Research Article

Computing Eccentricity-Based Topological Indices of 2-Power Interconnection Networks

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In a connected graph G with a vertex v , the eccentricity $\varepsilon(v)$ of v is the distance between v and a vertex farthest from v in the graph G . Among eccentricity-based topological indices, the eccentric connectivity index, the total eccentricity index, and the Zagreb index are of vital importance. The eccentric connectivity index of G is defined by $\xi(G) = \sum_{v \in V(G)} d(v)\varepsilon(v)$, where $d(v)$ is the degree of the vertex v and $\varepsilon(v)$ is the eccentricity of v in G . The topological structure of an interconnected network can be modeled by using graph explanation as a tool. This fact has been universally accepted and used by computer scientists and engineers. More than that, practically, it has been shown that graph theory is a very powerful tool for designing and analyzing the topological structure of interconnection networks. The topological properties of the interconnection network have been computed by Hayat and Imran (2014), Haynes et al. (2002), and Imran et al. (2015). In this paper, we compute the close results for eccentricity-based topological indices such as the eccentric connectivity index, the total eccentricity index, and the first, second, and third Zagreb eccentricity index of a hypertree, sibling tree, and X -tree for k -level by using the edge partition method.

1. Introduction

The advancement of large-scale integrated circuit technology has enabled the construction of complex interconnection networks. Graph theory provides a fundamental tool for designing and analyzing such networks. Graph theory and interconnection networks provide a thorough understanding of these interrelated topics [1–3]. The architecture of an interconnected network is represented by a graph, where nodes represent the processors and edges represent the links between processors. Electric power companies need to continually monitor the state of their systems as in the case of the voltage magnitude at loads and the machine phase angle at generators. In the electric power system, a vertex

represents an electric node and an edge represents a transmission line joining two electrical nodes [4, 5]. *Chemical graph theory* is a branch of mathematical chemistry in which we apply tools of graph theory to model the chemical phenomenon mathematically. This theory contributes a prominent role in the fields of chemical sciences. A molecular or chemical graph is a simple finite graph in which vertices denote the atoms and edges denote the chemical bonds in the underlying chemical structure. A *topological index* is actually a numeric quantity associated with chemical constitution purporting for correlation of the chemical structure with many physiochemical properties, chemical reactivity, and biological activities. Let G be a graph with u and v being the vertices of G , then the distance $d(u, v)$ is

defined as the shortest length between u and v . The *eccentricity* $\varepsilon(v)$ of a vertex $v \in V(G)$ is defined as

$$\varepsilon(v) = \max\{d(v, w) | w \in V(G)\}. \quad (1)$$

The minimum eccentricity in a graph G is known as the *radius* $\text{rad}(G)$, while the maximum eccentricity in a graph G is known as the *diameter* $2p - 2$. Topological descriptors play an important role in the quantitative structure-activity (QSAR) and structure-property (QSPR) study. Topological indices, which are based on eccentricity of the vertices in a graph G , are known as eccentricity-based topological indices. The *eccentric connectivity index* [6] of G is defined by

$$\xi(G) = \sum_{v \in V(G)} d(v)\varepsilon(v), \quad (2)$$

where $d(v)$ is the degree of the vertex v and $\varepsilon(v)$ is the eccentricity of v in G .

The *total eccentricity index* [7] is defined as the summation of eccentricity of each vertex in graph G . In other words, when the vertex degrees are not considered in the eccentric connectivity index, then we obtain the total eccentricity index:

$$\zeta(G) = \sum_{u \in V(G)} \varepsilon(u), \quad (3)$$

where $\varepsilon(v)$ is the eccentricity of u in G .

Zagreb indices [8] have been introduced more than thirty years ago. Let G be a graph with u and v being the vertices of G ; then the Zagreb indices are defined as follows:

$$\begin{aligned} M_1(G) &= \sum_{v \in V(G)} (d(v))^2, \\ M_2(G) &= \sum_{uv \in E(G)} (d(u)d(v)). \end{aligned} \quad (4)$$

Some new modified versions of Zagreb indices [9] are expressed in terms of eccentricity as follows:

$$\begin{aligned} M_1^*(G) &= \sum_{uv \in E(G)} (\varepsilon(u) + \varepsilon(v)), \\ M_1^{**}(G) &= \sum_{v \in V(G)} (\varepsilon(v))^2, \\ M_2^*(G) &= \sum_{uv \in E(G)} (\varepsilon(u)\varepsilon(v)). \end{aligned} \quad (5)$$

