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Research Article

An Approach to the Extremal Inverse Degree Index for Families of Graphs with Transformation Effect

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The inverse degree index is a topological index first appeared as a conjuncture made by computer program Graffiti in 1988. In this work, we use transformations over graphs and characterize the inverse degree index for these transformed families of graphs. We established bonds for different families of n-vertex connected graph with pendent paths of fixed length attached with fully connected vertices under the effect of transformations applied on these paths. Moreover, we computed exact values of the inverse degree index for regular graph specifically unicyclic graph.

1. Introduction and Preliminary Results

Graph theory has many applications in chemistry, physics, computer sciences, and other applied sciences. Topological indices are graph invariants used to study the topology of graphs. Along with the computer networks, graph theory considers as a powerful tool in other areas of research, such as in coding theory, database management system, circuit design, secret sharing schemes, and theoretical chemistry [1]. Cheminformatics is the combination of technology, graph theory, and chemistry. It develops a relationship between structure of organic substances and their physiochemical properties through some useful graph invariants with the help of their associated molecular graph. The molecular graph is the combination of vertices and edges which are representatives of atoms and bonds between atoms of corresponding substance, respectively. Theoretical study of underlying chemical structure by some useful graph invariants is an attractive area of research in mathematical chemistry due to its effective applications in the QSAR/QSPR investigation [2, 3]. Topological indices among these invariants have special place and used to estimate the physiochemical properties of chemical compound. A topological index can be considered as a function which maps a graph to a real number.

Throughout this work, we used standard notations, $G = G(V, E)$ for graph, $V(G)$ set of vertices, $E(G)$ the set of edges, d_{v_i} degree of vertex v_i (the number of edges incident to v_i), Δ and δ be the maximum and minimum degrees of fully connected vertices, vertices with degree one are pendent vertices, and path attached with fully connected vertices taken as a pendent paths.

In the last five decades, after the Wiener index, many topological indices had been introduced. Probably, the Randić connectivity index [4]

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}, \tag{1}$$

is one of the best predictive invariants among these topological indices. The accuracy in predictability of indices is the main interest of researchers which leads them to purpose a new topological index.

The zeroth-order general Randić index ${}^0R_\alpha(G) = \sum_{u\in V(G)} d_u^\alpha$ was conceived by Li and Zheng in their work [5]. ${}^0R_{1/2}(G) = \sum_{u\in V(G)} 1/\sqrt{d_u}$ equivalent to ${}^0R_\alpha(G)$ for $\alpha = -1/2$. Hu et al. in [6] and others [7–10] characterize ${}^0R_\alpha$ for different values of α. For $\alpha = -1, {}^0R_{-1} = \mathrm{ID}$ is modified total adjacency index or inverse degree, first appears in the conjecture over computer program Graffiti [11]. The

$$\mathrm{ID}(G) = \sum_{v\in V(G)} \frac{1}{d_v} = \sum_{uv\in E(G)} \frac{d_u^2 + d_v^2}{d_u^2 d_v^2}, \tag{2}$$

for graphs without isolated vertices are well discussed in [12, 13]. Extremal characterization and bonds of $\mathrm{ID}(G)$ also discussed at some extent in [14–18]. For more detail, one can review survey [19].

In this work, we investigated the effect of transformations over families of graphs for ID and established inequalities for these transformed graphs. Graph transformations are very important in chemistry, computer designing, and animations. Moreover, we determined the exact value of ID for some major families of graphs under the effect of transformations over pendent paths.

2. Results and Discussion

In this section, we present some transformations over pendent paths. These have solid effect over increase and decrease of $\mathrm{ID}(G)$. Through out this work, we considered n_0-vertex connected graph G_0. G_k^l be the graph G_0 with k pendent paths of length $l \geq 1$ having order $n = n_0 + kl$ with degree sequence $d_1 = \delta \leq d_2 \leq d_3 \leq \cdots \leq \Delta + 1$.

2.1. Graph Transformations. Let $E'(G) \subset E(G)$, the $G_1 = G - E'(G)$ be subgraph obtained by removing edges of $E(G')$, and $G_1' = G - V'(G)$ be the subgraph obtained by deleting vertices set $V'(G) \subset V(G)$ along with their incident edges. We give following transformations using these techniques which have solid effect on $\mathrm{ID}(G)$.

