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Research Article

Some New Results on Various Graph Energies of the Splitting Graph

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The energy of a simple connected graph G is equal to the sum of the absolute value of eigenvalues of the graph G where the eigenvalue of a graph G is the eigenvalue of its adjacency matrix $A(G)$. Ultimately, scores of various graph energies have been originated. It has been shown in this paper that the different graph energies of the regular splitting graph $S'(G)$ is a multiple of corresponding energy of a given graph G.

1. Introduction

Let G be a simple, finite, and undirected graph and its vertex set and edge set are denoted by $V(G) = \{v_1, v_2, v_3, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, e_3, \ldots, e_q\}$, respectively. Number of edges finishing at a vertex v of a graph G is named as degree of vertex v and is denoted by $d(v)$ or d_v.

The adjacency matrix of G, denoted by $A(G)$, is a square matrix $[a_{ij}]$ such that a_{ij} is equal to unity if $v_i v_j \in E(G)$ and is equal to zero otherwise. The eigenvalues of the adjacency matrix $A(G)$ are known as the eigenvalues of the graph G. Collection of eigenvalues of the graph G together with their multiplicities is called spectrum of the graph G.

Let $\mu_1, \mu_2, \mu_3, \ldots, \mu_p$ be eigenvalues of G and are assumed in nonincreasing order; then, Ivan Gutman in 1978 [1] defined the energy of the graph G as the sum of the absolute values of all eigenvalues of the graph G:

$$E(G) = \sum_{j=1}^{p} \left|\mu_j\right|. \tag{1}$$

The inspiration of description energy of graph happened from quantum Chemistry. During 1930s, E. Hückel presented chemical applications of graph theory in his molecular orbital theory where eigenvalues of graphs take place. In quantum chemistry, the skeleton of nonsaturated hydrocarbon is represented by a graph. The energy levels of electrons in such a molecule are eigenvalues of graph. The strength of particles is closely identified with the spectrum of its graph. The carbon atoms and chemical bond between them in a hydrocarbon system denote vertices and edges, respectively, in a molecular graph. A lot of work has been done on graph theory, special graph labeling [2–10], chemical graph theory and graph energies. In the thesis of Siraj [11], certain elementary results on the energy of the graph are also described.

The present work is considered to relate several energies of a graph to bigger graph acquired from the given graph with the help of some graph operations, namely, the splitting graph which is defined in [12]. For a graph G, the splitting graph $S'(G)$ is obtained by taking a new vertex v' corresponding to each vertex v of the graph G and then join v' to all vertices of G adjacent to v. In [13], it has been proven that $E(S'(G)) = \sqrt{5}E(G)$.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of order $a \times m$ and $b \times n$, respectively. Then, their tensor product, $A \otimes B$ is obtained from A when every element a_{ij} is replaced by the block $a_{ij}B$ and is of order $ab \times mn$.

Proposition 1 (see [14]). *Let $A \in M^q$ and $B \in M^p$. Also, let α be an eigenvalue of the matrix A with corresponding eigenvector y and β be an eigenvalue of the matrix B with corresponding eigenvector z. Then, $\alpha\beta$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector yz.*

In recent times, comparable energies are being considered, based on eigenvalues of a variety of other graph matrices. Numerous matrices can be related to a graph, and their spectrums provide certain helpful information about the graph [15–18].

2. Maximum Degree Energy

The maximum degree energy E_M of a simple connected graph G in [19] is defined as the sum of the absolute values of eigenvalues of the maximum degree matrix $M(G)$ of a graph G. Then, $M(G) = [M_{ij}]$ where

$$M_{ij} = \begin{cases} \max\left(d_i, d_j\right), & \text{if } v_i, v_j \in E(G), \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where d_i and d_j are the degrees of vertices v_i and v_j, respectively.

Theorem 1. *For a graph G,*

$$E_M(S'(G)) = 2E_M(G). \tag{3}$$

Proof. Let G be a graph with vertices $v_1, v_2, v_3, \ldots, v_p$. Then the maximum degree matrix $M(G)$ is

$$M(G) = \begin{array}{c|c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_p \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \end{array} & \begin{bmatrix} 0 & d_{k_{12}} & d_{k_{13}} & \cdots & d_{k_{1p}} \\ d_{k_{21}} & 0 & d_{k_{23}} & \cdots & d_{k_{2p}} \\ d_{k_{31}} & d_{k_{32}} & 0 & \cdots & d_{k_{3p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{k_{p1}} & d_{k_{p2}} & d_{k_{p3}} & \cdots & 0 \end{bmatrix} \end{array}, \tag{4}$$

where $d_{k_{ij}} = \max(d_i, d_j)$ and d_i and d_j are the degrees of vertices v_i and v_j, respectively, for $i = 1, 2, 3, \ldots, p$ and $j = 1, 2, 3, \ldots, p$.

Let $v_1', v_2', v_3', \ldots, v_p'$ be the vertices corresponding to $v_1, v_2, v_3, \ldots, v_p$ which are added in G to obtain $S'(G)$ such that $N(v_j) = N(v_j')$ for $j = 1, 2, 3, \ldots, p$. Then, the maximum degree matrix of $S'(G)$ is denoted by $M(S'(G))$ and can be written as a block matrix:

$$\begin{array}{c|c} & \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_p & v_1' & v_2' & v_3' & \cdots & v_p' \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \\ v_1' \\ v_2' \\ v_3' \\ \vdots \\ v_p' \end{array} & \begin{bmatrix} 0 & 2d_{k_{12}} & 2d_{k_{13}} & \cdots & 2d_{k_{1p}} & 0 & 2d_{k_{12}} & 2d_{k_{13}} & \cdots & 2d_{k_{1p}} \\ 2d_{k_{21}} & 0 & 2d_{k_{23}} & \cdots & 2d_{k_{2p}} & 2d_{k_{21}} & 0 & 2d_{k_{23}} & \cdots & 2d_{k_{2p}} \\ 2d_{k_{31}} & 2d_{k_{32}} & 0 & \cdots & 2d_{k_{3p}} & 2d_{k_{31}} & 2d_{k_{32}} & 0 & \cdots & 2d_{k_{3p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2d_{k_{p1}} & 2d_{k_{p2}} & 2d_{k_{p3}} & \cdots & 0 & 2d_{k_{p1}} & 2d_{k_{p2}} & 2d_{k_{p3}} & \cdots & 0 \\ 0 & 2d_{k_{12}} & 2d_{k_{13}} & \cdots & 2d_{k_{1p}} & 0 & 0 & 0 & \cdots & 0 \\ 2d_{k_{21}} & 0 & 2d_{k_{23}} & \cdots & 2d_{k_{2p}} & 0 & 0 & 0 & \cdots & 0 \\ 2d_{k_{31}} & 2d_{k_{32}} & 0 & \cdots & 2d_{k_{3p}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2d_{k_{p1}} & 2d_{k_{p2}} & 2d_{k_{p3}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}. \tag{5}$$