On further results for certain degree-based topological indices of networks and nanostructures, consult [1, 10–13]. A *tree* is a connected acyclic graph, and the common type of a tree is a binary tree. A *binary tree* is made of nodes, where each node contains a left reference, a right reference, and a data element. The top most node is called the root. The vertex of the binary tree has three fields. The first field represents data, while the second and third contain information of the left and right sons of the vertex. If each internal vertex/node has exactly two descendents, then the binary tree is said to be a complete binary tree as shown in Figure 1. The basic skeleton of the *hypertree*-level is a complete binary tree $HT(k)$. The nodes of the trees are labeled in such a way that root node has label 1. The root is at level 0. Labels

FIGURE 1: Binary tree.

of the left and right children are formed by appending 0 and 1, respectively, to the labels of the parent node. Here, the children of the nodes x are labeled as $2x$ and $2x + 1$. Additional links in a hypertree are horizontal, and two nodes in the same level of the tree are joined if their label difference is 2^{i-2} [14]. The hypertree k -level $HT(k)$ shown in Figure 2 has vertices $2^{k+1} - 1$ and edges $3(2^k - 1)$.

2. Main Results

In this section, we compute the close results for eccentricity-based topological indices such as the eccentric connectivity index, the total eccentricity index, and the first, second, and third Zagreb eccentricity index of a hypertree, sibling tree, and X -tree for k -level by using the edge partition method. The molecular topological descriptors of fullerenes and several interconnection networks have been already computed in the literature [1–3, 15].

In the next theorem, an exact expression for the eccentric connectivity index for a binary tree is computed.

Theorem 1. Consider the graph $G \cong HT(k)$, then the eccentric connectivity index is equal to

$$\xi(G) = 2k + 4 \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} 2^i p + 2^{k+1} (2k - 1). \quad (6)$$

Proof. In order to prove the above result, we use the formula of the eccentric connectivity index:

$$\xi(G) = \sum_{v \in V(G)} d(v)\varepsilon(v). \quad (7)$$

By using Table 1, we have

$$\xi(G) = 2 \sum_{p=k} \sum_{i=0} 2^i p + 4 \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} 2^i p + 2 \sum_{i=k} \sum_{p=k} (2p - 1) 2^i. \quad (8)$$

After an easy calculation, we get

FIGURE 2: k -level hypertree $HT(k)$.TABLE 1: Vertices partition of a hypertree (k -level) based on degree and eccentricity of each vertex with the existence of its frequencies.

$d(v)$	$\varepsilon(v)$	Frequency	Range of i	Range of p
2	p	2^i	$i = 0$	$p = k$
4	p	2^i	$1 \leq i \leq k-1$	$k \leq p \leq 2k-2$
2	$2p-1$	2^i	$i = k$	$p = k$

$$\xi(G) = 2k + 4 \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} 2^i p + 2^{k+1} (2k-1). \quad (9)$$

The total eccentricity index of a binary tree is computed in the following theorem. $\square$

Theorem 2. Consider the graph $G \cong HT(k)$, then the total eccentricity index is equal to

$$\zeta(G) = k + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} 2^i p + (2k-1)2^k. \quad (10)$$

Proof. Let G be a graph of a hypertree (k -level). To prove (10), we use the total eccentricity index formula:

$$\zeta(G) = \sum_{u \in V(G)} \varepsilon(u). \quad (11)$$

By using Table 1, we get

$$\zeta(G) = \sum_{p=k} \sum_{i=0} 2^i p + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p \times 2^i + \sum_{i=k} \sum_{p=k} 2^i (2p-1). \quad (12)$$

After an easy calculation, we get

$$\zeta(G) = k + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} 2^i p + (2k-1)2^k. \quad (13)$$

The first Zagreb eccentricity index of a binary tree is computed in the following theorem. $\square$

Theorem 3. Consider the graph $G \cong HT(k)$, then the first Zagreb eccentricity index is equal to

$$M_1^*(G) = 6k + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (2p+1)2^{i+1} + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (2p+2)2^i. \quad (14)$$

