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Research Article

Characterization of (Molecular) Graphs with Fractional Metric Dimension as Unity

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Distance-based dimensions provide the foreground for the identification of chemical compounds that are chemically and structurally different but show similarity in different reactions. The reason behind this similarity is the occurrence of a set $\mathbb{S}$ of atoms and their same relative distances to some ordered set $\mathbb{T}$ of atoms in both compounds. In this article, the aforementioned problem is considered as a test case for characterising the (molecular) graphs bearing the fractional metric dimension (FMD) as 1. For the illustration of the theoretical development, it is shown that the FMD of path graph is unity. Moreover, we evaluated the extremal values of fractional metric dimension of a tetrahedral diamond lattice.

1. Introduction

Day by day, the nexus of chemistry is progressing by the advancements in drug discovery, formation of chemical compounds, and development of testing kits for the diagnosis of different diseases and medical anomalies. Besides different concepts that arose as a result of the emergence of cheminformatics, distance-based dimensions also have their stake in this concern. Assume that, in a graph $\mathbb{C}$, the shortest path between the 2 vertices s, t is given by $d(s,t)$. Let $\mathbb{S} = \{s_1, s_2, s_3, \ldots, s_k\} \subseteq V(\mathbb{C})$ and $u \in V(\mathbb{C})$; then, the k-tuple metric form of $\mathbb{S}$ in terms of u is given by $r(u|S) = (d(u,s_1), d(u,s_2), d(u,s_3), \ldots, d(u,s_k))$. The set S becomes a resolving set having k elements for a graph $\mathbb{C}$ if each pairs of vertices in $\mathbb{C}$ bears distinct k-tuple metric forms. The resolving set with minimum cardinality in $\mathbb{C}$ forms its metric basis, and its cardinality represents its metric dimension.

The terminology of resolving sets was introduced by Slatter [1, 2] by naming them as locating sets. Harary and Melter [3] personally discovered these terminologies and called them as the metric dimension of $\mathbb{C}$. Afterward, many researchers have studied different graph structures for the calculation of metric dimensions. The results for the metric dimensions of path, cycle, Peterson, and generalized Peterson graphs can be found in [4–6]. For various results on metric dimensions of graphs, we refer to [7–9] and [10]. Chartrand et al. [11] employed metric dimension to find the solution of an integer programming problem (IPP). Subsequently, Currie and Oellermann introduced the concept of fractional metric dimension (FMD) and obtained the solution of IPP with higher accuracy [12]. Arumugam and Mathew [13] after discovering the hidden properties of FMD formally defined it. Since then, many researchers have tried their luck in this area by attacking different graph structures. The results for the FMD of graph structures as obtained from Cartesian, hierarchial, corona, lexicographic, and comb product of connected graph structures can be seen in [14–16] and [17, 18]. Recently, Liu et al. [19] calculated the fractional metric dimension of the generalized Jahangir graph $J_{5,k}$ and Raza et al. calculated the FMD of a metal organic network [20, 21]. Alisyah et al. presented the concept of local

fractional metric dimension (LFMD) and found the LFMD of the corona product of two connected networks [22]. Liu et al. calculated the LFMD of rotationally symmetric and planar networks [23]. Recently, Javaid et al. calculated the bounds for the LFMD of connected and cycle-related networks in [24, 25].

Johnson [26, 27] employed the concept of metric dimension for creating proficiency of large datasets of chemical graph structures. The mathematical study of chemical structures concerns the development of mathematical classification of chemical compounds. The graph-theoretic version of chemical compounds naturally exists. Despite having different chemical and structural aspects, two chemical compounds show similar behaviour during the reactions. The reason behind this peculiarity is the existence of certain common substructures within these compounds. If in two compounds, the elements of the set $\mathbb{S}$ of atoms and the elements of the ordered set $\mathbb{T}$ are relatively equidistant, then we call these compounds to be similar or equivalent [28]. Finding a $\mathbb{T}$ with minimum cardinality such that the ordered lists associated with every two distinct vertices of $\mathbb{S}$ are distinct has applications to classification problems in chemistry, as described in [11].

In this article, we are going to characterise the (molecular) graphs with FMDs as unity. As a test case, we have considered the allotropic form of carbon called by tetrahedral diamond developed by Ali et al. This article propels in the following manner: Section 1 is for introduction, Section 2 is devoted for the applications of FMD in chemistry, Section 3 is for preliminaries, Section 4 concerns with the development of a tool for the characterization of graphs with FMD as 1, and Section 5 deals with the resolving neighbourhood sets of $\mathbb{TD}(n)$. In Section 6, we have calculated the FMD of $\mathbb{TD}(n)$. Section 7 gives the conclusion.

2. Applications in Chemistry

In a molecular graph, atoms are denoted by nodes and bond between them by edges. The fraternity of chemists and pharmacists is always in search of finding out chemical compounds in some collection bearing physiochemical properties in common at some particular places. This objective is achieved by the identification of the substructure having the smallest number of atoms. In graph theory, this problem is the same as finding the FMD of the graph under consideration. In this way, druggists and chemists will be able to capture the aforementioned features of these compounds and comprehend whether they are responsible for some pharmacological activity for a newly developed drug. For more on the applications like these, see [11].

3. Preliminaries

For $c \in V(\mathbb{C})$ and $\{a,b\} \subseteq V(\mathbb{C})$, $\{a,b\}$ is said to be resolved by c if $d(a,c) \neq d(b,c)$. The set formed by the pair of nodes comprising nodes such as c is called resolving neighbourhood. The resolving neighbourhood (RN) of $\{a,b\}$ is mathematically given by $R\{a,b\} = \{c \in V(\mathbb{C}) | d(a,c) \neq d(b,c)\}$.

Suppose a connected network $\mathbb{C}(V(\mathbb{C}), E(\mathbb{C}))$ having order p. A function $\tau: V(\mathbb{C}) \longrightarrow [0,1]$ is known as the resolving function (RF) of $\mathbb{C}$ if $\tau(R\{a,b\}) \geq 1$ $\forall a,b \in V(\mathbb{C})$, where $\tau(R\{x,y\}) = \sum_{z \in R\{x,y\}} \tau(z)$. An RF η of $\mathbb{C}$ is known as a minimal resolving function (MRF) if any function $\phi: V(\mathbb{C}) \longrightarrow [0,1]$ such that $\phi \leq \eta$ and $\phi(z) \neq \eta(z)$ for at least one $z \in V(\mathbb{C})$ that is not an RF of $\mathbb{C}$. Then, the FMD of the network $\mathbb{C}$ is given by $\dim_f(\mathbb{C}) = \min\{|\eta| : \eta \textit{ is the MRF of } \mathbb{C}\}$, where $|\eta| = \sum_{z \in V(\mathbb{C})} \eta(z)$ [13].

3.1. Construction of Tetrahedral Diamond. The tetrahedral diamond graph is an n-dimensional lattice, comprising n_i layers where $1 \leq i \leq n$. Figures 1 and 2 show $\mathbb{TD}(n)$ for $3 \leq n \leq 5$.

Each n_i layer is having n_i^2 vertices, $((n_i - 2)(n_i - 1)/2)$ hexagons, and three pendent edges. The vertices of each layer are denoted by $v_j^{n_i}$ where $1 \leq j \leq n_i$. The first layer is isomorphic to K_1, and layer two is isomorphic to $K_{1.3}$, whereas for $1 \leq i \leq n$, each n_{i-1} layer is the subgraph of the n_i-th layer. Hence, the graph formed by each layer is denoted by $S_{n_i}^{n_i}$. Similarly, following are the subgraphs found to be in all the layers: $S_{n_j}^{n_i,p}$, $P_{n_i - n_{i-1}}^{n_i,p}$, $P_s^{n_i,p}$, and $K_1^{n_i,p}$, where $1 \leq j, s \leq n_i - 1$ and p describes their position that can be top, top right, top left, bottom, bottom right, bottom left, middle, middle right, middle left, and bottom denoted by $t, tr, tl, b, br, bl, m, mr$, and ml, respectively. Figure 3 shows all the subsets of $\mathbb{TD}(n)$.

It can be seen from the figure that, in each layer, $v_1^{n_i}$ is adjacent to $v_2^{n_i+1}$, $v_{n_i^2 - n_{i-1}^2}^{n_i}$ is adjacent to $v_{n_i^2 - n_{i-1}^2 - 1}^{n_i+1}$, and $v_{n_i^2}^{n_i}$ is adjacent to $v_{n_i^2 - 1}^{n_i+1}$. Apart from them, every vertex with an odd label in the $n_i - 1$ layer is adjacent to the vertex with an even label in the n_i layer and vice versa.

4. Characterization of Graphs with FMD as Unity

In this section of the article, we are giving generic criteria for identifying graphs with FMD as 1. These criteria have been shaped up as a theorem given below.

Theorem 1. *Let $\mathbb{C}$ be a connected graph and $R\{a,b\}$ be a resolving neighbourhood set of the pair of vertices a, b in $\mathbb{C}$. If $\cap R\{a,b\} \neq \Phi$, then*

$$\dim_{\text{frac}}(\mathbb{C}) = 1, \tag{1}$$

where $|V(\mathbb{C})| \geq 3$.

Proof. Assume that $R = R\{a,b\}$ is an arbitrary resolving neighbourhood set for $\{a,b\} \subset V(\mathbb{C})$ and $Y = \cap R$. Now, we define the function $\psi: V(\mathbb{C}) \longrightarrow [0,1]$ as $c = \sum_{x \in Y} \psi(x)$ and $1 - c = \sum_{x \in (R-Y) \cap X} \psi(x)$, where c is a real number that approaches to 1 and $X = V(\mathbb{C})$. For $a, b \in V(\mathbb{C})$ and $c \longrightarrow 1$,

$$\begin{aligned} \psi(R) &= \sum_{x \in R} \psi(x) = \sum_{x \in Y} \psi(x) + \sum_{x \in (R-Y) \cap X} \psi(x), \\ &= c + w(1 - c) \\ &\geq 1, \end{aligned} \tag{2}$$

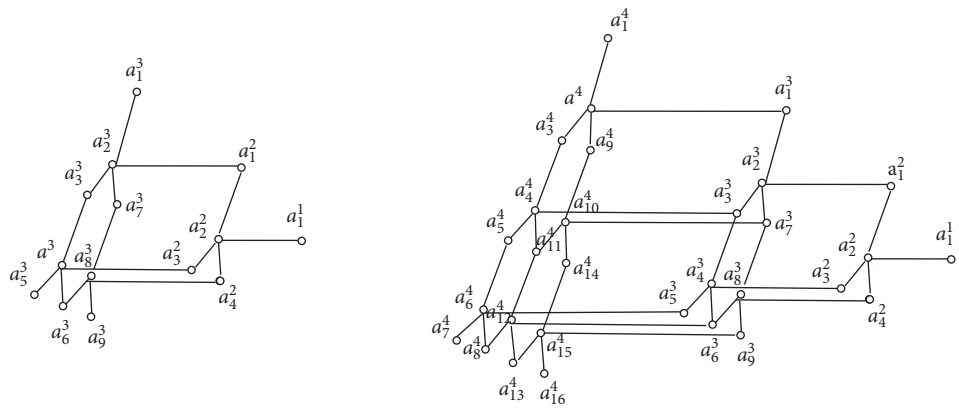

Figure 1: Tetrahedral diamond lattice with 3 (a) and 4 (b) layers.

