Research Article

Computing Analysis of Zagreb Indices for Generalized Sum Graphs under Strong Product

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Numerous studies based on mathematical models and tools indicate that there is a strong inherent relationship between the chemical properties of the chemical compounds and drugs with their molecular structures. In the last two decades, the graph-theoretic techniques are frequently used to analyse the various physicochemical and structural properties of the molecular graphs which play a vital role in chemical engineering and pharmaceutical industry. In this paper, we compute Zagreb indices of the generalized sum graphs in the form of the different indices of their factor graphs, where generalized sum graphs are obtained under the operations of subdivision and strong product of graphs. Moreover, the obtained results are illustrated with the help of particular classes of graphs and analysed to find the efficient subclass with dominant indices.

1. Introduction

In many fields (chemistry, physics, computer science, and electrical networks) various physicochemical and structural properties such as melting point, boiling point, chemical bonds, bond energy, solubility, surface tension, critical temperature, connectivity, stability, density, and polarizability are studied with the help of various TIs (degree-based, distance-based, and polynomial-based). Moreover, degree-based TIs have been used as a powerful approach to discover many new drugs, such as anineoplastics, anticonvulsants, antiallergics, antimalarials, and silico generation (see [1]). Therefore, this practice has proven that the TIs and the quantitative structure-activity (or structure-property) relationships (QSAR or QSPR) have presented a foundation stone in chemical engineering and pharmaceutical industry for the process of the drug design and discovery (see [2, 3]).

Let Ω be a collection of (molecular) graphs in which each graph is considered as a simple graph without multiedges and loops. A topological index (TI) is a function

$Top: \Omega \longrightarrow \mathbb{R}$ that assigns a real number to each element (graph) of Ω , where $\mathbb{R}$ is a set of real numbers. Moreover, for two graphs G_1 and G_2 , $Top(G_1) = Top(G_2)$ if and only if G_1 is isomorphic to G_2 . Mostly, TIs are computed for the hydrogen-suppressed molecular graphs in which the atoms are represented by nodes and bonds between them by edges. In 1947, Wiener index (path number), first distance-based TI, is utilized in the study of paraffin's boiling point [4]. Gutman and Trinajstić [5] calculated total π -electrons energy of the molecules through a degree-based TI called as the first Zagreb index (FZI). They also studied the various properties of the second Zagreb index (SZI) in the same paper. In chemical graph theory, many more TIs are introduced in [6], but degree-based TIs are prominent than others. For more details, we refer to [7, 8].

On the other hand, operations on graphs (addition, complement, deletion, switching, subdivision, union, intersection, and product) also play a very important role in the construction of new graphs and structures. Yan et al. [9] introduced the four operations S_1 , R_1 , Q_1 , and T_1 on the

subdivision of a graph and obtained the Wiener index of these resultant graphs $(S_1(G), R_1(G), Q_1(G), T_1(G))$. For $\Phi_1 \in \{S_1, R_1, Q_1, T_1\}$, Taeri and Eliasi [10] defined the Φ_1 -sum graphs $(G_{1+\Phi_1} G_2)$ using the Cartesian product on graphs $\Phi_1(G_1)$ and G_2 , where G_1 and G_2 are connected graphs. They also studied the Wiener index of these Φ_1 -sum graphs. Liu et al. [11] constructed Φ -sum graphs with the help of the Cartesian product on the graphs $\Phi(G_1)$ and G_2 and calculated the first general Zagreb index for these graphs, i.e., $M_1^\alpha((G_{1+\Phi_1} G_2))$. Liu et al. [12] introduced the generalized Φ -sum (Φ_k -sum) graphs with the help of the Cartesian product on the graphs $\Phi_k(G_1)$ and G_2 , where k represents some integral value. Moreover, they calculated the mathematical expressions of the Zagreb indices for these graphs, i.e., $M_1((G_{1+\Phi_1} G_2))$ and $M_2((G_{1+\Phi_1} G_2))$. Furthermore, Awais et al. [13, 14] computed the forgotten topological and hyper-Zagreb indices of generalized F -sum graphs based on Cartesian product in terms of its factor graphs. Recently, Awais et al. [15] computed the first general Zagreb index of F_k -sum graphs in terms of TIs of their factor graphs.

In the current study, we study the generalized Φ -sum graphs which are obtained under the operation of strong product on the graphs $\Phi_k(G_1)$ and G_2 , where $\Phi_k \in \{S_k, R_k, Q_k, T_k\}$ and k is some counting number. Mainly, we compute the Zagreb indices of these generalized Φ -sum graphs based on strong product such as $M_1((G_{1\otimes\Phi_k} G_2))$ and $M_2((G_{1\otimes\Phi_k} G_2))$. Moreover, a comparison is also organized of the generalized Φ -sum graphs $(G_{1\otimes S_k} G_2)$, $(G_{1\otimes R_k} G_2)$, $(G_{1\otimes Q_k} G_2)$, and $(G_{1\otimes T_k} G_2)$ with respect to both the Zagreb indices (M_1 and M_2). The rest of the paper is settled as follows: Section 2 covers basic notions, Section 3 predicated on main results, and conclusively Section 4 included the application and conclusion."

2. Preliminaries

A graph G_1 is a structure consisting of two finite sets of vertices $V(G)$ and edges $E(G)$ in which pairs of vertices are connected by edges. In particular, a graph will refer to a simple undirected graph if each edge connects two distinct vertices and there are no parallel edges. Throughout the paper, the order of G_1 is $|V(G_1)| = n_{G_1}$, and the size of a G_1 is $|E(G_1)| = e_{G_1}$. Given two vertices p and z , if $pz \in G_1$, then p and z are said to be adjacent. The strength of edges which are incident on any node $p \in V(G_1)$ is known as its degree $d_{G_1}(p)$ [16]. Here, we defined few topological indices.

Definition 1. Let G_1 be a simple undirected graph. The first Zagreb index ($M_1(G_1)$) and second Zagreb index ($M_2(G_1)$) are:

$$\begin{aligned} M_1(G_1) &= \sum_{p \in V(G_1)} [d_{G_1}(p)]^2 = \sum_{p \in z \in E(G_1)} [d_{G_1}(p) + d_{G_1}(z)], \\ M_2(G_1) &= \sum_{p \in z \in E(G_1)} [d_{G_1}(p) \times d_{G_1}(z)]. \end{aligned} \quad (1)$$

In 1972, Trinajsti and Gutman [5] introduced these two TIs which are used in study of structure-based properties of (molecular) graphs (see [17–19]). In 1960, Sabidussi [6] introduced the strong product $(G_1 \boxtimes G_2)$ for two graphs G_1 and G_2 with vertex set as Cartesian product $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ such that (p_1, p_2) and (z_1, z_2) will be adjacent in $G_1 \boxtimes G_2$ iff $p_1 = z_1$ and p_2 is adjacent to z_2 or $p_2 = z_2$ and p_1 is adjacent to z_1 or p_1 is adjacent to z_1 and p_2 is adjacent to z_2 . Strong product is union of tensor product and Cartesian product.

