Research Article

Zagreb-Type Indices of R -Vertex Join and R -Edge Join of Graphs

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There are various methods available which are used to search large chemical databases and to predict the physicochemical properties of molecular structures. Using molecular descriptors for this purpose is the simplest of these methods. The Zagreb indices are amongst the oldest molecular descriptors, and their properties have been extensively studied and applied in QSAR/QSPR studies. The Zagreb coindices were recently introduced, attracting the attention of researchers in mathematical chemistry. In this paper, we study Zagreb indices and several other Zagreb-type indices including the general Randić index, sum-connectivity index, F-index, and Zagreb coindices of R -vertex and edge join of two arbitrary graphs.

1. Introduction

The relation between the atoms and chemical bonds between atoms in a molecule can be represented by a molecular graph, commonly denoted by G . The atoms and their chemical bonds, respectively, play the role of vertices and edges of the corresponding molecular graph [1]. A graph G is a pair $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are, respectively, the cardinalities $|V(G)|$ and $|E(G)|$. The neighbourhood $N_G(v)$ of a vertex v in G is the set of vertices adjacent to v . The degree $d_G(v)$ of a vertex v in G is defined as $d_G(v) = |N_G(v)|$. Let $d_G(v) = |N_G(v)|$ and C_n , respectively, denote the path and cycle graphs of order n . For simplicity, we use the notation $d(u)$ instead of $d_G(u)$. Other important and undefined terminologies from graph theory can be found in [2].

A scalar quantity, which is necessarily a graph invariant, which can be used to represent a certain topological feature of a molecular graph is known as a topological index [3, 4]. These indices are used as functions assigning numerical values to the qualitative properties of molecules. The structural properties of molecules have

impact on their physicochemical properties. These properties are correlated by using topological indices in QSAR/QSPR studies [5]. In QSAR/QSPR studies, these indices are used in search for new chemicals which are then used in producing potent drugs [6–9]. These topological indices are also used for virtual screening of chemical libraries to search for new chemicals of specific needs in computational chemistry and pharmacology [10–13]. The first topological index was used as a numerical descriptor to correlate the boiling points of some paraffins by Wiener [14] in 1947. This descriptor was called Wiener number, but Hosoya [15] used the term topological index for the first time in 1971. He defined the Wiener index of a graph G as follows:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v). \quad (1)$$

Gutman and Trinajstić [16, 17] investigated the dependence of the total π -electron energy of an alternant hydrocarbon, where they encountered the terms denoted by M_1 and M_2 commonly known as Zagreb indices of the first and second kind and defined as follows:

$$M_1(G) = \sum_{u \in V(G)} d^2(u), \quad (2)$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v). \quad (3)$$

Randić [18] introduced the Randić connectivity index in 1975 as follows:

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}. \quad (4)$$

Bollobás and Erdos [19] defined the general Randić index as follows:

$$R_\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha. \quad (5)$$

The general sum-connectivity index was defined by Zhou and Trinajstić [20] as follows:

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha. \quad (6)$$

Furtula and Gutman [21] introduced the forgotten topological index as

$$F(G) = \sum_{u \in V(G)} d^3(u) = \sum_{uv \in E(G)} (d^2(u) + d^2(v)). \quad (7)$$

Došlić [22] defined the first and second Zagreb coindices as follows:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d(u) + d(v)], \quad (8)$$

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d(u)d(v). \quad (9)$$

Some details and applications of these indices can be found in [23–27] and a survey paper on the Randić index in [28]. The zeroth-order general Randić index is a general case of the first Zagreb index [29, 30]. Zagreb indices of some tree-like graphs containing hydrocarbons like ethane, propane, and butane are studied in [31].

In [32], the authors studied the first and second Zagreb indices of some graph operations including subdivision and total graphs. The Zagreb coindices were recently studied in [33–37], and details on the relations between Zagreb indices and coindices can be found in [38–44]. In [45], the upper and lower bounds of $M_1(G)$ and $M_2(G)$ are determined. An upper bound of $M_2(G)$ in k -apex trees is also studied, and the corresponding extremal k -apex trees are also characterized. Two distance-based indices of some graph operations were studied in [46].

2. R-Graphs

The R -graph [47] of a graph G (denoted by $R(G)$) is the graph obtained from G by adding a vertex $w_e \notin V(G)$ corresponding to every edge $e = uv \in E(G)$ and joining w_e to end vertices u and v for each edge $e = uv \in E(G)$. Let $I(G)$ denote the set of all new vertices $R(G)$, that is,

$$I(G) = \{w_e \mid e = uv \in E(G) \text{ where } u, v \in V(G)\}. \quad (10)$$

Similarly, let $J(G)$ denote the new edges in $R(G)$, that is,

$$J(G) = \{w_e u, w_e v \mid e = uv \in E(G) \text{ and } u, v \in V(G)\}. \quad (11)$$

Then, we have

$$V(R(G)) = V(G) \cup I(G), \quad (12)$$

$$E(R(G)) = E(G) \cup J(G). \quad (13)$$

The degree of vertex $u \in V(R(G))$ is given by

$$d_{R(G)}(u) = \begin{cases} 2d_G(u), & u \in V(G), \\ 2, & u \in I(G). \end{cases} \quad (14)$$

Based on graph $R(G)$, two new graph operations obtained from two arbitrary graphs G_1 and G_2 were defined in [48]. These operations are called R -vertex join and R -edge join of G_1 and G_2 , respectively, denoted by $G_1 \langle v \rangle G_2$ and $G_1 \langle e \rangle G_2$. For two vertex disjoint graphs G_1 and G_2 , the R -vertex join of G_1 and G_2 is obtained from $R(G_1)$ and G_2 by joining every vertex of $V(G_1)$ and $V(G_2)$ by an edge. Similarly, for two vertex disjoint graphs G_1 and G_2 , the R -edge join of G_1 and G_2 is obtained from $R(G_1)$ and G_2 by joining every vertex of $I(G_1)$ and $V(G_2)$ by an edge. Graphs $C_6 \langle v \rangle C_4$ and $C_6 \langle e \rangle C_4$ are shown in Figure 1.

