Research Article

Forgotten Index of Generalized Operations on Graphs

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In theoretical chemistry, several distance-based, degree-based, and counting polynomial-related topological indices (TIs) are used to investigate the different chemical and structural properties of the molecular graphs. Furtula and Gutman redefined the F -index as the sum of cubes of degrees of the vertices of the molecular graphs to study the different properties of their structure-dependency. In this paper, we compute F -index of generalized sum graphs in terms of various TIs of their factor graphs, where generalized sum graphs are obtained by using four generalized subdivision-related operations and the strong product of graphs. We have analyzed our results through the numerical tables and the graphical presentations for the particular generalized sum graphs constructed with the help of path (alkane) graphs.

1. Introduction

Throughout the paper, we consider a simple and undirected graph $H = (V(H), E(H))$ with vertex-set $V(H) = \{v_1, v_2, v_3, \dots, v_n\}$ and edge-set $E = \{e_1, e_2, \dots, e_m\} \subseteq V(H) \times V(H)$, where $n = |V(H)|$ and $m = |E(H)|$. A molecular graph is a connected and undirected graph in which atoms are presented by vertices, and chemical bonds between these atoms are shown by edges (see Figure 1). For a finite set of graphs $\mathcal{G}$ and set of real numbers $\mathbb{R}$, the function $I: \mathcal{G} \rightarrow \mathbb{R}$ defined by $I(H) = \sum_{u \in V(H)} f(u)$ is called a degree-based topological index (TI), where the graph H belongs to $\mathcal{G}$ and $f(u)$ is a degree-function from the vertex-set of H to the degree-set of its vertices. It is important to know that $I(H) = I(K)$ if and only if H is isomorphic to K . For more details, see [1, 2].

Graph-theoretic modeling of the molecular graphs plays a fundamental part in the analysis of the quantitative structures activity/property relationships (QSAR/QSPR). In chemistry, the study of structural relationships is used to characterize the various physicochemical properties of

organic molecules such as surface tension, density, melting, freezing point, solubility, heat of evaporation, and heat of formation [3]. In last two decades, many TIs are introduced, but degree-based TIs got much more attention of the researchers, see the latest survey [4]. In 1947, Winer introduced the first distance-based TI to compute the boiling point of paraffin [5]. Also, we refer [6].

In molecular graph theory, the different operations on a graph perform a fundamental role in the formation of different new classes of graphs. Yan et al. [7] introduced the four operations S_1 , R_1 , Q_1 , and T_1 on a graph H and computed the Wiener indices for the graphs $F_1(H)$, where $F_1 \in \{S_1, R_1, Q_1, T_1\}$. Eliasi and Taeri [8] defined the F_1 -sum graphs $(H_{1+F_1}H_2)$ using the Cartesian product on graphs $F_1(H_1)$ and H_2 , where H_1 and H_2 are any two simple and connected graphs. They also computed the Wiener indices for these F_1 -sum graphs. Furthermore, Deng et al. [9] calculated the M_1 and M_2 Zagreb indices, Imran and Shehnaz [10] computed the F -index, Liu et al. [11] computed the first general Zagreb, Chu et al. [12] calculated the bounds of first general Zagreb index and general Randic index, and

FIGURE 1: Clearly, structural formula of butane is isomorphic to P_4 .

Sarala et al. [13] computed the F -index for these F_1 -sum graphs under the Cartesian and strong products. We also refer [14–20].

Recently, for $k \geq 1$, Liu et al. [21] defined the generalized F -sum (F_k -sum) graphs using Cartesian product and computed their Zagreb indices. Moreover, Awais et al. computed the F -index [22], hyper-Zagreb [23], and FGZ index [24] for these generalized F -sum graphs. Recently, Javaid et al. [25] computed first and second Zagreb indices for the generalized F -sum graphs under the strong product.

In this paper, we compute F -index of generalized sum graphs in terms of various TIs of their factor graphs, where generalized sum graphs are obtained by using four generalized subdivision-related operations and the strong product of graphs. We have analyzed our results through the numerical tables and the graphical presentations for the particular generalized sum graphs constructed with the help of path (alkane) graphs. The remaining paper is managed as follows: Section 2 consists of elementary definitions, Section 3 includes main results, and Section 4 covers the application and conclusion.

2. Preliminaries

2.1. Degree-Based Topological Indices. In 1972, Trainajsi and Gutman defined first and second Zagreb indices that are utilized to find the π -electron energy of molecular graphs [26]. Let H be any graph (molecular structure), then the first and second Zagreb indices are defined as $M_1(H) = \sum_{v \in V(H)} d^2(v)$ and $M_2(H) = \sum_{uv \in E(H)} d(u)d(v)$.

