Research Article

Computing the Hosoya Polynomial of M -th Level Wheel and Its Subdivision Graph

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The determination of Hosoya polynomial is the latest scheme, and it provides an excellent and superior role in finding the Wiener and hyper-Wiener index. The application of Wiener index ranges from the introduction of the concept of information theoretic analogues of topological indices to the use as major tool in crystal and polymer studies. In this paper, we will compute the Hosoya polynomial for multiwheel graph and uniform subdivision of multiwheel graph. Furthermore, we will derive two well-known topological indices for the abovementioned graphs, first Wiener index, and second hyper-Wiener index.

1. Introduction

Let G be a finite connected graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The distance $d_{u,v}$ between $u, v \in V(G)$ is the length of the shortest path joining u, v . The diameter $d(G)$ of G is $\max_{(u,v)} d_{u,v}$. The terminologies not defined here can be seen in [1, 2]. The Wiener index W was first put forward in chemistry by Harold Wiener to compute the cardinality of the carbon-carbon bonds among all pairs of carbon atoms in alkane. For a molecular graph G , it is defined as

$$W(G) = \sum_{u,v \in V} d_{u,v}. \quad (1)$$

To read more about the chemical application of Wiener index, see [3–6], and for its mathematical properties, see [7, 8].

Milan Randic coined the term hyper-Wiener index $WW(G)$ of G [9] as

$$WW(G) = \frac{1}{2} \sum_{u,v \in V} (d_{u,v} + d_{u,v}^2). \quad (2)$$

To read more the properties of hyper-Wiener index, see [9–12]. Hosoya polynomial was first introduced by Hosoya [13] and it received the attention of a lot of researchers. The same notion was independently put forward by Sagan et al. [14] as Wiener polynomial G . The Hosoya polynomial $H(G, x)$ of G is defined as

$$H(G, x) = \sum_{u,v \in V} (x^{d_{u,v}}). \quad (3)$$

Let $\alpha(G, k)$ be the number of ordered pair (u, v) in V with $d_{u,v} = k$. Then, the above definition of Hosoya polynomial can be expressed as

$$H(G, x) = \sum_{k=0}^{d(G)} (\alpha(G, k) x^k) \quad (4)$$

The Hosoya polynomial has been investigated on polycyclic aromatic hydrocarbons [15], benzenoid chains [16], Fibonacci and Lucas cubes [17], zigzag polyhexnanotorus [9], carbon nanotubes [18], Hanoi graphs [19], and circumcoronene series [20]. A significant importance of $H(G, x)$ is that some distance-based topological indices (TIs) such as $W(G)$ and $WW(G)$ of G can be computed from the Hosoya polynomial as

$$W(G) = H'(G; 1), WW(G) = H'(G; 1) + \frac{1}{2}H''(G; 1). \quad (5)$$

The readers can see the following papers [21–25] for the results on distance-based TIs.

2. Hosoya Polynomial of M -th Level Wheel Graph

For $n \geq 2$, the join $K_1 \vee C_n$ is called a wheel graph denoted by W_{n+1} . The vertex that comes from the graph K_1 is called the core and is denoted by c . It has order $n + 1$ and size $2n$. A m -level wheel graph denoted by mW_n is the graph obtained by taking m copies of the cycle C_n and one copy of K_1 , such that all the vertices of each copy of C_n are adjacent with the core vertex c . The graph of mW_n is depicted in Figure 1. Note that mW_n has $mn + 1$ vertices and $2mn$ edges. If we label the vertices of cycle at the m -th level by $u_0^m, u_1^m, u_2^m, \dots, u_{n-1}^m$, then the $V(mW_n)$ and the $E(mW_n)$ can be written as

$$\begin{aligned} V(mW_n) &= \{c, u_i^j, 0 \leq i \leq n-1, 1 \leq j \leq m\}, \\ E(mW_n) &= \{cu_i^j, 0 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j, 0 \leq i \leq n-1, 1 \leq j \leq m\}. \end{aligned} \quad (6)$$

Next, the theorem gives the expression for the Hosoya polynomial of mW_n .