3. R-Vertex Join and R-Edge Join of Graphs

Let G_1 and G_2 be two graphs. The order and size of G_i are denoted by n_i and m_i , $i = 1, 2$. Let G_v and G_e , respectively, denote graphs $G_1 \langle v \rangle G_2$ and $G_1 \langle e \rangle G_2$. The vertex and edge sets of graphs G_v and G_e are given as follows:

$$V(G_1 \langle v \rangle G_2) = V(G_1 \langle e \rangle G_2) = V(G_1) \cup V(G_2) \cup I(G_1), \quad (15)$$

$$E(G_1 \langle v \rangle G_2) = E(G_1) \cup E(G_2) \cup J(G_1) \cup K_v, \quad (16)$$

where $K_v = \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Similarly,

$$E(G_1 \langle e \rangle G_2) = E(G_1) \cup E(G_2) \cup J(G_1) \cup K_e, \quad (17)$$

where $K_e = \{w_e u \mid w_e \in I(G_1), u \in V(G_2)\}$.

Then, the order n_v and size m_v of graph G_v can be observed as $|V(G_v)| = n_v = n_1 + n_2 + m_1$ and $|E(G_v)| = m_v = m_1 + m_2 + n_1 n_2 + 2m_1$. Similarly, the order

FIGURE 1: The R -vertex join (a) and R -edge join (b) of graphs C_6 and C_4 .

n_e and size m_e of graph G_e is given by $|V(G_e)| = n_e = n_1 + n_2 + m_1$ and $|E(G_e)| = m_e = m_1 + m_2 + 2m_1 + m_1n_2$. The order and size of the complements of the sets $E(G_v)$ and $E(G_e)$ are given by

$$\begin{aligned} |E(G_v)| &= \binom{n_1 + n_2 + m_1}{2} - (3m_1 + m_2 + n_1n_2), \\ |E(G_e)| &= \binom{n_1 + n_2 + m_1}{2} - (3m_1 + m_2 + m_1n_2). \end{aligned} \quad (18)$$

The degree of a vertex in G_v and G_e can be obtained from the following expressions:

$$\begin{aligned} d_{G_v}(w) &= \begin{cases} 2d_{G_1}(w) + n_2, & w \in V(G_1), \\ d_{G_2}(w) + n_1, & w \in V(G_2), \\ 2, & w \in I(G_1), \end{cases} \\ d_{G_e}(w) &= \begin{cases} 2d_{G_1}(w), & w \in V(G_1), \\ d_{G_2}(w) + m_1, & w \in V(G_2), \\ 2 + n_2, & w \in I(G_1). \end{cases} \end{aligned} \quad (19)$$

The following lemma will be used for calculating the Zagreb coindices of raphs G_v and G_e .

Lemma 1 (see [33]). *The first and second Zagreb coindices of graph G of order n and size m can be computed as*

$$\begin{aligned} \overline{M}_1(G) &= 2m(n-1) - M_1(G), \\ 2\overline{M}_2(G) &= 4m^2 - 2M_2(G_v) - M_1(G_v). \end{aligned} \quad (20)$$

In the next section, we compute the Zagreb indices of the graph $R(G)$ and R -vertex and R -edge join of graphs and the Zagreb coindices of $R(G)$ and R -vertex and R -edge join of graphs.

4. Zagreb Indices of R -Graphs and R -Vertex and Edge Join of Graphs

Theorem 1. *The first Zagreb index of graph $R(G)$ is given by*

$$M_1(R(G)) = 2M_1(G) + 4m. \quad (21)$$

Proof. Using the definition of the vertex set of graph $R(G)$ and by using (2), we get

$$\begin{aligned} M_1(R(G)) &= \sum_{w \in R(G)} d_{R(G)}^2(w) \\ &= \sum_{w \in V(G)} d_{R(G)}^2(w) + \sum_{w \in I(G)} d_{R(G)}^2(w) \\ &= \sum_{w \in V(G)} 2d_G^2(w) + \sum_{w \in I(G)} (2)^2 \\ &= 2 \sum_{w \in V(G)} d_G^2(w) + \sum_{w \in I(G)} 4 \\ &= 2M_1(G) + 4m. \end{aligned} \quad (22)$$

□

Theorem 2. *The second Zagreb index of graph $R(G)$ is given by*

$$M_2(R(G)) = 4(M_1(G) + M_2(G)). \quad (23)$$

Proof. Using the definition of the edge set of graph $R(G)$ and by using (3), we get

$$\begin{aligned}
M_2(R(G)) &= \sum_{uv \in E(R(G))} d_{R(G)}(u)d_{R(G)}(v) \\
&= \sum_{uv \in E(G)} d_{R(G)}(u)d_{R(G)}(v) \\
&\quad + \sum_{uv \in I(G)} d_{R(G)}(u)d_{R(G)}(v) \\
&= 4 \sum_{uv \in E(G)} d_G(u)d_{R(G)}(v) + \sum_{u \in V(G)} 4d_G^2(u) \\
&= 4M_2(G) + 4M_1(G) \\
&= (M_1(G) + M_2(G)).
\end{aligned} \tag{24}$$

Theorem 3. The first Zagreb index of graph G_v is given by

$$\begin{aligned}
M_1(G_v) &= 4M_1(G_1) + n_2^2 n_1 + 8n_2 m_1 \\
&\quad + M_2(G_2) + n_1^2 n_2 + 4n_1 m_2 + 4m_1.
\end{aligned} \tag{25}$$