Proof. In order to proof (14), we use the first Zagreb eccentricity index formula:

$$M_1^*(G) = \sum_{uv \in E(G)} \{\varepsilon(u) + \varepsilon(v)\}. \quad (15)$$

By using Table 2, we get

$$M_1^*(G) = 3 \sum_{p=k} \sum_{i=0} 2p2^i + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (2p+1)2^{i+1} + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (2p+2)2^i. \quad (16)$$

After an easy calculation, we get

$$M_1^*(G) = 6k + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (2p+1)2^{i+1} + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (2p+2)2^i. \quad (17)$$

$\square$

Theorem 4. Consider the graph $G \cong HT(k)$, then the second Zagreb eccentricity index is equal to

$$M_1^{**}(G) = k^2 + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p^2 2^i + (2k-1)^2 2^k. \quad (18)$$

Proof. Let G be a graph of a hypertree (k -level). The formula of the second Zagreb eccentricity index is given by

$$M_1^{**}(G) = \sum_{v \in V(G)} [\varepsilon(v)]^2. \quad (19)$$

By using Table 1, we get

$$M_1^{**}(G) = \sum_{p=k} \sum_{i=0} p^2 2^i + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p^2 2^i + \sum_{i=k} \sum_{p=k} (2p-1)^2 2^i. \quad (20)$$

After an easy calculation, we get

$$M_1^{**}(G) = k^2 + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p^2 2^i + (2k-1)^2 2^k. \quad (21)$$

$\square$

Theorem 5. Consider the graph $G \cong HT(k)$, then the third Zagreb eccentricity index is equal to

$$M_2^*(G) = 3k^2 + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p(p+1)2^{i+1} + \sum_{i=k} \sum_{p=k} (p+1)^2 2^i. \quad (22)$$

Proof. In order to prove (22), we use the formula of the third Zagreb eccentricity index:

$$M_2^*(G) = \sum_{uv \in E(G)} \varepsilon(u) \times \varepsilon(v). \quad (23)$$

By using Table 2, we get

TABLE 2: Edge partition of a hypertree (k -level) based on eccentricity of end vertices of each edge with existence of its frequencies.

$(\varepsilon(u), \varepsilon(v))$	Frequency	Range of i	Range of p
(p, p)	$3 \cdot 2^i$	$i = 0$	$p = k$
$(p, p+1)$	2^{i+1}	$1 \leq i \leq k-1$	$k \leq p \leq 2k-2$
$(p+1, p+1)$	2^i	$1 \leq i \leq k-1$	$k \leq p \leq 2k-2$

$$M_2^*(G) = 3 \sum_{p=k} \sum_{i=0} p^2 2^i + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p(p+1) 2^{i+1} + \sum_{i=k} \sum_{p=k} (p+1)^2 2^i. \quad (24)$$

After an easy calculation, we get

$$M_2^*(G) = 3k^2 + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} p(p+1) 2^{i+1} + \sum_{i=1}^{k-1} \sum_{p=k}^{2k-2} (p+1)^2 2^i. \quad (25)$$

The 1-rooted *sibling tree* ST_k^1 shown in Figure 3 is obtained from the 1-rooted complete binary tree T_k^1 by adding edges (sibling edges) between the left and right children of the same parent node [16].

An *X-tree* $XT(k)$ shown in Figure 4 is obtained from a complete binary tree on $2^{k+1} - 1$ vertices of height $2^i - 1$ and adding paths P_i left to right through all the vertices at level i , $1 \leq i \leq k$.

A graph $G_1 = (V_1, E_1)$ is said to be isomorphic to the graph $G_2 = (V_2, E_2)$, if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1 , then the corresponding edge e_2 in G_2 has its end points in the vertices u_2 and v_2 in G_2 which correspond to u_1 and v_1 , respectively. Such a pair of correspondences is called the *graph isomorphism*. $\square$

Remark 1. $HT(k) \not\cong ST_k^1$, but their topological eccentricity-based indices are equal:

$$\begin{aligned} \xi(HT(k)) &= \xi(ST_k^1), \quad \zeta(HT(k)) = \zeta(ST_k^1), \quad M_1^*(HT(k)) \\ &= M_1^*(ST_k^1), \\ M_1^{**}(HT(k)) &= M_1^{**}(ST_k^1), \quad M_2^*(HT(k)) = M_2^*(ST_k^1). \end{aligned} \quad (26)$$