Theorem 1. Let $m, n \geq 1$, then $H(mW_n; x)$ is of the form

$$\begin{aligned} H(mW_n; x) &= (mn + 1) + (2mn)x \\ &+ \left[mn(n-3) + \frac{(n(m-1))^2(m)}{2} \right] x^2. \end{aligned} \quad (7)$$

Proof. It is easy to observe that the diameter of mW_n is 2. In order to derive the $H(mW_n; x)$, we compute the coefficients $\alpha(mW_n, k)$ for $k = 0, 1, 2$. By definition, we have $\alpha(mW_n, 0) = mn + 1$ and $\alpha(mW_n, 1) = 2mn$. To compute $\alpha(mW_n, 2)$, we use the following notation:

$$\alpha_A = \text{number of pair of vertices in set } A. \quad (8)$$

The cardinality of order pairs in $V(mW_n)$ with distance 2 can be characterized by the following two sets:

$$\begin{aligned} A_1 &= \{(u_i^j, u_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 2 \leq l \leq n-2\}, \\ A_2 &= \{(u_i^j, u_l^h), \quad 0 \leq i \leq n-1, 0 \leq l \leq n-1, 1 \leq j \leq m-1, j+1 \leq h \leq m\}. \end{aligned} \quad (9)$$

The cardinality of the above sets is $\alpha_{A_1} = mn(n-3)$ and $\alpha_{A_2} = (n(m-1))^2(m)/2$ and hence the coefficient $\alpha(mW_n, 2)$ is equal to $\alpha(mW_n, 2) = \alpha_{A_1} + \alpha_{A_2} = mn(n-3) +$

$(n(m-1))^2(m)/2$. Now, using the values of $\alpha(mW_n, 0)$, $\alpha(mW_n, 1)$, and $\alpha(mW_n, 2)$, we get the desired result. $\square$

FIGURE 1: Multiwheel graph mW_n .

Corollary 1. Let $m, n \geq 1$, then $W(mW_n)$ and $WW(mW_n)$ are given as

$$\begin{aligned} W(mW_n) &= (2mn) + 2 \left[mn(n-3) + \frac{(n(m-1))^2(m)}{2} \right], \\ WW(mW_n) &= (2mn) + 3 \left[mn(n-3) + \frac{(n(m-1))^2(m)}{2} \right]. \end{aligned} \quad (10)$$

3. Hosoya Polynomial of Subdivision of M-th Level Wheel Graph

The subdivision graph $S(mW_n)$ of mW_n is constructed from mW_n by adding a vertex into each edge of mW_n . In other words, we replace each edge of mW_n by a path of length 2. The graph of $S(mW_n)$ is depicted in Figure 2. If we label the new vertices that we insert in the cycle at the j -th level by $x_1^j, x_2^j, \dots, x_{n-1}^j$ for $j = 1, 2, \dots, m$, then the vertex set and edge set of $S(mW_n)$ can be written as

$$\begin{aligned} V(S(mW_n)) &= \{c, u_i^j, v_i^j, x_i^j, \quad 0 \leq i \leq n-1, 1 \leq j \leq m\}, \\ E(S(mW_n)) &= \{cv_i^j, v_i^j u_i^j, x_i^j u_i^j, u_i^j x_{i+1}^j, \quad 0 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j, \quad 0 \leq i \leq n-1, 1 \leq j \leq m\}. \end{aligned} \quad (11)$$

It is easy to observe that order and size of are $3mn + 1$ and $4mn$, respectively. In the next theorem, we give the analytic formula to derive the $H(S(mW_n); x)$.

Theorem 2. Let $m, n \geq 1$, then the $H(S(mW_n); x)$ is of the form

$$\begin{aligned} H(S(mW_n); x) &= (3mn + 1) + (4mn)x + [5mn + mn(n-1)]x^2 \\ &+ \left[\frac{(mn)^2(m-1)}{2} + 3mn + mn(n-1) \right] x^3 + [m((m-1)n)^2 + mn(2n-5) + mn] x^4 \\ &+ \left[mn(n-4) + \frac{m((m-1)n)^2}{2} \right] x^5 + \left[mn(n-5) + \frac{m((m-1)n)^2}{2} \right] x^6. \end{aligned} \quad (12)$$

Proof. It is easy to observe that the diameter of mW_n is 6. In order to derive the $H(S(mW_n); x)$, we find the coefficients $\alpha(S(mW_n), k)$ for $k = 0, 1, 2, \dots, 6$. By definition, we have $\alpha(S(mW_n), 0) = 3mn + 1$ and $\alpha(S(mW_n), 1) = 4mn$. To

compute $\alpha(S(mW_n), j)$ for $j = 2, 3, 4, 5, 6$, we use the following notation:

$$\alpha_A = \text{number of pair of vertices in set } A. \quad (13)$$

FIGURE 2: Uniform subdivision of multiwheel graph mS_n .