Proof. The definition of the vertex set of graph G_v and the use of (2) and (3) yield the following:

$$\begin{aligned}
M_1(G_v) &= \sum_{w \in V(G_v)} d_{G_v}^2(w) \\
&= \sum_{w \in V(G_1)} d_{G_v}^2(w) + \sum_{w \in V(G_2)} d_{G_v}^2(w) \\
&\quad + \sum_{w \in I(G_1)} d_{G_v}^2(w) \\
&= \sum_{w \in V(G_1)} (2d_{G_1}(w) + n_2)^2 \\
&\quad + \sum_{w \in V(G_2)} (d_{G_2}(w) + n_1)^2 + \sum_{w \in I(G_1)} (2)^2 \\
&= \sum_{w \in V(G_1)} 4d_{G_1}^2(w) + \sum_{w \in V(G_1)} n_2^2 \\
&\quad + \sum_{w \in V(G_1)} 4n_2 d_{G_1}(w) + \sum_{w \in V(G_2)} d_{G_2}^2(w) \\
&\quad + \sum_{w \in V(G_2)} n_1^2 + \sum_{w \in V(G_2)} 2n_1 d_{G_2}(w) + \sum_{w \in I(G_1)} (4) \\
&= 4M_1(G_1) + n_2^2 n_1 + 4n_2(2m_1) + M_2(G_2) \\
&\quad + n_1^2 n_2 + 2n(2m_2) + 4m_1 \\
&= 4M_1(G_1) + n_1 n_2^2 + 8n_2 m_1 + M_2(G_2) \\
&\quad + n_1^2 n_2 + 4n_1 m_2 + 4m_1.
\end{aligned} \tag{26}$$

Theorem 4. The second Zagreb index of graph G_v is given as

$$\begin{aligned}
M_2(G_v) &= 4M_2(G_1) + 2n_2 M_1(G_1) + n_2^2 m_1 \\
&\quad + M_2(G_2) + n_1 M_1(G_2) + n_1^2 m_2 \\
&\quad + 4M_1(G_1) + 8m_1 m_2 \\
&\quad + (2n_1 + n_2)(2n_2 m_1 + 2n_1 m_2) + 4n_2 m_1 + n_1^2 n_2^2.
\end{aligned} \tag{27}$$

Proof. Using the definition of the edge set of graph G_v and by using (2) and (3), we get the following:

$$\begin{aligned}
M_2(G_v) &= \sum_{uv \in V(G_v)} d_{G_v}(u)d_{G_v}(v) \\
&= \sum_{uv \in E(G_1)} d_{G_v}(u)d_{G_v}(v) + \sum_{uv \in E(G_2)} d_{G_v}(u) \times d_{G_v}(v) \\
&\quad + \sum_{uv \in I(G_1)} d_{G_v}(u)d_{G_v}(v) + \sum_{uv \in K_v} d_{G_v}(u)d_{G_v}(v) \\
&= \sum_{uv \in E(G_1)} (2d_{G_1}(u) + n_2)(2d_{G_1}(v) + n_2) \\
&\quad + \sum_{uv \in E(G_2)} (d_{G_2}(u) + n_1)(d_{G_2}(v) + n_1) \\
&\quad + \sum_{u \in V(G_1)} 2d_{G_1}^2(u) \times 2 + 4n_2 m_1 \\
&\quad + \sum_{uv \in K_v} (2d_{G_1}(u) + n_2)(d_{G_2}(v) + n_1) \\
&= \sum_{uv \in E(G_1)} (4d_{G_1}(u)d_{G_1}(v) + 2d_{G_1}(u)n_2 + 2n_2 d_{G_1}(v) + n_2^2) \\
&\quad + \sum_{uv \in E(G_2)} (d_{G_2}(u)d_{G_2}(v) + n_1(d_{G_1}(u) + d_{G_2}(u) + n_1^2)) \\
&\quad + 4 \sum_{u \in V(G_1)} d_{G_1}^2(u) + 4n_2 m_1 + 2 \sum_{uv \in K_v} d_{G_1}(u)d_{G_2}(v) \\
&\quad + 2n_1 \sum_{uv \in K_v} d_{G_1}(u) + n_2 \sum_{uv \in K_v} d_{G_2}(v) + (n_1 n_2)^2 \\
&= 4 \sum_{uv \in E(G_1)} d_{G_1}(u)d_{G_1}(v) + 2n_2 \sum_{uv \in E(G)} (d_{G_1}(u) + d_{G_1}(v)) \\
&\quad + n_2^2 m_1 + \sum_{uv \in E(G_2)} d_{G_2}(u)d_{G_2}(v) \\
&\quad + n_1 \sum_{uv \in E(G_2)} d_{G_2}(u)d_{G_2}(v) + n_1^2 m_2 + 4M_1(G_1) + 4n_2 m_1 \\
&\quad + 2(2m_1 2m_2) + (2n_1 + n_2)(n_2 2m_1 + n_1 2m_2) + n_1^2 n_2^2 \\
&= 4M_2(G_1) + 2n_2 \sum_{u \in V(G_1)} d_{G_1}^2(u) + n_2^2 m_1 \\
&\quad + M_2(G_2) + n_1 \sum_{u \in V(G_2)} d_{G_2}^2(u) + n_1^2 m_2 + 4M_1(G_1) \\
&\quad + 4n_2 m_1 + 8m_1 m_2 + (2n_1 + n_2)(2n_2 m_1 + 2n_1 m_2) + n_1^2 n_2^2 \\
&= 4M_2(G_1) + 2n_2 M_1(G_1) + n_2^2 m_1 + M_2(G_2) \\
&\quad + n_1 M_1(G_2) + n_1^2 m_2 + 4M_1(G_1) \\
&\quad + 8m_1 m_2 + (2n_1 + n_2)(2n_2 m_1 + 2n_1 m_2) + 4n_2 m_1 + n_1^2 n_2^2.
\end{aligned} \tag{28}$$

Theorem 5. The first Zagreb index of graph G_e is calculated as

$$M_1(G_e) = 4M_1(G_1) + M_1(G_2) + 4m_1m_2 + m_1^2n_2 + 4m_1 + n_2^2m_1 + 4n_2m_1. \quad (29)$$