That is

$$\begin{aligned} M(S'(G)) &= \begin{bmatrix} 2M(G) & 2M(G) \\ 2M(G) & 0 \end{bmatrix} \\ \text{or } &= \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \otimes M(G). \end{aligned} \tag{6}$$

Here, the maximum degree spectrum of $S'(G)$ is

$$\begin{pmatrix} (1+\sqrt{5})\mu_j & (1-\sqrt{5})\mu_j \\ p & p \end{pmatrix}, \tag{7}$$

where μ_j for $j = 1, 2, 3, \ldots, p$ are the eigenvalues of $M(G)$and $1 \pm \sqrt{5}$ are the eigenvalues of $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$.

Here,

$$\begin{aligned} E_M(S'(G)) &= \sum_{j=1}^{j=p} \left|(1 \pm \sqrt{5})\mu_j\right| \\ &= \sum_{j=1}^{j=p} \left|\mu_j\right| (1 + \sqrt{5} + 1 - \sqrt{5}) \\ &= 2\sum_{j=1}^{j=p} \left|\mu_j\right| \\ &= 2E_M(G), \end{aligned} \tag{8}$$

which completes the proof. □

3. Minimum Degree Energy

In [20], the minimum degree energy E_m of a simple connected graph G is defined as the sum of the absolute values of eigenvalues of minimum degree matrix $m(G)$ of a graph G. Here, $m(G) = [m_{ij}]$ where

$$m_{ij} = \begin{cases} \min\left(d_i, d_j\right), & \text{if } v_i, v_j \in E(G), \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where d_i and d_j are the degrees of vertices v_i and v_j, respectively.

Theorem 2. *For a graph G,*

$$E_m\left(S'(G)\right) = 2E_m(G). \tag{10}$$

Proof. Let G be a graph with vertices $v_1, v_2, v_3, \ldots, v_p$. Then, the minimum degree matrix $m(G)$ is

$$m(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & \ldots & v_p \end{array} \\ \begin{bmatrix} 0 & d_{k_{12}} & d_{k_{13}} & \ldots & d_{k_{1p}} \\ d_{k_{21}} & 0 & d_{k_{23}} & \ldots & d_{k_{2p}} \\ d_{k_{31}} & d_{k_{32}} & 0 & \ldots & d_{k_{3p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{k_{p1}} & d_{k_{p2}} & d_{k_{p3}} & \ldots & 0 \end{bmatrix} \end{array}, \tag{11}$$

where $d_{k_{ij}} = \min(d_i, d_j)$ and d_i and d_j are the degrees of vertices v_i and v_j, respectively, for $i = 1, 2, 3, \ldots, p$ and $j = 1, 2, 3, \ldots, p$. Let $v_1', v_2', v_3', \ldots, v_p'$ be the vertices corresponding to $v_1, v_2, v_3, \ldots, v_p$ which are added in G to obtain $S'(G)$ such that $N(v_j) = N(v_j')$ for $j = 1, 2, 3, \ldots, p$. Then, the minimum degree matrix of splitting graph of G, denoted by $m(S'(G))$, can be defined as a block matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \\ v_1' \\ v_2' \\ v_3' \\ \vdots \\ v_p' \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \ldots & v_p & v_1' & v_2' & v_3' & \ldots & v_p' \end{array} \\ \begin{bmatrix} 0 & 2d_{k_{12}} & 2d_{k_{13}} & \ldots & 2d_{k_{1p}} & 0 & d_{k_{12}} & d_{k_{13}} & \ldots & d_{k_{1p}} \\ 2d_{k_{21}} & 0 & 2d_{k_{23}} & \ldots & 2d_{k_{2p}} & d_{k_{21}} & 0 & d_{k_{23}} & \ldots & d_{k_{2p}} \\ 2d_{k_{31}} & 2d_{k_{32}} & 0 & \ldots & 2d_{k_{3p}} & d_{k_{31}} & d_{k_{32}} & 0 & \ldots & d_{k_{3p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2d_{k_{p1}} & 2d_{k_{p2}} & 2d_{k_{p3}} & \ldots & 0 & d_{k_{p1}} & d_{k_{p2}} & d_{k_{p3}} & \ldots & 0 \\ 0 & d_{k_{12}} & d_{k_{13}} & \ldots & d_{k_{1p}} & 0 & 0 & 0 & \ldots & 0 \\ d_{k_{21}} & 0 & d_{k_{23}} & \ldots & d_{k_{2p}} & 0 & 0 & 0 & \ldots & 0 \\ d_{k_{31}} & d_{k_{32}} & 0 & \ldots & d_{k_{3p}} & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{k_{p1}} & d_{k_{p2}} & d_{k_{p3}} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \end{array}. \tag{12}$$

That is

$$m\left(S'(G)\right) = \begin{bmatrix} 2m(G) & m(G) \\ m(G) & 0 \end{bmatrix} \quad \text{or} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \otimes m(G). \tag{13}$$

Here, the minimum degree spectrum of $S'(G)$ is

$$\begin{pmatrix} (1+\sqrt{2})\nu_j & (1-\sqrt{2})\nu_j \\ p & p \end{pmatrix}, \tag{14}$$

where ν_j for $j = 1, 2, 3, \ldots, p$ are the eigenvalues of $m(G)$ and $1 \pm \sqrt{2}$ are the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.

Here,

$$\begin{aligned} E_m\left(S'(G)\right) &= \sum_{j=1}^{j=n} \left|(1 \pm \sqrt{2})\nu_j\right| \\ &= \sum_{j=1}^{j=p} \left|\nu_j\right| (1 + \sqrt{2} + 1 - \sqrt{2}) \\ &= 2\sum_{j=1}^{j=n} \left|\nu_j\right| \\ &= 2E_m(G), \end{aligned} \tag{15}$$

which is the required result. □

4. Randić Energy

The randić energy E_R of a simple connected graph G in [21] is the sum of the absolute values of eigenvalues of the randić matrix $R(G)$. Here, $R(G) = [r_{ij}]$ where

$$r_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i d_j}}, & \text{if } v_i, v_j \in E(G), \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

Here, d_i and d_j are the degrees of vertices v_i and v_j, respectively.