The cardinality of order pairs in $V(S(mW_n))$ at distance 2 can be characterized by the following sets:

$$\begin{aligned} B_1 &= \{(c, u_i^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}, \\ B_2 &= \{(v_i^j, v_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 1 \leq l \leq n-1\}, \\ B_3 &= \{(v_i^j, x_{i+1}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}, \\ B_4 &= \{(v_i^j, x_i^j), \quad 0 \leq i \leq n-1, 1 \leq j \leq m\}, \\ B_5 &= \{(u_i^j, u_{i+1}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}, \\ B_6 &= \{(x_i^j, x_{i+1}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}. \end{aligned} \quad (14)$$

The cardinality of the above sets is $\alpha_{B_1} = \alpha_{B_3} = \alpha_{B_4} = \alpha_{B_5} = \alpha_{B_6} = mn$, $\alpha_{B_2} = mn(n-1)$, and hence

$$\begin{aligned} \alpha(S(mW_n), 2) &= \alpha_{B_1} + \alpha_{B_2} + \alpha_{B_3} + \alpha_{B_4} + \alpha_{B_5} + \alpha_{B_6} \\ &= 5mn + mn(n-1). \end{aligned} \quad (15)$$

The cardinality of order pairs in $V(S(mW_n))$ at distance 3 can be characterized by the following sets:

$$\begin{aligned} C_1 &= \{(c, x_i^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}, \\ C_2 &= \{(v_i^j, u_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 1 \leq l \leq n-1\}, \\ C_3 &= \{(v_i^j, u_1^{j+k}), \quad 1 \leq j \leq m, 1 \leq k \leq m-j, 0 \leq i \leq n-1, 0 \leq l \leq n-1\}, \\ C_4 &= \{(u_i^j, x_{i+2}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}, \\ C_5 &= \{(x_i^j, u_{i+1}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}. \end{aligned} \quad (16)$$

The cardinality of the above sets is $\alpha_{C_1} = \alpha_{C_4} = \alpha_{C_5} = mn$, $\alpha_{C_2} = mn(n-1)$, $\alpha_{C_3} = (mn)^2(m-1)/2$, and hence

$$\alpha(S(mW_n), 3) = \alpha_{C_1} + \alpha_{C_2} + \alpha_{C_3} + \alpha_{C_4} + \alpha_{C_5}$$

$$= \frac{(mn)^2(m-1)}{2} + 3mn + mn(n-1). \quad (17)$$

The cardinality of order pairs in $V(S(mW_n))$ at distance 4 can be characterized by the following sets:

$$\begin{aligned} D_1 &= \{(v_i^j, x_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 2 \leq l \leq n-1\}, \\ D_2 &= \{(v_i^j, x_l^{j+k}), \quad 1 \leq j \leq m-1, 1 \leq k \leq m-j, 0 \leq i \leq n-1, 0 \leq l \leq n-1\}, \\ D_3 &= \{(u_i^j, u_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 2 \leq l \leq n-2\}, \\ D_4 &= \{(u_i^j, u_l^{j+k}), \quad 1 \leq j \leq m-1, 1 \leq k \leq m-j, 0 \leq i \leq n-1, 0 \leq l \leq n-1\}, \\ D_5 &= \{(x_i^j, x_{i+2}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1\}. \end{aligned} \quad (18)$$

The cardinality of the above sets is $\alpha_{D_1} = mn(n-2)$, $\alpha_{D_2} = m((m-1)n)^2/2$, $\alpha_{D_3} = mn(n-3)$, $\alpha_{D_4} = m((m-1)n)^2/2$, $\alpha_{D_5} = mn$, and hence

$$\begin{aligned} \alpha(S(mW_n), 4) &= \alpha_{D_1} + \alpha_{D_2} + \alpha_{D_3} + \alpha_{D_4} + \alpha_{D_5} \\ &= m((m-1)n)^2 + mn(2n-4). \end{aligned} \quad (19)$$