In 2015, Furtula and Gutman [22] redefined a TI called by forgotten TI [27]. The forgotten TI of a (molecular) graph H is defined as

$$F(H) = \sum_{v \in V(H)} d^3(v). \quad (1)$$

They also verified that the different predictive abilities of F -index and first Zagreb index are same. In particular, both the indices yield the values of correlation coefficients for entropy and acentric factor more than 0.95. For more results on its mathematical properties and chemical applications, see [23].

2.2. Four Generalized Operations. Let H_1 be a (molecular) graph, then for the integral value $k \geq 1$, the graphs $(F_k(H_1))$ under the four generalized operations ($F_k \in \{S_k, R_k, Q_k, T_k\}$) on H_1 are defined as follows [21]:

- (i) k -subdivision graph: we add k new vertices in every edge of H_1 and obtain the new k -subdivision graph $S_k(H_1)$
- (ii) k -semitotal point graph: the graph k -semitotal point $(R_k(H_1))$ is defined from the graph $S_k(H_1)$ by joining the vertices of $S_k(H_1)$ which were adjacent in H_1
- (iii) k -semitotal line graph: the k -semitotal line graph $Q_k(H_1)$ is defined from the graph $S_k(H_1)$ by joining the newly added k vertices for each incident pair of edges of H_1
- (iv) k -total point graph: the k -total point graph is defined from the graph $S_k(H_1)$ by applying both the operations R_k and Q_k , respectively. For more details, see Figures 2–4.

2.3. Generalized Sum Operation for Strong Product. Let H_1 and H_2 be two graphs, $F_k \in \{S_k, R_k, Q_k, T_k\}$ presents generalized operations and $F_k(H_1)$ is obtained after applying F_k on H_1 having edge-set $E(F_k(H_1))$ and node-set $V(F_k(H_1))$. The generalized F -sum graphs $(H_{1F_k} \boxtimes H_2)$ under the operation of strong product is a graph having a vertex-set:

FIGURE 2: (a) $H_1 \cong C_3$. (b) $S_3(H_1)$. FIGURE 3: (a) $R_3(H_1)$. (b) $Q_3(H_1)$. FIGURE 4: $T_3(H_1)$.

$$\begin{aligned} V(H_{1F_k} \boxtimes H_2) &= V(F_k(H_1)) \times V(H_2) \\ &= (V(H_1) \cup kE(H_1)) \times V(H_2), \end{aligned} \quad (2)$$

such that two vertices (s_1, t_1) and (s_2, t_2) of $V(H_{1F_k} \boxtimes H_2)$ are adjacent iff $s_1 = s_2 \in V(H_1)$ and $(t_1, t_2) \in E(H_2)$ or $t_1 = t_2 \in V(H_2)$ and $(s_1, s_2) \in E(F_k(H_1))$ or $(t_1, t_2) \in E(H_2)$ and $(s_1, s_2) \in E(F_k(H_1))$, where $k \geq 1$ is natural number. Furthermore, the generalized F -sum graphs $(H_{1F_k} \boxtimes H_2)$ contain $|V(H_2)|$ copies of graphs $F_k(H_1)$ that are labeled

with the vertices of H_2 . For more explanation, see Figures 5–7.

3. Methodology

This section presents the main results.

Theorem 1. Let H_1 and H_2 be two connected graphs, then

$$\begin{aligned} F(H_{1F_k} \boxtimes H_2) &= [|V(H_2)| + 3M_1(H_2) + 6|E(H_2)|]F(H_1) + [|V(H_1)| + 3M_1(H_1) + 14|E(H_1)|]F(H_2) \\ &\quad + 6|E(H_2)|M_1(H_1) + 30|E(H_1)|M_1(H_2) + 6M_1(H_1)M_1(H_2) + 48|E(H_1)||E(H_2)| \\ &\quad + 8(k-1)|E(H_1)|[6|E(H_2)| + |V(H_2)| + F(H_2) + 3M_1(H_2)] + F(H_1)F(H_2) \\ &\quad + 8|V(H_2)||E(H_1)| \end{aligned} \quad (3)$$

Proof. Let $d(s, t) = d_{H_{1F_k} \boxtimes H_2}(s, t)$ is the degree of the vertex (s, t) in $H_{1F_k} \boxtimes H_2$, then