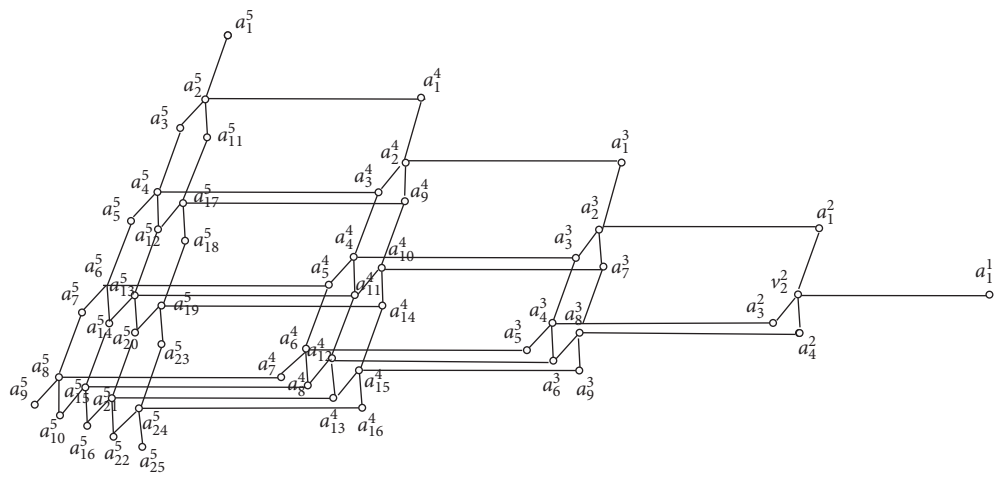

Figure 2: Tetrahedral diamond lattice with 5 layers.

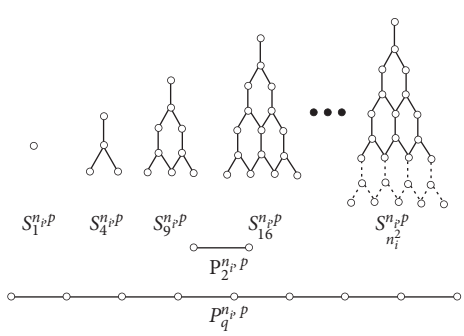

Figure 3: Subsets of the tetrahedral diamond lattice.

where $w = (|V(\mathbb{C})||V(\mathbb{C}) - 1|/2)$. It implies that ψ is a resolving function. To check that ψ is a minimal resolving function, assume that there is another minimal resolving function τ such that $\tau \leq \psi$. By definition, $\tau(x) < \psi(x)$ for some $x \in X$. Now, for some resolving neighbourhood set R, we have

$$\begin{aligned}\tau(R) &= \sum_{x\in Y} \tau(x) + \sum_{x\in (R-Y)\cap X} \tau(x)\\ &< \sum_{x\in Y} \psi(x) + \sum_{x\in (R-Y)\cap X} \psi(x),\\ &= c + (1-c) = 1.\end{aligned} \tag{3}$$

Consequently, $\tau(R) < 1$ which implies that τ is not a resolving function. Thus, ψ is a minimal resolving function. Let $\overline{\psi}$ be another minimal resolving function of $\mathbb{C}$. Now, we have the following possibilities:

(i) $\overline{\psi}(x) < \psi(x)\ \forall x \in X$

(ii) $\overline{\psi}(x) \geq \psi(x)\ \forall x \in X$

(iii) $\overline{\psi}(x) < \psi(x)$ for some $x \in X$ □

Case 1. If $\overline{\psi}(x) < \psi(x)$ for all $x \in X$, then for each resolving neighbourhood set R, $\overline{\psi}(R) < 1 \Rightarrow \overline{\psi}$ is not a resolving function; therefore, this case does not hold.

Case 2. If $\overline{\psi}(x) \geq \psi(x)$ for all $x \in X$, then we have the following subcases:

Subcase A: for $1 \leq r, s \leq w$ and $R_r - Y \cap R_s - Y = \Phi$, we have

$$\begin{aligned} |\overline{\psi}| &= \sum_{x \in Y} \overline{\psi}(x) + \sum_{x \in (R-Y) \cap X} \overline{\psi}(x) \\ &> \sum_{x \in Y} \psi(x) + \sum_{x \in (R-Y) \cap X} \psi(x), \\ &= c + w(1-c) = |\psi|. \end{aligned} \tag{4}$$

As $c \longrightarrow 1$, $\dim_{\text{frac}}(\mathbb{C}) = |\psi| = c + w(1-c) = 1$.

Subcase B: for $1 \leq r, s \leq w$ and $R_r - Y \cap R_s - Y \neq \Phi$, we have

$$|\overline{\psi}| = \sum_{x \in Y} \overline{\psi}(x) + \sum_{x \in \overline{R}_t} \overline{\psi}(x), \tag{5}$$

where $\overline{R}_t = R_t \cap [X - \cap_{j=1}^{t-1} R_t - Y]$. Then,

$$\begin{aligned} |\overline{\psi}| &= \sum_{x \in Y} \overline{\psi}(x) + \sum_{x \in \overline{R}_t} \overline{\psi}(x) \\ &\geq \sum_{x \in Y} \psi(x) + \sum_{x \in \overline{R}_t} \psi(x), \\ &= c + \sum_{x \in \overline{R}_t} \psi(x). \end{aligned} \tag{6}$$

So,

$$\begin{aligned} \dim_{\text{frac}}(\mathbb{C}) &= c + \sum_{x \in \overline{R}_t} \psi(x) \\ &\leq c + \sum_{x \in (R-Y) \cap X} \psi(x), \\ &= c + w(1-c), \\ &= 1. \end{aligned} \tag{7}$$

Thus, $\dim_{\text{frac}}(\mathbb{C}) \leq 1$. But, by definition, $\dim_{\text{frac}}(\mathbb{C}) \geq 1$. Therefore,

$$\dim_{\text{frac}}(\mathbb{C}) = 1. \tag{8}$$

Case 3. If $\overline{\psi}(x) < \psi(x)$ for some $x \in X$, this case is a consequent of the abovementioned two cases (Case I and II); therefore, we have $\dim_{\text{frac}}(\mathbb{C}) = 1$.

Consequently, from Case 1–3, we arrive at the following conclusion:

$$\dim_{\text{frac}}(\mathbb{C}) = 1. \tag{9}$$

Using the result presented above, we are now going to prove the following fact:

Proposition 1. *Suppose that, for any $n \geq 3$, $\mathbb{G} \cong P_n$; then, $\dim_{frac}(\mathbb{G}) = 1$.*

Proof

Case 1: for $n = 3$: the resolving neighbourhood sets for the current case are $R_1 = R\{a_1, a_2\} = \{a_1, a_2, a_3\}$, $R_2 = R\{a_2, a_3\} = \{a_1, a_2, a_3\}$, and $R_3 = R\{a_1, a_3\} = \{a_1, a_3\}$. It can be seen that $\cap_{t=1}^{3} R_t = \{a_1, a_3\} \neq \Phi$. Therefore, from Theorem 1, we arrive at the conclusion that$\dim_{\text{frac}}(P_3) = 1$.

Case 2: for $n \geq 4$: the resolving neighbourhood sets of P_n are $R\{a_i, a_{i+p}\} = V(P_n) - \{a_{(2i+p/2)}\}$ and $R\{a_i, a_{i+s}\} = V(\mathbb{P}_n)$, where $p, s \geq 1$, $1 \leq i \leq n$, $p \equiv 0 (\text{mod} 2)$, and $s \equiv 1 (\text{mod} 2)$.

It can be seen that $\cap_{t=1}^{\binom{n}{2}} R = \{a_1, a_n\} \neq \Phi$. Therefore, from Theorem 1, it implies that

$$\dim_{\text{frac}}(P_n) = 1. \tag{10}$$

□

Remark 1. The abovementioned proposition strengthens the result proved in [13].

5. Resolving Neighbourhood Sets of $\mathbb{TD}(n)$

In this section, we present some the results regarding the resolving neighbourhood sets of $\mathbb{TD}(n)$. Lemma 1 deals with the resolving neighbourhoods of $\mathbb{TD}(n)$ having minimum cardinality followed by Lemma 2 and Lemma 3 that are concerned with resolving neighbourhood sets of maximum cardinalities.

Lemma 1. *Suppose that $\mathbb{C} \cong \mathbb{TD}(n)$ is an n-dimensional tetrahedral diamond lattice. Then, the minimum resolving neighbourhood sets are as follows:*

(a) For $n \geq 4$, $n \equiv 0 (\text{mod} 2)$, $\alpha = (n/2), \beta = \alpha + 1$, $\gamma = 2\alpha - 1$, and $\eta = 2\beta - 1$, $|R_t| = |R\{a_1^\alpha, a_1^\beta\}\}| = |R\{a_\gamma^\alpha, a_\eta^\beta\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}| = (n(n^2 + 3n + 2)/12)$ and $|\cup_{t=1}^{3} R_t| = (n(n^2 + 3n + 2)/6)$.

(b) For $n \geq 3$, $n \equiv 1 (\text{mod} 2)$, $\alpha = (n - 1/2)$, $\beta = (n + 1/2)$, $\gamma = (n + 3/2)$, $\eta = 2\alpha - 1$, $\lambda = 2\beta - 1$, and $\mu = 2\gamma - 1$, $|R_t| = |R\{a_1^\alpha, a_1^\beta\}\}| = |R\{a_1^\beta, a_1^\gamma\}| = |R\{a_\eta^\alpha, a_\lambda^\beta\}| = |R\{a_\lambda^\beta, a_\mu^\gamma\}\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}| = |R\left\{a_{\beta^2}^\beta, a_{\gamma^2}^\gamma\right\}| = (n^3 + 3n^2 + 5n + 3/12)$ and $|\cup_{t=1}^{6} R_t| = (n^3 + 6n^2 + 11n - 30/6)$.

Proof

(a) The resolving neighbourhood sets of $R\{a_1^\alpha, a_1^\beta\}$, $R\{a_\gamma^\alpha, a_\eta^\beta\}$, and $R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}$ are $R\{a_1^\alpha, a_1^\beta\} = V(\mathbb{C}) - \cup_{j=\beta}^{n} (S_{j^2}^j - S_{(j-2)^2}^{j,t})$, $R\{a_\gamma^\alpha, a_\eta^\beta\} = V(\mathbb{C}) - \cup_{j=\beta}^{n} (S_{j^2}^j - S_{(j-2)^2}^{j,bl})$, and $R\left\{a_{\gamma^2}^\alpha, a_{\eta^2}^\beta\right\} = V(\mathbb{C}) - \cup_{j=\beta}^{n} (S_{j^2}^j - S_{(j-2)^2}^{j,br})$. We note that $|R_t| = |R\{a_1^\alpha, a_1^\beta\}\}| = |R\{a_\gamma^\alpha, a_\eta^\beta\}| = |R\left\{a_{(\alpha)^2}^\alpha, a_{(\alpha+1)^2}^\beta\right\}| = (n(n^2 + 3n + 2)/12)$, $\cup_{t=1}^{3} R_t = \cup_{j=1}^{\alpha} S_{j^2}^j \cup \cup_{j=\alpha+1}^{n} (S_{(j-\alpha)^2}^{j,t} \cup S_{(j-\alpha)^2}^{j,br} \cup S_{(j-\alpha)^2}^{j,bl})$, and $|\cup_{t=1}^{3} R_t| = (n(n^2 + 3n + 2)/6)$.