Definition 2. The four generalized operations related to the subdivision of graphs defined in [15] are given as follows:

- (i) k -subdivision operation $S_k = S_k(G_1)$ can be made by adding k new node in each $uv \in E(G_1)$ of G_1 , where $k \geq 1$ is an integral value
- (ii) k -semitotal-point graph $R_k = R_k(G_1)$ with node set $V(R_k) = V(G_1) \cup kE(G_1)$ and link set $E(R_k) = E(S_k) \cup E(G_1)$
- (iii) k -semitotal line graph $Q_k = Q_k(G_1)$ with node set $V(Q_k) = V(G_1) \cup kE(G_1)$ and link set $E(Q_k) = E(S_k) \cup E(L_k)$.
- (iv) k -total point graph $T_k = T_k(G_1)$ with node set $V(T_k) = V(G_1) \cup kE(G_1)$ and link set $E(T_k) = E(S_k) \cup E(L_k) \cup E(G_1)$ (for more details, see Figure 1).

Definition 3. Let G_1 and G_2 be two graphs, $\Phi_k \in \{S_k, R_k, Q_k, T_k\}$ is an operation, and $\Phi_k(H_1)$ is obtained after applying Φ_k on G_1 having edge-set $E(\Phi_k(G_1))$ and node set $V(\Phi_k(G_1))$. The generalized Φ -sum graph $(G_{1\otimes\Phi_k} G_2)$ is a graph having node set:

$$\begin{aligned} V(G_{1\otimes\Phi_k} G_2) &= V(\Phi_k(G_1)) \times V(G_2) \\ &= (V(G_1) \cup kE(G_1)) \times V(G_2), \end{aligned} \quad (2)$$

such that two nodes (p_1, z_1) & (p_2, z_2) of $V(G_{1\otimes\Phi_k} G_2)$ are adjacent iff $p_1 = p_2 \in V(G_1)$ and $(z_1, z_2) \in E(G_2)$ or $z_1 = z_2 \in V(G_2)$ and $(p_1, p_2) \in E(\Phi_k(G_1))$ or $[(z_1, z_2) \in E(G_2) \text{ and } (p_1, p_2) \in E(\Phi_k(G_1))]$, where $k \geq 1$ is a positive number. We noticed that the generalized Φ -sum graphs $(G_{1\otimes\Phi_k} G_2)$ contain $|V(G_2)|$ copies of graphs $\Phi_k(G_1)$ that are labeled with the nodes of G_2 . For more details, see Figures 2 and 3.

3. Main Results

Now, we will prove the key results of $M_1(G_{1\otimes\Phi_k} G_2)$ and $M_2(G_{1\otimes\Phi_k} G_2)$ in terms of its factor graphs G_1 and G_2 . We assume that G_1 and G_2 be two simple, undirected, and connected graphs with order and size, $|V(G_1)| = n_{G_1}$ & $|E(G_1)| = e_{G_1}$ and $|V(G_2)| = n_{G_2}$ & $|E(G_2)| = e_{G_2}$ respectively.

Theorem 1. Let G_1 and G_2 be two connected graphs such that $|V(G_1)| \geq 3$, $|V(G_2)| \geq 2$. For $k \geq 1$,

FIGURE 1: (a) $G_1 \cong P_4$. (b) $S_3(G_1)$. (c) $R_3(Gp1)$. (d) $Q_3(G_1)$. (e) $T_3(G_1)$. FIGURE 2: Right: $= P_{3\otimes S_2}P_4$. Left: $= P_{3\otimes R_2}P_4$. FIGURE 3: Right: $= P_{3\otimes Q_2}P_4$. Left: $= P_{3\otimes T_2}P_4$.

FIGURE 4: (a) Graphical representation of $M_1(P_{n_{s_1}} \boxtimes P_m)$, $M_1(P_{n_{q_1}} \boxtimes P_m)$, $M_1(P_{n_{r_1}} \boxtimes P_m)$, and $M_1(P_{n_{t_1}} \boxtimes P_m)$ by red, green, orange, and purple colour, respectively. (b) $M_2(P_{n_{s_1}} \boxtimes P_m)$, $M_2(P_{n_{q_1}} \boxtimes P_m)$, $M_2(P_{n_{r_1}} \boxtimes P_m)$, and $M_2(P_{n_{t_1}} \boxtimes P_m)$ by purple, orange, pink, and green colour, respectively.

$$\begin{aligned}
 \text{(a)} \quad M_1(G_1 \boxtimes_{S_k} G_2) &= M_1(S_1 G_1) [n_{G_2} + 2e_{G_2}] + 8(2k-1)e_{G_1}e_{G_2} + [M_1(G_1) + 4e_{G_1}] [M_1(G_2) + 2e_{G_2}] \\
 &\quad + M_1(G_2) [4e_{G_1} + n_{G_1}] + 4(k-1)e_{G_1} [n_2 + M_1(G_2)], \\
 \text{(b)} \quad M_2(G_1 \boxtimes_{S_k} G_2) &= [M_2(G_1) + 4e_{G_1}] [5e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] + M_1(G_1) [e_{G_2} + M_1(G_2) + M_2(G_2)] \\
 &\quad + 4(k-1)e_{G_1} [6e_{G_2} + 3M_1(G_2) + M_2(G_2) + n_{G_2}] + M_2(G_2) [12e_{G_1} + n_{G_1}] \\
 &\quad + 8e_{G_1}e_{G_2} + M_1(G_2) [14e_{G_1}].
 \end{aligned} \tag{3}$$

Proof. Let $d(p, z) = d_{G_1 \boxtimes_{S_k} G_2}(p, z)$ be the degree of a vertex (p, z) in the graph $G_1 \boxtimes_{S_k} G_2$.

$$\begin{aligned}
 M_1(G_1 \boxtimes_{S_k} G_2) &= \sum_{(p,z) \in V(G_1 \boxtimes_{S_k} G_2)} d^2(p, z) = \sum_{(p_1, z_1)(p_2, z_2) \in E(G_1 \boxtimes_{S_k} G_2)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 p_2 \in E(G_2)} [d(p, z_1) + d(p, z_2)] + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &\quad + \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{z_1 z_2 \in V(G_2)} [d(p_1, z_1) + d(p_2, z_2)] = \sum 1 + \sum 2 + \sum 3.
 \end{aligned} \tag{4}$$

Consider

$$\begin{aligned}
 \sum 1 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) + d(p, z_2)] = \sum_{p \in V(G)} \sum_{z_1 z_2 \in E(H)} [2d(p) + d(z_1) + d(z_2) + d(p)(d(z_1) + d(z_2))] \\
 &= 4e_{G_1} e_{G_2} + M_1(G_2) n_{G_1} + 2M_1(G_2) e_{G_1}. \\
 \sum 2 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) + d(p_2, z)] = \sum_{z \in V(G_2)} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - G_1)} [d(p_1, z) + d(p_2, z)] \\
 &\quad + \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in V(S_k(G_1) - G_1)} [d(p_1, z) + d(p_2, z)] = \sum 2' + \sum 2'', \\
 \sum 2' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [(d(p_1) + d(p_2)) + d(z) + (d(p_1) + d(p_2))d(z)].
 \end{aligned} \tag{5}$$