Theorem 6. Consider the graph $G \cong XT(k)$ and $k \geq 3$, then the eccentric connectivity index is equal to

$$\begin{aligned} \xi(G) &= 2k + 8 \sum_{p=k}^{2k-2} p + 4(2k-1) + 5 \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} 2^i p + 3(2k-2) 2^{k-1} \\ &\quad + 5 \sum_{p=k+2, k \geq 3}^{2k-2} \sum_{i=1, k \geq 3}^{k-3} (2(2^i - 1)p) + 3(2k-1)(2(2^{k-2} - 1)). \end{aligned} \quad (27)$$

FIGURE 3: 1-rooted sibling tree ST_k^1 .

FIGURE 4: k -level X-tree $XT(k)$.

Proof. In order to prove the above result, we use the formula of the eccentric connectivity index:

$$\xi(G) = \sum_{v \in V(G)} d(v) \varepsilon(v). \quad (28)$$

By using Table 3, we have

$$\begin{aligned} \xi(G) &= 2 \sum_{p=k, k \geq 3} p \sum_{i=0} 2^i + 4 \sum_{i=1}^{2k-2} 2^i \sum_{p=k, k \geq 3} p \\ &\quad + 2 \sum_{i=1}^{2k-2} 2^i \sum_{p=k, k \geq 3} (2p-1) + 5 \sum_{p=k, k \geq 3}^{2k-3} p \sum_{i=1, k \geq 3}^{k-2} 2^i \\ &\quad + 5 \sum_{p=k+2, k \geq 3}^{2k-2} p \sum_{i=1, k \geq 3}^{k-3} 2(2^i - 1) \\ &\quad + 3 \sum_{p=k, k \geq 3} (2p-2) \sum_{i=k-1, k \geq 3} 2^i. \end{aligned} \quad (29)$$

After an easy calculation, we get

$$\begin{aligned} \xi(G) &= 2k + 8 \sum_{p=k}^{2k-2} p + 4 \sum_{p=k}^{2k-2} (2p-1) + 5 \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} 2^i p + 3(2k-2) 2^{k-1} \\ &\quad + 5 \sum_{p=k+2, k \geq 3}^{2k-2} \sum_{i=1, k \geq 3}^{k-3} (2(2^i - 1)p) + 3(2k-1)(2(2^{k-2} - 1)). \end{aligned} \quad (30)$$

Theorem 7. Consider the graph $G \cong XT(k)$ and $k \geq 3$, then the total eccentricity index is equal to

$$\begin{aligned} \zeta(G) &= k + 2 \sum_{p=k}^{2k-2} p + 2(2k-1) + \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} 2^i p \\ &\quad + \sum_{p=k+2, k \geq 3}^{2k-2} \sum_{i=1, k \geq 3}^{k-3} (2(2^i - 1)p) \\ &\quad + (2k-1)(2(2^{k-2} - 1)) + (2k-2) 2^{k-1}. \end{aligned} \quad (31)$$

TABLE 3: Vertices partition of an X -tree (k -level) based on degree and eccentricity of each vertex with the existence of its frequencies.

$d(v)$	$\varepsilon(v)$	Frequency	Range of i	Range of p
2	p	2^i	$i = 0$	$p = k$
4	p	2^i	$i = 1$	$k \leq p \leq 2k - 2$
2	$2p - 1$	2^i	$i = 1$	$p = k$
5	p	2^i	$1 \geq i \geq k - 2$	$k \geq p \geq 2k - 3$
5	p	$2(2^i - 1)$	$1 \geq i \geq k - 3, k > 3$	$k + 2 \geq p \geq 2k - 2, k > 3$
3	$2p - 1$	$2(2^i - 1)$	$i = k - 2$	$p = k$
3	$2p - 2$	2^i	$i = k - 1$	$p = k$