(b) The resolving neighbourhood sets of $R\{a_1^\alpha, a_1^\beta\}\}$, $R\{a_1^\beta, a_1^\gamma\}$, $R\{a_\eta^\alpha, a_\lambda^\beta\}$, $R\{a_\lambda^\beta, a_\mu^\gamma\}$, $R\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\}$, $R\{a_{\beta^2}^\beta, a_{\gamma^2}^\gamma\}$ are $R\{a_1^\alpha, a_1^\beta\} = V(\mathbb{C}) - \cup_{j=\beta}^n (S_{j^2}^j - S_{(j-2)^2}^{j,t})$, $R\{a_\gamma^\alpha, a_\eta^\beta\} = V(\mathbb{C}) - \cup_{j=\beta}^n (S_{j^2}^j - S_{(j-2)^2}^{j,bl})$, and $R\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\} = V(\mathbb{C}) - \cup_{j=\beta}^n (S_{j^2}^j - S_{(j-2)^2}^{j,br})$ and $R\{a_1^\beta, a_1^\gamma\} = V(\mathbb{C}) - \cup_{j=\gamma}^n (S_{j^2}^j - S_{(j-2)^2}^{j,t})$, $R\{a_\lambda^\beta, a_\mu^\gamma\} = V(\mathbb{C}) - \cup_{j=\gamma}^n (S_{j^2}^j - S_{(j-2)^2}^{j,bl})$, and $R\left\{a_{\beta^2}^\beta, a_{\gamma^2}^\gamma\right\} = V(\mathbb{C}) - \cup_{j=\gamma}^n (S_{j^2}^j - S_{(j-2)^2}^{j,br})$ .We note that $|R_t| = |R\{a_1^\alpha, a_1^\beta\}\}| = |R\{a_1^\beta, a_1^\gamma\}| = |R\{a_\eta^\alpha, a_\lambda^\beta\}| = |R\{a_\lambda^\beta, a_\mu^\gamma\}\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}| = |R\left\{a_{\beta^2}^\beta, a_{\gamma^2}^\gamma\right\}| = (n^3 + 3n^2 + 5n + 3/12)$, $\cup_{t=1}^6 R_t = \cup_{j=1}^\gamma S_{j^2}^j \cup \cup_{j=\gamma+1}^n (S_{(j-\gamma)^2}^{j,t} \cup S_{(j-\gamma)^2}^{j,br} \cup S_{(j-\gamma)^2}^{j,bl})$, and $|\cup_{t=1}^6 R_t| = (n^3 + 6n^2 + 11n - 30/6)$. □

Lemma 2. *Suppose that $\mathbb{C} \cong \mathbb{TD}(n)$ is an n-dimensional tetrahedral diamond lattice with $n \geq 3$ and $n \equiv 1 (\mathrm{mod} n)$. Then,*

(a) For $1 \leq \alpha \leq n, \beta = \alpha + 1$, $\gamma = 2\alpha - 1$, and $\eta = 2\beta - 1$, $|R_t| < |R\{a_1^\alpha, a_1^\beta\}| = |R\{a_\gamma^\alpha, a_\eta^\beta\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}|$ and $|R\{a_1^\alpha, a_1^\beta\} \cap \cup_{t=1}^6 R_t| = |R\{a_\gamma^\alpha, a_\eta^\beta\} \cap \cup_{t=1}^6 R_t| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\} \cap \cup_{t=1}^6 R_t| \geq |R_t|$

(b) For $1 \leq \alpha \leq n$, $\beta = \alpha + 1$ and $2 \leq \gamma \leq 2\alpha - 1$, $|R_t| < |R\{a_\gamma^\alpha, a_\gamma^\beta\}|$ and $|R\{a_\gamma^\alpha, a_\gamma^\beta\} \cup_{t=1}^6 R_t| \leq |R_t|$

(c) For $2 \leq \alpha \leq n$, $\beta = 2\alpha - 1, \gamma = 2\alpha + 2, \eta = \alpha^2 - 3$, and $\mu = \alpha^2$, $|R_t| < |R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\}| = |R\{a_\eta^\alpha, a_\mu^\alpha\}|$ and $|R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\} \cup_{t=1}^6 R_t| = |R\{a_i^i, a_{i+1}^{i+1}\} \cup_{t=1}^6 R_t| = |R\{a_\eta^\alpha, a_\mu^\alpha\} \cup_{t=1}^6 R_t| \geq |R_t|$

(d) For any $\left\{a_\alpha^\alpha, a_\beta^\beta\right\} \in E(\mathbb{C})$, $\gamma \geq 1$, $\eta \geq 2$, $\gamma \equiv 1 (\mathrm{mod} 2)$, and $\eta \equiv 0 (\mathrm{mod} 2)$, $|R_t| < |R\left\{a_\alpha^\alpha, a_\beta^\beta\right\}| = |R\{a_\alpha^\alpha, a_{\alpha+\gamma}^{\alpha+\eta}\}| = |V(\mathbb{C})| = (n(2n^2 + 3n + 1)/6)$ and $|R\left\{a_\alpha^\alpha, a_\beta^\beta\right\} \cap \cup_{t=1}^6 R_t| \geq |R_t|$

Proof

(a) The resolving neighbourhood sets of $R\{a_1^1, a_1^2\}$, $\{a_1^1, a_3^2\}$, $\{a_1^1, a_4^2\}$, $\{a_1^\alpha, a_1^\beta\}$, and $R\{a_\gamma^\alpha, a_\eta^\beta\}$ are $R\{a_1^\alpha, a_1^\beta\} = V(\mathbb{C}) - \cup_{j=\beta}^n (S_{j^2}^j - S_{(j-1)^2}^{j,t})$, $R\{a_\gamma^\alpha, a_\eta^\beta\} = V(\mathbb{C}) - \cup_{j=2}^\beta (S_{j^2}^j - S_{(j-1)^2}^{j,bl})$, and $R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\} = V(\mathbb{C}) - \cup_{j=\beta}^n (S_{j^2}^j - S_{(j-1)^2}^{j,br})$, respectively.
Since $|R\{a_1^\alpha, a_1^\beta\}| = |R\{a_\gamma^\alpha, a_\eta^\beta\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}| = (2n^3 + 3n^2 + n + 6\alpha(n+1)(\alpha - n) - 6\alpha\beta(\alpha - \beta - 1)/6)$, thus $|R_t| = (2n^3 + 3n^2 + n + 6\alpha(n+1)(\alpha - n) - 6\alpha\beta(\alpha - \beta - 1)/6)$. Therefore, $|R\{a_1^\alpha, a_1^\beta\} \cap \cup_{t=1}^6 R_t| = |R\{a_\gamma^\alpha, a_\eta^\beta\} \cap \cup_{t=1}^6 R_t| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\} \cap \cup_{t=1}^6 R_t| \geq |R_t|$.

(b) First of all, we introduce a notation for simplification.

$$\mu = \begin{cases} \dfrac{\gamma}{2}, & \text{for} n \equiv 0 (\mathrm{mod} n), \\ \dfrac{\gamma - 1}{2}, & \text{for} n \equiv 1 (\mathrm{mod} n). \end{cases} \tag{11}$$

The resolving neighbourhood for $R\{a_\gamma^\alpha, a_\gamma^\beta\}$ is $R\{a_\gamma^\alpha, a_\gamma^\beta = V(\mathbb{C}) - \cup_{j=\alpha-\mu+1}^\alpha S_{j^2}^j \cup \cup_{j=} \beta^n S_{j^2}^{j,t} \cup \cup_{j=\mu+1}^{\mu+n-\alpha} S_{j^2}^{j,t}$. It can be seen that $|R\{a_\gamma^\alpha, a_\gamma^\beta| = (n - 1 - 4(\mu + 1)^3 + 6(\mu+1)^2 - \mu + 2b^3 - 3b^2 + b + 2(\mu + n - \alpha + 1)^3 - (\mu + n - \alpha + 1)^2 - \alpha/6) > |R_t|$. Therefore, $|R\{a_3^2, a_4^2\} \cup_{t=1}^6 R_t| \geq |R_t|$. Also, by the symmetry of the network, $|R_t| < |R\{a_\gamma^\alpha, a_\gamma^\beta|$ and $|R\{a_\gamma^\alpha, a_\gamma^\beta \cup_{t=1}^6 R_t| \geq |R_t|$.

(c) The resolving neighbourhood for $R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\}$ and $R\{a_\eta^\alpha, a_\mu^\alpha\}$ are $R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\} = V(\mathbb{C}) - S_1^1 \cup \cup_{j=3}^\alpha P_j^{j,l} \cup \cup_{j=\alpha+1}^n (P_j^{j,l} - S_{(j-\alpha)^2}^{j,l})$ and $R\{a_\eta^\alpha, a_\mu^\alpha\} = V(\mathbb{C}) - S_1^1 \cup \cup_{j=} 3^\alpha P_j^{j,r} \cup \cup_{j=\alpha+1}^n (P_j^{j,r} - S_{(j-\alpha)^2}^{j,r})$. It can be seen that $|R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\}| = |R\{a_\eta^\alpha, a_\mu{}^\alpha\}| = (2(n+1)^3 - 3(n+1)^2 + 2n + 8 - 7\alpha + 6(2\alpha - 2)(n - \alpha + 1) + 2(n - \alpha + 1)^3 - 3(n - \alpha + 1)^2/6)$. Hence, $|R_t| < (2(n+1)^3 - 3(n+1)^2 + 2n + 8 - 7\alpha + 6(2\alpha - 2)(n - \alpha + 1) + 2(n - \alpha + 1)^3 - 3(n - \alpha + 1)^2/6)$ and $|R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\} \cup_{t=1}^6 R_t| = |R\{a_\eta^\alpha, a_\mu^\alpha\} \cup_{t=1}^6 R_t| \geq |R_t|$.

(d) The resolving neighbourhood sets $R\left\{a_\alpha^\alpha, a_\beta^\beta\right\}$ and $R\{a_\alpha^\alpha, a_{a^{\alpha+\gamma}}^{\alpha+\eta}\}$ are $R\left\{a_\alpha^\alpha, a_\beta^\beta\right\} = V(\mathbb{C}) = R\{a_\alpha^\alpha, a_{a^{\alpha+\gamma}}^{\alpha+\eta}\}$. Clearly, $|R\{a_\alpha^\alpha, a_\beta{}^\beta\}| = |R\{a_\alpha^\alpha, a_{a^{\alpha+\gamma}}^{\alpha+\eta}\}| = (n(n+1)(2n+1)/6) > |R_t|$. Also, $|R\left\{a_\alpha^\alpha, a_\beta^\beta\right\} \cup_{t=1}^6 R_t| = |R\{a_\alpha^\alpha, a_{a^{\alpha+\gamma}}^{\alpha+\eta} \cup_{t=1}^6 R_t| \geq |R_t|$. □

Lemma 3. *Suppose that $\mathbb{C} \cong \mathbb{TD}(n)$ is an n-dimensional tetrahedral diamond lattice with $n \geq 4$ and $n \equiv 0 (\mathrm{mod} n)$. Then,*

(a) For $1 \leq \alpha \leq n, \beta = \alpha + 1$, $\gamma = 2\alpha - 1$, and $\eta = 2\beta - 1$, $|R_t| < |R\{a_1^\alpha, a_1^\beta\}| = |R\{a_\gamma^\alpha, a_\eta^\beta\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}|$ and $|R\{a_1^\alpha, a_1^\beta\} \cap \cup_{t=1}^3 R_t| = |R\{a_\gamma^\alpha, a_\eta^\beta\} \cap \cup_{t=1}^3 R_t| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\} \cap \cup_{t=1}^3 R_t| \geq |R_t|$