Proof. Using the vertex set of graph G_e and by using (2) and (3), we get the following:

$$\begin{aligned} M_1(G_e) &= \sum_{w \in V(G_e)} d_{G_e}^2(w) \\ &= \sum_{w \in V(G_1)} d_{G_e}^2(w) + \sum_{w \in V(G_2)} d_{G_e}^2(w) + \sum_{w \in I(G_1)} d_{G_e}^2(w) \\ &= \sum_{w \in V(G_1)} (2d_{G_1}(w))^2 + \sum_{w \in V(G_2)} (d_{G_2}(w) + m_1)^2 + \sum_{w \in I(G_1)} (2 + n_2)^2 \\ &= 4 \sum_{w \in V(G_1)} d_{G_1}^2(w) + \sum_{w \in V(G_2)} d_{G_2}^2(w) + 2m_1 \sum_{w \in V(G_2)} d_{G_2}(w) + m_1^2n_2 + 4m_1 + n_2^2m_1 + 4n_2m_1 \\ &= 4M_1(G_1) + M_1(G_2) + 2m_12m_2 + m_1^2n_2 + 4m_1 + n_2^2m_1 + 4n_2m_1. \end{aligned} \quad (30)$$

Theorem 6. The second Zagreb index of graph G_e is given as

$$\begin{aligned} M_2(G_e) &= 4M_2(G_1) + M_2(G_2) + m_1M(G_2) + m_1^2m_2 \\ &\quad + m_1(m_1m_2(n_2 + 2) + m_2(2n_2 + 4) + 4n_2 + 8). \end{aligned} \quad (31)$$

Proof. Using the edge set of graph G_e and by using (2) and (3), we get

$$\begin{aligned} M_2(G_e) &= \sum_{uv \in E(G_e)} d_{G_e}(u)d_{G_e}(v) \\ &= \sum_{uv \in E(G_1)} d_{G_e}(u)d_{G_e}(v) + \sum_{uv \in E(G_2)} d_{G_e}(u)d_{G_e}(v) + \sum_{uv \in J(G_1)} d_{G_e}(u)d_{G_e}(v) + \sum_{uv \in K_e} d_{G_e}(u)d_{G_e}(v) \\ &= \sum_{uv \in E(G_1)} 2 \times 2d_{G_1}(u)d_{G_1}(v) + \sum_{uv \in E(G_2)} (d_{G_2}(u + m_1))(d_{G_2}(v) + m_1) \\ &\quad + \sum_{u \in V(G_1)} (2d_{G_1}^2(u))(2 + n_2) + m_1d_{G_e}(w_e) \sum_{v \in V(G_2)} d_{G_e}(v) \\ &= 4 \sum_{uv \in E(G_1)} d_{G_1}(u)d_{G_1}(v) + \sum_{uv \in E(G_2)} (d_{G_2}(u)d_{G_2}(v) + m_1(d_{G_2}(u) + d_{G_2}(v)) + m_1^2) \\ &\quad + (4 + 2n_2) \sum_{u \in V(G_1)} d_{G_1}^2(u) + m_1(2 + n_2) \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1) \\ &= 4M_2(G_1) + M_2(G_2) + m_1M(G_2) + m_1^2m_2 + (4 + 2n_2)2m_1 + m_1(2 + n_2)2m_2 + m_1(2 + n_2)m_1m_2. \end{aligned} \quad (32)$$

It can be observed that, for graph G and $v \in V(G)$, we have $d_G^-(v) = |V(G)| - 1 - d_G(v)$. Using this relation and the definitions of Zagreb coindices, we obtain the following results.

Theorem 7. The first Zagreb coindex of graph G_v is given by

$$\overline{M}_1(G_v) = 2 \left[\binom{n_1 + n_2 + m_1}{2} - (3m_1 + m_2 + n_1 n_2) \right] \\ (n_1 + n_2 + m_1 - 1) - (4M_1(G_1) + n_2^2 n_1 + 8n_2(m_1) + M_2(G_2) + n_1^2 n_2 + 4n(m_2) + 4m_1). \quad (33)$$

Proof. Using Lemma 1, Theorem 3, and (8), we obtain the following:

$$\overline{M}_1(G_v) = 2 \left[\binom{n_1 + n_2 + m_1}{2} - (m_1 + m_2 + n_1 n_2 + 2m_1) \right] \\ \times (n_1 + n_2 + m_1 - 1) - M_1(G_v) \\ = 2 \left[\binom{n_1 + n_2 + m_1}{2} - (m_1 + m_2 + n_1 n_2 + 2m_1) \right] \\ \times (n_1 + n_2 + m_1 - 1) - (4M_1(G_1) + n_2^2 n_1 \\ + 4n_2(2m_1) + M_2(G_2) + n_1^2 n_2 + 2n(2m_2) + 4m_1). \quad (34)$$

□

Theorem 8. The first Zagreb coindex of graph G_e is given as

$$\overline{M}_1(G_e) = 2 \left[\binom{n_1 + n_2 + m_1}{2} - (3m_1 + m_2 + n_1 n_2) \right] \\ \times (n_1 + n_2 + m_1 - 1) - (4M_1(G_1) + M_1(G_2) \\ + 4m_1 m_2 + m_2^2 n_2 + 4m_1 + n_2^2 m_1 + 4n_2 m_1). \quad (35)$$

Proof. By substituting Theorem 5 in (8) and using Lemma 1, we obtain

$$\overline{M}_1(G_e) = 2 \left[\binom{n_1 + n_2 + m_1}{2} - (m_1 + m_2 + 2m_1 + n_1 n_2) \right] \times (n_1 + n_2 + m_1 - 1) - M_1(G_e) \\ = 2 \left[\binom{n_1 + n_2 + m_1}{2} - (m_1 + m_2 + 2m_1 + n_1 n_2) \right] \\ \times (n_1 + n_2 + m_1 - 1) - 4M_1(G_1) + M_1(G_2) + 2m_1 2m_2 + m_2^2 n_2 + 4m_1 + n_2^2 m_1 + 4n_2 m_1. \quad (36)$$