Since in this case $|E(S_1(G))| = 2|E(G)|$, we have

$$\begin{aligned}
 &= M_1(S_1(G_1)) n_{G_2} + 4e_{G_1} e_{G_2} + 2M_1(S_1(G_1)) e_{G_2}, \\
 \sum 2'' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1, p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1, p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1) + d(p_1)d(z) + d(p_2) + d(p_2)d(z)].
 \end{aligned} \tag{6}$$

Since in this case $|E(S_k(G_1))| = (k-1)|E(G_1)|$, we have

$$\begin{aligned}
 &= \sum_{z \in V(G_2)} (4(k-1)e_{G_1} + 4d(z)(k-1)e_{G_1}) = 4(k-1)e_{G_1} n_{G_2} + 8e_{G_1}(k-1)e_{G_2}, \\
 \sum 3 &= \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{z_1 z_2 \in V(G_2)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
 &\quad + \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1, p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z_1) + d(p_2, z_2)] = \sum 3' + \sum 3'', \\
 \sum 3' &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [(d(p_1) + d(p_2)) + d(z_1) + d(p_1)d(z_1) + d(p_2)d(z_2)] \\
 &= 2e_{G_2} [M_1(G_1) + 4e_{G_1}] + 2e_{G_1} M_1(G_2) + M_1(G_2) [M_1(G_1) + 4e_{G_1}], \\
 \sum 3'' &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1, p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1, p_2 \in V(S_k(G_1) - V(G_1))} [4 + 2(d(z_1) + d(z_2))] = 8(k-1)e_{G_1} e_{G_2} + 4(k-1)e_{G_1} M_1(G_2).
 \end{aligned} \tag{7}$$

Consequently, we get

$$\begin{aligned}
 M_1(G_1 \boxtimes_{S_k} G_2) &= M_1(S_1 G_1) [n_{G_2} + 2e_{G_2}] + 8(2k-1)e_{G_1}e_{G_2} + [M_1(G_1) + 4e_{G_1}] [M_1(G_2) + 2e_{G_2}] \\
 &\quad + M_1(G_2) [4e_{G_1} + n_{G_1}] + 4(k-1)e_{G_1} [n_2 + M_1(G_2)], \\
 \text{(b)} M_2(G_1 \boxtimes_{S_k} G_2) &= \sum_{(p,z) \in V(G_1 \boxtimes_{S_k} G_2)} d^2(p, z) = \sum_{(p_1, z_1) (p_2, z_2) \in E(G_1 \boxtimes_{S_k} G_2)} [d(p_1, z_1) d(p_2, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) d(p, z_2)] + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &\quad + \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{z_1 z_2 \in V(G_2)} [d(p_1, z_1) d(p_2, z_2)] = \sum 1 + \sum 2 + \sum 3.
 \end{aligned} \tag{8}$$

Consider

$$\begin{aligned}
 \sum 1 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) d(p, z_2)] = \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p) + d(z_1)] [d(p) + d(z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d^2(p) + d^2(p)(d(z_1) + d(z_2)) + d(p)(d(z_1) + d(z_2)) + 2d(z_1)d(z_2) + d(z_1)d(z_2)] \\
 &= M_1(G_1)e_{G_2} + M_1(G_1)[M_1(G_2) + M_2(G_2)] + 2e_{G_1}[M_1(G_2) + 2M_2(G_2)] + M_2(G_2)n_{G_1}, \\
 \sum 2 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &\quad + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum 2' + \sum 2'', \\
 \sum 2' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [(d(p_1) + d(z) + d(p_1)d(z))(d(p_2) + d(p_2)d(z))] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1)d(p_2) + 2d(p_1)d(p_2)d(z) + d(p_2)d(y) + d(p_2)d^2(z) \\
 &\quad + d(p_1)d(p_2)d^2(z)] \\
 &= [M_2(G_1) + 4e_{G_1}] [n_{G_2} + 4e_{G_2} + M_1(G_2)] + 8e_{G_1}e_{G_2} + 4e_{G_1}M_1(G_2), \\
 \sum 2'' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [(d(p_1) + d(p_1)d(z))(d(p_2) + d(p_2)d(z))] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [4 + 8d(z) + 4d^2(z)] = 4(k-1)e_{G_1} [n_{G_2} + 4e_{G_2} + M_1(G_2)], \\
 \sum 3 &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] \\
 &\quad + \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] = \sum 3' + \sum 3'',
 \end{aligned}$$

$$\begin{aligned}
\sum 3' &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1)) - V(G_1)} (d(p_1) + d(z_1) + d(p_1) d(z_1)) (d(p_2) + d(p_2) d(z_2)) \\
&= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1)) - V(G_1)} \\
&\quad \cdot [d(p_1) d(p_2) + d(p_1) d(p_2) d(z_2) + d(p_2) d(z_1) + d(p_1) d(z_1) d(z_2) + d(p_1) d(p_2) d(z_1) + d(p_1) d(p_2) d(z_1) d(z_2)] \\
&= [M_2(G_1) + 4e_{G_1}] [e_{G_2} + 2M_1(G_2) + 2M_2(G_2)] + 8e_{G_1} M_1(G_2) + 8e_{G_1} M_2(G_2), \\
\sum 3'' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{z_1, p_2 \in V(S_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(S_k(G_1))} \sum_{p_1, p_2 \in V(S_k(G_1)) - V(G_1)} [(d(p_1) + d(p_1) d(z_1)) (d(p_2) + d(p_2) d(z_2))] \\
&= \sum_{z \in V(G_2)} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1)) - V(G_1)} [4 + 4(d(z_2) + d(z_1)) + 4d(z_1) d(z_2)] = 8(k-1)e_{G_1} [e_{G_2} + M_1(G_2) + M_2(G_2)].
\end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned}
M_2(G_1 \boxtimes_{S_k} G_2) &= [M_2(G_1) + 4e_{G_1}] [5e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] + M_1(G_1) [e_{G_2} + M_1(G_2) + M_2(G_2)] \\
&\quad + 4(k-1)e_{G_1} [6e_{G_2} + 3M_1(G_2) + M_2(G_2) + n_{G_2}] + M_2(G_2) [12e_{G_1} + n_{G_1}] \\
&\quad + 8e_{G_1} e_{G_2} + M_1(G_2) [14e_{G_1}].
\end{aligned} \tag{10}$$

Theorem 2. Let G_1 and G_2 be two connected graphs such that $|V(G_1)|, |V(G_2)| \geq 4$. For $k \geq 1$,

$$\begin{aligned}
\text{(a)} \quad M_1(G_1 \boxtimes_{R_k} G_2) &= M_1(G_2) [n_{G_1} + 12e_{G_1} + 4M_1(G_1)] + M_1(G_1) [12e_{G_2} + 2n_{G_2}] + 24e_{G_1} e_{G_2} \\
&\quad + [M_1(R_1(G_1)) - 2M_1(G_1)] [n_{G_2} + 2e_{G_2}] + 4(k-1)e_{G_1} [n_{G_2} + 4e_{G_2} + M_1(G_2)], \\
\text{(b)} \quad M_2(G_1 \boxtimes_{R_k} G_2) &= M_1(G_1) [8e_{G_2} + 12M_1(G_2) + 4M_2(G_2)] + 4M_2(G_1) [n_{G_2} + 6e_{G_2} + 3M_1(G_2)] \\
&\quad + [M_2(R_1(G_1)) - 4M_2(G_1) + 4(k-1)e_{G_1}] [6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] \\
&\quad + M_2(G_2) [n_{G_1} + 2e_{G_1}] + 4e_{G_1} [3M_1(G_2) + 3M_2(G_2) + 4e_{G_2}] + e_{G_1} M_1(G_2).
\end{aligned} \tag{11}$$