(b) For $1 \leq \alpha \leq n$, $\beta = \alpha + 1$, and $2 \leq \gamma \leq 2\alpha - 1$, $|R_t| < |R\{a_\gamma^\alpha, a_\gamma^\beta\}|$ and $|R\{a_\gamma^\alpha, a_\gamma^\beta\} \cup_{t=1}^3 R_t| \leq |R_t|$

(c) For $2 \leq \alpha \leq n$, $,\beta = 2\alpha - 1, \gamma = 2\alpha + 2, \eta = \alpha^2 - 3$, and $\mu = \alpha^2$, $|R_t| < |R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\}| = |R\{a_\eta^\alpha, a_\mu^\alpha\}|$ and $|R\left\{a_\beta^\alpha, a_\gamma^\alpha\right\} \cup_{t=1}^3 R_t| = |R\{a_i^i, a_{i+1}^{i+1}\} \cup_{t=1}^3 R_t| = |R\{a_\eta^\alpha, a_\mu^\alpha\} \cup_{t=1}^3 R_t| \geq |R_t|$

(d) For any $\left\{a_\alpha^\alpha, a_\beta^\beta\right\} \in E(\mathbb{C})$, $\gamma \geq 1$, $\eta \geq 2$, $\gamma \equiv 1 (\mathrm{mod} 2)$, and $\eta \equiv 0 (\mathrm{mod} 2)$, $|R_t| < |R\left\{a_\alpha^\alpha, a_\beta^\beta\right\}| =$

$$|R\left\{a_{\alpha}^{\alpha}, a_{\alpha+\gamma}^{\alpha+\eta}\right\}| = |V(\mathbb{C})| = (n(2n^2+3n+1)/6) \quad \textit{and}$$
$$|R\left\{a_{\alpha}^{\alpha}, a_{\beta}^{\beta}\right\} \cap \cup_{t=1}^{3} R_t| \geq |R_t|$$

Proof. The proof is the same as that of Lemma 2. □

6. Fractional Metric Dimension of $\mathbb{TD}(n)$

In this section, the FMD of $\mathbb{TD}(n)$ is calculated and the criterion of their evaluation is devised by the following result.

Theorem 2. *If $\mathbb{C} \cong \mathbb{TD}(n)$ is an n-dimensional tetrahedral diamond lattice with $n \geq 3$ and $n \equiv 1(\mathrm{mod}2)$, then*

$$1 < \dim_{\mathrm{frac}}(\mathbb{C}) \leq 2\left(\frac{n^3+6n^2+11n-30}{n^3+3n^2+5n+3}\right). \tag{12}$$

Proof

Case 1: when $n = 3$.

The resolving neighbourhood sets are as shown in Tables 1–4.

The resolving neighbourhood sets that are equal due to symmetry are given by the following:

Now, for $uv \in E(\mathbb{C})$ and $47 \leq L \leq 62$, we have

$$R\{uv\} = V(\mathbb{C}). \tag{13}$$

In the same manner, the pairwise resolving neighbourhood sets that equals $V(\mathbb{C})$ are as follows:

As we can see, Table 3 shows the resolving neighbourhood sets of $\mathbb{TD}(3)$ having the maximum cardinality of 13 and $\cup_{L=63}^{84} R_L = V(\mathbb{TD}(3))$. Table 4, on the other hand, shows the resolving neighbourhood sets with minimum cardinality of 6 and $\cup_{t=1}^{6} R_t = V(\mathbb{TD}(3))$. Suppose that, for $63 \leq L \leq 84$, $|R_L| = \gamma$, for $1 \leq t \leq 6$, $|R_t| = \lambda$, $\eta = |\cup_{L=63}^{84} R_L| = |V(\mathbb{TD}(3))| = 13$, and $\delta = |\cup_{t=1}^{6} R_t| = |V(\mathbb{TD}(3))| = 13$. Now, we define a mapping κ: $V(\mathbb{TD}(3)) \longrightarrow [0,1]$ such that $\kappa(a) = (1/13)$ for all $a \in \cup_{L=63}^{84} R_L$; assigning the value of $(1/13)$ to all the elements of $\cup_{L=63}^{84} R_L$ and summing up all the labels, we get $|\kappa| = \sum_{a \in \cup_{L=63}^{84} R_L} \kappa(a) = (13/13) = 1$; thus, $\dim_{\mathrm{frac}}(T\ D)(3) = |\kappa| = 1$, as all the R_L for $63 \leq L \leq 84$ are all having $V(\mathbb{TD}(3))$ in common. Similarly, we define another mapping τ: $V(\mathbb{TD}(3)) \longrightarrow [0,1]$ such that $\tau = (1/\lambda) = (1/6)$ for all $a \in \cup_{t=1}^{6} R_t$, and giving labels to the elements of $\cup_{t=1}^{6} R_t$ and later on summing them up, we get $|\tau| = \sum_{a \in \cup_{t=1}^{6} R_t} \tau(a) = (13/6) = 2.33$; hence, $\dim_{\mathrm{frac}}(\mathbb{C}) < 2.33$ as all the R_t for $1 \leq t \leq 6$ are pairwise overlapping. Therefore, we arrive at the following conclusion:

$$1 < \dim_{\mathrm{frac}}(\mathbb{TD}(3)) \leq 2.33. \tag{14}$$

Case 2: when $n \geq 5$.

The required minimum resolving neighbourhood sets are $R\left\{a_1^{\alpha}, a_1^{\beta}\right\}, R\left\{a_1^{\beta}, a_1^{\gamma}\right\}, R\left\{a_{\eta}^{\alpha}, a_{\lambda}^{\beta}\right\}, R\left\{a_{\lambda}^{\beta}, a_{\mu}^{\gamma}\right\}$, $R\left\{a_{\alpha^2}^{\alpha}, a_{\beta^2}^{\beta}\right\}$, and $R\left\{a_{\beta^2}^{\beta}, a_{\gamma^2}^{\gamma}\right\}$, where $\alpha = (n-1/2)$, $\beta = (n+1/2)$, $\gamma = (n+3/2)$, $\eta = 2\alpha - 1$, $\lambda = 2\beta - 1$, and $\mu = 2\gamma - 1$. Lemma 1 clarifies that $|R_t| = |R\left\{a_1^{\alpha}, a_1^{\beta}\right\}| = |R\left\{a_1^{\beta}, a_1^{\gamma}\right\}| = |R\left\{a_{\eta}^{\alpha}, a_{\lambda}^{\beta}\right\}| = |R\left\{a_{\lambda}^{\beta}, a_{\mu}^{\gamma}\right\}| = |R\left\{a_{\alpha^2}^{\alpha}, a_{\beta^2}^{\beta}\right\}| = |R\left\{a_{\beta^2}^{\beta}, a_{\gamma^2}^{\gamma}\right\}| = (n^3+3n^2+5n+3/12) \leq |R\{a,b\}|$ for all $a, b \in V(\mathbb{C})$ and $\cup_{t=1}^{6} R_t = \cup_{j=1}^{\gamma} S_{j^2}^{j} \cup \cup_{j=\gamma+1}^{n} (S_{(j-\gamma)^2}^{j,t} \cup S_{(j-\gamma)^2}^{j,br} \cup S_{(j-\gamma)^2}^{j,bl})$. Also, the resolving neighbourhood sets with maximum cardinality of $|V(\mathbb{C})|$, as clarified by Lemma 2, are $R_1 = R\{a_{\alpha}^{\alpha}, a_{a^{\alpha+\gamma}}^{\alpha+\eta}\}$ and $R_1 = R\left\{a_{\alpha}^{\alpha}, a_{\beta}^{\beta}\right\}$ and $R_1 = R\{a_{\alpha}^{\alpha}, a_{a^{\alpha+\gamma}}^{\alpha+\eta}\}$, respectively. Moreover, $\cup_{L=1}^{2} R_L = V(\mathbb{C})$. Let

Table 1: Resolving neighbourhood sets of $\mathbb{TD}(3)$.

Resolving neighbourhood sets	Elements
$R_1 = R\{a_1^1, a_2^1\}$	$V(\mathbb{C}) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\}$
$R_2 = R\{a_1^1, a_2^3\}$	$V(\mathbb{C}) - \{a_1^2, a_2^2, a_4^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\}$
$R_3 = R\{a_1^1, a_4^2\}$	$V(\mathbb{C}) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\}$
$R_4 = R\{a_1^1, a_3^1\}$	$V(\mathbb{C}) - \{a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3, a_9^3\}$
$R_5 = R\{a_2^2, a_5^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_6^3, a_7^3, a_8^3, a_9^3\}$
$R_6 = R\{a_4^2, a_9^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3\}$
$R_1 = R\{a_1^2, a_3^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_4^2\} \cup \{a_3^3, a_8^3, a_9^3\}$
$R_2 = R\{a_1^2, a_4^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_3^2\} \cup \{a_4^3, a_5^3, a_7^3\}$
$R_3 = R\{a_1^2, a_3^3\}$	$V(\mathbb{C}) - \{a_3^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\}$
$R_4 = R\{a_1^2, a_5^3\}$	$V(\mathbb{C}) - \{a_5^2\} \cup \{a_3^3, a_8^3, a_9^3\}$
$R_5 = R\{a_1^2, a_6^3\}$	$V(\mathbb{C}) - \{a_3^2, a_4^2\} \cup \{a_3^3, a_7^3\}$
$R_6 = R\{a_1^2, a_7^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\}$
$R_7 = R\{a_1^2, a_9^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_4^3, a_5^3, a_7^3\}$
$R_8 = R\{a_2^2, a_2^3\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_1^3, a_2^3, a_3^3, a_7^3\}$
$R_9 = R\{a_2^2, a_4^3\}$	$V(\mathbb{C}) - \{a_3^2\} - \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\}$
$R_{10} = R\{a_2^2, a_8^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\}$
$R_{11} = R\{a_3^2, a_1^3\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_3^3, a_8^3, a_9^3\}$
$R_{12} = R\{a_3^2, a_3^3\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\}$
$R_{13} = R\{a_3^2, a_5^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_6^3, a_7^3, a_8^3, a_9^3\}$
$R_{14} = R\{a_3^2, a_6^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\}$
$R_{15} = R\{a_3^2, a_7^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_3^3, a_6^3\}$
$R_{16} = R\{a_3^2, a_9^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_4^3, a_5^3, a_7^3\}$
$R_{17} = R\{a_4^2, a_1^3\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_4^3, a_5^3, a_7^3\}$
$R_{18} = R\{a_4^2, a_3^3\}$	$V(\mathbb{C}) - \{a_1^2, a_5^2\} \cup \{a_6^3, a_7^3, a_7^3\}$
$R_{19} = R\{a_4^2, a_5^3\}$	$V(\mathbb{C}) - \{a_3^2\} \cup \{a_1^3, a_2^3, a_6^3\}$
$R_{20} = R\{a_4^2, a_6^3\}$	$V(\mathbb{C}) - \cup \{a_1^3, a_2^3, a_8^3, a_9^3\}$
$R_{21} = R\{a_4^2, a_7^3\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\}$
$R_{22} = R\{a_1^1, a_6^3\}$	$V(\mathbb{C}) - \{a_3^2, a_4^2\} \cup \{a_1^3, a_2^3\}$
$R_{23} = R\{a_1^1, a_7^3\}$	$V(\mathbb{C}) - \{a_1^2, a_4^2\} \cup \{a_4^3, a_5^3\}$
$R_{24} = R\{a_2^2, a_4^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2\} \cup \{a_1^3, a_2^3, a_6^3\}$
$R_{25} = R\{a_3^3, a_9^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_6^3, a_7^3\}$
$R_{26} = R\{a_4^3, a_7^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_3^3, a_6^3\}$
$R_{27} = R\{a_5^3, a_6^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3\}$
$R_{28} = R\{a_6^3, a_9^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_4^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3\}$
$R_{29} = R\{a_1^2, a_3^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_4^2\} \cup \{a_3^3, a_8^3, a_9^3\}$
$R_{30} = R\{a_1^2, a_4^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_3^2\} \cup \{a_4^3, a_5^3, a_7^3\}$
$R_{31} = R\{a_1^3, a_3^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_4^2\} \cup \{a_2^3, a_7^3, a_8^3\}$
$R_{32} = R\{a_1^3, a_6^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_3^3, a_7^3\}$
$R_{43} = R\{a_1^3, a_7^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_3^2\} \cup \{a_2^3, a_3^3, a_4^3, a_5^3\}$
$R_{33} = R\{a_2^3, a_4^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_4^2\} \cup \{a_3^3, a_8^3, a_9^3\}$
$R_{34} = R\{a_2^3, a_8^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_3^2\} \cup \{a_4^3, a_5^3, a_7^3\}$
$R_{35} = R\{a_5^3, a_7^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_3^3, a_6^3\}$