$$\begin{aligned} F(H_{1F_k} \boxtimes H_2) &= \sum_{(s,t) \in V(H_{1F_k} \boxtimes H_2)} d(s, t)^3 = \sum_{(s_1, t_1), (s_2, t_2) \in E(H_{1F_k} \boxtimes H_2)} [d(s_1, t_1)^2 + d(s_2, t_2)^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1, t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] + \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in E(S_k(H_1))} [d(s_1, t)^2 + d(s_2, t)^2] \\ &\quad + \sum_{t_1, t_2 \in E(H_2)} \sum_{s_1, s_2 \in E(S_k(H_1))} [d(s_1, t_1)^2 + d(s_2, t_2)^2] = \sum_A + \sum_B + \sum_C. \\ \sum_A &= \sum_{s \in V(H_1)} \sum_{t_1, t_2 \in E(H_2)} [(d(s) + d(t_1) + d(s)d(t_1))^2 + (d(s) + d(t_2) + d(s)d(t_2))^2] \\ &= \sum_{s \in V(H_1)} \sum_{t_1, t_2 \in E(H_2)} [2d(s)^2 + (d(t_1)^2 + d(t_2)^2) + d^2(s)(d^2(t_1) + d^2(t_2))] \\ &\quad + 2d(s)(d(t_1) + d(t_2)) + 2d^2(s)(d(t_1) + d(t_2)) + 2d(s)(d^2(t_1) + d^2(t_2))] \\ &= 2|E(H_2)|M_1(H_1) + |V(H_1)|F(H_2) + 4|E(H_1)|M_1(H_2) + M_1(H_1)F(H_2) \\ &\quad + 2M_1(H_1)M_1(H_2) + 4|E(H_1)|F(H_2), \\ \sum_B &= \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in E(S_k(H_1))} [d^2(s_1, t) + d^2(s_2, t)] = \sum_{t \in V(H_2)} \sum_{s_1 \in V(H_1), s_2 \in V(S_k(H_1)-H_1)} [d^2(s_1, t) + d^2(s_2, t)] \\ &\quad + \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in V(S_k(H_1)-H_1)} [d^2(s_1, t) + d^2(s_2, t)] = \sum_{B_1} + \sum_{B_2}. \\ \sum_{B_1} &= \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in E(S_k(H_1)), s_1 \in V(H_1), s_2 \in V(S_k(H_1)-H_1)} [(d(s_1) + d(t) + d(s_1)d(t))^2 + (d(s_2) + d(s_2)d(t))^2] \\ &= \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in E(S_k(H_1)), s_1 \in V(H_1), s_2 \in V(S_k(H_1)-H_1)} [d(s_1)^2 + d(s_2)^2 + d(t)^2(d(s_1)^2 + d(s_2)^2)] \\ &\quad + 2d(t)(d(s_1)^2 + d(s_2)^2) + 2d(t)(d(s_1) + 2d^2(t))(d(s_1) + d^2(t))]. \end{aligned} \quad (4)$$

FIGURE 5: (a) $H_1 = C_3$, (b) $H_2 = P_2$. FIGURE 6: (a) $H_{1_{s_2}} \boxtimes H_2$, (b) $H_{1_{r_2}} \boxtimes H_2$. FIGURE 7: (a) $H_{1_{Q_2}} \boxtimes H_2$, (b) $H_{1_{r_2}} \boxtimes H_2$.

Since $s_1 \in V(H_1)$ and $s_2 \in V(S_k(H_1) - H_1)$, therefore

$$= |V(H_2)|F(S_1(H_1)) + 2|E(H_1)|M_1(H_2) + 4|E(H_2)M_1(H_1) + 4|E(H_2)F(S_1(H_1)) + M_1(H_2)F(S_1(H_1)) + 2M_1(H_1)M_1(H_2). \quad (5)$$

We know that $F(S_1(H_1)) = F(H_1) + 8|E(H_1)|$.

$$\begin{aligned} &= [|V(H_2)| + 4|E(H_2) + M_1(H_2)|][F(H_1) + 8|E(H_1)|] + 2|E(H_1)|M_1(H_2) + 4|E(H_2)M_1(H_1) \\ &+ 4|E(H_2) + 2M_1(H_1)M_1(H_2), \\ \sum_{B_2} &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1}, \\ s_2 \in V(S_k(H_1) - H_1)}} [(d(s_1) + d(s_1)d(t))^2 + (d(s_2) + d(s_2)d(t))^2] \\ &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1}, \\ s_2 \in V(S_k(H_1) - H_1)}} [d(s_1)^2 + d(s_2)^2 + d(t)^2(d(s_1)^2 + d(s_2)^2) + 2d(t)(d(s_1)^2 + d(s_2)^2)] \\ &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1}, \\ s_2 \in V(S_k(H_1) - H_1)}} [(2)^2 + (2)^2 + 8d_{H_2}(t)^2 + 16d_{H_2}(t)]. \end{aligned} \quad (6)$$