Proof. Let $d(p, z) = d_{G_1 \boxtimes_{R_k} G_2}(p, z)$ be the degree of a vertex (p, z) in the graph $G_1 \boxtimes_{R_k} G_2$.

$$\begin{aligned}
 M_1(G_1 \boxtimes_{R_k} G_2) &= \sum_{(p,z) \in V(G_1 \boxtimes_{R_k} G_2)} d^2(x, z) = \sum_{(p_1, z_1) (p_2, z_2) \in E(G_1 \boxtimes_{R_k} G_2)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1, z_2 \in E(G_2)} [d(p, z_1) + d(p, z_2)] + \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &\quad + \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{z_1, z_2 \in V(G_2)} [d(p_1, z_1) + d(p_2, z_2)] = \sum 1 + \sum 2 + \sum 3, \\
 \sum 1 &= \sum_{p \in V(G_1)} \sum_{z_1, z_2 \in E(G_2)} [d(p, z_1) + d(p, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1, z_2 \in E(G_2)} [2d(p) + d(z_1) + 2d(p)d(z_1) + 2d(p) + d(z_2) + 2d(p)d(z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1, z_2 \in E(G_2)} [4d(p) + d(z_1) + d(z_2) + 2d(p)(d(z_1) + d(z_2))] \\
 &= 8e_{G_1}e_{G_2} + M_1(G_2)n_{G_1} + 4M_1(G_2)e_{G_1}, \\
 \sum 2 &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} [d(p_1, z) + d(p_2, z)] = \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1, z) + d(p_2, z)] \\
 &\quad + \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - G_1)} [d(p_1, y) + d(p_2, z)] + \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - G_1)} \\
 &\quad \cdot [d(p_1, z) + d(p_2, y)] \\
 &= \sum 2' + \sum 2'' + \sum 2''', \\
 \sum 2' &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1, z) + d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1) + d(z) + d(p_1)d(z) + d(p_2) + d(z) + d(p_2)d(z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1) + d(p_2) + 2d(z) + d(y)(d(p_1) + d(p_2))] \\
 &= 2n_{G_2}M_1(G_1) + 4e_{G_1}e_{G_2} + 4e_{G_2}M_1(G_1), \\
 \sum 2'' &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1, z) + d(p_2, y)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(S_k(G_1) - V(G_1))} [d(p_1) + d(p_2) + d(z) + (d(p_1) + d(p_2))d(z)] \\
 &= [M_1(R_1(G_1)) - 2M_1(G_1)][n_{G_2} + 2e_{G_2}] + 4e_{G_1}e_{G_2}, \\
 \sum 2''' &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - V(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - V(G_1))} [d(p_1) + d(p_2) + (d(p_1) + d(p_2))d(z)] \\
 &= 4(k-1)e_{G_1}n_{G_2} + 8(k-1)e_{G_1}e_{G_2}, \\
 \sum 3 &= \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{z_1, z_2 \in V(G_2)} [d(p_1, z_1) + d(p_2, z_2)] = \sum_{z_1, z_2 \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &\quad + \sum_{z_1, z_2 \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - V(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
 &\quad + \sum_{z_1, z_2 \in V(G_2)} \sum_{p_1, p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - V(G_1))} [d(p_1, z_1) + d(p_2, z_2)] = \sum 3' + \sum 3'' + \sum 3''',
 \end{aligned}$$

$$\begin{aligned}
\sum 3' &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1) + d(z_1) + d(p_1)d(z_1) + d(p_2) + d(z_2) + d(p_2)d(z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1) + d(p_2) + d(z_1) + d(z_2) + d(p_1)d(z_1) + d(p_2)d(z_2)] \\
&= 4e_{G_2}M_1(G_1) + 2M_1(G_2)e_{G_1} + 2M_1(G_2)M_1(G_1), \\
\sum 3'' &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 \varepsilon V(G_1), p_2 \varepsilon V(R_k(G_1)) - V(G_1)} [d(p_1) + d(z_1) + d(p_1)d(z_1) + d(p_2) + d(p_2 z_2)] \\
&= 4e_{G_2}M_1(G_1) + 2M_1(G_2)e_{G_1} + 2M_1(G_2)M_1(G_1) + 8e_{G_1}e_{G_2} + 4e_{G_1}M_1(G_2), \\
\sum 3''' &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) + d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1) + d(p_2) + d(p_1)d(z_1) + d(p_2)d(z_2)] \\
&= 8(k-1)e_{G_1}e_{G_2} + 4(k-1)e_{G_1}M_1(G_2). \tag{12}
\end{aligned}$$

Hence,

$$\begin{aligned}
M_1(G_1 \boxtimes_{R_k} G_2) &= M_1(G_2)[n_{G_1} + 12e_{G_1} + 4M_1(G_1)] + M_1(G_1)[12e_{G_2} + 2n_{G_2}] + 24e_{G_1}e_{G_2} \\
&\quad + [M_1(R_1(G_1)) - 2M_1(G_1)][n_{G_2} + 2e_{G_2}] + 4(k-1)e_{G_1}[n_{G_2} + 4e_{G_2} + M_1(G_2)], \\
\text{(b)} M_2(G_1 \boxtimes_{R_k} G_2) &= \sum_{(x, z) \varepsilon V(G_1 \boxtimes_{R_k} G_2)} d^2(p, z) = \sum_{(p_1, z_1)(p_2, z_2) \varepsilon E(G_1 \boxtimes_{R_k} G_2)} [d(p_1, z_1)d(p_2, z_2)] \\
&= \sum_{p \varepsilon V(G_1)} \sum_{z_1 z_2 \varepsilon E(G_2)} [d(p, z_1)d(p, z_2)] + \sum_{z \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z)d(p_2, z)] \\
&\quad + \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} \sum_{z_1 z_2 \varepsilon V(G_2)} [d(p_1, z_1)d(p_2, z_2)] = \sum 1 + \sum 2 + \sum 3. \tag{13}
\end{aligned}$$