2.1.1. Transformation A. Let $w_j \in V(G_0)$, $d_{w_j} \geq 3$, $j = 1, 2, 3, \ldots, k \leq n$ and $\{w_j u_j^1, u_j^1 u_j^2, u_j^2 u_j^3, \ldots, u_j^{l-1} u_j^l\}$ be the pendent paths attached with fully connected vertex w_j of G_0 forms G_k^l. Then,

$$\begin{aligned} A\left(G_k^l\right) = G_1 = G_0 &- \sum_{j=1}^{k} \{u_j^2 u_j^3, u_j^3 u_j^4, \ldots, u_j^{l-1} u_j^l\} \\ &+ \sum_{j=1}^{k} \{w_j u_j^2, u_j^2 u_j^3, \ldots, u_j^{l-1} u_j^l\}. \end{aligned} \tag{3}$$

Figure 1 depicts successive application of transformation A as A_i, $i = 1, 2, 3, \ldots, l-1$.

2.1.2. Transformation B. Let $w_j \in V(G_0), d_{w_j} \geq 3$, $j = 1, 2, 3, \ldots, k \leq n$ and $\{w_j u_j^1, w_j u_j^1, w_j u_j^3, \ldots, w_j u_j^{l-i}\}$ be the leafs attached with fully connected vertex w_j of G. Then, for fixed vertex w_1,

$$\begin{aligned} G_j' = G - &\{u_j^1, u_j^2, u_j^3, \ldots, u_j^{l-q}\} \cup \{u_j^{l-(q-1)} u_j^{l-(q-2)}, u_j^{l-(q-2)} u_j^{l-(q-3)}, \ldots, u_j^{l-1} u_j^l\} \\ &+ \{w_1 u_j^1, w_1 u_j^1, w_1 u_j^3, \ldots, w_1 u_j^{l-q}\} \cup \{w_1 u_j^{l-(q-1)}, u_j^{l-(q-1)} u_j^{l-(q-2)}, u_j^{l-(q-2)} u_j^{l-(q-3)}, \ldots, u_j^{l-1} u_j^l\}. \end{aligned} \tag{4}$$

Theorem 1. *Let G_0 be the graph of order n_1 with maximum degrees Δ and minimum δ. Then,*

$$\begin{aligned} \mathrm{ID}\left(G_k^l\right) &\leq \mathrm{ID}\left(A\left(G_k^l\right)\right), \\ \mathrm{ID}\left(G_k^l\right) &\leq \mathrm{ID}\left(B\left(G_k^l\right)\right). \end{aligned} \tag{5}$$

Proof. Let G_k^l be the graph of order $n = n_1 + kl$, minimum degree δ, and maximum degree $\Delta + 1$. G_k^l is the composition of G_0 and k pendent paths of length l. In G_k^l, there are at least k vertices of degree 1 and $k(l-1)$ having 2 and n_1 vertices with degree $d_{v_s} + 1, \delta \leq d_{v_s} + 1 \leq \Delta + 1$:

$$\mathrm{ID}\left(G_k^l\right) = \sum_{s=1}^{n-k(l+1)} \frac{1}{d_{v_s}} + \sum_{s=1}^{k} \frac{1}{d_{v_s} + 1} + k + \frac{k(l-1)}{2}. \tag{6}$$

The transformation A transforms k vertices from degree 2 to 1 and another k vertices from $d_s + 1$ to $d_s + 2$ which have an effect in ID as

$$\mathrm{ID}\left(A\left(G_k^l\right)\right) = \sum_{s=1}^{n-k(l+1)} \frac{1}{d_{v_s}} + \sum_{s=1}^{k} \frac{1}{d_{v_s} + 2} + 2k + \frac{k(l-2)}{2}. \tag{7}$$

So, from equations (6) and (7), we have

$$\begin{aligned} \mathrm{ID}\left(G_k^l\right) - \mathrm{ID}\left(A\left(G_k^l\right)\right) &= \sum_{s=1}^{k} \left(\frac{1}{d_{v_s} + 1} - \frac{1}{d_{v_s} + 2}\right) - k + \frac{k}{2} \\ &= \sum_{s=1}^{k} \frac{1}{\left(d_{v_s} + 1\right)\left(d_{v_s} + 2\right)} - \frac{k}{2}. \end{aligned} \tag{8}$$

Replace d_{v_s} with minimum degree δ. It maximizes the term $\sum_{s=1}^{k} 1/((d_{v_s} + 1)(d_{v_s} + 2))$, which implies

$$\begin{aligned} &= \sum_{s=1}^{k} \frac{1}{(\delta + 1)(\delta + 2)} - \frac{k}{2} = \frac{k}{(\delta + 1)(\delta + 2)} - \frac{k}{2} \\ &= \frac{k(2 - (\delta + 1)(\delta + 2))}{(\delta + 1)(\delta + 2)} = \frac{k\left(-\delta^2 - 3\delta\right)}{(\delta + 1)(\delta + 2)}. \end{aligned} \tag{9}$$