Proof. Let G be a graph of an X -tree (k -level). In order to prove, we use the total eccentricity index formula:

$$\zeta(G) = \sum_{u \in V(G)} \varepsilon(u). \quad (32)$$

By using Table 3, we get

$$\begin{aligned} \zeta(G) = & \sum_{p=k} p \sum_{i=0} 2^i + \sum_{i=1} 2^i \sum_{p=k} p + \sum_{i=1} 2^i \sum_{p=k} (2p - 1) \\ & + \sum_{p=k} p \sum_{i=1}^{2k-3} 2^i + \sum_{p=k+2, k>3}^{2k-2} p \sum_{i=1, k>3}^{2k-2} (2(2^i - 1)) \\ & + \sum_{p=k} (2p - 1) \sum_{i=k-2} (2(2^i - 1)) \\ & + \sum_{p=k} (2p - 2) \sum_{i=k-1} 2^i. \end{aligned} \quad (33)$$

After an easy calculation, we get

$$\begin{aligned} \zeta(G) = & k + 2 \sum_{p=k}^{2k-2} p + 2(2k - 1) + \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} 2^i p \\ & + \sum_{p=k+2, k>3}^{2k-2} \sum_{i=1, k>3}^{k-3} (2(2^i - 1)p) \\ & + (2k - 1)(2(2^{k-2} - 1)) + (2k - 2)2^{k-1}. \end{aligned} \quad (34)$$

□

Theorem 8. Consider the graph $G \cong XT(k)$ and $k \geq 3$, then the first Zagreb eccentricity index is equal to

$$\begin{aligned} M_1^*(G) = & 12k + \sum_{i=1}^{k-2} \sum_{p=k+1}^{2k-2} (2p)(3(2^i - 1)) \\ & + (4k - 3)(2^{k-1} + 2) \\ & + \sum_{i=1}^{k-2} \sum_{p=k}^{2k-3} (2p + 1)(2(3(2^{i-1}) + 1)) \\ & + (4k - 2)(2(2^{k-2} - 1)). \end{aligned} \quad (35)$$

Proof. In order to prove the above result, we use the first Zagreb eccentricity index formula:

$$M_1^*(G) = \sum_{uv \in E(G)} \{\varepsilon(u) + \varepsilon(v)\}. \quad (36)$$

By using Table 4, we get

$$\begin{aligned} M_1^*(G) = & \sum_{p=k} (p + p) \sum_{i=0} 6(2^i) + \sum_{i=1}^{k-2} 3(2^i - 1) \sum_{p=k+1}^{2k-2} (p + p) \\ & + \sum_{i=1}^{k-2} (2(3(2^{i-1}) + 1)) \sum_{p=k}^{2k-3} (p + p + 1) \\ & + \sum_{p=k} (2p - 2 + 2p - 1) \sum_{i=k-2} (2^{i+1} + 2) \\ & + \sum_{p=k} (2p - 1 + 2p - 1) \sum_{i=k-2} (2(2^i - 1)). \end{aligned} \quad (37)$$

After an easy calculation, we get

$$\begin{aligned} M_1^*(G) = & 12k + \sum_{i=1}^{k-2} \sum_{p=k+1}^{2k-2} (2p)(3(2^i - 1)) \\ & + (4k - 3)(2^{k-1} + 2) \\ & + \sum_{i=1}^{k-2} \sum_{p=k}^{2k-3} (2p + 1)(2(3(2^{i-1}) + 1)) \\ & + (4k - 2)(2(2^{k-2} - 1)). \end{aligned} \quad (38)$$

□

Theorem 9. Consider the graph $G \cong XT(k)$ and $k \geq 3$, then the second Zagreb eccentricity index is equal to

$$\begin{aligned} M_1^{**}(G) = & k^2 + 2 \sum_{p=k}^{2k-2} p^2 + 2(2k - 1)^2 + \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} 2^i p^2 \\ & + \sum_{p=k+2, k>3}^{2k-2} \sum_{i=1}^{k-3} (2(2^i - 1)p^2) \\ & + (2k - 1)^2(2(2^{k-2} - 1)) + (2k - 2)^2 2^{k-1}. \end{aligned} \quad (39)$$

Proof. Let G be a graph of an X -tree (k -level) and $k \geq 3$. The formula of the second Zagreb eccentricity index is given by

$$M_1^{**}(G) = \sum_{v \in V(G)} [\varepsilon(v)]^2. \quad (40)$$

By using Table 3, we get the following expression:

$$\begin{aligned} M_1^{**}(G) = & \sum_{p=k} p^2 \sum_{i=0} 2^i + \sum_{i=1} 2^i \sum_{p=k} p^2 + \sum_{i=1} 2^i \sum_{p=k} (2p - 1)^2 \\ & + \sum_{i=1}^{k-2} \sum_{p=k}^{2k-3} p^2 + \sum_{i=1, k>3}^{2k-3} (2(2^i - 1)) \sum_{p=k+2, k>3}^{2k-2} p^2 \\ & + \sum_{i=k-2} (2(2^i - 1)) \sum_{p=k} (2p - 1)^2 \\ & + \sum_{i=k-1} 2^i \sum_{p=k} (2p - 2)^2. \end{aligned} \quad (41)$$

TABLE 4: Edge partition of an X -tree (k -level) based on eccentricity of end vertices of each edge with existence of its frequencies.