Consider

$$\begin{aligned}
 \sum 1 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) d(p, z_2)] = \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p) + d(z_1)] [d(p) + d(z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d^2(p) + d^2(p)(d(z_1) + d(z_2)) + d^2(p)(d(z_1) + d(z_2)) + 2d(p)d(z_1)d(z_2) + d(z_1)d(z_2)] \\
 &= \sum_{z_1 z_2 \in E(G_2)} [4M_1(G_1) + 4M_1(G_1)] [d(z_1) + d(z_2) + d(z_1)d(z_2)] + 4e_{G_1} [d(z_1) + d(z_2) + 2d(z_1)d(z_2)] \\
 &\quad + d(z_1)d(z_2)(n_{G_1}) = 4M_1(G_1)[e_{G_2} + M_1(G_2) + M_2(G_2)] + 4e_{G_1}[M_1(G_2) + 2M_2(G_2)] + M_2(G_2)n_{G_1}, \\
 \sum 2 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} [d(p_1, z) d(p_2, z)] + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - G_1)} [d(p_1, z) d(p_2, z)] \\
 &\quad + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - G_1)} [d(p_1, z) d(p_2, z)] = \sum 2' + \sum 2'' + \sum 2''', \\
 \sum 2' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} [d(p_1) + d(z) + d(p_1)d(z)] [d(p_2) + d(z) + d(p_2)d(z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} [4d(p_1)d(p_2) + 2d(p_1)d(z) + 4d(p_1)d(p_2)d(z) + 2d(p_2)d(z) \\
 &\quad + d^2(z) + 2d(p_2)d^2(z) + 4d(p_1)d(p_2)d(z) + 4d(p_1)d(p_2)d^2(z)] \\
 &= 4M_2(G_1)[n_{G_2} + 4e_{G_2} + M_1(G_2) + 2M_1(G_1)[M_1(G_2) + 2e_{G_2} + e_{G_1}M_1(G_2)], \\
 \sum 2'' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - V(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - V(G_1))} [(d(p_1) + d(z) + d(p_1)d(z))(d(p_2) + d(p_2)d(z))] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - V(G_1))} [2d(p_1)[1 + d(z) + d(y) + d^2(z) + 2d(z) + 2d^2(y)] \\
 &= [M_2(R_1(G_1)) - 4M_2(G_1)][n_{G_2} + 4e_{G_2} + M_1(G_2) + 8e_{G_1}e_{G_2} + 4e_{G_1}M_1(G_2)], \\
 \sum 2''' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - V(G_1))} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(R_k(G_1))} \sum_{p_1, p_2 \in V(R_k(G_1) - V(G_1))} [(d(p_1) + d(p_1)d(z))(d(p_2) + d(p_2)d(z))] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 \in V(G_1), p_2 \in V(R_k(G_1) - V(G_1))} [4 + 8d(y) + 4d^2(z)] = 4(k-1)e_{G_1}[n_{G_2} + 4e_{G_2} + M_1(G_2)],
 \end{aligned}$$

$$\begin{aligned}
\sum 3 &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] \\
&+ \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] \\
&+ \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] = \sum 3' + \sum 3'' + \sum 3''', \\
\sum 3' &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [4d(p_1)d(p_2) + 2[d(p_1)d(z_2) + d(p_2)d(z_1)] + d(z_1)d(z_2) \\
&\quad + 4d(p_1)d(p_2)d(z_1)d(z_2) + 4d(p_1)d(p_2)[d(z_1) + d(z_2)] + 2[d(p_1) + d(p_2)]d(z_1)d(z_2)] \\
&= 8M_2(G_1)e_{G_2} + 2M_1(G_1)M_1(G_2) + 2M_2(G_2)e_{G_1} + 8M_2(G_1)[M_2(G_2) + M_1(G_2)] + 4M_1(G_1)M_2(G_2), \\
\sum 3'' &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [(d(p_1) + d(z_1) + d(p_1)d(z_1))(d(p_2) + d(p_2)d(z_2))] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [4d(p_1)[1 + d(z_2) + d(z_1) \\
&\quad + d(z_1)d(z_2)] + 2[d(z_1) + d(z_1)d(z_2)]] \\
&= [M_2(R_1(G_1)) - 4M_2(G_1)][2e_{G_2} + 2M_1(G_2) + 2M_2(G_2)] + 4e_{G_1}[M_1(G_2) + M_2(G_2)], \\
\sum_3''' &= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [(d(p_1) + d(p_1)d(z_1))(d(p_2) + d(p_2)d(z_2))] \\
&= \sum_{z_1 z_2 \varepsilon V(G_2)} \sum_{p_1 p_2 \varepsilon E(R_k(G_1))} [d(p_1)d(p_2)[1 + (d(y_2) + d(z_1)) + d(z_1)d(z_2)]] \\
&= 8(k-1)e_{G_1}[e_{G_2} + M_1(G_2) + M_2(G_2)]. \tag{14}
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
M_2(G_1 \boxtimes_{R_k} G_2) &= M_1(G_1)[8e_{G_2} + 12M_1(G_2) + 4M_2(G_2)] + 4M_2(G_1)[n_{G_2} + 6e_{G_2} + 3M_1(G_2)] \\
&\quad + [M_2(R_1(G_1)) - 4M_2(G_1) + 4(k-1)e_{G_1}][6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] \\
&\quad + M_2(G_2)[n_{G_1} + 2e_{G_1}] + 4e_{G_1}[3M_1(G_2) + 3M_2(G_2) + 4e_{G_2}] + e_{G_1}M_1(G_2). \tag{15}
\end{aligned}$$

□

Theorem 3. Let G_1 and G_2 be two connected graphs such that $|V(G_1)|, |V(G_2)| \geq 4$. For $k \geq 1$,

$$\begin{aligned}
 \text{(a)} \quad M_1(G_1 \boxtimes_{Q_k} G_2) &= k[M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)][n_{G_2} + 4e_{G_2} + M_1(G_2)] + 8e_{G_1}e_{G_2} \\
 &\quad + M_1(G_1)[2(k-1) + 3][n_{G_2} + 4e_{G_2} + M_1(G_2)] + M_1(G_2)[n_{G_1} + 4e_{G_1}], \\
 \text{(b)} \quad M_2(G_1 \boxtimes_{Q_k} G_2) &= k[n_{G_2} + 6e_{G_2} + 3M_1(G_2) + 2M_2(G_2)] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) \right] \\
 &\quad + \left[\sum_{uv \in V(G_1)} rd_{G_1}(u)d_{G_1}(v) + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{u \in V(G_1)} d_{G_1}(u) - 2M_2(G_1) \right. \\
 &\quad \left. + M_2(G_2)[4e_{G_1} + n_{G_1}] + 2e_{G_1}M_1(G_2) + M_1(G_1)[5e_2 + 5M_1(G_2) + 5M_2(G_2)] \right. \\
 &\quad \left. + k[M_3(G_1) + 2M_2(G_1)][6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] \right]. \tag{16}
 \end{aligned}$$

Proof. Let $d(p, z) = d_{G_1 \boxtimes_{Q_k} G_2}(p, z)$ be the degree of a vertex (p, z) in the graph $G_1 \boxtimes_{Q_k} G_2$.

$$\begin{aligned}
 M_1(G_1 \boxtimes_{Q_k} G_2) &= \sum_{(p,z) \in V(G_1 \boxtimes_{Q_k} G_2)} d^2(p, z) = \sum_{(p_1, z_1)(p_2, z_2) \in E(G_1 \boxtimes_{Q_k} G_2)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) + d(p, z_2)] + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &\quad + \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{z_1 z_2 \in V(G_2)} [d(p_1, z_1) + d(p_2, z_2)] = \sum 1 + \sum 2 + \sum 3, \\
 \sum 1 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) + d(p, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p) + d(z_1) + d(p)d(z_1) + d(p) + d(z_2) + d(p)d(z_2)] \\
 &= \sum_{p \in V(G)} \sum_{z_1 z_2 \in E(G_2)} [2d(p) + d(z_1) + d(z_2) + d(p)[d(z_1) + d(z_2)]] \\
 &= 4e_{G_1}e_{G_2} + M_1(G_2)n_{G_1} + 2M_1(G_2)e_{G_1}, \\
 \sum 2 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} [d(p_1, z) + d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1) - G_1)} [d(p_1, z) + d(p_2, z)] \\
 &\quad + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1) - G_1)} [d(p_1, y) + d(p_2, z)] = \sum 2' + \sum 2'',
 \end{aligned}$$

$$\begin{aligned}
\sum 2' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z) + d(p_2, z)] \\
&= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) + d(z) + d(p_1)d(z) + d(p_2) + d(p_2)d(z)] \\
&= M_1(G_1)n_{G_2} + 4e_{G_1}e_{G_2} + 6M_1(G_1)e_{G_2} + 2n_{G_2}M_1(G_1), \\
\sum 2'' &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z)d(p_2, z)]. \tag{17}
\end{aligned}$$