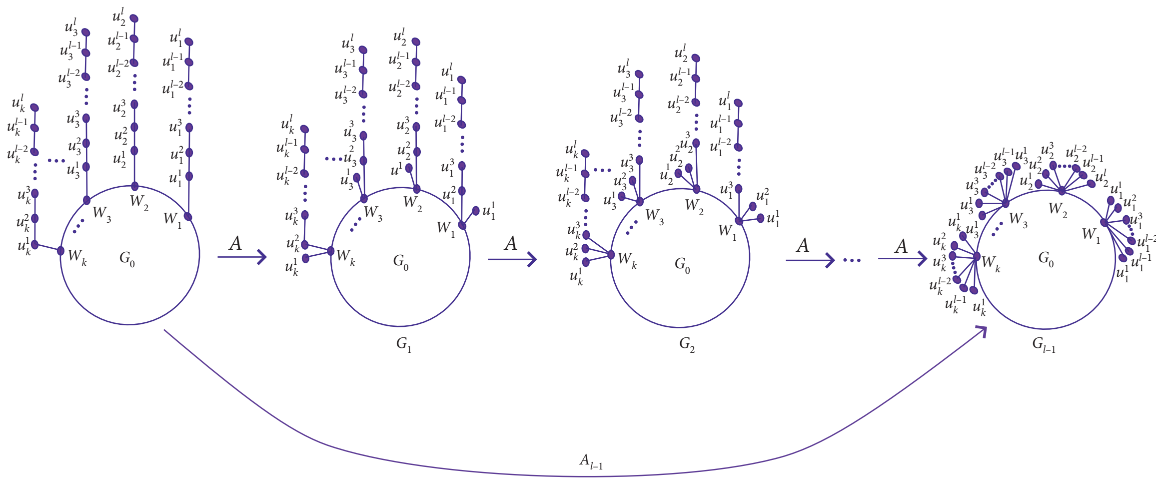


Figure 1: Transformation A.

It is clear from (9) that $\mathrm{ID}(G_k^l) - \mathrm{ID}(A(G_k^l)) \leq 0$. Hence,

$$\mathrm{ID}\left(G_k^l\right) \leq \mathrm{ID}\left(A\left(G_k^l\right)\right). \tag{10}$$

The transformation B shown in Figure 2 decreases the degree of one vertex and makes the same increase into the degree of fixed selected vertex:

$$\begin{aligned}\mathrm{ID}\left(B\left(G_k^l\right)\right) &= \sum_{s=1}^{n-k(l+1)} \frac{1}{d_{v_s}} + \sum_{s=1}^{k-2} \frac{1}{d_{v_s}+1} + \frac{1}{d_{v_k}} \\ &\quad + \frac{1}{d_{v_{(k-1)}}+2} + \frac{k(l-2)}{2} + k.\end{aligned} \tag{11}$$

So, from equations (6) and (11), we get

$$\begin{aligned}\mathrm{ID}\left(G_k^l\right) - \mathrm{ID}\left(B\left(G_k^l\right)\right) &= \sum_{s=1}^{k} \frac{1}{d_{v_s}+1} - \sum_{s=1}^{k-2} \frac{1}{d_{v_s}+1} \\ &\quad - \frac{1}{d_{v_k}} - \frac{1}{d_{v_{(k-1)}}+2},\end{aligned} \tag{12}$$

to maximize the fraction involved in above expression replace d_{v_s}, $0 \leq s \leq k$ with δ, and we get

$$\begin{aligned}\mathrm{ID}\left(G_k^l\right) - \mathrm{ID}\left(B\left(G_k^l\right)\right) &= \frac{2}{\delta+1} - \frac{1}{\delta} - \frac{1}{\delta+2} \\ &= \frac{-2\delta - 2\delta^2 - 2}{(\delta)(\delta+1)(\delta+2)} \leq 0.\end{aligned} \tag{13}$$

The equation (13) implies

$$\mathrm{ID}\left(G_k^l\right) \leq \mathrm{ID}\left(B\left(G_k^l\right)\right). \tag{14}$$

□

2.1.3. Transformation A_i^j. The transformation A_i^j is composition of A_i, $0 \leq i \leq l-1$ and B_j, $0 \leq j \leq k-1$ which is shown in Figure 3. Here, A_i, $0 \leq i \leq l-1$ be the repetition of transformation A and B_j, $0 \leq j \leq k-1$ be the repetition of transformation B.

For main results related to the transformation A_i^j shown in Figure 3, we need to prove Propositions 1 and 2.