Theorem 3. *For a graph G,*

$$E_R(S'(G)) = \frac{3}{2} E_R(G). \quad (17)$$

Proof. Let $x_1, x_2, x_3, \ldots, x_p$ be vertices of a graph G. Then, the randić matrix of G is denoted by $R(G)$ and is given as

$$R(G) = \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_p \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{array} & \begin{bmatrix} 0 & \frac{1}{d_1 d_2} & \frac{1}{d_1 d_3} & \cdots & \frac{1}{d_1 d_p} \\ \frac{1}{d_2 d_1} & 0 & \frac{1}{d_2 d_3} & \cdots & \frac{1}{d_2 d_p} \\ \frac{1}{d_3 d_1} & \frac{1}{d_3 d_2} & 0 & \cdots & \frac{1}{d_3 d_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{d_p d_1} & \frac{1}{d_p d_2} & \frac{1}{d_p d_3} & \cdots & 0 \end{bmatrix} \end{array}. \quad (18)$$

Let $x'_1, x'_2, x'_3, \ldots, x'_p$ be the vertices corresponding to $x_1, x_2, x_3, \ldots, x_p$ which are added in G to obtain $S'(G)$ such that $N(x_j) = N(x'_j)$ for $j = 1, 2, 3, \ldots, p$. Then, the randić matrix of $S'(G)$ is denoted by $R(S'(G))$ and can be written as a block matrix as follows:

$$\begin{array}{cc} & \begin{array}{cccccccccc} x_1 & x_2 & x_3 & \cdots & x_p & x'_1 & x'_2 & x'_3 & \cdots & x'_p \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \\ x'_1 \\ x'_2 \\ x'_3 \\ \vdots \\ x'_p \end{array} & \begin{bmatrix} 0 & \frac{1}{d_1 d_2} & \frac{1}{d_1 d_3} & \cdots & \frac{1}{d_1 d_p} & 0 & \frac{1}{d_1 d_2} & \frac{1}{d_1 d_3} & \cdots & \frac{1}{d_1 d_p} \\ \frac{1}{d_2 d_1} & 0 & \frac{1}{d_2 d_3} & \cdots & \frac{1}{d_2 d_p} & \frac{1}{d_2 d_1} & 0 & \frac{1}{d_2 d_3} & \cdots & \frac{1}{d_2 d_p} \\ \frac{1}{d_3 d_1} & \frac{1}{d_3 d_2} & 0 & \cdots & \frac{1}{d_3 d_p} & \frac{1}{d_3 d_1} & \frac{1}{d_3 d_2} & 0 & \cdots & \frac{1}{d_3 d_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{d_p d_1} & \frac{1}{d_p d_2} & \frac{1}{d_p d_3} & \cdots & 0 & \frac{1}{d_p d_1} & \frac{1}{d_p d_2} & \frac{1}{d_p d_3} & \cdots & 0 \\ 0 & \frac{1}{d_1 d_2} & \frac{1}{d_1 d_3} & \cdots & \frac{1}{d_1 d_p} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{d_2 d_1} & 0 & \frac{1}{d_2 d_3} & \cdots & \frac{1}{d_2 d_p} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{d_3 d_1} & \frac{1}{d_3 d_2} & 0 & \cdots & \frac{1}{d_3 d_p} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{d_p d_1} & \frac{1}{d_p d_2} & \frac{1}{d_p d_3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}. \quad (19)$$

That is,

$$R(S'(G)) = \begin{bmatrix} \frac{1}{2} R(G) & \frac{1}{\sqrt{2}} R(G) \\ \frac{1}{\sqrt{2}} R(G) & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \otimes R(G). \quad (20)$$

Here, the randić spectrum of $S'(G)$ is

$$\begin{pmatrix} \left(\frac{-1}{2}\right)\rho_j & (1)\rho_j \\ p & p \end{pmatrix}, \quad (21)$$

where ρ_j for $j = 1, 2, 3, \ldots, p$ are the eigenvalues of $R(G)$and $-1/2$ and 1 are the eigenvalues of $\begin{bmatrix} 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{bmatrix}$.

Here,

$$\begin{aligned} E_R(S'(G)) &= \sum_{j=1}^{j=p} \left| \left(\frac{-1}{2} + 1 \right) \rho_j \right| \\ &= \sum_{j=1}^{j=p} \left| \rho_j \right| \left(\frac{1}{2} + 1 \right) \\ &= \frac{3}{2} \sum_{j=1}^{j=p} \left| \rho_j \right| \\ &= \frac{3}{2} E_R(G). \end{aligned} \tag{22}$$

□

5. Seidel Energy

In [22], Haemers defined the Seidel energy E_{SE} of a simple connected graph G as the sum of the absolute values of eigenvalues of the seidel matrix $SE(G)$ of G. Here, $SE(G) = [s_{ij}]$ where

$$s_{ij} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent and } i \neq j, \\ 1, & \text{if } v_i \text{ and } v_j \text{ are non adjacent and } i \neq j, \\ 0, & \text{if } i = j. \end{cases} \tag{23}$$

Theorem 4. *For a s-regular graph G,*

$$E_{SE}(S'(G)) \geq E_{SE}(G). \tag{24}$$

Proof. Let G be a graph with vertices $v_1, v_2, v_3, \ldots, v_p$. Then, the seidel matrix $SE(G)$ of G is

$$SE(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_p \end{array} \\ \begin{bmatrix} 0 & s_{12} & s_{13} & \cdots & s_{1p} \\ s_{21} & 0 & s_{23} & \cdots & s_{2p} \\ s_{31} & s_{32} & 0 & \cdots & s_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & s_{p3} & \cdots & 0 \end{bmatrix} \end{array}. \tag{25}$$