$|R\{a_\alpha^\alpha, a_{a^{\alpha+\gamma}}^{\alpha+\eta}\}| = |R\left\{a_\alpha^\alpha, a_\beta^\beta\right\}| = \gamma$, $|R_t| = \lambda$, $|\cup_{L=1}^{2} \acute{R}_L| = \eta$, and $|\cup_{t=1}^{6} R_t| = \delta$. To find the minimum value for the $\dim_{\text{frac}}(\mathbb{C})$, we define a mapping κ: $V(\mathbb{C}) \longrightarrow [0,1]$ such that

$$\kappa(a) = \begin{cases} \dfrac{1}{\kappa}, & \text{for } a \in \bigcup_{L=1}^{2} \acute{R}_L, 0, \text{for } a \in V(\mathbb{C}) - \bigcup_{L=1}^{2} \acute{R}_L, \end{cases} \tag{15}$$

where $\gamma = \eta = (n(2n^2 + 3n + 1)/6)$. Assigning the labels to the elements of $\bigcup \cup_{L=1}^{2} R_L$ and summing them up, we get $|\kappa| = \sum_{a \in \cup_{L=1}^{2} \acute{R}_L} (1/\gamma) = 1$.

Similarly, for the maximum value of $\dim_{\text{frac}}(\mathbb{C})$, we define another mapping τ: $V(\mathbb{C}) \longrightarrow [0,1]$ such that

$$\tau(a) = \begin{cases} \dfrac{1}{\lambda}, & \text{for } a \in \bigcup_{t=1}^{6} R_t, 0, \text{for } a \in V(\mathbb{C}) - \bigcup_{t=1}^{6} R_t. \end{cases} \tag{16}$$

It can be seen that τ is a resolving function for $\mathbb{C}$ with $n \geq 3$ because $\tau(R\{u,v\}) \geq 1 \forall u, v \in V(\mathbb{C})$. On the contrary, assume that there is another resolving function ρ such that $\rho(u) \leq \tau(u)$, for at least one $u \in V(\mathbb{C}) \rho(u) \neq \tau(u)$. As a consequence, $\rho(R\{u,v\}) < 1$, where $R\{u,v\}$ is a resolving neighbourhood of $\mathbb{C}$ with minimum cardinality λ. It shows that ρ is not a resolving function which is a contradiction. Therefore, τ is a minimal resolving function that attains minimum $|\tau|$ for $\mathbb{C}$. Since all the R_t have nonempty intersection, there is another minimal resolving function of $\overline{\tau}$ of $\mathbb{C}$ such that $|\overline{\tau}| \leq |\tau|$. Hence, assigning $(1/\lambda)$ to the vertices of $\mathbb{C}$ in $\cup_{t=1}^{3} R_t$ and calculating the summation of all the weights, we get

$$\dim_{\text{frac}}(\mathbb{C}) = \sum_{t=1}^{\delta} \frac{1}{\lambda} \leq 2 \frac{n^3 + 6n^2 + 11n - 30}{n^3 + 3n^2 + 5n + 3}. \tag{17}$$

In the end, we arrive at the following finding:

$$1 < \dim_{\text{frac}}(\mathbb{C}) \leq 2\left(\frac{n^3 + 6n^2 + 11n - 30}{n^3 + 3n^2 + 5n + 3}\right). \tag{18}$$

□

Theorem 3. *If $\mathbb{C} \cong \mathbb{TD}(n)$ is an n-dimensional tetrahedral diamond lattice with $n \geq 4$ and $n \equiv 0(\text{mod}2)$, then*

$$1 < \dim_{\text{frac}}(\mathbb{C}) \leq 2. \tag{19}$$

Proof

Case 1: when $n = 4$.

The resolving neighbourhood sets are as shown in Tables 4–6.

The resolving neighbourhood sets that are equal due to symmetry are given by the following:

For $109 \leq L \leq 149$ and $\{uv\} \in V(\mathbb{C})$, we have

$$R_L = R\{uv\} = V(\mathbb{C}). \tag{20}$$

Similarly, the pairwise resolving neighbourhood sets that equals $V(\mathbb{C})$ are as follows:

As we can see, Table 7 shows the resolving neighbourhood sets of $\mathbb{C}$ having the maximum cardinality of 30 and $\cup_{L=150}^{240} \acute{R}_L = V(\mathbb{C})$. Table 8, on the other hand, shows the resolving neighbourhood sets with a minimum cardinality of 10 and $\cup_{t=1}^{3} R_t = V(\mathbb{C}) - \{a_2^3, a_3^3, a_4^3, a_6^3, a_7^3, a_8^3\} \cup \{a_4^4, a_{10}^4, a_{11}^4, a_{12}^4\}$. Suppose that, for $150 \leq L \leq 240$, $|R_L| = \gamma$, for $1 \leq t \leq 3$, $|R_t| = \lambda$, $\eta = |\cup_{L=63}^{84} R_L| = |V(\mathbb{TD}(3))| = 30$, and $\delta = |\cup_{t=1}^{6} R_t| = |V(\mathbb{TD}(3))| = 20$. Now, we define a

Table 2: Resolving neighbourhood sets of $\mathbb{TD}(3)$ equal due to symmetry.

Resolving neighbourhood sets	Elements
$R_{36} = R\{a_1^1, a_1^3\}$	$R\{a_2^2, a_2^3\}$
$R_{38} = R\{a_1^1, a_5^3\}$	$R\{a_2^2, a_4^3\}$
$R_{40} = R\{a_3^3, a_7^3\}$	$R\{a_3^2, a_4^2\}$
$R_{42} = R\{a_6^3, a_7^3\}$	$R\{a_1^2, a_3^2\}$
$R_{44} = R\{a_3^3, a_6^3\}$	$R\{a_1^2, a_4^2\}$
$R_{46} = R\{a_1^3, a_9^3\}$	$R\{a_1^2, a_4^2\}$
$R_{37} = R\{a_1^1, a_3^3\}$	$R\{a_2^2, a_4^3\}$
$R_{39} = R\{a_1^1, a_9^3\}$	$R\{a_2^2, a_8^3\}$,
$R_{41} = R\{a_4^3, a_8^3\}$	$R\{a_3^2, a_4^2\}$
$R_{43} = R\{a_7^3, a_9^3\}$	$R\{a_1^2, a_3^2\}$
$R_{45} = R\{a_1^3, a_5^3\}$	$R\{a_1^2, a_3^2\}$

Table 3: Resolving neighbourhood sets of $\mathbb{TD}(3)$ equal to $V(\mathbb{TD}(3))$.

Resolving neighbourhood sets	Elements
$R_{63} = R\{a_1^1, a_2^3\}$	$V(\mathbb{C})$
$R_{65} = R\{a_1^1, a_8^3\}$	$V(\mathbb{C})$
$R_{67} = R\{a_1^2, a_8^3\}$	$V(\mathbb{C})$
$R_{69} = R\{a_2^2, a_3^3\}$	$V(\mathbb{C})$
$R_{71} = R\{a_2^2, a_6^3\}$	$V(\mathbb{C})$
$R_{73} = R\{a_2^2, a_9^3\}$	$V(\mathbb{C})$
$R_{75} = R\{a_3^2, a_4^3\}$	$V(\mathbb{C})$
$R_{77} = R\{a_4^2, a_2^3\}$	$V(\mathbb{C})$
$R_{79} = R\{a_4^2, a_8^3\}$	$V(\mathbb{C})$
$R_{81} = R\{a_1^3, a_8^3\}$	$V(\mathbb{C})$
$R_{83} = R\{a_2^3, a_6^3\}$	$V(\mathbb{C})$
$R_{64} = R\{a_1^1, a_4^3\}$	$V(\mathbb{C})$
$R_{66} = R\{a_1^2, a_4^3\}$	$V(\mathbb{C})$
$R_{68} = R\{a_2^2, a_1^3\}$	$V(\mathbb{C})$
$R_{70} = R\{a_2^2, a_5^3\}$	$V(\mathbb{C})$
$R_{72} = R\{a_2^2, a_7^3\}$	$V(\mathbb{C})$
$R_{74} = R\{a_3^2, a_2^3\}$	$V(\mathbb{C})$
$R_{76} = R\{a_3^2, a_8^3\}$	$V(\mathbb{C})$
$R_{78} = R\{a_4^2, a_4^3\}$	$V(\mathbb{C})$
$R_{80} = R\{a_1^3, a_4^3\}$	$V(\mathbb{C})$
$R_{82} = R\{a_2^3, a_5^3\}$	$V(\mathbb{C})$
$R_{84} = R\{a_2^3, a_9^3\}$	$V(\mathbb{C})$

Table 4: Resolving neighbourhood sets of $\mathbb{TD}(3)$ with minimum cardinality.

Resolving neighbourhood sets	Elements
$R_1 = R\{a_1^1, a_2^1\}$	$V(\mathbb{C}) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\}$
$R_2 = R\{a_1^1, a_2^3\}$	$V(\mathbb{C}) - \{a_1^2, a_2^2, a_4^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\}$
$R_3 = R\{a_1^1, a_4^2\}$	$V(\mathbb{C}) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\}$
$R_4 = R\{a_2^1, a_3^1\}$	$V(\mathbb{C}) - \{a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3, a_9^3\}$
$R_5 = R\{a_2^2, a_5^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_6^3, a_7^3, a_8^3, a_9^3\}$
$R_6 = R\{a_4^2, a_9^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3\}$

Table 5: Resolving neighbourhood sets of $\mathbb{TD}(4)$ that are not equal to each other.