Proposition 1. *Let $g\colon N \times W \longrightarrow Q$ defined as $g(\eta,\zeta) = 1/(\eta+\zeta)$. Then,*

(1) $g(\eta,\zeta) + 1 \geq g(\eta,\zeta-1) + (1/2)$ for $\zeta \geq 1$

(2) For $\alpha,\beta \geq 0$, $g[\eta,(\alpha+1)(\beta+1)] + g(\eta,0) \geq [g[\eta,\alpha(\beta+1)] + g(\eta,\beta+1)]$

Proof. (1) If transformation A applied on pendent path attached with vertex w_j of G having degree $\eta+\zeta$. The degree of vertex w_j increased by one with change of vertex having degree 2 to leaf attached to w_j. This change has effect on ID in the following way.

Let $g(\eta,\zeta) = 1/(\eta+\zeta)$. Then,

$$\begin{aligned}g(\eta,\zeta) + 1 - \left[g(\eta,\zeta-1) + \frac{1}{2}\right] &= \frac{1}{\eta+\zeta} + 1 - \frac{1}{\eta+\zeta-1} - \frac{1}{2} \\ &= \frac{1}{2} - \frac{1}{(\eta+\zeta)(\eta+\zeta-1)}.\end{aligned} \tag{15}$$

It is clear from basic calculus that $(1/\alpha) \geq (1/(\alpha+\beta))$; $\beta \geq 0$. So,

$$= \frac{1}{2} - \frac{1}{(\eta+\zeta)(\eta+\zeta-1)} \geq 0, \tag{16}$$

implies $g(\eta,\zeta) + \zeta \geq g(\eta,\zeta-1)$.

(2) The 2nd part of this preposition is related to the effect of transformation A_i^j shown in Figure 3:

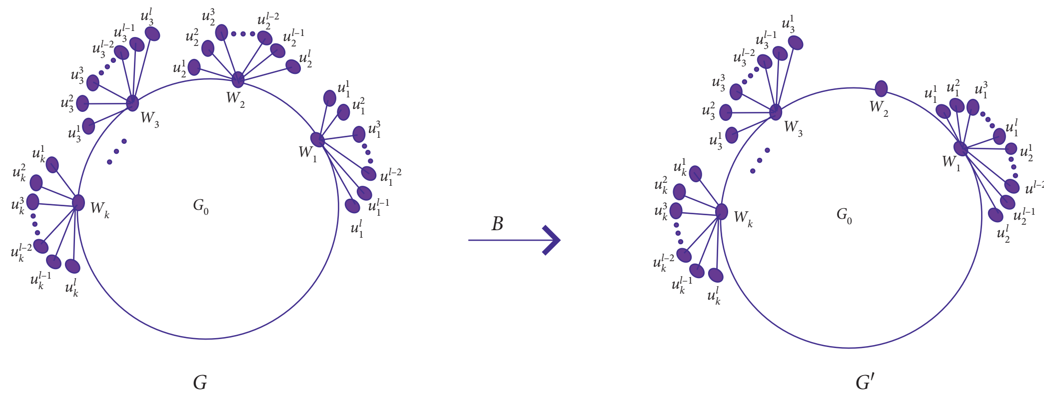


Figure 2: Transformation B for $q = 1$.

$$g[\eta, (\alpha+1)(\beta+1)] + g(\eta,0) - g[\eta,\alpha(\beta+1)] + g(\eta,\beta+1)$$
$$= \frac{1}{(\alpha+1)\beta+\eta} + \frac{1}{\eta} - \frac{1}{\alpha*\beta+\eta} - \frac{1}{\beta+\eta+1} \tag{17}$$
$$= \frac{\alpha^2\beta^3 + 3\alpha^2\beta^2 + 3\alpha^2\beta + \alpha^2 + \alpha\beta^3 + 2\alpha\beta^2\eta + 3\alpha\beta^2 + 4\alpha\beta\eta + 3\alpha\beta + 2a\eta + \alpha}{\eta(\beta+\eta+1)(\alpha(\beta+1)+\eta)(\eta+b+1)} \geq 0.$$

Thus, $g[\eta, (\alpha+1)(\beta+1)] + g(\eta,0) \geq g[\eta,\alpha(\beta+1)] + g(\eta,\beta+1)$. □

Proposition 2. *Let $f(\eta) = (1/\eta) + (3/2)$ and $g(\eta) = (1/(\eta+1)) + 2$, then for $\eta \geq 1$, $g(\eta) \geq f(\eta)$.*