Let $u_1, u_2, u_3, \ldots, u_p$ be the vertices corresponding to $v_1, v_2, v_3, \ldots, v_p$ which are added in G to obtain $S'(G)$. Then, the seidel matrix of $S'(G)$ is denoted by $SE(S'(G))$ and can be written as a block matrix as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_p \end{array} \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_p & u_1 & u_2 & u_3 & \cdots & u_p \end{array} \\ \begin{bmatrix} 0 & s_{12} & s_{13} & \cdots & s_{1p} & 1 & s_{12} & s_{13} & \cdots & s_{1p} \\ s_{21} & 0 & s_{23} & \cdots & s_{2p} & s_{21} & 1 & s_{23} & \cdots & s_{2p} \\ s_{31} & s_{32} & 0 & \cdots & s_{3p} & s_{31} & s_{32} & 1 & \cdots & s_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & s_{p3} & \cdots & 0 & s_{p1} & s_{p2} & s_{p3} & \cdots & 1 \\ 1 & s_{12} & s_{13} & \cdots & s_{1p} & 0 & 1 & 1 & \cdots & 1 \\ s_{21} & 1 & s_{23} & \cdots & s_{2p} & 1 & 0 & 1 & \cdots & 1 \\ s_{31} & s_{32} & 1 & \cdots & s_{3p} & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & s_{p3} & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \end{bmatrix} \end{array}. \tag{26}$$

That is,

$$\begin{aligned} SE(S'(G)) &= \begin{bmatrix} SE(G) & SE(G) + I_p \\ SE(G) + I_p & J_p - I_p \end{bmatrix} \\ \text{or } &= \begin{bmatrix} SE(G) & SE(G) \\ SE(G) & 0 \end{bmatrix} + \begin{bmatrix} 0I_p & I_p \\ I_p & J_p - I_p \end{bmatrix}. \end{aligned} \tag{27}$$

Hence,

$$SE(S'(G)) \geq \begin{bmatrix} SE(G) & SE(G) \\ SE(G) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes SE(G). \tag{28}$$

Here, the seidel spectrum of $S'(G)$ is greater than or equal to the spectrum

$$\begin{pmatrix} \frac{1}{2}(1 + \sqrt{5})s_j & \frac{1}{2}(1 - \sqrt{5})s_j \\ p & p \end{pmatrix}, \tag{29}$$

where s_j for $j = 1, 2, 3, \ldots, p$ are the eigenvalues of $SE(G)$and $1/2(1 \pm \sqrt{5})$ are the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Thus,

$$\begin{aligned} E_{SE}(S'(G)) &\geq \sum_{j=1}^{j=p} \left| \left(\frac{1}{2}(1 \pm \sqrt{5}) \right) s_j \right| \\ &= \sum_{j=1}^{j=p} \left| s_j \right| \left(\frac{1}{2} + \frac{\sqrt{5}}{2} + \frac{1}{2} - \frac{\sqrt{5}}{2} \right) \\ &= \left(\frac{1}{2} + \frac{1}{2} \right) \sum_{j=1}^{j=p} \left| s_j \right| \\ &= E_{SE}(G). \end{aligned} \tag{30}$$

Hence,

$$E_{SE}\left(S'(G)\right) \geq E_{SE}(G). \tag{31}$$

□

6. Sum-Connectivity Energy

The sum-connectivity energy E_{SC} of a simple connected graph G in [23] is defined as the sum of the absolute values of eigenvalues of the sum-connectivity matrix $SC(G)$. Here, $SC(G) = [sc_{ij}]$ where

$$sc_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i + d_j}}, & \text{if } v_i, v_j \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Here, d_i and d_j are the degrees of vertices v_i and v_j, respectively.

Theorem 5. *For a regular graph G,*

$$E_{SC}\left(S'(G)\right) = \frac{1}{\sqrt{2}} E_{SC}(G). \tag{33}$$

Proof. Let G be a graph with vertices $z_1, z_2, z_3, \ldots, z_p$. Then the sum-connectivity matrix of G is denoted by $SC(G)$ and is defined as

$$SC(G) = \begin{array}{cc} & \begin{array}{ccccc} z_1 & z_2 & z_3 & \cdots & z_p \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_p \end{array} & \begin{bmatrix} 0 & \frac{1}{\sqrt{d_1+d_2}} & \frac{1}{\sqrt{d_1+d_3}} & \cdots & \frac{1}{\sqrt{d_1+d_p}} \\ \frac{1}{\sqrt{d_2+d_1}} & 0 & \frac{1}{\sqrt{d_2+d_3}} & \cdots & \frac{1}{\sqrt{d_2+d_p}} \\ \frac{1}{\sqrt{d_3+d_1}} & \frac{1}{\sqrt{d_3+d_2}} & 0 & \cdots & \frac{1}{\sqrt{d_3+d_p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{d_p+d_1}} & \frac{1}{\sqrt{d_p+d_2}} & \frac{1}{\sqrt{d_p+d_3}} & \cdots & 0 \end{bmatrix} \end{array}, \tag{34}$$

where d_j is the degree of vertex z_j for $j = 1, 2, 3, \ldots, p$. Let $z'_1, z'_2, z'_3, \ldots, z'_p$ be the vertices that are added in G to acquire $S'(G)$ such that $N(z_j) = N(z'_j)$. Then the sum-connectivity matrix of $S'(G)$ is denoted by $SC(S'(G))$ and is defined as a block matrix as follows:

$$\begin{array}{cc} & \begin{array}{cccccccccc} z_1 & z_2 & z_3 & \cdots & z_p & z'_1 & z'_2 & z'_3 & \cdots & z'_p \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_p \\ z'_1 \\ z'_2 \\ z'_3 \\ \vdots \\ z'_p \end{array} & \begin{bmatrix} 0 & \frac{1}{\sqrt{d_1+d_2}} & \frac{1}{\sqrt{d_1+d_3}} & \cdots & \frac{1}{\sqrt{d_1+d_p}} & 0 & \frac{1}{\sqrt{d_1+d'_2}} & \frac{1}{\sqrt{d_1+d'_3}} & \cdots & \frac{1}{\sqrt{d_1+d'_p}} \\ \frac{1}{\sqrt{d_2+d_1}} & 0 & \frac{1}{\sqrt{d_2+d_3}} & \cdots & \frac{1}{\sqrt{d_2+d_p}} & \frac{1}{\sqrt{d_2+d'_1}} & 0 & \frac{1}{\sqrt{d_2+d'_3}} & \cdots & \frac{1}{\sqrt{d_2+d'_p}} \\ \frac{1}{\sqrt{d_3+d_1}} & \frac{1}{\sqrt{d_3+d_2}} & 0 & \cdots & \frac{1}{\sqrt{d_3+d_p}} & \frac{1}{\sqrt{d_3+d'_1}} & \frac{1}{\sqrt{d_3+d'_2}} & 0 & \cdots & \frac{1}{\sqrt{d_3+d'_p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{d_p+d_1}} & \frac{1}{\sqrt{d_p+d_2}} & \frac{1}{\sqrt{d_p+d_3}} & \cdots & 0 & \frac{1}{\sqrt{d_p+d'_1}} & \frac{1}{\sqrt{d_p+d'_2}} & \frac{1}{\sqrt{d_p+d'_3}} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d'_1+d_2}} & \frac{1}{\sqrt{d'_1+d_3}} & \cdots & \frac{1}{\sqrt{d'_1+d_p}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{d'_2+d_1}} & 0 & \frac{1}{\sqrt{d'_2+d_3}} & \cdots & \frac{1}{\sqrt{d'_2+d_p}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{d'_3+d_1}} & \frac{1}{\sqrt{d'_3+d_2}} & 0 & \cdots & \frac{1}{\sqrt{d'_3+d_p}} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{d'_p+d_1}} & \frac{1}{\sqrt{d'_p+d_2}} & \frac{1}{\sqrt{d'_p+d_3}} & \cdots & \text{0\&0} & 0 & 0 & \cdots & 0 & \end{bmatrix} \end{array}, \tag{35}$$