Resolving neighbourhood sets	Elements
$R_1 = R\{a_2^1, a_3^1\}$	$V(\mathbb{C}) - \{a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_4^4, a_5^4, a_6^4, a_7^4, a_8^4, a_{10}^4, a_{11}^4, a_{12}^4, a_{13}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_2 = R\{a_2^2, a_5^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_9^4, a_{10}^4, a_{11}^4, a_{12}^4, a_{13}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_3 = R\{a_4^2, a_9^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4, a_8^4, a_9^4, a_{10}^4, a_{11}^4, a_{12}^4, a_{13}^4\}$
$R_1 = R\{a_1^1, a_2^1\}$	$V(\mathbb{C}) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\} \cup \{a_6^4, a_7^4, a_8^4, a_{12}^4, a_{13}^4, a_{15}^4, a_{16}^4\}$
$R_2 = R\{a_1^1, a_2^3\}$	$V(\mathbb{C}) - \{a_1^2, a_2^2, a_4^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\} \cup \{a_4^1, a_4^2, a_4^9, a_4^{10}, a_4^{14}, a_4^{15}, a_4^{16}\}$
$R_3 = R\{a_1^1, a_4^2\}$	$V(\mathbb{C}) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_4 = R\{a_1^2, a_3^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_4^2\} \cup \{a_3^3, a_8^3, a_9^3\} \cup \{a_3^4, a_4^4, a_5^4, a_{11}^4, a_{15}^4, a_{16}^4\}$
$R_5 = R\{a_1^2, a_4^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_3^2\} \cup \{a_4^3, a_5^3, a_7^3\} \cup \{a_6^4, a_7^4, a_9^4, a_{10}^4, a_{11}^4, a_{14}^4\}$
$R_6 = R\{a_1^2, a_3^3\}$	$V(\mathbb{C}) - \{a_3^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\} \cup \{a_1^4, a_2^4, a_9^4, a_{10}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_{12} = R\{a_1^2, a_5^3\}$	$V(\mathbb{C}) - \{a_3^2\} \cup \{a_3^3, a_8^3, a_9^3\} \cup \{a_3^4, a_4^4, a_5^4, a_{11}^4, a_{15}^4, a_{16}^4\}$
$R_{13} = R\{a_1^2, a_6^3\}$	$V(\mathbb{C}) - \{a_3^2, a_4^2\} \cup \{a_3^3, a_7^3\} \cup \{a_3^4, a_4^4, a_5^4, a_9^4, a_{10}^4, a_{14}^4\}$
$R_{14} = R\{a_1^2, a_7^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{15} = R\{a_1^2, a_9^3\}$	$V(\mathbb{C}) - \{a_4^2\} \cup \{a_4^3, a_5^3, a_7^3\} \cup \{a_6^4, a_7^4, a_9^4, a_{10}^4, a_{11}^4, a_{14}^4\}$
$R_{16} = R\{a_2^2, a_2^3\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\} \cup \{a_6^4, a_7^4, a_8^4, a_{12}^4, a_{13}^4, a_{15}^4, a_{16}^4\}$
$R_{17} = R\{a_1^3, a_1^4\}$	$V(\mathbb{C}) - \{a_2^4, a_3^4, \ldots, a_{16}^4\}$
$R_{18} = R\{a_1^3, a_3^4\}$	$V(\mathbb{C}) - \{a_3^3, a_4^3, a_5^3, a_6^3\} \cup \{a_1^4, a_2^4, \ldots, a_7^4\}$
$R_{19} = R\{a_1^3, a_5^4\}$	$V(\mathbb{C}) - \{a_3^3, a_4^3, a_5^3, a_6^3\} \cup \{a_{10}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_{20} = R\{a_1^3, a_7^4\}$	$V(\mathbb{C}) - \{a_4^3, a_6^3\} \cup \{a_4^4, a_{11}^4, a_{15}^4, a_{16}^4\}$
$R_{21} = R\{a_1^3, a_8^4\}$	$V(\mathbb{C}) - \{a_4^3, a_8^3\} \cup \{a_4^4, a_{10}^4, a_{14}^4\}$
$R_{22} = R\{a_1^3, a_9^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_1^4, a_2^4, \ldots, a_7^4\}$
$R_{23} = R\{a_1^3, a_{11}^4\}$	$V(\mathbb{C}) - \{a_3^3, a_4^3, a_5^3, a_7^3, a_8^3, a_9^3\} \cup \{a_3^4, a_9^4\}$
$R_{24} = R\{a_1^3, a_{14}^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{25} = R\{a_1^3, a_{16}^4\}$	$V(\mathbb{C}) - \{a_6^3, a_8^3\} \cup \{a_6^4, a_7^4, a_{10}^4, a_{11}^4\}$
$R_{26} = R\{a_2^3, a_2^4\}$	$V(\mathbb{C}) - \{a_1^3\} \cup \{a_4^4, a_5^4, \ldots, a_8^4, a_{10}^4 \ldots, a_{10}^4\}$
$R_{27} = R\{a_2^3, a_4^4\}$	$V(\mathbb{C}) - \{a_3^3, a_4^3, a_5^3, a_6^3,\} \cup \{a_1^4, a_2^4, a_9^4, a_{10}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_{28} = R\{a_2^3, a_6^4\}$	$V(\mathbb{C}) - \{a_4^3, a_6^3\} \cup \{a_4^4, a_{11}^4, a_{15}^4, a_{16}^4\}$
$R_{29} = R\{a_2^3, a_{10}^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3, a_9^3,\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{30} = R\{a_2^3, a_{12}^4\}$	$V(\mathbb{C}) - \{a_4^3, a_5^3, a_8^3, a_9^3\} \cup \{a_4^4, a_{10}^4, a_{14}^4\}$
$R_{31} = R\{a_2^3, a_{15}^4\}$	$V(\mathbb{C}) - \{a_6^3, a_8^3\} \cup \{a_5^4, a_6^4, a_{10}^4, a_{11}^4\}$
$R_{32} = R\{a_3^3, a_1^4\}$	$V(\mathbb{C}) - \{a_1^3\} \cup \{a_3^4, a_{15}^4, a_{16}^4\}$
$R_{33} = R\{a_3^3, a_5^4\}$	$V(\mathbb{C}) - \{a_5^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, \ldots, a_9^4, \ldots, a_{16}^4\}$
$R_{34} = R\{a_3^3, a_7^4\}$	$V(\mathbb{C}) - \{a_5^3\} \cup \{a_5^4, a_{15}^4, a_{16}^4\}$
$R_{35} = R\{a_3^3, a_8^4\}$	$V(\mathbb{C}) - \{a_4^3, a_5^3, a_6^3, a_8^3, a_9^3\} \cup \{a_4^4, a_5^4, a_{10}^4, a_{11}^4, a_{14}^3\}$
$R_{36} = R\{a_3^3, a_9^4\}$	$V(\mathbb{C}) - \{a_1^3, a_7^3, a_8^3, a_9^3\} \cup \{a_8^4, a_{11}^4, a_{12}^4, a_{13}^4\}$
$R_{37} = R\{a_3^3, a_{11}^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_1^4, a_2^4, \ldots, a_7^4\}$
$R_{38} = R\{a_3^3, a_{13}^4\}$	$V(\mathbb{C}) - \{a_6^3, a_8^3\} \cup \{a_6^4, a_7^4, a_{10}^4, a_{11}^4\}$
$R_{39} = R\{a_3^3, a_{14}^4\}$	$V(\mathbb{C}) - \{a_8^3\} \cup \{a_8^4, a_{11}^4, a_{12}^4, a_{13}^4\}$
$R_{40} = R\{a_3^3, a_{16}^4\}$	$V(\mathbb{C}) - \{a_8^3\} \cup \{a_8^4, a_{12}^4\}$
$R_{41} = R\{a_4^3, a_2^4\}$	$V(\mathbb{C}) - \{a_2^3, a_7^3\} \cup \{a_4^4, a_5^4, a_{15}^4, a_{16}^4\}$
$R_{42} = R\{a_4^3, a_4^4\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_7^3\} \cup \{a_6^4, a_7^4, a_8^4, a_{12}^4, a_{13}^4, a_{15}^4, a_{16}^4\}$
$R_{43} = R\{a_4^3, a_6^4\}$	$V(\mathbb{C}) - \{a_5^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, \ldots, a_9^4, \ldots, a_{16}^4\}$
$R_{44} = R\{a_4^3, a_{10}^4\}$	$V(\mathbb{C}) - \{a_8^3, a_9^3\} \cup \{a_4^4, a_5^4\}$
$R_{45} = R\{a_4^3, a_{15}^4\}$	$V(\mathbb{C}) - \{a_7^3, a_8^3\} \cup \{a_{11}^4, a_{12}^4\}$
$R_{46} = R\{a_5^3, a_1^4\}$	$V(\mathbb{C}) - \{a_2^3, a_7^3\} \cup \{a_4^4, a_{11}^4, a_{15}^4, a_{16}^4\}$
$R_{47} = R\{a_5^3, a_3^4\}$	$V(\mathbb{C}) - \{a_2^3, a_7^3\} \cup \{a_5^4, a_{12}^4, a_{13}^4\}$
$R_{48} = R\{a_5^3, a_5^4\}$	$V(\mathbb{C}) - \{a_2^2\} \cup \{a_1^3, a_2^3, a_3^3, a_7^3\} \cup \{a_1^4, a_2^4, \ldots, a_5^4, a_9^4, a_{10}^4, a_{11}^4, a_{14}^4\}$
$R_{49} = R\{a_5^3, a_7^4\}$	$V(\mathbb{C}) - \{a_1^4, a_2^4, \ldots, a_6^4, a_8^4 \ldots, a_{16}^4\}$
$R_{50} = R\{a_5^3, a_8^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{51} = R\{a_5^3, a_9^4\}$	$V(\mathbb{C}) - \{a_8^3, a_9^3\} \cup \{a_4^4, a_{12}^4, a_{13}^4\}$
$R_{52} = R\{a_5^3, a_{11}^4\}$	$V(\mathbb{C}) - \{a_8^3, a_9^3\} \cup \{a_4^4, a_{12}^4, a_{13}^4\}$
$R_{53} = R\{a_5^3, a_{13}^4\}$	$V(\mathbb{C}) - \{a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_8^4\}$
$R_{54} = R\{a_5^3, a_{14}^4\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_8^3\} \cup \{a_3^4, a_4^4, a_8^4\}$
$R_{55} = R\{a_5^3, a_{16}^4\}$	$V(\mathbb{C}) - \{a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_{11}^4, a_{12}^4\}$
$R_{56} = R\{a_6^3, a_1^4\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_2^3\} \cup \{a_4^4, a_5^4, a_{10}^4, a_{15}^4\}$
$R_{57} = R\{a_9^3, a_{16}^4\}$	$V(\mathbb{C}) - \{a_1^4, a_2^4 \ldots, a_{15}^4\}$
$R_{58} = R\{a_3^2, a_4^2\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_8^4\}$
$R_{59} = R\{a_5^3, a_6^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_8^4\}$
$R_{60} = R\{a_2^4, a_{12}^4\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2\} \cup \{a_3^3, a_7^3\} \cup \{a_4^4, a_{10}^4\}$
$R_{61} = R\{a_2^3, a_4^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_4^2\} \cup \{a_3^3, a_8^3, a^3, a_9^3\} \cup \{a_3^4, a_4^4, a_5^4, a_{11}^4, a_{15}^4, a_{16}^4\}$
$R_{62} = R\{a_2^3, a_8^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2, a_3^2\} \cup \{a_4^3, a_5^3, a^3, a_7^3\} \cup \{a_6^4, a_7^4, a_9^4, a_{10}^4, a_{11}^4, a_{14}^4\}$

Table 5: Continued.