Proof

$$g(\eta) - f(\eta) = \frac{1}{\eta+1} + 2 - \left(\frac{1}{\eta} + \frac{3}{2}\right) = \frac{1}{\eta+1} - \frac{1}{\eta} + \frac{1}{2}$$
$$= \frac{2\eta - 2(\eta+1) + \eta(\eta+1)}{2\eta(\eta+1)} \tag{18}$$
$$= \frac{(\eta-1) + (\eta^2-1)}{2\eta(\eta+1)} \geq 0.$$

This fraction is nonnegative for all $\eta \geq 1$ which implies that $g(\eta) \geq f(\eta)$. □

Theorem 2. *Let G be the graph of order n having p pendent vertices. G_k^l is the graph having k pendent paths attached to the fully connected vertices with maximum degree of a vertex $\Delta + 1$. Then, for $0 \leq i \leq l-1$ and $\alpha \leq \beta$,*

$$\mathrm{ID}\left(A_i^\alpha\left(G_k^l\right)\right) \leq \mathrm{ID}\left(A_i^\beta\left(G_k^l\right)\right). \tag{19}$$

Proof. Let G_k^l be the graph with order $n = n + kl$, minimum degree δ, and maximum degree $\Delta + 1$. Using the fact of A_i^j over G_k^l, we get

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) = k(i+1) + \frac{k[l-(i+1)]}{2} + \sum_{r=1}^{k-(j+1)} \frac{1}{d_{v_r} + (i+1)}$$
$$+ \sum_{s=1}^{n-p-k+j} \frac{1}{d_{v_s}} + \frac{1}{d_{v_r} + (j+1)(i+1)} + p, \tag{20}$$

for $1 \leq d_r, d_s \leq \Delta \leq n-1$ and $0 \leq i \leq l-1, 0 \leq j \leq k-1$. Then, for $\alpha \leq \beta$,

$$\mathrm{ID}\left(A_i^\alpha\left(G_k^l\right)\right) - \mathrm{ID}\left(A_i^\beta\left(G_k^l\right)\right)$$
$$= \sum_{r=1}^{k-(\alpha+1)} \frac{1}{d_{v_r} + i + 1} + \sum_{s=1}^{n-p-k+\alpha} \frac{1}{d_{v_s}} + \frac{1}{d_{v_r} + (\alpha+1)(i+1)}$$
$$- \left[\sum_{r=1}^{k-(\beta+1)} \frac{1}{d_{v_r} + i + 1} + \sum_{s=1}^{n-p-k+\beta} \frac{1}{d_{v_s}} + \frac{1}{d_{v_r} + (\beta+1)(i+1)}\right]$$
$$= \sum_{r=k-\beta}^{k-\alpha-1} \frac{1}{d_{v_r} + i + 1} - \sum_{s=n-p-k+\alpha+1}^{n-p-k+\beta} \frac{1}{d_{v_s}}$$
$$+ \frac{(i+1)(\beta-\alpha)}{\left(d_{v_r} + (i+1)(\alpha+1)\right)\left(d_{v_r} + (i+1)(\beta+1)\right)}. \tag{21}$$

So, by using Proposition 1 and replacing Δ with d_{v_r} and δ with d_{v_s}, it is clear that Δ minimizes the positive terms and δ maximizes the negative term. After simplification, we get

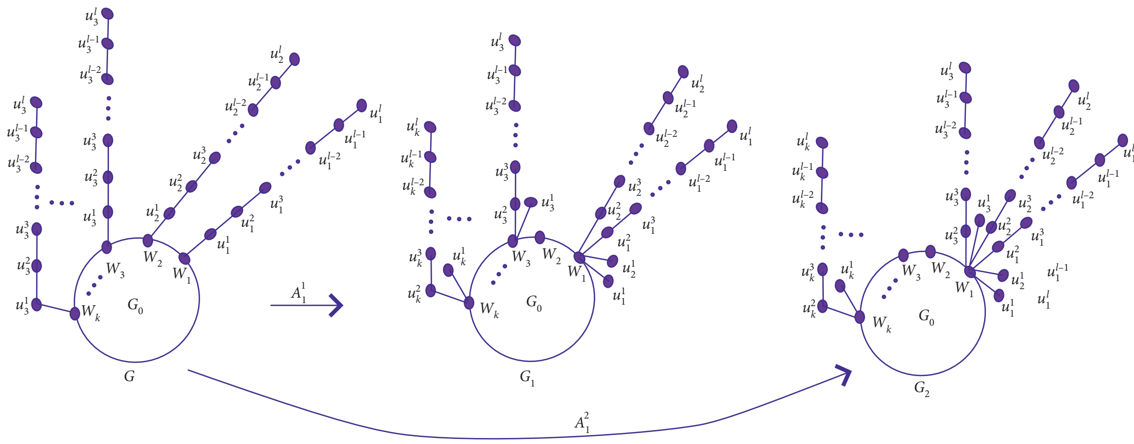


Figure 3: Transformation A_i^j.

