ((3/4)t^2 + (9/4)t + (3/2))n^2 - ((3/4)t^2 + (3/4)t + (1/4))n - ((3/2)t^2 + (7/2)t + 2)$, a contradiction.

If $G \subseteq N_n^t$: note that $WW(\overline{N}_n^t) = 3n^2 - 10tn - 3n + 9t^2 + 5t$. Then, if $G \subseteq N_n^t$, we have $WW(\overline{G}) \leq WW(\overline{N}_n^t) < (1/4)n^4 - ((1/2)t + 1)n^3 + ((3/4)t^2 + (9/4)t + (3/2))n^2 - ((3/4)t^2 + (3/4)t + (1/4))n - ((3/2)t^2 + (7/2)t + 2)$, a contradiction. $\square$

Theorem 15. Let $\overline{G}$ be a connected graph with $n(\geq 6t+5)$ vertices for some integer t . If $\delta(G) \geq t$ and $H(\overline{G}) < (((2t+2)n^2 - (3t^2 + 11t + 8)n + 6t^2 + 14t + 8)/(2n-2))$, then G is Hamiltonian.

Proof. Since $H(\overline{G}) \leq (((2t+2)n^2 - (3t^2 + 11t + 8)n + 6t^2 + 14t + 8)/(2n-2))$, by Lemma 6, we get $e(G) > \binom{n-t-1}{2} + t^2 + 2t + 1$. By Lemma 11, we obtain that G is Hamiltonian unless $G \subseteq L_n^t$ or N_n^t .

If $G \subseteq L_n^t$: note that $H(\overline{L}_n^t) = (1/4)(n^2 - 3n + 2tn - 2t^2 - 2t + 2)$. Then, if $G \subseteq L_n^t$, we have $H(\overline{G}) \geq H(\overline{L}_n^t) > (((2t+2)n^2 - (3t^2 + 11t + 8)n + 6t^2 + 14t + 8)/(2n-2))$, a contradiction.

If $G \subseteq N_n^t$: note that $H(\overline{N}_n^t) = (1/4)(n^2 - n - 3t^2 + t)$. Then, if $G \subseteq N_n^t$, we have $H(\overline{G}) \geq H(\overline{N}_n^t) > (((2t+2)n^2 - (3t^2 + 11t + 8)n + 6t^2 + 14t + 8)/(2n-2))$, a contradiction. $\square$

8. Hamilton-Connected of Graphs

Lemma 12 (see [20]). Let G be a t -connected graph with n vertices, where $t \geq 2$. If $e(G) > ((n(n-1) - t(n-t-1))/2)$, then G is Hamilton-connected.

By Lemmas 4, 5, 6, and 12 and by direct computations, we get Theorems 16, 17, and 18, respectively.

Theorem 16. Let G be a t -connected graph with n vertices and $\overline{G}$ be a connected graph, where $t \geq 2$. If

$$W(\overline{G}) > \frac{1}{2}n^3 - \frac{1}{2}(t+2)n^2 + \frac{1}{2}(t^2 + 3t + 1)n - t^2 - t, \quad (39)$$

then G is Hamilton-connected.

Theorem 17. Let G be a t -connected graph with n vertices and $\overline{G}$ be a connected graph, where $t \geq 2$. If

$$WW(\overline{G}) > \frac{1}{4}n^4 - \frac{1}{4}(t+2)n^3 + \frac{1}{4}(t^2 + 2t + 1)n^2 - \frac{1}{4}(t^2 - t)n - \frac{1}{2}t^2 - \frac{1}{2}t, \quad (40)$$

then G is Hamilton-connected.

Theorem 18. Let G be a t -connected graph with n vertices and $\overline{G}$ be a connected graph, where $t \geq 2$. If

$$H(\overline{G}) < \frac{(t+1)n^2 - (t^2 + 3t + 1)n + 2t^2 + 2t}{2(n-1)}, \quad (41)$$

then G is Hamilton-connected.

9. Traceable from Every Vertex of Graphs

Lemma 13 (see [20]). Let G be a t -connected graph with n vertices, where $t \geq 2$. If $e(G) > ((n(n-1) - t(n-t))/2)$, then G is traceable from every vertex.

By Lemmas 4, 5, 6, and 13 and by direct computation, we get Theorems 19, 20, and 21, respectively.

Theorem 19. Let G be a t -connected graph with n vertices and $\overline{G}$ be a connected graph, where $t \geq 2$. If

$$W(\overline{G}) > \frac{1}{2}n^3 - \left(\frac{1}{2}t + 1\right)n^2 + \frac{1}{2}(t^2 + 2t + 1)n - t^2, \quad (42)$$

then G is traceable from every vertex.

Theorem 20. Let G be a t -connected graph with n vertices and $\overline{G}$ be a connected graph, where $t \geq 2$. If

$$WW(\overline{G}) > \frac{1}{4}n^4 - \frac{1}{4}(t+2)n^3 + \frac{1}{4}(t^2 + t + 1)n^2 - \frac{1}{4}(t^2 - 2t)n - \frac{1}{2}t^2, \quad (43)$$

then G is traceable from every vertex.

Theorem 21. Let G be a t -connected graph with n vertices and $\overline{G}$ be a connected graph, where $t \geq 2$. If

$$H(\overline{G}) < \frac{(t+1)n^2 + (-t^2 - 2t - 1)n + 2t^2}{2n-2}, \quad (44)$$

then G is traceable from every vertex.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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