Now, we split this sum into two parts for the vertices, p_1 and p_2 , where $p_1 p_2 \in V(Q_k(G_1)) - V(G_1)$. Assume that $\sum 2'' = \sum 2''a + \sum 2''b$, where $\sum 2''a$ cover the edges of

$Q_k(G_1)$ which are in the same edges of G_1 and $\sum 2''b$ of $Q_k(G_1)$ in two different adjacent edges of G_1 .

$$\begin{aligned}
\sum 2''a &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z) + d(p_2, z)] \\
&= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) + d(p_1)d(z) + d(p_2) + d(p_2)d(z)] \\
&= 2(k-1)n_{G_2}M_1(G_1) + 4(k-1)e_{G_2}M_1(G_1), \\
\sum 2''b &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, y) + d(p_2, z)] \\
&= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) + d(p_1)d(z) + d(p_2) + d(p_2)d(z)] \\
&= \sum_{z \in V(G_2)} \sum_{u, v, w \in V(G_1)} \sum_{uv, vw \in E(G_1)} [d(u) + d(v) + d(v) + d(w) + [d(u) + d(v) + d(v) + d(w)]d(z)] \\
&= k[M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)][n_{G_2} + 2e_{G_2}], \\
\sum 3 &= \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{z_1 z_2 \in V(G_2)} [d(p_1, z_1) + d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) + d(p_2, z_2)] \\
&\quad + \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) + d(p_2, z_2)] \\
&= \sum 3' + \sum 3'', \\
\sum 3' &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1, z_1) + d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(G_1)} [d(p_1) + d(z_1) + d(p_1)d(z_1) + d(p_2) + d(z_2) + d(p_2)d(z_2)] \\
&= 6e_{G_2}M_1(G_1) + 2M_1(G_2)e_{G_1} + 3M_1(G_2)M_1(G_1). \tag{18}
\end{aligned}$$

Now, we split this sum into two parts for the vertices, p_1 and p_2 , where $p_1 p_2 \in V(Q_k(\Gamma_1)) - V(\Gamma_1)$. Assume that $\sum 3'' = \sum 3''a + \sum 3''b$, where $\sum 3''a$ cover the edges of

$Q_k(G_1)$ which are in the same edges of G_1 and $\sum 3''b$ of $Q_k(G_1)$ in two different adjacent edges of G_1 .

$$\begin{aligned}
 \sum 3''a &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) + d(p_1)d(z_1) + d(p_2) + d(p_2)d(z_2)] \\
 &= 4(k-1)e_{G_2}M_1(G_1) + 2(k-1)M_1(G_2)M_1(G_1), \\
 \sum 3''b &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) + d(p_2, z_2)] \\
 &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) + d(p_2) + d(p_1)d(z_1) + d(p_2)d(z_2)] \\
 &= k[M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)][2e_{G_2} + M_1(G_2)].
 \end{aligned} \tag{19}$$

Consequently, we have

$$\begin{aligned}
 M_1(G_1 \boxtimes_{Q_k} G_2) &= k[M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)][n_{G_2} + 4e_{G_2} + M_1(G_2)] + 8e_{G_1}e_{G_2} \\
 &\quad + M_1(G_1)[2(k-1) + 3][n_{G_2} + 4e_{G_2} + M_1(G_2)] + M_1(G_2)[n_{G_1} + 4e_{G_1}].
 \end{aligned} \tag{20}$$

Next,

$$\begin{aligned}
 (b) M_2(G_1 \boxtimes_{Q_k} G_2) &= \sum_{(p,z) \in V(G_1 \boxtimes_{Q_k} G_2)} d^2(p, z) = \sum_{(p_1, z_1)(p_2, z_2) \in E(G_1 \boxtimes_{Q_k} G_2)} [d(p_1, z_1)d(p_2, z_2)] \\
 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1)d(p, z_2)] + \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} [d(p_1, z)d(p_2, z)] \\
 &\quad + \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{z_1 z_2 \in V(G_2)} [d(p_1, z_1)d(p_2, z_2)] = \sum 1 + \sum 2 + \sum 3.
 \end{aligned} \tag{21}$$

Consider

$$\begin{aligned}
 \sum 1 &= \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p, z_1) d(p, z_2)] = \sum_{p \in V(G_1)} \sum_{z_1 z_2 \in E(G_2)} [d(p) + d(z_1)][d(p) + d(z_2)] \\
 &= M_1(G_1) e_{G_2} + M_1(G_1) [M_1(G_2) + M_2(G_2)] + 2e_{G_1} [M_1(G_2) + 2M_2(G_2)] + M_2(G_2) n_{G_1}, \\
 \sum 2 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} [d(p_1, z) d(p_2, z)] = \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 \in V(G_1) p_2 \in V(Q_k(G_1)) - V(G_2)}} [d(p_1, z) d(p_2, z)] \\
 &\quad + \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1, z) d(p_2, z)] = ' \sum 2 + '' \sum 2, \\
 ' \sum 2 &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 \in V(G_1) p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1, z) d(p_2, z)] \\
 &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 \in V(G_1) p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1) + d(z) + d(p_1) d(z)][d(p_2) + d(z) + d(p_2) d(z)] \\
 &= \sum_{z \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} [d(p_1) d(p_2) (1 + 2d(z) + d^2(z)) + d(p_2) (d(z) + d^2(z))] \\
 &= [M_3(G_1) + 2M_2(G_1)] [n_{G_2} + 4e_{G_2} + M_1(G_2)] + 2M_1(G_1) [2e_{G_2} + M_1(G_2)], \\
 \sum 2'' &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1, z) d(p_2, z)].
 \end{aligned} \tag{22}$$

Now, we split this sum into two parts for the vertices, p_1 and p_2 , where $p_1 p_2 \in V(Q_k(G_1)) - V(G_1)$. Assume that

$\sum 2'' a$ which are in the same edges of G_1 and $\sum 2'' b$ of $Q_k(G_1)$ in two different adjacent edges of G_1 .