$$= \frac{(\beta-\alpha)(\delta-\Delta)(\Delta+(i+1)(\beta+1))(\Delta+(i+1)(\alpha+1))+(\delta-\Delta)(\Delta+i+1)(i+1)-(i+1)^2(\alpha\Delta+(\Delta+(i+1)(\alpha+1))(\beta+1))}{(\Delta+l(\alpha+1))(\Delta+l(\beta+1))\delta(\Delta+1)}. \tag{22}$$

It is clear that for $\alpha \leq \beta$, the nominator is a negative number and denominator is positive which implies that

$$\mathrm{ID}\left(A_i^{\alpha}\left(G_k^l\right)\right) - \mathrm{ID}\left(A_i^{\beta}\left(G_k^l\right)\right) \leq 0. \tag{23}$$

Thus, for $\beta \geq \alpha$,

$$\mathrm{ID}\left(A_i^{\alpha}\left(G_k^l\right)\right) \leq \mathrm{ID}\left(A_i^{\beta}\left(G_k^l\right)\right). \tag{24}$$

□

In the following theorem, we determined bonds of ID for graph G_k^l under the effect of transformation A_i^j by using Propositions 1 and 2.

Theorem 3. *Let G be the graph of order n having p pendent vertices. G_k^l is the graph with maximum degree $\Delta+1$ having order $n+kl$ with $k \leq n-p$ of pendent paths of length l. Then, for $ID(A_i^j(G_k^l))$; $0 \leq i \leq l-1, 0 \leq j \leq k-1$,*

$$\frac{(\Delta+1)[\Delta(l+1)+2(p\Delta+n-p-1)]+2\Delta}{2\Delta(\Delta+1)} \leq \mathrm{ID}\left(A_i^j\left(G_k^l\right)\right). \tag{25}$$

Equality holds for r-regular graph with $i=0$, $j=0$, and $k=1$. And

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) \leq \frac{(\delta+(n-p)l)[\delta l(n-p)+n-1+p\delta]+\delta}{((n-p)l+\delta)\delta}, \tag{26}$$

equality holds if under consideration graph is r-regular with $k=n-p$ pendent paths of length l and $i=l-1$, $j=k-1$.

Proof. Let G be the graph having order $n \geq 3$ with $0 \leq p \leq n-1$ pendent vertices with minimum degree δ and maximum degree Δ. G_k^l is the graph with maximum degree $\Delta+1$ and maximum number of pendent paths $k=n-p$ of length l. Then, by using equation (20),

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) = k(i+1) + \frac{k[l-(i+1)]}{2} + \sum_{r=1}^{k-(j+1)} \frac{1}{d_{v_r}+(i+1)} + \sum_{s=1}^{n-p-k+j} \frac{1}{d_{v_s}} + \frac{1}{d_{v_r}+(j+1)(i+1)} + p, \tag{27}$$

where $1 \leq d_r, d_s \leq \Delta \leq n-1$, and $0 \leq i \leq l-1, 0 \leq j \leq k-1$.

The order of G_k^l is fixed. So, the increase in pendent paths causes to decrease their lengths l. This fact increases the number of pendent paths and decreases the vertices of degree two. So, Proposition 2 clears that $\mathrm{ID}(A_i^j(G_k^l))$ increases with the increase in k. It is clear from Theorems 1 and 2 and Propositions 1 and 2 that the least value of $ID(A_i^j(G_k^l))$ was obtained by setting $i, j=0$, $d_r=d_s=\Delta$, and $k=1$:

$$\mathrm{ID}\left(A_0^0\left(G_1^l\right)\right) \geq 1 + \frac{1[l-1]}{2} + \frac{1-1}{\Delta+1} + \frac{n-p-1}{\Delta} + \frac{1}{\Delta+1} + p. \tag{28}$$

After simplification, we get

$$\frac{(\Delta+1)[\Delta(l+1)+2(p\Delta+n-p-1)]+2\Delta}{2\Delta(\Delta+1)} \leq \mathrm{ID}\left(A_i^j\left(G_k^l\right)\right), \tag{29}$$

and equality holds in (28) for r-regular graph with the $k=1$ pendent path of length l and $i=0$, $j=0$.