Resolving neighbourhood sets	Elements
$R_{63} = R\{a_5^3, a_7^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_3^3, a_6^3\} \cup \{a_3^4, a_4^4, a_{15}^4, a_{16}^4\}$
$R_{64} = R\{a_3^3, a_9^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_2^2\} \cup \{a_6^3, a_7^3\} \cup \{a_6^4, a_7^4, a_9^4, a_{10}^4\}$
$R_{65} = R\{a_1^3, a_3^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_4^2\} \cup \{a_2^3, a_7^3, a_8^3\} \cup \{a_3^4, a_{10}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_{66} = R\{a_1^3, a_7^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_3^2\} \cup \{a_2^3, a_3^3, a_4^3, a_5^3\} \cup \{a_4^4, a_5^4, a_6^4, a_7^4, a_9^4\}$
$R_{67} = R\{a_1^4, a_3^4\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_4^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3, a_9^3\} \cup \{a_2^4, a_9^4, a_{10}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_{68} = R\{a_1^4, a_7^4\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_3^2\} \cup \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3\} \cup \{a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{69} = R\{a_6^3, a_9^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2, a_4^2\} \cup \{a_1^3, a_2^3, a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_9^4, a_{10}^4, a_{13}^4\}$
$R_{70} = R\{a_5^3, a_9^3\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2\} \cup \{a_1^3, a_2^3, a_6^3\} \cup \{a_1^4, a_2^4, a_{11}^4, a_{12}^4\}$
$R_{71} = R\{a_5^4, a_{14}^4\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2\} \cup \{a_1^3, a_2^3, a_6^3\} \cup \{a_1^4, a_2^4, a_{11}^4, a_{12}^4\}$
$R_{72} = R\{a_7^4, a_{16}^4\}$	$V(\mathbb{C}) - \{a_1^1\} \cup \{a_1^2, a_2^2\} \cup \{a_1^3, a_2^3, a_6^3\} \cup \{a_1^4, a_2^4, a_{11}^4, a_{12}^4\}$
$R_{73} = R\{a_1^1, a_1^4\}$	$V(\mathbb{C}) - \{a_2^3, a_3^3, a_7^3\} \cup \{a_6^4, a_7^4, a_8^4, a_{12}^4, a_{13}^4, a_{15}^4, a_{16}^4\}$
$R_{74} = R\{a_1^1, a_7^4\}$	$V(\mathbb{C}) - \{a_3^3, a_4^3, a_6^3\} \cup \{a_1^4, a_2^4, a_9^4, a_{10}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_{75} = R\{a_1^1, a_9^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{76} = R\{a_2^3, a_9^4\}$	$V(\mathbb{C}) - \{a_6^3, a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4\}$
$R_{77} = R\{a_9^3, a_{14}^4\}$	$V(\mathbb{C}) - \{a_1^2\} \cup \{a_1^3, a_2^3, a_3^3, a_7^3\} \cup \{a_1^4, a_2^4, \ldots, a_5^4, a_9^4, a_{10}^4, a_{11}^4, a_{14}^4\}$
$R_{78} = R\{a_2^3, a_2^4\}$	$V(\mathbb{C}) - \{a_1^3\} \cup \{a_4^4, a_5^4, \ldots, a_8^4, a_{10}^4, \ldots, a_{16}^4\}$

mapping $\kappa: V(\mathbb{TD}(3)) \longrightarrow [0,1]$ such that $\kappa(a) = (1/30)$ for all $a \in \cup_{L=63}^{84} R_L$, and assigning the value of $(1/30)$ to all the elements of $\cup_{L=63}^{84} R_L$ and summing up all the labels, we get $|\kappa| = \sum_{a \in \cup_{L=63}^{84} R_L} \kappa(a) = (30/30) = 1$; thus, $\dim_{\text{frac}}(\mathbb{C}) = |\kappa| = 1$, as all the R_L for $150 \le L \le 240$ are all having $V(\mathbb{TD}(3))$ in common. Similarly, we define another mapping $\tau: V(\mathbb{TD}(3)) \longrightarrow [0,1]$ such that $\tau = (1/\lambda) = (1/10)$ for all $a \in \cup_{t=1}^{3} R_t$, and giving labels to the elements of $\cup_{t=1}^{3} R_t$ and latter on summing them up, we get $|\tau| = \sum_{a \in \cup_{t=1}^{3} R_t} \tau(a) = (20/10) \le 2$; hence, $\dim_{\text{frac}}(\mathbb{C}) < 2$ as all the R_t for $1 \le t \le 6$ are pairwise overlapping. Therefore, we arrive at the following conclusion:

$$1 < \dim_{\text{frac}}(\mathbb{TD}(3)) \le 2. \tag{21}$$

Case 2: when $n \ge 6$.

The required minimum resolving neighbourhood sets are $R\left\{a_1^\alpha, a_1^\beta\right\}, R\left\{a_\gamma^\alpha, a_\eta^\beta\right\}$, and $R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right\}$, where $\alpha = (n/2)$, $\beta + 1$, $\gamma = 2\alpha - 1$, and $\eta = 2\beta - 1$. As it is evident from Lemma 1, $|R_t| = |R\left\{a_1^\alpha, a_1^\beta\right\}| = |R\left\{a_\gamma^\alpha, a_\eta^\beta\right\}| = |R\left\{a_{\alpha^2}^\alpha, a_{\beta^2}^\beta\right.| = (n(n^2 + 3n + 2)/12) \le \|\text{acute}R\{a,b\}|$ for all $a, b \in V(\mathbb{C})$ and $\cup_{t=1}^{3} R_t = \cup_{j=1}^{\alpha} S_{j^2}^j \cup \cup_{j=\alpha+1}^{n} (S_{(j-\alpha)^2}^{j,t} \cup S_{(j-\alpha)^2}^{j,br} \cup S_{(j-\alpha)^2}^{j,bl})$. Moreover, $|R\{u,v\} \cap \cup_{t=1}^{3} R_t| \ge |R_t|$ for all $u, v \in V(\mathbb{C})$. Also, by Lemma 3 (d), the resolving neighbourhood sets with maximum cardinality of $|V(\mathbb{C})| = (n(2n^2 + 3n + 1)/6)$ are $R_1 = R\left\{a_\alpha^\alpha, a_\beta^\beta\right\}$ and $R_2 = R\left\{a_\alpha^\alpha, a_{\alpha+\gamma}^{\alpha+\eta}\right\}$ with $\cup_{L=1}^{2} R_L = V(\mathbb{C})$. Let $|R_t| = \lambda$, $|R_L| = \gamma$, $|\cup_{L=1}^{2} R_L| = \eta$, and $|\cup_{t=1}^{3} R_t| = \delta$; to find the minimum value of $\dim_{\text{frac}}(\mathbb{C})$, we define a mapping $\kappa: V(\mathbb{C}) \longrightarrow [0,1]$ such that

$$k(a) = \begin{cases} \dfrac{1}{k}, & \text{for } a \in \bigcup_{L=1}^{3} R_t, \\ 0, & \text{for } a \in V(\mathbb{C}) - \bigcup_{t=1}^{3} R_L. \end{cases} \tag{22}$$

Table 6: Resolving neighbourhood sets of $\mathbb{TD}(4)$ that are equal due to symmetry.

Resolving neighbourhood sets	Equality	Resolving neighbourhood sets	Equality
$R_{79} = R\{a_1^3, a_{13}^4\}$	$R\{a_1^3, a_8^4\}$	$R_{80} = R\{a_3^3, a_6^3\}$	$R\{a_1^2, a_4^2\}$
$R_{81} = R\{a_5^4, a_8^4\}$	$R\{a_1^2, a_4^2\}$	$R_{82} = R\{a_7^5, a_{10}^5\}$	$R\{a_1^2, a_4^2\}$
$R_{83} = R\{a_6^3, a_7^3\}$	$R\{a_1^2, a_3^2\}$	$R_{84} = R\{a_{13}^4, a_{14}^4\}$	$R\{a_1^2, a_3^2\}$
$R_{85} = R\{a_2^2, a_4^3\}$	$R\{a_1^2, a_3^3\}$	$R_{86} = R\{a_2^2, a_8^3\}$	$R\{a_1^2, a_7^3\}$
$R_{87} = R\{a_3^2, a_3^3\}$	$R\{a_2^2, a_2^3\}$	$R_{88} = R\{a_4^2, a_7^3\}$	$R\{a_2^2, a_2^3\}$
$R_{89} = R\{a_3^2, a_6^3\}$	$R\{a_1^2, a_7^3\}$	$R_{90} = R\{a_4^2, a_6^3\}$	$R\{a_1^2, a_3^3\}$
$R_{91} = R\{a_5^4, a_{11}^4\}$	$R\{a_5^3, a_6^3\}$	$R_{92} = R\{a_5^4, a_{11}^4\}$	$R\{a_5^3, a_6^3\}$
$R_{93} = R\{a_7^4, a_{13}^4\}$	$R\{a_5^3, a_6^3\}$	$R_{94} = R\{a_{11}^4, a_{14}^4\}$	$R\{a_6^3, a_9^3\}$
$R_{95} = R\{a_{12}^4, a_{15}^4\}$	$R\{a_6^3, a_9^3\}$	$R_{96} = R\{a_8^4, a_{16}^4\}$	$R\{a_6^3, a_9^3\}$
$R_{97} = R\{a_5^4, a_{14}^4\}$	$R\{a_5^3, a_9^3\}$	$R_{98} = R\{a_5^4, a_{14}^4\}$	$R\{a_5^3, a_9^3\}$
$R_{99} = R\{a_7^4, a_{13}^4\}$	$R\{a_5^3, a_6^3\}$	$R_{100} = R\{a_8^4, a_{16}^4\}$	$R\{a_6^3, a_9^3\}$
$R_{101} = R\{a_2^4, a_{10}^4\}$	$R\{a_1^3, a_7^3\}$	$R_{102} = R\{a_2^4, a_4^4\}$	$R\{a_1^3, a_3^3\}$
$R_{103} = R\{a_3^3, a_4^3\}$	$R\{a_2^3, a_2^4\}$	$R_{104} = R\{a_3^4, a_{11}^4\}$	$R\{a_1^3, a_7^3\}$
$R_{105} = R\{a_9^4, a_{11}^4\}$	$R\{a_1^3, a_3^3\}$	$R_{106} = R\{a_6^3, a_{11}^4\}$	$R\{a_9^3, a_{14}^4\}$
$R_{107} = R\{a_5^3, a_5^4\}$	$R\{a_9^3, a_{14}^4\}$	$R_{108} = R\{a_3^3, a_3^4\}$	$R\{a_2^3, a_2^4\}$

where $\gamma = \eta = (n(2n^2 + 3n + 1)/6)$. Assigning the labels to the elements of $\bigcup \cup_{L=1}^{2} R_L$ and summing them up, we get $|\kappa| = \sum_{a \in \cup_{L=1}^{2} R_L} (1/\gamma) = 1$.

Similarly, for the maximum value of $\dim_{\text{frac}}(\mathbb{C})$, we define another mapping $\tau: V(\mathbb{C}) \longrightarrow [0,1]$ such that

$$\tau(a) = \begin{cases} \dfrac{1}{\lambda}, & \text{for } a \in \bigcup_{t=1}^{3} R_t, \\ 0, & \text{for } a \in V(\mathbb{C}) - \bigcup_{t=1}^{3} R_t. \end{cases} \tag{23}$$

It can be seen that τ is a resolving function for $\mathbb{C}$ with $n \ge 3$ because $\tau(R\{u,v\}) \ge 1 \forall u, v \in V(\mathbb{N})$. On the contrary, we assume that there is another resolving function ρ such that $\rho(u) \le \tau(u)$, for at least one $u \in V(\mathbb{C}) \rho(u) \ne \tau(u)$. As a consequence, $\rho(R\{u,v\}) < 1$, where $R\{u,v\}$ is a resolving neighbourhood of $\mathbb{C}$ with minimum cardinality κ. It shows

Table 7: Resolving neighbourhood sets of $\mathbb{TD}(4)$ that are equal to $V(\mathbb{TD}(4))$.