□

Theorem 9. The second Zagreb coindex of graph G_v is as follows:

$$\overline{M}_2(G_v) = \frac{1}{2} \left[4m_v^2 - 2(4M_2(G_1) + 2n_2 M_1(G_1) + n_2^2 m_1 + M_2(G_2) + n_1 M_1(G_2) + n_1^2 m_2 + 4M_1(G_1) + 8(m_1 m_2) \right. \\ \left. + (2n_1 + n_2)(2n_2 m_1 + 2n_1 m_2 + 4n_2 m_1 + n_1^2 n_2^2) - (4M_1(G_1) + n_2^2 n_1 + 8n_2 m_1 + M_2(G_2) + n_1^2 n_2 + 4n m_2 + 4m_1)) \right]. \quad (37)$$

Proof. Using (9) and substituting Theorems 3 and 4 in Lemma 1, we obtain

$$\overline{M}_2(G_v) = \frac{1}{2} \left[4m_v^2 - 2M_2(G_v) - M_1(G_v) \right] \\ = \frac{1}{2} \left[4m_v^2 - 2(4M_2(G_1) + 2n_2 M_1(G_1) + n_2^2 m_1 + M_2(G_2) + n_1 M_1(G_2) \right. \\ \left. + n_1^2 m_2 + 4M_1(G_1) + 2(2m_1 2m_2) + (2n_1 + n_2)(n_2 2m_1 + n_1 2m_2 + 4n_2 m_1 + n_1^2 n_2^2) \right. \\ \left. + 8n_2 m_1 - (4M_1(G_1) + n_2^2 n_1 + 4n_2(2m_1) + M_2(G_2) + n_1^2 n_2 + 2n(2m_2) + 4m_1)) \right]. \quad (38)$$

□

Theorem 10. The second Zagreb coindex of graph G_e is given as

$$\begin{aligned} \overline{M}_2(G_e) = \frac{1}{2} [4m_e^2 - 2(4M_2(G_1) + M_2(G_2) + M_1(G_2) + m_1^2 2m_2 + (4 + 2n_2)2m_1 \\ + m_1(2 + n_2)2m_2 + m_1(2 + n_2)m_1 m_2) - (4M_1(G_1) + M_1(G_2) + 4m_1 m_2 + m_2^2 n_2 + 4m_1 + n_2^2 m_1 + 4n_2 m_1))] \end{aligned} \quad (39)$$

Proof. Using (9) and substituting Theorems 5 and 6 in Lemma 1, we obtain

$$\begin{aligned} \overline{M}_2(G_e) &= \frac{1}{2} [4m_e^2 - 2M_2(G_e) - M_1(G_e)] \\ &= \frac{1}{2} [4m_e^2 - 2(4M_2(G_1) + M_2(G_2) + M_1(G_2) + m_1^2 2m_2 + (4 + 2n_2)2m_1 + m_1(2 + n_2)2m_2 \\ &\quad + m_1(2 + n_2)m_1 m_2 - (4M_1(G_1) + M_1(G_2) + 2m_1 2m_2 + m_2^2 n_2 + 4m_1 + n_2^2 m_1 + 4n_2 m_1))] \end{aligned} \quad (40)$$

□

5. General Randić Index of R-Vertex and Edge Join of Graphs

In this section, we compute the general Randić index R_α , for $\alpha = 2$, of the R-vertex join and R-edge join of graphs.

Theorem 11. The general Randić index R_α , for $\alpha = 2$, of G_v is given as

$$\begin{aligned} R_2(G_v) &= 16R_2(G_1) + 4n_2^2 F(G_1) + 4n_2^3 M_1(G_1) \\ &\quad + 16n_2 ReM_3(G_1) + (4n_2)^2 M_2(G_1) + n_2^4 \\ &\quad + R_2(G_2) + n_1^2 F(G_2) + 2n_1^2 M_1(G_2) + 2n_1 \\ &\quad \times Re(M_3(G_2)) + (2n_1)^2 M_2(G_2) + n_1^4 + 16F(G_1) \\ &\quad + 8n_2^2 m_1 + 32n_2 M_1(G_1) + 8m_1 M_1(G_2) \\ &\quad + 8m_1 n_1^2 n_2 + 32m_1 m_2 n_1 + n_2^2 n_1 M_1(G_1) \\ &\quad + n_2^3 n_1^3 + 4n_1^2 n_2^2 m_2 + 8m_1 n_2 M_1(G_2) + 8n_1^2 n_2^2 m_1 \\ &\quad + 32m_1 m_2 n_1 n_2. \end{aligned} \quad (41)$$

Proof. Using (4) and the edge set of G_v , we obtain the following:

$$\begin{aligned}
R_2(G_v) &= \sum_{uv \in E(G_v)} (d_{G_v}(u)d_{G_v}(v))^2 \\
&= \sum_{uv \in E(G_1)} (2d_{G_1}(u) + n_2)^2 (2d_{G_1}(v) + n_2)^2 + \sum_{uv \in E(G_2)} (d_{G_2}(u) + n_1)^2 (d_{G_2}(v) + n_1)^2 \\
&\quad + \sum_{u \in V(G_1)} d_{G_1}(u) (2d_{G_1}(u) + n_2)^2 \times 2^2 + \sum_{uv \in K_v, u \in V(G_1), v \in V(G_2)} (2d_{G_1}(u) + n_2)^2 (d_{G_2}(v) + n_1)^2 \\
&= \sum_{uv \in E(G_1)} [16d_{G_1}^2(u)d_{G_1}^2(v) + 4n_2^2d_{G_1}^2(u) + n_2^2 \times 4n_2d_{G_1}(v) + n_2^2 \times 4n_2d_{G_1}(u) + 4n_2d_{G_1}(u) \times 4d_{G_1}^2(v) \\
&\quad + (4n_2)^2d_{G_1}(u)d_{G_1}(v) + 4n_2d_{G_1}(u) \times 4d_{G_1}^2(v) + (4n_2)^2d_{G_1}(u)d_{G_1}(v) + 4d_{G_1}^2(u) \times 4n_2d_{G_1}(v) + 4n_2^2d_{G_1}^2(v) + n_2^2n_2^2] \\
&\quad + \sum_{uv \in E(G_2)} [(d_{G_2}^2(u)d_{G_2}^2(v)) + n_1^2d_{G_2}^2(v) + n_1^2 \times d_{G_2}^2(u) + 2n_1^2 \times d_{G_2}(u) + 2n_1d_{G_2}(u)d_{G_2}^2(v) \\
&\quad + (2n_1)^2 \times d_{G_2}(u)d_{G_2}(v) + 2n_1^2d_{G_2}(v) + 2n_1d_{G_2}(v)d_{G_2}^2(u) + n_1^2n_1^2] \\
&\quad + \sum_{u \in V(G_1)} 4d_{G_1}(u) [4d_{G_1}^2(u) + n_2^2 + 8d_{G_1}(u)n_2] + \sum_{u \in V(G_1)} (2d_{G_1}(u) + n_2)^2 \times \sum_{v \in V(G_2)} (d_{G_2}(v) + n_1)^2 \\
&= \sum_{uv \in E(G_1)} 16d_{G_1}^2(u)d_{G_1}^2(v) + 4n_2^2 \sum_{uv \in E(G_1)} d_{G_1}^2(u) + d_{G_1}^2(v) + 4n_2^3 \sum_{uv \in E(G_1)} d_{G_1}(u) + d_{G_1}(v) + \\
&\quad 16n_2 \sum_{uv \in E(G_1)} (d_{G_1}(u)d_{G_1}^2(v)) + (d_{G_1}^2(u)d_{G_1}(v)) + (4n_2)^2 \sum_{uv \in E(G_1)} (d_{G_1}(u)d_{G_1}^2(v)) + n_2^4 + \sum_{uv \in E(G_2)} d_{G_2}^2(u) \times d_{G_2}^2(v) \\
&\quad + n_1^2 \sum_{uv \in E(G_2)} d_{G_2}^2(u) + d_{G_2}^2(v) + 2n_1^2 \times \sum_{uv \in E(G_2)} d_{G_2}(u) + d_{G_2}(v) \\
&\quad + 2n_1 \sum_{uv \in E(G_2)} d_{G_2}(u)d_{G_2}^2(v) + d_{G_2}^2(u)d_{G_2}(v) \\
&\quad + (2n_1)^2 \sum_{uv \in E(G_2)} d_{G_2}(u)d_{G_2}^2(v) + (n_1)^4 + 16 \sum_{u \in V(G_1)} d_{G_1}^3(u) + 8n_2^2m_1 + 32n_2 \sum_{u \in V(G_1)} d_{G_1}^2(u) \\
&\quad + \sum_{u \in V(G_1)} 4d_{G_1}(u) + n_2^2 + 4n_2d_{G_1}(u) + \sum_{u \in V(G_2)} (d_{G_2}^2(v) + n_1^2 + 2n_1d_{G_2}(v)) \\
&= 16 \sum_{uv \in E(G_1)} ((d_{G_1}(u)d_{G_1}(v))^2) + 4n_2^2F(G_1) + 4n_2^3M_1(G_1) + 16n_2 \sum_{uv \in E(G_1)} d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v)) \\
&\quad + (4n_2)^2M_2(G_1) + n_2^4 + R_2(G_2) + n_1^2F(G_2) + 2n_1^2M_1(G_2) + 2n_1Re(M_3(G_2)) + (2n_1)^2M_2(G_2) + n_1^4 + 16F(G_1) \\
&\quad + 8n_2^2m_1 + 32n_2M_1(G_1) + [4(2m_1) + n_2^2n_1 + 4n_2(2m_1)] \times [M_1(G_2) + n_1^2n_2 + 2n_1(2m_2)] \\
&= 16R_2(G_1) + 4n_2^2F(G_1) + 4n_2^3M_1(G_1) + 16n_2Re(M_3(G_1)) + (4n_2)^2M_2(G_1) + n_2^4 + R_2(G_2) + n_1^2F(G_2) + 2n_1^2M_1(G_2) \\
&\quad + 2n_1Re(M_3(G_2)) + (2n_1)^2M_2(G_2) + n_1^4 + 16F(G_1) + 8n_2^2m_1 + 32n_2M_1(G_1) + 8m_1M_1(G_2) + 8m_1n_1^2n_2 \\
&\quad + 32m_1m_2n_1 + n_2^2n_1M_1(G_1) + n_2^3n_1^3 + 4n_1^2n_2^2m_2 + 8m_1n_2M_1(G_2) + 8n_1^2n_2^2m_1 + 32m_1m_2n_1n_2.
\end{aligned}$$

(42)

□

Theorem 12. General Randić index of graph G_e for $\alpha = 2$ is

$$\begin{aligned}
R_2(G_e) &= 16R_2(G_1) + R_2(G_2) + m_1^2 F(G_2) + (2m_1)^2 M_2(G_2) \\
&\quad + m_1^2 m_2^2 + 4m_1 M_2(G_2) + 2m_1^2 M_1(G_2) \\
&\quad + 8m_1 + 4M_1(G_1) + (4n_2^2 + 16n_2) F(G_1) \\
&\quad + 4M_1(G_1) M_1(G_2) \\
&\quad + m_1(2 + n_2)^2 \times (M_1(G_2) + m_1^2 n_2 + 4m_1 m_2).
\end{aligned}
\tag{43}$$