Now again from (20), setting $i=l-1$, $j=k-1$, $d_r=d_s=\delta$, $k=n-p$ and using Proposition 1 and Theorems 1, 2, we get maximal value of $\mathrm{ID}(A_i^j(G_k^l))$ as

$$\begin{aligned}\mathrm{ID}\left(A_{l-1}^{k-1}\left(G_k^l\right)\right) \leq{}& (n-p)(l-1+1) + \frac{(n-p)[l-(l-1+1)]}{2} + \sum_{s=1}^{n-(n-p)+(n-p)-1} \frac{1}{\delta} \\ &+ \sum_{r=1}^{(n-p)-((n-p)-1+1)} \frac{1}{\delta+(l-1+1)} + \frac{1}{[\delta+((n-p)-1+1)(l-1+1)]} + p.\end{aligned} \tag{30}$$

After simplification, we get maximal value of $\mathrm{ID}(A_i^j(G_k^l))$ as

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) \leq \frac{(\delta+(n-p)l)[\delta l(n-p)+n-1+p\delta]+\delta}{((n-p)l+\delta)\delta}, \tag{31}$$

in which equality holds for r-regular graph with $k=n-p$ pendent paths of length l and $i=l-1$, $j=k-1$.

Inequalities (29) and (31) complete the proof. □

Theorem 4. *Let G be the graph without pendent vertices and G_k^l for $k>1$ be the graph with maximum degree $\Delta+1$. Then, for $0 \leq i \leq l-1, 0 \leq j \leq k-1$, the lower bond of $ID(A_i^j(G_k^l))$ is*

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) \geq \frac{(\Delta+1)[\Delta k(l+1)+2(n-k)]+2k\Delta}{2\Delta(\Delta+1)}. \tag{32}$$

Equality holds for r-regular graph with k pendent paths of length l and $i=0$, $j=0$.

Proof. Let G be the graph of order n without pendent vertices having minimum degree δ and maximum degree Δ. G_k^l be the graph with maximum degree $\Delta+1$ and $k \geq 1$ be the count of pendent paths of length l. Then, by using (20),

$$\begin{aligned}\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) ={}& k(i+1) + \frac{k[l-(i+1)]}{2} + \sum_{r=1}^{k-(j+1)} \frac{1}{d_{v_r}+(i+1)} \\ &+ \sum_{s=1}^{n-k+j} \frac{1}{d_{v_s}} + \frac{1}{d_{v_r}+(j+1)(i+1)},\end{aligned} \tag{33}$$

where $1 \leq d_r, d_s \leq \Delta \leq n-1$, and $0 \leq i \leq l-1, 0 \leq j \leq k-1$. Using Propositions 1 and 2 and setting $i, j=0, d_r=d_s=\Delta$, we get least value of $\mathrm{ID}(A_i^j(G_k^l))$ as $\mathrm{ID}(A_0^0(G_k^l))$:

$$\mathrm{ID}\left(A_0^0\left(G_k^l\right)\right) = k + \frac{k[l-1]}{2} + \frac{k-1}{\Delta+1} + \frac{n-k}{\Delta} + \frac{1}{\Delta+1}. \tag{34}$$

After simplification, we get minimal value as

$$\frac{(\Delta+1)[\Delta k(l+1)+2(n-k)]+2k\Delta}{2\Delta(\Delta+1)} \leq \mathrm{ID}\left(A_i^j\left(G_k^l\right)\right). \tag{35}$$

Equality for equation (3) holds for r-regular graph with k pendent paths of length l and $i=0$, $j=0$. □

Theorem 5. *Let G be the Δ-regular graph. $ID(A_i^j(G_k^l))$ be the graph with k pendent paths of length l. Then, for $0 \leq i \leq l-1, 0 \leq j \leq k-1$:*

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) = \frac{[\Delta+(i+1)j](\Delta+i+1)[2(n-k+j)+k(i+l+1)][2\Delta(\Delta+(i+1)j)](k-j)+\Delta(k-j-1)+\Delta(\Delta+l)}{2\Delta(\Delta+i+1)(\Delta+j(i+1))}. \tag{36}$$