$$\begin{aligned}
 \sum 2'' a &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [(d(p_1) + d(p_1) d(z))(d(p_2) + d(p_2) d(z))] \\
 &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1) d(p_2) + 2d(p_1) d(p_2) d(z) + d(p_1) d(p_2) d^2(z)] \\
 &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1) d(p_2)] [1 + 2d(z) + d^2(z)] \\
 &= (k-1) [M_3(G_1) + 2M_2(G_1)] [n_{G_2} + 4e_{G_2} + M_1(G_2)], \\
 \sum 2'' b &= \sum_{z \in V(G_2)} \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [(d(p_1) + d(p_1) d(z))(d(p_2) + d(p_2) d(z))] \\
 &= k \sum_{z \in V(G_2)} [1 + 2d(z) + d^2(z)] \sum_{\substack{p_1 p_2 \in E(Q_k(G_1)) \\ p_1 p_2 \in V(Q_k(G_1)) - V(G_1)}} [d(p_1) d(p_2)] \\
 &= k [n_{G_2} + 4e_{G_2} + M_1(G_2)] \sum_{\substack{uv \in E(G_1) \\ vw \in E(G_2)}} [d_{G_1}(u) + d_{G_1}(v)] [d_{G_1}(v) + d_{G_1}(w)],
 \end{aligned} \tag{23}$$

where p_1 is the added vertex in the edge uv and p_2 is added vertex in the edges vw of G_1 :

$$= (k) \left[n_{G_2} + 4e_{G_2} + M_1(G_2) \right] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) + \sum_{uv \in V(G_1)} r d_{G_1}(u) d_{G_1}(v) \right] \\ + \left[\sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{\substack{u \in V(G_1) \\ uv \in E(G_1)}} d_{G_1}(u) - 2M_2(G_1) \right], \quad (24)$$

where r is the number of neighbors which are common vertices of u and v in (G_1) .

$$\sum 3 = \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] \\ + \sum_{y_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] = \sum 3' + \sum 3'', \\ \sum 3' = \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] \\ = \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [(d(p_1) + d(z_1) + d(p_1) d(z_1)) (d(p_2) + d(p_2) d(z_2))] \quad (25) \\ = [M_3(G_1) + 2M_2(G_1)] [2e_{G_2} + 2M_1(G_2) + 2M_2(G_2)] + 2M_1(G_1) [2M_2(G_2) + M_1(G_2)], \\ \sum 3'' = \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z) d(p_2, z)].$$

Now, we split this sum into two parts for the vertices, p_1 and p_2 , where $p_1 p_2 \in V(Q_k(G_1)) - V(G_1)$. Assume that

$\sum 3'' a$ which are in the same edges of G_1 and $\sum 3'' b$ of $Q_k(G_1)$ in two different adjacent edges of G_1 .

$$\begin{aligned}
\sum 3''a &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [(d(p_1) + d(p_1) d(z_1)) (d(p_2) + d(p_2) d(z_2))] \\
&= \sum_{z_1 z_2 \in V(G_2)} [1 + d(z_2) + d(z_1) + d(z_1) d(z_2)] \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) d(p_2)] \\
&= (k-1) [2e_{G_2} + 2M_1(G_2) + 2M_2(G_2)] [M_3(G_1) + 2M_2(G_1)], \\
\sum 3''b &= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1, z_1) d(p_2, z_2)] \\
&= \sum_{z_1 z_2 \in V(G_2)} \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1, p_2 \in V(Q_k(G_1)) - V(G_1)} [(d(p_1) + d(p_1) d(z_1)) (d(p_2) + d(p_2) d(z_2))] \\
&= \sum_{z_1 z_2 \in V(G_2)} [1 + d(z_2) + d(z_1) + d(z_1) d(z_2)] \sum_{p_1 p_2 \in E(Q_k(G_1))} \sum_{p_1 \in V(G_1), p_2 \in V(Q_k(G_1)) - V(G_1)} [d(p_1) d(p_2)] \\
&= k [2e_{G_2} + 2M_1(G_2) + 2M_2(G_2)] \sum_{\substack{uv \in E(G_1) \\ vw \in E(G_2)}} [d_{G_1}(u) + d_{G_1}(v)] [d_{G_1}(v) + d_{G_1}(w)],
\end{aligned} \tag{26}$$

where p_1 is the added vertex in the edge uv and p_2 is added vertex in the edges vw of G_1 :

$$\begin{aligned}
&= (k) [2e_{G_2} + 2M_1(G_2) + 2M_2(G_2)] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) + \sum_{uv \in V(G_1)} r d_{G_1}(u) d_{G_1}(v) \right. \\
&\quad \left. + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{\substack{u \in V(G_1) \\ uv \in E(G_1)}} d_{G_1}(u) - 2M_2(G_1) \right],
\end{aligned} \tag{27}$$

where r is the number of neighbors which are common vertices of u and v in (G_1) .

Consequently, we have

$$\begin{aligned}
M_2(G_1 \boxtimes_{Q_k} G_2) &= k [n_{G_2} + 6e_{G_2} + 3M_1(G_2) + 2M_2(G_2)] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) \right. \\
&\quad \left. + \sum_{uv \in V(G_1)} r d_{G_1}(u) d_{G_1}(v) + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{\substack{u \in V(G_1) \\ uv \in E(G_1)}} d_{G_1}(u) - 2M_2(G_1) \right] \\
&\quad + M_2(G_2) [4e_{G_1} + n_{G_1}] + 2e_{G_1} M_1(G_2) + M_1(G_1) [5e_2 + 5M_1(G_2) + 5M_2(G_2)] \\
&\quad + k [M_3(G_1) + 2M_2(G_1)] [6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}].
\end{aligned} \tag{28}$$

□

Theorem 4. Let G_1 and G_2 be two connected graphs such that $|V(G_1)|, |V(G_2)| \geq 4$. For $k \geq 1$,

$$\begin{aligned}
 (a) \ M_1(G_1 \boxtimes_{T_k} G_2) &= M_1(G_2) [8e_{G_1} + n_{G_1} + 6M_1(G_1)] + 2(k-1)M_1(G_1) [2e_{G_2} + M_1(G_2)] \\
 &\quad + k[M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)] [4e_{G_2} + n_{G_2} + M_1(G_2)] + 16e_{G_1}e_{G_2} \\
 &\quad + M_1(G_1) [2(11+k)e_{G_2} + 2(k+2)n_{G_2}], \\
 (b) \ M_2(G_1 \boxtimes_{T_k} G_2) &= k[n_{G_2} + 6e_{G_2} + 3M_1(G_2) + 2M_2(G_2)] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) + \sum_{uv \in E(G_1)} rd_{G_1}(u)d_{G_1}(v) \right. \\
 &\quad \left. + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{\substack{ue \in V(G_1) \\ ue \in E(G_1)}} d_{G_1}(u) - 2M_2(G_1) \right] + M_2(G_1) [4n_{G_2} + 24e_{G_2} + 8M_2(G_2) + 12M_1(G_2)] \\
 &\quad + k[M_3(G_1) + 2M_2(G_1)] [6e_{G_2} + n_{G_2} + 3M_1(G_2) + 2M_2(G_2)] + 5M_1(G_2)e_{G_1} \\
 &\quad + M_2(G_2) [10e_{G_1} + n_{G_1}] + M_1(G_1) [10e_{G_2} + 11M_1(G_2) + 10M_2(G_2)].
 \end{aligned} \tag{29}$$