Proof. Using (4) and the edge set of G_e , we obtain the following:

$$\begin{aligned}
R_2(G_e) &= \sum_{uv \in E(G_e)} (d_{G_e}(u) d_{G_e}(v))^2 \\
&= \sum_{uv \in E(G_1)} (2d_{G_1}(u) \times 2d_{G_1}(v))^2 + \sum_{uv \in E(G_2)} (d_{G_2}(u) + m_1)^2 (d_{G_2}(v) + m_1)^2 + \sum_{u \in V(G_1)} d_{G_1}(u) + (2 + n_2)^2 (2d_{G_1}(u))^2 \\
&\quad + \sum_{uv \in K_e, u \in I(G_1), v \in V(G_2)} (2 + n_2)^2 \times (d_{G_2}(v) + m_1)^2 \\
&= \sum_{uv \in E(G_1)} 16(d_{G_1}(u) d_{G_1}(v))^2 \\
&\quad + \sum_{uv \in E(G_2)} (d_{G_2}^2(u) \times d_{G_2}^2(v) + d_{G_2}^2(u) m_1^2 + m_1^2 d_{G_2}^2(v) + m_1^2 m_2^2 + 2m_1 d_{G_2}(u) \times d_{G_2}(v) \\
&\quad + 2m_1^2 d_{G_2}(u) + 2m_1 d_{G_2}(u) 2m_1 d_{G_2}(v) + 2m_1 d_{G_2}(v) d_{G_2}(u) + 2m_1^2 d_{G_2}(v)) \\
&\quad + \sum_{u \in V(G_1)} (4d_{G_1}(u) \times 4d_{G_1}^2(u) + n_2^2 d_{G_1}(u) \times 4d_{G_1}^2(u) + 4n_2 d_{G_1}(u) \times 4d_{G_1}^2(u)) \\
&\quad + \sum_{u \in I(G_1)} (2 + n_2)^2 \sum_{v \in V(G_2)} (d_{G_2}(v) + m_1)^2 \\
&= 16R_2(G_1) + R_2(G_2) + m_1^2 \sum_{uv \in E(G_2)} (d_{G_2}^2(u) + d_{G_2}^2(v) + (2m_1)^2 \sum_{uv \in E(G_2)} (d_{G_2}(u) d_{G_2}(v)) \\
&\quad + m_1^2 m_2^2 + 4m_1 \times \sum_{uv \in E(G_2)} (d_{G_2}(u) d_{G_2}(v)) + 2m_1^2 \times \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v)) + 4 \times 2m_1 + 4 \times \\
&\quad \sum_{u \in V(G_1)} d_{G_1}^2(u) + (4n_2^2 + 16n_2) \times \sum_{u \in V(G_1)} d_{G_1}^3(u) + m_1(2 + n_2)^2 \times (M_1(G_2) + m_1^2 n_2 + 2m_1(2m_2)) \\
&= 16R_2(G_1) + R_2(G_2) + m_1^2 F(G_2) + (2m_1)^2 M_2(G_2) + m_1^2 m_2^2 + 4m_1 M_2(G_2) + 2m_1^2 M_1(G_2) + 8m_1 \\
&\quad + 4M_1(G_1) + (4n_2^2 + 16n_2) F(G_1) + 4M_1(G_1) M_1(G_2) + m_1(2 + n_2)^2 \times M_1(G_2) + m_1^2 n_2 + 4m_1 m_2.
\end{aligned}$$

(44)
□

6. Forgotten Index of R-Vertex and Edge Join of Graphs

Theorem 13. The forgotten index of G_v is given as

In this section, we calculate the forgotten index of the R-vertex and R-edge join of graphs.

$$\begin{aligned} F(G_v) &= 4F(G_1) + 2n_1^2n_2 + 4n_2M_1(G_1) + F(G_2) + 2n_1^2m_2 \\ &\quad + 2n_1M_1(G_2) + 4F(G_1) + n_2^2 \times 2m_1 + 4n_2M_1(G_1) + 8m_1 + 4M_1(G_1) + n_2^2n_1 + 8n_2m_1 + M_1(G_2) \\ &\quad + n_1^2n_2 + 4n_1m_2. \end{aligned} \quad (45)$$

Proof. Using (7) and the edge set of G_v , we get

$$\begin{aligned} F(G_v) &= \sum_{uv \in E(G_v)} (d_{G_v}^2(u) + d_{G_v}^2(v)) \\ &= \sum_{uv \in E(G_1)} (2d_{G_1}(u) + n_2)^2 + (2d_{G_1}(v) + n_2)^2 \\ &\quad + \sum_{uv \in E(G_2)} (d_{G_2}(u) + n_1)^2 + (d_{G_2}(v) + n_1)^2 \\ &\quad + \sum_{u \in V(G_1)} ((2d_{G_1}(u) + n_2)^2 + 4) \\ &\quad + \sum_{\substack{uv \in K_v, u \in V(G_1) \\ v \in V(G_2)}} ((2d_{G_1}(u) + n_2)^2 + (d_{G_2}(v) + n_1)^2) \\ &= \sum_{uv \in E(G_1)} [4d_{G_1}^2(u) + n_2^2 + 4n_2d_{G_1}(u) + 4d_{G_1}^2(v) + n_2^2 + 4n_2d_{G_1}(v)] \\ &\quad + \sum_{uv \in E(G_2)} [d_{G_2}^2(u) + n_1^2 + 2n_1d_{G_2}(u) + d_{G_2}^2(v) + n_1^2 + 2n_1d_{G_2}(v)] \\ &\quad + \sum_{u \in V(G_1)} [d_{G_1}(u) \times 4d_{G_1}^2(u) + d_{G_1}(u) \times n_2^2 + 4n_2d_{G_1}(u)d_{G_1}(u) + 4d_{G_1}(u)] \\ &\quad + \sum_{u \in V(G_1)} [4d_{G_1}^2(u) + n_2^2 + 4n_2d_{G_1}(u)] \\ &\quad + \sum_{u \in V(G_1)} [d_{G_2}^2(v) + n_1^2 + 2n_1d_{G_2}(v)] \\ &= 4 \sum_{uv \in E(G_1)} (d_{G_1}^2(u) + d_{G_1}^2(v)) + 2n_2^2m_1 + 4n_2 \\ &\quad \times \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) + \sum_{uv \in E(G_2)} (d_{G_2}^2(u) + d_{G_2}^2(v)) + 2n_1^2m_2 + 2n_1 \sum_{uv \in E(G_2)} (d_{G_2}(u) + d_{G_2}(v)) \\ &\quad + \sum_{u \in V(G_1)} 4d_{G_1}^3(u) + n_2^2 \times 2m_1 + 4n_2 \times \sum_{u \in V(G_1)} d_{G_1}^2(u) + 4 \times 2m_1 \\ &\quad + 4 \sum_{u \in V(G_1)} d_{G_1}^2(u) + n_2^2n_1 + 4n_2 \times 2m_1 + \sum_{v \in V(G_2)} d_{G_2}^2(v) + n_1^2n_2 + 2n_1 \times 2m_2 \\ &= 4F(G_1) + 2n_1^2n_2 + 4n_2M_1(G_1) + F(G_2) + 2n_1^2m_2 + 2n_1M_1(G_2) + 4F(G_1) + n_2^2 \times 2m_1 + 4n_2M_1(G_1) \\ &\quad + 8m_1 + 4M_1(G_1) + n_2^2n_1 + 8n_2m_1 + M_1(G_2) + n_1^2n_2 + 4n_1m_2. \end{aligned} \quad (46)$$