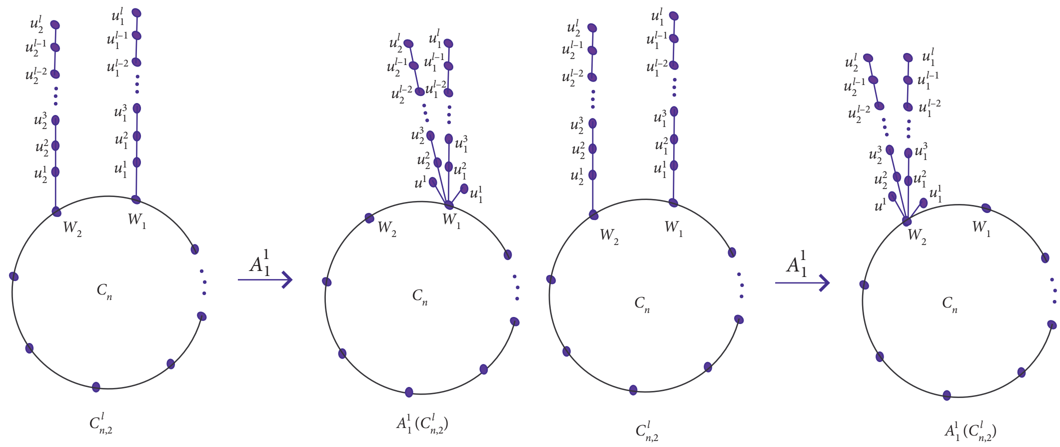
Proof. Let G be the Δ-regular graph and k be the count of pendent paths of length l. Then, G_k^l is the graph with maximum degree $\Delta+1$. Then, for $0 \leq i \leq l-1, 0 \leq j \leq k-1$, equation (20) takes the form

$$\begin{aligned}\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) ={}& k(i+1) + \frac{k[l-(i+1)]}{2} + \sum_{r=1}^{k-(j+1)} \frac{1}{\Delta+(i+1)} \\ &+ \sum_{s=1}^{n-k+j} \frac{1}{\Delta} + \frac{1}{[\Delta+(j+1)(i+1)]}.\end{aligned} \tag{37}$$

After simplification, we get required result:

$$\mathrm{ID}\left(A_i^j\left(G_k^l\right)\right) = \frac{[\Delta+(i+1)j](\Delta+i+1)[2(n-k+j)+k(i+l+1)][2\Delta(\Delta+(i+1)j)](k-j)+\Delta(k-j-1)+\Delta(\Delta+l)}{2\Delta(\Delta+i+1)(\Delta+j(i+1))}. \tag{38}$$

□



(a) (b)

FIGURE 4: Transformation A_i^j effect over $G_k^l = C_{n,2}^l$: (a) transformed graph $A_1^1(C_{n,2}^l)$ with fixed vertex w_1; (b) transformed graph $A_1^1(C_{n,2}^l)$ with fixed vertex w_2.

In Corollary 1, we determined exact values of ID for unicyclic graphs with k pendent paths of length l under transformation A_i^j. Figure 4(a) depicts transformed graph $A_i^j(C_{n,k}^l)$ for $i = j = 1$, $k = 2$ with fixed vertex w_1 and Figure 4(b) with fixed vertex w_2.

Corollary 1. *Let C_n be the unicyclic graph of order n. $C_{n,k}^l$ is the graph with k pendent paths of length l. Then, for* $0 \le i \le l-1, 0 \le j \le k-1,$

$$\mathrm{ID}\left(A^j\left(G_k^l\right)\right) = \frac{(2+(i+1)j)(3+i)[2(n-k+j)+k(i+l+1)]4(2+(i+1)j)(k-j)+2(k-j-1)+2(2+l)}{4(3+i)(2+j(i+1))}. \tag{39}$$

Proof. C_n is the unicyclic graph of order n. C_n is 2-regular graph. Then, we get required result by replacing Δ by 2 in Theorem 5:

$$\mathrm{ID}\left(A^j\left(G_k^l\right)\right) = \frac{(2+(i+1)j)(3+i)[2(n-k+j)+k(i+l+1)]4(2+(i+1)j)(k-j)+2(k-j-1)+2(2+l)}{4(3+i)(2+j(i+1))}. \tag{40}$$

3. Conclusions

Topological indices and graph transformations play a significant role in modern chemistry and computer networks. It is an interesting problem to determine the bonds of the topological index for different families of graphs [5, 6, 9, 16]. In this work, we give graph transformations, A, B, and A_i^j for variable values of i and j over pendent paths attached with the fully connected vertices of graphs and characterized ID for these transformed graphs. At first, we determined the effect of transformations A and B over increase and decrease of ID individually. Then, we established result for A_i^j for arbitrary values of i and j which provides moving graphs such as animation. We also determined the exact result for Δ-regular graph under transformation effect. Moreover, we computed the exact formula for the family of unicyclic graphs with pendent paths under the action of transformation as an application of proved results.

Data Availability

No data were used to support this study.

Disclosure

The paper has not been published elsewhere, and it will not be submitted anywhere else for publication.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors have equal contribution.

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