where d'_j is the degree of vertex z'_j for $j = 1, 2, 3, \ldots, p$. Thus,

$$\mathrm{SC}(S'(G)) = \begin{bmatrix} \frac{1}{\sqrt{2}}\mathrm{SC}(G) & \sqrt{\frac{2}{3}}\mathrm{SC}(G) \\ \sqrt{\frac{2}{3}}\mathrm{SC}(G) & 0 \end{bmatrix}$$
$$\text{or} \Longrightarrow \mathrm{SC}(S'(G)) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & 0 \end{bmatrix} \otimes \mathrm{SC}(G). \tag{36}$$

Here, the sum-connectivity spectrum of $S'(G)$ is

$$\begin{pmatrix} \left(\frac{3\sqrt{2}+\sqrt{114}}{12}\right)\beta_j & \left(\frac{3\sqrt{2}-\sqrt{114}}{12}\right)\beta_j \\ p & p \end{pmatrix}, \tag{37}$$

where β_j, for $j = 1, 2, 3, \ldots, p$ are the eigenvalues of $\mathrm{SC}(G)$ and $(3\sqrt{2} \pm \sqrt{114})/12$ are the eigenvalues of $\begin{bmatrix} 1/\sqrt{2} & \sqrt{2/3} \\ \sqrt{2/3} & 0 \end{bmatrix}$.

Hence,

$$\begin{aligned} E_{\mathrm{SC}}(S'(G)) &= \sum_{j=1}^{j=p} \left| \left(\frac{3\sqrt{2} \pm \sqrt{114}}{12}\right)\beta_j \right| \\ &= \sum_{j=1}^{j=p} \left|\beta_j\right| \left(\frac{3\sqrt{2}+\sqrt{114}}{12} + \frac{3\sqrt{2}-\sqrt{114}}{12}\right) \\ &= \frac{6\sqrt{2}}{12} \sum_{j=1}^{j=p} \left|\beta_j\right| = \frac{1}{\sqrt{2}} E_{\mathrm{SC}}(G), \end{aligned} \tag{38}$$

which completes the proof. □

7. Degree Sum Energy

In [24], the degree sum energy E_{DS} of a simple connected graph G is defined as the sum of the absolute values of eigenvalues of the degree sum matrix $\mathrm{DS}(G)$ of G. Here, $\mathrm{DS}(G) = \mathrm{ds}_{ij}$ where

$$\mathrm{ds}_{ij} = \begin{cases} d_i + d_j, & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

Here, d_i and d_j are the degrees of vertices v_i and v_j, respectively.

Theorem 6. *For a s-regular graph G with order p,*

$$E_{\mathrm{DS}}(S'(G)) \leq 6s(3p-2). \tag{40}$$

Proof. Let G be a s-regular graph with vertices $w_1, w_2, w_3, \ldots, w_p$. Then the degree sum matrix of G is denoted by $\mathrm{DS}(G)$ and is defined as

$$\mathrm{DS}(G) = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_p \end{array} \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & \cdots & w_p \end{array} \\ \begin{bmatrix} 0 & d_1+d_2 & d_1+d_3 & \cdots & d_1+d_p \\ d_2+d_1 & 0 & d_2+d_3 & \cdots & d_2+d_p \\ d_3+d_1 & d_3+d_2 & 0 & \cdots & d_3+d_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p+d_1 & d_p+d_2 & d_p+d_3 & \cdots & 0 \end{bmatrix} \end{array}, \tag{41}$$

where d_j is degree of vertex w_j for $j = 1, 2, 3, \ldots, p$. Let $w'_1, w'_2, w'_3, \ldots, w'_p$ be the vertices corresponding to vertices $w_1, w_2, w_3, \ldots, w_p$ that are added in G to get the splitting graph $S'(G)$. Then the degree sum matrix of $S'(G)$ is given as

$$\begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_p \\ w'_1 \\ w'_2 \\ w'_3 \\ \vdots \\ w'_p \end{array} \begin{array}{c} \begin{array}{cccccccccc} w_1 & w_2 & w_3 & \cdots & w_p & w'_1 & w'_2 & w'_3 & \cdots & w'_p \end{array} \\ \begin{bmatrix} 0 & d_1+d_2 & d_1+d_3 & \cdots & d_1+d_p & 0 & d_1+d'_2 & d_1+d'_3 & & d_1+d'_p \\ d_2+d_1 & 0 & d_2+d_3 & \cdots & d_2+d_p & d_2+d'_1 & 0 & d_2+d'_3 & \cdots & d_2+d'_p \\ d_3+d_1 & d_3+d_2 & 0 & \cdots & d_3+d_p & d_3+d'_1 & d_3+d'_2 & 0 & \cdots & d_3+d'_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p+d_1 & d_p+d_2 & d_p+d_3 & \cdots & 0 & d_p+d'_1 & d_p+d'_2 & d_p+d'_3 & \cdots & 0 \\ 0 & d'_1+d_2 & d'_1+d_3 & \cdots & d'_1+d_p & 0 & d'_1+d'_2 & d'_1+d'_3 & \cdots & d'_1+d'_p \\ d'_2+d_1 & 0 & d'_2+d_3 & \cdots & d'_2+d_p & d'_2+d'_1 & 0 & d'_2+d'_3 & \cdots & d'_2+d'_p \\ d'_3+d_1 & d'_3+d_2 & 0 & \cdots & d'_3+d_p & d'_3+d'_1 & d'_3+d'_2 & 0 & \cdots & d'_3+d'_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d'_p+d_1 & d'_p+d_2 & d'_p+d_3 & \cdots & 0 & d'_p+d'_1 & d'_p+d'_2 & d'_p+d'_3 & \cdots & 0 \end{bmatrix} \end{array}, \tag{42}$$

where d'_j is the degree of vertex w'_j for $j = 1, 2, 3, \ldots, p$.