□

Theorem 14. The forgotten index of graph G_e is given as follows:

$$\begin{aligned} F(G_e) &= 4F(G_1) + F(G_2) + 2m_1^2m_2 + 2m_1M_1(G_2) \\ &\quad + 4F(G_1) + (2+n_2)^2 \times 2m_1 + m_1 \times 2m_2 \quad (47) \\ &\quad + m_1^3n_2 + 2m_1^2 \times 2m_2 + (2+n_2^2)m_1n_2. \end{aligned}$$

Proof. Using (7) and the edge set of G_e , we get

$$\begin{aligned} F(G_e) &= \sum_{uv \in E(G_e)} (d_{G_e}^2(u) + d_{G_e}^2(v)) = \sum_{uv \in E(G_1)} \left[(2d_{G_1}(u))^2 + (2d_{G_1}(v))^2 \right] \\ &\quad + \sum_{uv \in E(G_2)} \left[(d_{G_2}(u) + m_1)^2 + (d_{G_2}(v) + m_1)^2 \right] + \sum_{u \in V(G_1)} \left[(2d_{G_1}(u))^2 + (2+n_2)^2 \right] d_{G_1}(u) \\ &\quad + \sum_{u \in V(G_2)} \left[(d_{G_2}(u) + m_1)^2 + (2+n_2)^2 \right] m_1 \\ &= 4 \sum_{uv \in E(G_1)} (d_{G_1}^2(u) + d_{G_1}^2(v)) + \sum_{uv \in E(G_2)} (d_{G_2}^2(u) + d_{G_2}^2(v)) + \sum_{uv \in E(G_2)} (m_1^2 + m_1^2) + \sum_{uv \in E(G_2)} 2m_1 \times (d_{G_2}(u) + d_{G_2}(v)) \\ &\quad + \sum_{u \in V(G_1)} [4d_{G_1}^3(u) + (2+n_2)^2 \times d_{G_1}(u)] + \sum_{u \in V(G_2)} [m_1d_{G_2}(u) + m_1^3 + 2m_1^2d_{G_2}(u) + (2+n_2)^2m_1] \\ &= 4F(G_1) + F(G_2) + 2m_1^2m_2 + 2m_1M_1(G_2) + 4F(G_1) \\ &\quad + (2+n_2)^2 \times 2m_1 + m_1 \times 2m_2 + m_1^3n_2 + 2m_1^2 \times 2m_2 + (2+n_2^2)m_1n_2. \end{aligned} \quad (48)$$

□

7. General Sum-Connectivity Index of R-Vertex and Edge Join of Graphs

In this section, we obtain the general sum-connectivity index of the R-vertex and R-edge join of graphs.

Theorem 15. The general sum-connectivity index of graph G is given by

$$HM(G) = F(G) + 2M_2(G). \quad (49)$$

Proof. Using (6), we get

$$\begin{aligned} \chi_2(G) &= \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2 \\ &= \sum_{uv \in E(G)} (d_G^2(u) + d_G^2(v) + 2d_G(u)d_G(v)) \\ &= \sum_{uv \in E(G)} (d_G^2(u) + d_G^2(v)) + 2 \sum_{uv \in E(G)} d_G(u)d_G(v) \\ &= \sum_{u \in E(G)} d_{G_v}^3(u) + 2 \sum_{uv \in E(G)} d_G(u)d_G(v) \\ &= F(G) + 2M_2(G). \end{aligned} \quad (50)$$

When $G = G_v$, substituting expressions from Theorems 13 and 4, we get

$$\begin{aligned}\chi_2(G_v) = & \left[4F(G_1) + 2n_1^2n_2 + 4n_2M_1(G_1) + F(G_2) + 2n_1^2m_2 + 2n_1M_1(G_2) + 4F(G_1) + n_2^2 \times 2m_1 \right. \\ & + 4n_2M_1(G_1) + 8m_1 + 4M_1(G_1) + n_2^2n_1 + 8n_2m_1 + M_1(G_2) + n_1^2n_2 + 4n_1m_2 \Big] \\ & + 2 \left[4M_2(G_1) + 2n_2M_1(G_1) + n_2^2m_1 + M_2(G_2) + n_1M_1(G_2) + n_1^2m_2 \right. \\ & \left. + 4M_1(G_1) + 8m_1m_2 + (2n_1 + n_2)(2n_2m_1 + 2n_1m_2) + 4n_2m_1 + n_1^2n_2^2 \right].\end{aligned}\quad (51)$$

Similarly, when $G = G_e$, we obtain $SCI(G_e)$ by substituting expressions from Theorems 14 and 6 as follows:

$$\begin{aligned}\chi_2(G_e) = & \left[4F(G_1) + F(G_2) + 2m_1^2m_2 + 2m_1M_1(G_2) + 4F(G_1) + (2 + n_2)^2 \times 2m_1 \right. \\ & + m_1 \times 2m_2 + m_1^3n_2 + 2m_1^2 \times 2m_2 + (2 + n_2^2)m_1n_2 \Big] \\ & + \left[4M_2(G_1) + M_2(G_2) + m_1M(G_2) + m_1^2m_2 + (4 + 2n_2)2m_1 + m_1(2 + n_2)2m_2 + m_1(2 + n_2)m_1m_2 \right].\end{aligned}\quad (52)$$

□

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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