Resolving neighbourhood sets	Elements	Resolving neighbourhood sets	Elements
$R_{150} = R\{a_1^1, a_2^3\}$	$V(\mathbb{C})$	$R_{151} = R\{a_1^1, a_4^3\}$	$V(\mathbb{C})$
$R_{152} = R\{a_1^1, a_8^3\}$	$V(\mathbb{C})$	$R_{153} = R\{a_1^1, a_2^4\}$	$V(\mathbb{C})$
$R_{154} = R\{a_1^1, a_4^4\}$	$V(\mathbb{C})$	$R_{155} = R\{a_1^1, a_6^4\}$	$V(\mathbb{C})$
$R_{156} = R\{a_1^1, a_{10}^4\}$	$V(\mathbb{C})$	$R_{157} = R\{a_1^1, a_{12}^4\}$	$V(\mathbb{C})$
$R_{158} = R\{a_1^1, a_{15}^4\}$	$V(\mathbb{C})$	$R_{159} = R\{a_1^2, a_2^3\}$	$V(\mathbb{C})$
$R_{160} = R\{a_1^2, a_4^3\}$	$V(\mathbb{C})$	$R_{161} = R\{a_1^2, a_8^3\}$	$V(\mathbb{C})$
$R_{162} = R\{a_2^2, a_1^3\}$	$V(\mathbb{C})$	$R_{163} = R\{a_2^2, a_3^3\}$	$V(\mathbb{C})$
$R_{164} = R\{a_2^2, a_5^3\}$	$V(\mathbb{C})$	$R_{165} = R\{a_2^2, a_6^3\}$	$V(\mathbb{C})$
$R_{166} = R\{a_2^2, a_7^3\}$	$V(\mathbb{C})$	$R_{167} = R\{a_2^2, a_9^3\}$	$V(\mathbb{C})$
$R_{168} = R\{a_3^2, a_2^3\}$	$V(\mathbb{C})$	$R_{169} = R\{a_3^2, a_4^3\}$	$V(\mathbb{C})$
$R_{170} = R\{a_3^2, a_8^3\}$	$V(\mathbb{C})$	$R_{171} = R\{a_4^2, a_2^3\}$	$V(\mathbb{C})$
$R_{172} = R\{a_4^2, a_4^3\}$	$V(\mathbb{C})$	$R_{173} = R\{a_4^2, a_8^3\}$	$V(\mathbb{C})$
$R_{174} = R\{a_1^3, a_4^3\}$	$V(\mathbb{C})$	$R_{175} = R\{a_1^3, a_8^3\}$	$V(\mathbb{C})$
$R_{176} = R\{a_2^3, a_5^3\}$	$V(\mathbb{C})$	$R_{177} = R\{a_2^3, a_6^3\}$	$V(\mathbb{C})$
$R_{178} = R\{a_2^3, a_9^3\}$	$V(\mathbb{C})$	$R_{179} = R\{a_1^4, a_4^4\}$	$V(\mathbb{C})$
$R_{180} = R\{a_1^4, a_{10}^4\}$	$V(\mathbb{C})$	$R_{181} = R\{a_1^4, a_6^4\}$	$V(\mathbb{C})$
$R_{182} = R\{a_1^4, a_{12}^4\}$	$V(\mathbb{C})$	$R_{183} = R\{a_1^4, a_{15}^4\}$	$V(\mathbb{C})$
$R_{184} = R\{a_2^4, a_5^4\}$	$V(\mathbb{C})$	$R_{185} = R\{a_2^4, a_{11}^4\}$	$V(\mathbb{C})$
$R_{186} = R\{a_2^4, a_{14}^4\}$	$V(\mathbb{C})$	$R_{187} = R\{a_2^4, a_7^4\}$	$V(\mathbb{C})$
$R_{188} = R\{a_2^4, a_8^4\}$	$V(\mathbb{C})$	$R_{189} = R\{a_2^4, a_{13}^4\}$	$V(\mathbb{C})$
$R_{190} = R\{a_2^4, a_{16}^4\}$	$V(\mathbb{C})$	$R_{191} = R\{a_3^4, a_6^4\}$	$V(\mathbb{C})$
$R_{192} = R\{a_3^4, a_{12}^4\}$	$V(\mathbb{C})$	$R_{193} = R\{a_3^4, a_{15}^4\}$	$V(\mathbb{C})$
$R_{194} = R\{a_4^4, a_7^4\}$	$V(\mathbb{C})$	$R_{195} = R\{a_4^4, a_8^4\}$	$V(\mathbb{C})$
$R_{196} = R\{a_4^4, a_{13}^4\}$	$V(\mathbb{C})$	$R_{197} = R\{a_4^4, a_{16}^4\}$	$V(\mathbb{C})$
$R_{198} = R\{a_1^1, a_2^4\}$	$V(\mathbb{C})$	$R_{199} = R\{a_1^1, a_4^4\}$	$V(\mathbb{C})$
$R_{200} = R\{a_1^1, a_6^4\}$	$V(\mathbb{C})$	$R_{201} = R\{a_1^1, a_{12}^4\}$	$V(\mathbb{C})$
$R_{202} = R\{a_1^1, a_{15}^4\}$	$V(\mathbb{C})$	$R_{203} = R\{a_1^2, a_2^4\}$	$V(\mathbb{C})$
$R_{204} = R\{a_1^2, a_4^4\}$	$V(\mathbb{C})$	$R_{205} = R\{a_1^2, a_6^4\}$	$V(\mathbb{C})$
$R_{206} = R\{a_1^2, a_{12}^4\}$	$V(\mathbb{C})$	$R_{207} = R\{a_1^2, a_4^4\}$	$V(\mathbb{C})$
$R_{208} = R\{a_1^2, a_6^4\}$	$V(\mathbb{C})$	$R_{209} = R\{a_1^2, a_{12}^4\}$	$V(\mathbb{C})$
$R_{210} = R\{a_2^2, a_1^4\}$	$V(\mathbb{C})$	$R_{211} = R\{a_2^2, a_3^4\}$	$V(\mathbb{C})$
$R_{212} = R\{a_2^2, a_5^4\}$	$V(\mathbb{C})$	$R_{213} = R\{a_2^2, a_7^4\}$	$V(\mathbb{C})$
$R_{214} = R\{a_2^2, a_8^4\}$	$V(\mathbb{C})$	$R_{215} = R\{a_2^2, a_{11}^4\}$	$V(\mathbb{C})$
$R_{216} = R\{a_2^2, a_{13}^4\}$	$V(\mathbb{C})$	$R_{217} = R\{a_2^2, a_{16}^4\}$	$V(\mathbb{C})$
$R_{218} = R\{a_3^2, a_2^4\}$	$V(\mathbb{C})$	$R_{219} = R\{a_3^2, a_4^4\}$	$V(\mathbb{C})$
$R_{220} = R\{a_3^2, a_6^4\}$	$V(\mathbb{C})$	$R_{221} = R\{a_3^2, a_{10}^4\}$	$V(\mathbb{C})$
$R_{222} = R\{a_3^2, a_{12}^4\}$	$V(\mathbb{C})$	$R_{223} = R\{a_3^2, a_{15}^4\}$	$V(\mathbb{C})$
$R_{224} = R\{a_4^2, a_2^4\}$	$V(\mathbb{C})$	$R_{225} = R\{a_4^2, a_4^4\}$	$V(\mathbb{C})$
$R_{226} = R\{a_4^2, a_6^4\}$	$V(\mathbb{C})$	$R_{227} = R\{a_4^2, a_{10}^4\}$	$V(\mathbb{C})$
$R_{228} = R\{a_4^2, a_{12}^4\}$	$V(\mathbb{C})$	$R_{229} = R\{a_4^2, a_{15}^4\}$	$V(\mathbb{C})$
$R_{230} = R\{a_1^3, a_2^4\}$	$V(\mathbb{C})$	$R_{231} = R\{a_1^3, a_4^4\}$	$V(\mathbb{C})$
$R_{232} = R\{a_1^3, a_6^4\}$	$V(\mathbb{C})$	$R_{233} = R\{a_1^3, a_{12}^4\}$	$V(\mathbb{C})$
$R_{234} = R\{a_1^3, a_{15}^4\}$	$V(\mathbb{C})$	$R_{235} = R\{a_2^3, a_1^4\}$	$V(\mathbb{C})$
$R_{236} = R\{a_2^3, a_3^4\}$	$V(\mathbb{C})$	$R_{237} = R\{a_2^3, a_5^4\}$	$V(\mathbb{C})$
$R_{238} = R\{a_2^3, a_7^4\}$	$V(\mathbb{C})$	$R_{239} = R\{a_2^3, a_8^4\}$	$V(\mathbb{C})$
$R_{240} = R\{a_2^3, a_{11}^4\}$	$V(\mathbb{C})$		

Table 8: Resolving neighbourhood sets of $\mathbb{TD}(4)$ that are not equal to each other.

Resolving neighbourhood sets	Elements
$R_1 = R\{a_2^1, a_3^1\}$	$V(\mathbb{C}) - \{a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_4^4, a_5^4, a_6^4, a_7^4, a_8^4, a_{10}^4, a_{11}^4, a_{12}^4, a_{13}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_2 = R\{a_3^2, a_5^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_6^3, a_7^3, a_8^3, a_9^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_9^4, a_{10}^4, a_{11}^4, a_{12}^4, a_{13}^4, a_{14}^4, a_{15}^4, a_{16}^4\}$
$R_3 = R\{a_4^2, a_9^3\}$	$V(\mathbb{C}) - \{a_1^3, a_2^3, a_3^3, a_4^3, a_5^3, a_6^3, a_7^3, a_8^3\} \cup \{a_1^4, a_2^4, a_3^4, a_4^4, a_5^4, a_6^4, a_7^4, a_8^4, a_9^4, a_{10}^4, a_{11}^4, a_{12}^4, a_{13}^4\}$

that ρ is not a resolving function which is a contradiction. Therefore, τ is a minimal resolving function that attains minimum $|\tau|$ for $\mathbb{C}$. Since $1 \leq t \leq 3$ R_t has a nonempty intersection, thus there exists another minimal resolving function $\overline{\tau}$ of $\mathbb{C}$ such that $|\overline{\tau}| \leq |\tau|$. Thus, assigning $(1/\lambda)$ to the vertices of $\mathbb{C}$ in $\cup_{t=1}^{3} R_t$ and calculating the summation of all the weights, we get $\dim_{\text{frac}}(\mathbb{C}) = \sum_{t=1}^{\delta} (1/\lambda) \leq (12n(n^2 + 3n + 2)/6n(n^2 + 3n + 2)) = 2$. Therefore, we arrive at the following result:

$$1 < \dim_{\text{frac}}(\mathbb{C}) \leq 2. \tag{24}$$

□

7. Conclusions

We conclude our discussion by the following remarks:

(i) In this article, we have made a characterization of graphs having the FMD as unity

(ii) It is computed that the FMD of the path is 1 that strengthens the result proved in [13]

(iii) We have calculated the extremal values of FMD of $\mathbb{TD}(n)$ as (i) for $n \equiv 0(\text{mod}2)$, $1 < \dim_{\text{frac}}(\mathbb{C}) \leq 2$ and (ii) for $n \equiv 1(\text{mod}2)$, $1 < \dim_{\text{frac}}(\mathbb{C}) \leq 2(n^3 + 6n^2 + 11n - 30/n^3 + 3n^2 + 5n + 3)$

(iv) Now, we close our discussion with the open problem that investigates the families of graphs other than P_n having FMD as unity

Data Availability

All the data are included within this article. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors have no conflicts of interest.

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