Note that

$$\mathrm{DS}(S'(G)) = \begin{bmatrix} 4s\left[J_p - I_p\right] & 3s\left[J_{p\times p}\right] \\ 3s\left[J_{p\times p}\right] & 2s\left[J_p - I_p\right] \end{bmatrix}. \tag{43}$$

Thus,

$$E_{\mathrm{DS}}(S'(G)) \leq \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 4s\left[J_p - I_p\right] & 0I_p \\ 0I_p & 0I_p \end{bmatrix}\right| + \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 3sJ_{p\times p} \\ 3sJ_{p\times p} & 0I_p \end{bmatrix}\right| + \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 0I_p \\ 0I_p & 4s\left[J_p - I_p\right] \end{bmatrix}\right|. \tag{44}$$

As

$$\begin{aligned} \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 4s\left[J_p - I_p\right] & 0I_p \\ 0I_p & 0I_p \end{bmatrix}\right| &= 8s(p-1), \\ \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 3sJ_{p\times p} \\ 3sJ_{p\times p} & 0I_p \end{bmatrix}\right| &= 6sp, \\ \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 0I_p \\ 0I_p & 4s\left[J_p - I_p\right] \end{bmatrix}\right| &= 4s(p-1). \end{aligned} \tag{45}$$

Hence,

$$\begin{aligned} E_{\mathrm{DS}}(S'(G)) &\leq 8s(p-1) + 6sp + 4s(p-1) \\ &= 12s(p-1) + 6sp \\ &= 18sp - 12s \\ &= 6s(3p-2), \end{aligned} \tag{46}$$

which is the required result. □

8. Degree Square Sum Energy

The degree square sum energy $E_{\mathrm{DSS}}(G)$ of a simple connected graph G in [25] is defined as the sum of the absolute values of eigenvalues of the degree square sum matrix $\mathrm{DSS}(G)$. Here, $\mathrm{DSS}(G) = [\mathrm{dss}_{ij}]$ where

$$\mathrm{dss}_{ij} = \begin{cases} d_i^2 + d_j^2, & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

Here, d_i and d_j are degrees of vertices v_i and v_j, respectively.

Theorem 7. *For a s-regular graph G with p vertices,*

$$E_{\mathrm{DSS}}(S'(G)) \leq 2s^2(14p-9). \tag{48}$$

Proof. Let $x_1, x_2, x_3, \ldots, x_p$ be vertices of a s-regular graph G and $x_1', x_2', x_3', \ldots, x_p'$ be the vertices added in G corresponding to $x_1, x_2, x_3, \ldots, x_p$ to get the splitting graph $S'(G)$ such that for $j = 1, 2, 3, \ldots, p$, $N(x_j) = N(x_j')$. Then, the degree square sum matrix of G and $S'(G)$ are given as

$$\mathrm{DSS}(\mathrm{G}) = \begin{array}{c|c} & \begin{array}{ccccc} x_1 & x_2 & x_3 & \cdots & x_p \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{array} & \begin{bmatrix} 0 & d_1^2 + d_2^2 & d_1^2 + d_3^2 & \cdots & d_1^2 + d_p^2 \\ d_2^2 + d_1^2 & 0 & d_2^2 + d_3^2 & \cdots & d_2^2 + d_p^2 \\ d_3^2 + d_1^2 & d_3^2 + d_2^2 & 0 & \cdots & d_3^2 + d_p^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p^2 + d_1^2 & d_p^2 + d_2^2 & d_p^2 + d_3^2 & \cdots & 0 \end{bmatrix} \end{array},$$

$$\begin{array}{c|c} & \begin{array}{cccccccccc} x_1 & x_2 & x_3 & \cdots & x_p & x_1' & x_2' & x_3' & \cdots & x_p' \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \\ x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_p' \end{array} & \begin{bmatrix} 0 & d_1^2 + d_2^2 & d_1^2 + d_3^2 & \cdots & d_1^2 + d_p^2 & 0 & d_1^2 + d_2'^2 & d_1^2 + d_3'^2 & \cdots & d_1^2 + d_p'^2 \\ d_2^2 + d_1^2 & 0 & d_2^2 + d_3^2 & \cdots & d_2^2 + d_p^2 & d_2^2 + d_1'^2 & 0 & d_2^2 + d_3'^2 & \cdots & d_2^2 + d_p'^2 \\ d_3^2 + d_1^2 & d_3^2 + d_2^2 & 0 & \cdots & d_3^2 + d_p^2 & d_3^2 + d_1'^2 & d_3^2 + d_2'^2 & 0 & \cdots & d_3^2 + d_p'^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p^2 + d_1^2 & d_p^2 + d_2^2 & d_p^2 + d_3^2 & \cdots & 0 & d_p^2 + d_1'^2 & d_p^2 + d_2'^2 & d_p^2 + d_3'^2 & \cdots & 0 \\ 0 & d_1'^2 + d_2^2 & d_1'^2 + d_3^2 & \cdots & d_1'^2 + d_p^2 & 0 & d_1'^2 + d_2'^2 & d_1'^2 + d_3'^2 & \cdots & d_1'^2 + d_p'^2 \\ d_2'^2 + d_1^2 & 0 & d_2'^2 + d_3^2 & \cdots & d_2'^2 + d_p^2 & d_2'^2 + d_1'^2 & 0 & d_2'^2 + d_3'^2 & \cdots & d_2'^2 + d_p'^2 \\ d_3'^2 + d_1^2 & d_3'^2 + d_2^2 & 0 & \cdots & d_3'^2 + d_p^2 & d_3'^2 + d_1'^2 & d_3'^2 + d_{2'}^2 & 0 & \cdots & d_3'^2 + d_p'^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p'^2 + d_1^2 & d_p'^2 + d_2^2 & d_p'^2 + d_3^2 & \cdots & 0 & d_p'^2 + d_1'^2 & d_p'^2 + d_2'^2 & d_p'^2 + d_{3'}^2 & \cdots & 0 \end{bmatrix} \end{array}, \tag{49}$$

where d_i is the degree of vertex x_i and d_i' is the degree of vertex x_i', for $i = 1, 2, 3, \ldots, p$.