The cardinality of order pairs in $V(S(mW_n))$ at distance 5 can be characterized by the following sets:

$$\begin{aligned} E_1 &= \{(u_i^j, x_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 3 \leq l \leq n-2\}, \\ E_2 &= \{(u_i^j, x_l^{j+k}), \quad 1 \leq j \leq m-1, 1 \leq k \leq m-j, 0 \leq i \leq n-1, 0 \leq l \leq n-1\}. \end{aligned} \quad (20)$$

The cardinality of the above sets is $\alpha_{E_1} = mn(n-4)$, $\alpha_{E_2} = m((m-1)n)^2/2$, and hence

$$\begin{aligned} \alpha(S(mW_n), 5) &= \alpha_{E_1} + \alpha_{E_2} \\ &= mn(n-4) + \frac{m((m-1)n)^2}{2}. \end{aligned} \quad (21)$$

The cardinality of order pairs in $V(S(mW_n))$ at distance 6 can be characterized by the following sets:

$$\begin{aligned} F_1 &= \{(x_i^j, x_{i+l}^j), \quad 1 \leq j \leq m, 0 \leq i \leq n-1, 3 \leq l \leq n-3\}, \\ F_2 &= \{(x_i^j, x_l^{j+k}), \quad 1 \leq j \leq m-1, 1 \leq k \leq m-j, 0 \leq i \leq n-1, 0 \leq l \leq n-1\}. \end{aligned} \quad (22)$$

The cardinality of the above sets is $\alpha_{F_1} = mn(n-5)$, $\alpha_{F_2} = m((m-1)n)^2/2$, and hence

$$\begin{aligned} \alpha(S(mW_n), 6) &= \alpha_{F_1} + \alpha_{F_2} \\ &= mn(n-5) + \frac{m((m-1)n)^2}{2}. \end{aligned} \quad (23)$$

Now, using the values of $\alpha(S(mW_n), 0)$, $\alpha(S(mW_n), 1)$, $\alpha(S(mW_n), 2)$, $\alpha(S(mW_n), 3)$, $\alpha(S(mW_n), 4)$, $\alpha(S(mW_n), 5)$, and $\alpha(S(mW_n), 6)$, we get the desired result. $\square$

Corollary 2. Let $m, n \geq 1$, then the $W(mS_n)$ and $WW(mS_n)$ are

$$\begin{aligned}
W(mS_n) &= (4mn) + 2[5mn + mn(n-1)] + 3\left[\frac{(mn)^2(m-1)}{2} + 3mn + mn(n-1)\right] \\
&\quad + 4\left[m((m-1)n)^2 + mn(2n-5) + mn\right] + 5\left[mn(n-4) + \frac{m((m-1)n)^2}{2}\right] \\
&\quad + 6\left[mn(n-5) + \frac{m((m-1)n)^2}{2}\right], \\
WW(mS_n) &= (4mn) + 3[5mn + mn(n-1)] + 6\left[\frac{(mn)^2(m-1)}{2} + 3mn + mn(n-1)\right] \\
&\quad + 10\left[m((m-1)n)^2 + mn(2n-5) + mn\right] + 15\left[mn(n-4) + \frac{m((m-1)n)^2}{2}\right] \\
&\quad + 21\left[mn(n-5) + \frac{m((m-1)n)^2}{2}\right].
\end{aligned} \tag{24}$$

4. Conclusion

We examined the Hosoya polynomial and two vastly studied TIs $W(G)$ and $WW(G)$ for multiwheel graph mW_n and subdivision of multiwheel graph mS_n .

Data Availability

No data were used for this study.

Disclosure

Mathematics subject classification: 05C09, 05C92, 92E10.