$(\varepsilon(u), \varepsilon(v))$	Frequency	Range of i	Range of p
(p, p)	$6(2^i)$	$i = 0$	$p = k$
(p, p)	$3(2^i - 1)$	$1 \leq i \leq k - 2$	$k + 1 \leq p \leq 2k - 2$
$(p, p + 1)$	$2(3(2^{i-1}) + 1)$	$1 \leq i \leq k - 2$	$k \leq p \leq 2k - 3$
$(2p - 2, 2p - 1)$	$2^{i+1} + 2$	$i = k - 2$	$p = k$
$(2p - 1, 2p - 1)$	$2(2^i - 1)$	$i = k - 2$	$p = k$

After an easy calculation, we get

$$\begin{aligned}
 M_1^{**}(G) &= k^2 + 2 \sum_{p=k}^{2k-2} p^2 + 2(2k-1)^2 + \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} 2^i p^2 \\
 &+ \sum_{p=k+2, k>3}^{2k-2} \sum_{i=1}^{k-3} (2(2^i - 1)p^2) \\
 &+ (2k-1)^2(2(2^{k-2} - 1)) + (2k-2)^2 2^{k-1}.
 \end{aligned} \quad (42)$$

□

Theorem 10. Consider the graph $G \cong XT(k)$ and $k \geq 3$, then the third Zagreb eccentricity index is equal to

$$\begin{aligned}
 M_2^*(G) &= 6k^2 + \sum_{p=k+1}^{2k-2} \sum_{i=1}^{k-2} 3(2^i - 1)p^2 \\
 &+ \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} p(p+1)(2(3(2^{i-1}) + 1)) \\
 &+ (2k-1)(2k-2)(2^{k-1} + 2) \\
 &+ (2k-1)^2(2(2^{k-2} - 1)).
 \end{aligned} \quad (43)$$

Proof. In order to prove the above result, we use the formula of the third Zagreb eccentricity index:

$$M_2^*(G) = \sum_{uv \in E(G)} \varepsilon(u) \times \varepsilon(v). \quad (44)$$

By using Table 4, we get

$$\begin{aligned}
 M_2^*(G) &= \sum_{p=k} (p)^2 \sum_{i=0} 6(2^i) + \sum_{i=1}^{k-2} (3(2^i - 1)) \sum_{p=k+1}^{2k-2} (p)^2 \\
 &+ \sum_{i=1}^{k-2} (2(3(2^{i-1}) + 1)) \sum_{p=k}^{2k-3} p(p+1) \\
 &+ \sum_{i=k-2} (2^{i+1} + 2) \sum_{p=k} (2p-2)(2p-1) \\
 &+ \sum_{i=k-2} (2(2^i - 1)) \sum_{p=k} (2p-1)(2p-1).
 \end{aligned} \quad (45)$$

After an easy calculation, we get

$$\begin{aligned}
 M_2^*(G) &= 6k^2 + \sum_{p=k+1}^{2k-2} \sum_{i=1}^{k-2} 3(2^i - 1)p^2 \\
 &+ \sum_{p=k}^{2k-3} \sum_{i=1}^{k-2} p(p+1)(2(3(2^{i-1}) + 1)) \\
 &+ (2k-1)(2k-2)(2^{k-1} + 2) \\
 &+ (2k-1)^2(2(2^{k-2} - 1)).
 \end{aligned} \quad (46)$$

□

3. Conclusion

In this paper, we have computed the eccentricity-based topological indices such as the eccentric connectivity index, the total eccentricity index, and the first, second, and third Zagreb eccentricity index for certain interconnection networks such as a hypertree, sibling tree, and X -tree k -level. These results are useful in topological characterization of these important chemical networks.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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