Note that

$$\mathrm{DSS}(S'(G)) = \begin{bmatrix} 8s^2\left[J_p - I_p\right] & 5s^2[J(p\times p)] \\ 5s^2[J(p\times p)] & 2s^2\left[J_p - I_p\right] \end{bmatrix}. \tag{50}$$

Thus,

$$\begin{aligned} E_{\mathrm{DSS}}(S'(G)) \le & \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 8s^2\left[J_p - I_p\right] & 0I_p \\ 0I_p & 0I_p \end{bmatrix}\right| \\ & + \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 5s^2[J(p\times p)] \\ 5s^2[J(p\times p)] & 0I_p \end{bmatrix}\right| + \sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 0I_p \\ 0I_p & 2s^2\left[J_p - I_p\right] \end{bmatrix}\right|. \end{aligned} \tag{51}$$

As

$$\begin{aligned} &\sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 8s^2\left[J_p - I_p\right] & 0I_p \\ 0I_p & 0I_p \end{bmatrix}\right| = 16s^2(p-1), \\ &\sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 5s^2[J(p\times p)] \\ 5s^2[J(p\times p)] & 0I_p \end{bmatrix}\right| = 10s^2 p, \\ &\sum_{j=1}^{2p}\left|\mu_j\begin{bmatrix} 0I_p & 0I_p \\ 0I_p & 2s^2\left[J_p - I_p\right] \end{bmatrix}\right| = 4s^2(p-1). \end{aligned} \tag{52}$$

Hence,

$$\begin{aligned} E_{\mathrm{DSS}}(S'(G)) &\le 16s^2(p-1) + 10s^2 p + 4s^2(p-1) \\ &= 20s^2(p-1) + 10s^2 p \\ &= 30s^2 p - 20s^2 \\ &= 10s^2(3p-2), \end{aligned} \tag{53}$$

which completes the proof. □

9. First Zagreb Energy

In [26], the First zagreb energy ZE_1 of a simple connected graph G is defined as the sum of the absolute values of eigenvalues of first zagreb matrix $Z^{(1)}(G)$ of G where $Z^{(1)}(G) = [z_{ij}^{(1)}]$ where

$$z_{ij}^{(1)} = \begin{cases} d_i + d_j, & \text{if } v_i, v_j \epsilon E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{54}$$

Here, d_i and d_j are degrees of vertices v_i and v_j, respectively.

Theorem 8. *For a regular graph G,*

$$\mathrm{ZE}_1(S'(G)) = 2\mathrm{ZE}_1(G). \tag{55}$$

Proof. Let G be a graph with vertices $z_1, z_2, z_3, \ldots, z_p$. Then, the first zagreb matrix of G is denoted by $Z^{(1)}(G)$ and is defined as

$$Z^{(1)}(G) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_p \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} z_1 & z_2 & z_3 & \cdots & z_p \end{array} \\ \begin{bmatrix} 0 & d_1+d_2 & d_1+d_3 & \cdots & d_1+d_p \\ d_2+d_1 & 0 & d_2+d_3 & \cdots & d_2+d_p \\ d_3+d_1 & d_3+d_2 & 0 & \cdots & d_3+d_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_p+d_1 & d_p+d_2 & d_p+d_3 & \cdots & 0 \end{bmatrix} \end{array}, \tag{56}$$

where d_j is the degree of vertex z_j for $j = 1, 2, 3, \ldots, p$. Let $z_1', z_2', z_3', \ldots, z_p''$ be vertices added in G corresponding to $z_1, z_2, z_3, \ldots, z_p$ to get $S'(G)$ such that $N(z_i) = N(z_i')$. Then, the first zagreb matrix of $S'(G)$ can be written as a block matrix as follows:

$$\begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_p \\ z_1' \\ z_2' \\ z_3' \\ \vdots \\ z_p' \end{array}
\begin{array}{c} \begin{array}{cccccccccc} z_1 & z_2 & z_3 & \dots & z_p & z_1' & z_2' & z_3' & \dots & z_p' \end{array} \\
\begin{bmatrix}
0 & d_1+d_2 & d_1+d_3 & \dots & d_1+d_p & 0 & d_1+d_2' & d_1+d_3' & \dots & d_1+d_p' \\
d_2+d_1 & 0 & d_2+d_3 & \dots & d_2+d_p & d_2+d_1' & 0 & d_2+d_3' & \dots & d_2+d_p' \\
d_3+d_1 & d_3+d_2 & 0 & \dots & d_3+d_p & d_3+d_1' & d_3+d_2' & 0 & \dots & d_3+d_p' \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
d_p+d_1 & d_p+d_2 & d_p+d_3 & \dots & 0 & d_p+d_1' & d_p+d_2' & d_p+d_3' & \dots & 0 \\
0 & d_1'+d_2 & d_1'+d_3 & \dots & d_1'+d_p & 0 & 0 & 0 & \dots & 0 \\
d_2'+d_1 & 0 & d_2'+d_3 & \dots & d_2'+d_p & 0 & 0 & 0 & \dots & 0 \\
d_3'+d_1 & d_3'+d_2 & 0 & \dots & d_3'+d_p & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
d_p'+d_1 & d_p'+d_2 & d_p'+d_3 & \dots & 0 & 0 & 0 & 0 & \dots & 0
\end{bmatrix} \end{array}, \quad (57)$$

where d_j' is the degree of vertex z_j' for $j = 1, 2, 3, \dots, p$.