Conflicts of Interest

The authors hereby declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] G. Chartrand, *Introduction to Graph Theory*, Tata McGraw-Hill Education, New York, NY, USA, 2006.
- [2] R. Diestel, *Graph Theory*, Springer-Verlag, Berlin, Germany, 1997.
- [3] Z. Mihalic and N. Trinajstic, "A graph theoretical approach to structure property relationships," *Journal of Chemical Education*, vol. 69, pp. 701–712, 1992.
- [4] I. Gutman, Y. N. Yeh, S. L. Lee, and Y. L. Luo, "Some recent results in the theory of the wiener number," *Indian Journal of Chemistry*, vol. 32A, pp. 651–661, 1993.
- [5] I. Gutman and J. H. Potgieter, "Wiener index and intermolecular forces," *Journal of the Serbian Chemical Society*, vol. 62, pp. 185–192, 1997.
- [6] D. H. Rouvray, "The rich legacy of half a century of the wiener index," in *Topology in Chemistry Discrete Mathematics of Molecules*, D. H. Rouvray and R. B. King, Eds., Horwood, Chichester, UK, pp. 16–37, 2002.
- [7] A. A. Dobrynin, R. Entringer, and I. Gutman, "Wiener index of trees: theory and applications," *Acta Applicandae Mathematica*, vol. 66, no. 3, pp. 211–249, 2001.
- [8] A. A. Dobrynin, I. Gutman, S. Klavžar, and P. Žigert, "Wiener index of hexagonal systems," *Acta Applicandae Mathematica*, vol. 72, no. 3, pp. 247–294, 2002.
- [9] M. Randic, "Novel molecular descriptor for structure property studies," *Chemical Physics Letters*, vol. 211, pp. 478–483, 1993.
- [10] D. J. Klein, I. Lukovits, and I. Gutman, "On the definition of the hyper-wiener index for cycle-containing structures," *Journal of Chemical Information and Computer Sciences*, vol. 35, no. 1, pp. 50–52, 1995.
- [11] I. Gutman, "Relation between hyper-wiener and wiener index," *Chemical Physics Letters*, vol. 364, no. 3–4, pp. 352–356, 2002.
- [12] I. Gutman and B. Furtula, "Hyper-wiener index vs. wiener index. two highly correlated structure-descriptors," *Monatshefte für Chemie—Chemical Monthly*, vol. 134, no. 7, pp. 975–981, 2003.
- [13] H. Hosoya, "On some counting polynomials in chemistry," *Discrete Applied Mathematics*, vol. 19, no. 1–3, pp. 239–257, 1988.
- [14] B. E. Sagan, Y.-N. Yeh, and P. Zhang, "The wiener polynomial of a graph," *International Journal of Quantum Chemistry*, vol. 60, no. 5, pp. 959–969, 1996.
- [15] S. Nikolic, N. Trinajstic, and Z. Mihalic, "The Wiener index: development and applications," *Croatica Chemica Acta*, vol. 68, pp. 105–129, 1995.
- [16] W. Gao, Y. Wang, B. Jamil, and M. Kamran, "Characteristics studies of molecular structures in drugs," *Saudi Pharmaceutical Journal*, vol. 25, no. 4, pp. 580–586, 2017.
- [17] N. PrabhakaraRao and A. LaxmiPrasanna, "On the Wiener index of pentachains," *Applied Mathematical Sciences*, vol. 2, pp. 2443–2457, 2008.
- [18] A. A. Ali and A. M. Ali, "Hosoya polynomials of pentachains," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 65, pp. 807–819, 2011.
- [19] I. Gutman, S. Klavžar, M. Petkovsek, and P. Žigert, "On Hosoya polynomials of benzenoid graphs," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 43, pp. 49–66, 2001.
- [20] M. Eliasi and B. Taeri, "Hosoya polynomial of zigzag polyhex nanotorus," *Journal of the Serbian Chemical Society*, vol. 73, no. 3, pp. 311–319, 2008.

- [21] M. Arockiaraj, S. R. J. Kavitha, K. Balasubramanian, and I. Gutman, "Hyper-Wiener and Wiener polarity indices of silicate and oxide frameworks," *Journal of Mathematical Chemistry*, vol. 56, no. 5, pp. 1493–1510, 2018.
- [22] W. Gao, Z. Iqbal, M. Ishaq, R. Sarfraz, M. Aamir, and A. Aslam, "On eccentricity-based topological indices study of a class of porphyrin-cored dendrimers," *Biomolecules*, vol. 8, no. 3, p. 71, 2018.
- [23] Z. Iqbal, M. Ishaq, A. Aslam, and W. Gao, "On eccentricity based topological descriptors of water soluble dendrimers," *Zeitschrift für Naturforschung C*, vol. 74, 2018.
- [24] W. Gao, Z. Iqbal, M. Ishaq, A. Aslam, and R. Sarfraz, "Topological aspects of dendrimers via distance based descriptors," *IEEE Access*, vol. 7, 2019.
- [25] Y. Rao, A. Aslam, M. U. Noor, A. Othman Almatroud, and Z. Shao, "Bond incident degree indices of catacondensed pentagonal systems," *Complexity*, vol. 2020, Article ID 4935760, 7 pages, 2020.