4. Applications and Conclusion

In this section, we have computed the first and second Zagreb indices of generalized Φ -sum graphs based on strong product as application of Theorem 1 to Theorem 4 for $k = 1$

given as follows. We also find the subclass with better Zagreb indices as presented in Tables 1 and 2 and Figure 4

(i) S_1 -sum:

$$\begin{aligned}
 (a) \ M_1(G_1 \boxtimes_{S_1} G_2) &= M_1(S_1 G_1) [n_{G_2} + 2e_{G_2}] + 8e_{G_1}e_{G_2} + [M_1(G_1) + 4e_{G_1}] [M_1(G_2) + 2e_{G_2}] \\
 &\quad + M_1(G_2) [4e_{G_1} + n_{G_1}], \\
 (b) \ M_2(G_1 \boxtimes_{S_1} G_2) &= [M_2(G_1) + 4e_{G_1}] [5e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] + 14e_{G_1}M_1(G_2) \\
 &\quad + M_2(G_2) [12e_{G_1} + n_{G_1}] + M_1(G_1) [e_{G_2} + M_1(G_2) + M_2(G_2)] + 8e_{G_1}e_{G_2}.
 \end{aligned} \tag{30}$$

(ii) R_1 -sum:

$$\begin{aligned}
 (a) \ M_1(G_1 \boxtimes_{R_1} G_2) &= 24e_{G_1}e_{G_2} + M_1(G_2) [n_{G_1} + 12e_{G_1} + M_1(G_1)] [12e_{G_2} + 2n_{G_2}] \\
 &\quad + [M_1(R_1(G_1)) - 2M_1(G_1)] [n_{G_2} + 2e_{G_2}], \\
 (b) \ M_2(G_1 \boxtimes_{R_1} G_2) &= 2M_1(G_1) [4e_{G_2} + 6M_1(G_2) + 2M_2(G_2)] + 4e_{G_1} [3M_1(G_2) + 3M_2(G_2) + 4e_{G_2}] \\
 &\quad + [M_2(R_1(G_1)) - 4M_2(G_1)] [6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}] + e_{G_1}M_1(G_2) \\
 &\quad + 4M_2(G_1) [n_{G_2} + 6e_{G_2} + 3M_1(G_2)] + M_2(G_2) [n_{G_1} + 2e_{G_1}].
 \end{aligned} \tag{31}$$

TABLE 1: $M_1(P_n \boxtimes_{\Phi_1} P_m)$ for path graph.

$[m, n]$	$M_1(P_n \boxtimes_{S_1} P_m)$	$M_1(P_n \boxtimes_{R_1} P_m)$	$M_1(P_n \boxtimes_{Q_1} P_m)$	$M_1(P_n \boxtimes_{T_1} P_m)$
(3, 3)	336	722	472	892
(4, 4)	804	1776	1220	2204
(5, 5)	1086	3294	2296	4108
(6, 6)	1854	7040	3700	8368
(7, 7)	2606	9878	5432	11848

TABLE 2: $M_2(P_n \boxtimes_{\Phi_1} P_m)$ for path graph.

$[m, n]$	$M_2(P_n \boxtimes_{S_1} P_m)$	$M_2(P_n \boxtimes_{R_1} P_m)$	$M_2(P_n \boxtimes_{Q_1} P_m)$	$M_2(P_n \boxtimes_{T_1} P_m)$
(3, 3)	824	2384	1535	2899
(4, 4)	2262	5230	5002	9700
(5, 5)	4404	11636	9674	17818
(6, 6)	6738	17546	15818	28696
(7, 7)	10740	25816	23026	40942

(iii) Q_1 -sum:

$$\begin{aligned}
 \text{(a)} \quad M_1(G_1 \boxtimes_{Q_1} G_2) &= [M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)][n_{G_2} + 4e_{G_2} + M_1(G_2)] + 8e_{G_1}e_{G_2} \\
 &\quad + 3M_1(G_1)[n_{G_2} + 4e_{G_2} + M_1(G_2)] + M_1(G_2)[n_{G_1} + 4e_{G_1}], \\
 \text{(b)} \quad M_2(G_1 \boxtimes_{Q_1} G_2) &= [n_{G_2} + 6e_{G_2} + 3M_1(G_2) + 2M_2(G_2)] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) \right. \\
 &\quad \left. + \sum_{uv \in V(G_1)} rd_{G_1}(u)d_{G_1}(v) + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{\substack{u \in V(G_1) \\ uv \in E(G_1)}} d_{G_1}(u) - 2M_2(G_1) \right] \\
 &\quad + M_2(G_2)[4e_{G_1} + n_{G_1}] + 2e_{G_1}M_1(G_2) + M_1(G_1)[5e_2 + 5M_1(G_2) + 5M_2(G_2)] \\
 &\quad + [M_3(G_1) + 2M_2(G_1)][6e_{G_2} + 3M_1(G_2) + 2M_2(G_2) + n_{G_2}],
 \end{aligned} \tag{32}$$

(iv) T_1 -sum:

$$\begin{aligned}
 \text{(a)} \quad M_1(G_1 \boxtimes_{T_1} G_2) &= M_1(G_2)[8e_{G_1} + n_{G_1} + 6M_1(G_1)] + M_1(G_1)[24e_{G_2} + 6n_{G_2}] \\
 &\quad + [M_3(G_1) + 2M_2(G_1) - 2M_1(G_1)][4e_{G_2} + n_{G_2} + M_1(G_2)] + 16e_{G_1}e_{G_2}, \\
 \text{(b)} \quad M_2(G_1 \boxtimes_{T_1} G_2) &= [n_{G_2} + 6e_{G_2} + 3M_1(G_2) + 2M_2(G_2)] \left[\frac{1}{2} \sum_{v \in V(G_1)} (d_{G_1}^4(v) - d_{G_1}^3(v)) \right. \\
 &\quad \left. + \sum_{uv \in V(G_1)} rd_{G_1}(u)d_{G_1}(v) + \sum_{v \in V(G_1)} d_{G_1}^2(v) \sum_{\substack{u \in V(G_1) \\ uv \in E(G_1)}} d_{G_1}(u) - 2M_2(G_1) \right] \\
 &\quad + [M_3(G_1) + 2M_2(G_1)][6e_{G_2} + n_{G_2} + 3M_1(G_2) + 2M_2(G_2)] + 5M_1(G_2)e_{G_1} \\
 &\quad + M_2(G_2)[10e_{G_1} + n_{G_1}] + M_1(G_1)[10e_{G_2} + 11M_1(G_2) + 10M_2(G_2)] \\
 &\quad + M_2(G_1)[4n_{G_2} + 24e_{G_2} + 8M_2(G_2) + 12M_1(G_2)].
 \end{aligned} \tag{33}$$

Now, we close our discussion with the conclusion that both the Zagreb indices of the generalized T-sum graph are dominant among the Zagreb indices of all the generalized sum graphs as shown in Figure 4. We also conclude that, in the generalized T-sum graph, the number of vertices (atoms) and edges (bonds) between them are more than the other graphs of this family for each integral values of k . Thus, the role of Zagreb indices for generalized T-sum graph remains dominant for each integral values of k . However, the problem is still open to find different degree- and distance-based TIs for the generalized sum graphs obtained under the various operations of product of graphs.

Data Availability

The data used to support the findings of this study are included within the article. Additional data can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

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