Here,

$$Z^{(1)}(S'(G)) = \begin{bmatrix} 2Z^{(1)}(G) & \frac{3}{2}Z^{(1)}(G) \\ \frac{3}{2}Z^{(1)}(G) & 0 \end{bmatrix}$$

$$\text{or } Z^{(1)}(S'(G)) = \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{bmatrix} \otimes Z^{(1)}(G). \quad (58)$$

Here, the first zagreb spectrum of $S'(G)$ is

$$\begin{pmatrix} \left(\frac{2-\sqrt{13}}{2}\right)\zeta_j & \left(\frac{2+\sqrt{13}}{2}\right)\zeta_j \\ p & p \end{pmatrix}, \quad (59)$$

where ζ_j for $j = 1, 2, 3, \dots, p$ are the eigenvalues of $Z^{(1)}(G)$and $((2 \pm \sqrt{13})/2)$ are the eigenvalues of $\begin{bmatrix} 2 & 3/2 \\ 3/2 & 0 \end{bmatrix}$. Hence,

$$\begin{aligned} ZE_1(S'(G)) &= \sum_{j=1}^{j=p} \left| \left(\frac{2 \pm \sqrt{13}}{2}\right)\zeta_j \right| \\ &= \sum_{j=1}^{j=p} |\zeta_j| \left(\frac{2-\sqrt{13}}{2} + \frac{2+\sqrt{13}}{2}\right) \\ &= 2ZE_1(G). \end{aligned} \quad (60)$$

□

10. Second Zagreb Energy

The second zagreb energy ZE_2 of a simple connected graph G is defined in [26] as the sum of the absolute values of eigenvalues of the second zagreb matrix $Z^{(2)}(G)$ of G where $Z^{(2)}(G) = [z_{ij}^{(2)}]$, where

$$z_{ij}^{(2)} = \begin{cases} d_i \cdot d_j, & \text{if } v_i, v_j \epsilon E(G), \\ 0, & \text{otherwise.} \end{cases} \quad (61)$$

Here, d_i and d_j are the degrees of vertices v_i and v_j, respectively.

Theorem 9. *For a regular graph G,*

$$ZE_2(S'(G)) = 4ZE_2(G). \quad (62)$$

Proof. Let G be a graph with vertices $z_1, z_2, z_3, \dots, z_p$. Then, the second zagreb matrix of G is denoted by $Z^{(2)}(G)$ and is defined as

$$Z^{(2)}(G) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_p \end{array}
\begin{array}{c} \begin{array}{ccccc} z_1 & z_2 & z_3 & \cdots & z_p \end{array} \\
\begin{bmatrix}
0 & d_1 \cdot d_2 & d_1 \cdot d_3 & \cdots & d_1 \cdot d_p \\
d_2 \cdot d_1 & 0 & d_2 \cdot d_3 & \cdots & d_2 \cdot d_p \\
d_3 \cdot d_1 & d_3 \cdot d_2 & 0 & \cdots & d_3 \cdot d_p \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
d_p \cdot d_1 & d_p \cdot d_2 & d_p \cdot d_3 & \cdots & 0
\end{bmatrix} \end{array}, \quad (63)$$

where d_j is the degree of vertex z_j for $j = 1, 2, 3, \dots, p$. Let $z_1', z_2', z_3', \dots, z_p'$ be vertices added in G corresponding to $z_1, z_2, z_3, \dots, z_p$ to get $S'(G)$ such that $N(z_j) = N(z_j')$. Then, the second Zagreb matrix of $S'(G)$ is denoted by $Z^{(2)}(S'(G))$ and can be written as a square matrix as follows:

$$\begin{array}{c|c}
 & \begin{matrix} z_1 & z_2 & z_3 & \cdots & z_p & z_1' & z_2' & z_3' & \cdots & z_p' \end{matrix} \\
\begin{matrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_p \\ z_1' \\ z_2' \\ z_3' \\ \vdots \\ z_p' \end{matrix} &
\begin{bmatrix}
0 & d_1\cdot d_2 & d_1\cdot d_3 & \cdots & d_1\cdot d_p & 0 & d_1\cdot d_2' & d_1\cdot d_3' & \cdots & d_1\cdot d_p' \\
d_2\cdot d_1 & 0 & d_2\cdot d_3 & \cdots & d_2\cdot d_p & d_2\cdot d_1' & 0 & d_2\cdot d_3' & \cdots & d_2\cdot d_p' \\
d_3\cdot d_1 & d_3\cdot d_2 & 0 & \cdots & d_3\cdot d_p & d_3\cdot d_1' & d_3\cdot d_2' & 0 & \cdots & d_3\cdot d_p' \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
d_p\cdot d_1 & d_p\cdot d_2 & d_p\cdot d_3 & \cdots & 0 & d_p\cdot d_1' & d_p\cdot d_2' & d_p\cdot d_3' & \cdots & 0 \\
0 & d_1'\cdot d_2 & d_1'\cdot d_3 & \cdots & d_1'\cdot d_p & 0 & 0 & 0 & \cdots & 0 \\
d_2'\cdot d_1 & 0 & d_2'\cdot d_3 & \cdots & d_2'\cdot d_p & 0 & 0 & 0 & \cdots & 0 \\
d_3'\cdot d_1 & d_3'\cdot d_2 & 0 & \cdots & d_3'\cdot d_p & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
d_p'\cdot d_1 & d_p'\cdot d_2 & d_p'\cdot d_3 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\end{array}, \tag{64}$$

where d_j' is the degree of vertex z_j' for $j = 1, 2, 3, \ldots, p$.

Here,

$$\begin{aligned} Z^{(2)}(S'(G)) &= \begin{bmatrix} 4Z^{(2)}(G) & 2Z^{(2)}(G) \\ 2Z^{(2)}(G) & 0 \end{bmatrix}, \\ Z^{(2)}(S'(G)) &= \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix} \otimes Z^{(2)}(G). \end{aligned} \tag{65}$$

Here, the second zagreb spectrum of $S'(G)$ is

$$\begin{pmatrix} (2+2\sqrt{2})\eta_j & (2-2\sqrt{2})\eta_j \\ p & p \end{pmatrix}, \tag{66}$$

where η_j for $j = 1, 2, 3, \ldots, p$ are the eigenvalues of $Z^{(2)}(G)$and $2 \pm 2\sqrt{2}$ are the eigenvalues of $\begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}$. Hence,

$$\begin{aligned} \mathrm{ZE}_2(S'(G)) &= \sum_{j=1}^{j=p} \left|(2 \pm 2\sqrt{2})\eta_j\right| \\ &= \sum_{j=1}^{j=p} |\eta_i| (2 + 2\sqrt{2} + 2 - 2\sqrt{2}) \\ &= \sum_{j=1}^{j=p} |\eta_i| (4) \\ &= 4\mathrm{ZE}_2(G). \end{aligned} \tag{67}$$

□

11. Conclusion

The energy of a graph is one of the important idea of spectral graph theory. This idea links organic chemistry to mathematics. Numerous graph energies established from the eigenvalues of a variety of graph matrices and their bounds has been discovered. In this paper, we give a relation of various graph energies between the regular graph and its splitting graph. It is interesting to compute graph energies for the families of graphs considered in [27–31].

Data Availability

All data are included within this paper.

Conflicts of Interest

The authors declare no conflicts of interests

Authors' Contributions

All authors contributed equally to this work.

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