Since in this case $|E(S_k(H_1))| = |E(H_1)|(k-1)$, so

$$\begin{aligned} &= 8(k-1)|V(H_2)||E(H_1)| + 8(k-1)M_1(H_2)|E(H_1)| + 32(k-1)|E(H_2)||E(H_1)|, \\ \sum_C &= \sum_{t_1 t_2 \in E(H_2)} \sum_{s_1 s_2 \in E(S_k(H_1))} [d^2(s_1, t_1) + d^2(s_2, t_2)] = \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 \in V(H_1), \\ s_2 \in V(S_k(H_1) - H_1)}} [d^2(s_1, t_1) + d^2(s_2, t_2)] \\ &+ \sum_{t_1 t_2 \in E(H_2)} \sum_{s_1 s_2 \in V(S_k(H_1) - H_1)} [d^2(s_1, t_1) + d^2(s_2, t_2)] = \sum_{C_1} + \sum_{C_1} \\ \sum_{C_1} &= \sum_{t_1 t_2 \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1 \in V(H_1)}, \\ s_2 \in V(S_k(H_1) - H_1)}} [(d(s_1) + d(t_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \\ &= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1 \in V(H_1)}, \\ s_2 \in V(S_k(H_1) - H_1)}} [d(s_1)^2 + d(s_2)^2 + d(t_1)^2 d(s_1)^2 + d(t_2)^2 d(s_2)^2] \\ &+ 2d(t_1)d(s_1)^2 + 2d(t_2)d(s_2)^2 + 2d(t_1)(d(s_1) + 2d^2(t_1))(d(s_1) + d_{H_2}^2(t_1))] \\ &= 2|E(H_2)||F(G_1) + 16|E(H_1)||E(H_2)| + 2|E(H_1)|F(G_2) + F(G_1)F(G_2) + 2M_1(H_1)M_1(H_2) \\ &+ 2F(G_1)M_1(H_2) + 2M_1(H_1)F(G_2) + 8|E(H_1)|F(G_2) + 16|E(H_1)|M_1(H_2), \\ \sum_{C_2} &= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1}, \\ s_2 \in V(S_k(H_1) - H_1)}} [(d(s_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \\ &= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(S_k(H_1))_{s_1}, \\ s_2 \in V(S_k(H_1) - H_1)}} [8 + 4(d^2(t_1) + d^2(t_2)) + 8(d(t_1) + d(t_2))] \\ &= 16(k-1)|E(H_2)||E(H_1)| + 8(k-1)F(H_2)|E(H_1)| + 16(k-1)|E(H_1)|M_1(H_2). \end{aligned} \quad (7)$$

We arrive at our desired result.

□ **Theorem 2.** Let H_1 and H_2 be two connected graphs, then

$$\begin{aligned}
F(H_{1_{R_k}} \boxtimes H_2) &= 8[|V(H_2)| + 6|E(H_2)|]F(H_1) + [|V(H_1)| + 20|E(H_1)|]F(H_2) + 8F(H_1)F(H_2) \\
&+ 24|E(H_2)|M_1(H_1) + 36|E(H_1)|M_1(H_2) + 24M_1(H_1)M_1(H_2) + 24F(H_1)M_1(H_2) \\
&+ 8|V(H_2)||E(H_1)| + 8(k-1)|E(H_1)|[|V(H_2)| + F(H_2) + 4|E(H_2)| + 3M_1(H_2)] \\
&+ 48|E(H_1)||E(H_2)| + 12F(H_2)M_1(H_1).
\end{aligned} \tag{8}$$

Proof. Let $d(s, t) = d_{H_{1_{R_k}} \boxtimes H_2}(s, t)$ is the degree of the vertex (s, t) in $H_{1_{R_k}} \boxtimes H_2$, then

$$\begin{aligned}
F(H_{1_{R_k}} \boxtimes H_2) &= \sum_{(s,t) \in V(H_{1_{R_k}} \boxtimes H_2)} d(s, t)^3 = \sum_{(s_1, t_1)(s_2, t_2) \in E(H_{1_{R_k}} \boxtimes H_2)} [d(s_1, t_1)^2 + d(s_2, t_2)^2] \\
&= \sum_{s \in V(H_1)} \sum_{t_1, t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] + \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in E(R_k(H_1))} [d(s_1, t)^2 + d(s_2, t)^2] \\
&+ \sum_{t_1, t_2 \in E(H_2)} \sum_{s_1, s_2 \in E(R_k(H_1))} [d(s_1, t_1)^2 + d(s_2, t_2)^2] = \sum_A + \sum_B + \sum_C. \\
\sum_A &= \sum_{s \in V(H_1)} \sum_{t_1, t_2 \in E(H_2)} [d(s, t_1)^2 + d(s, t_2)^2] = 8|E(H_2)||M_1(H_1)| + |V(H_1)|F(H_2) \\
&+ 4M_1(H_1)F(H_2) + 8|E(H_1)||M_1(H_2)| + 8M_1(H_1)M_1(H_2) + 8|E(H_1)|F(H_2), \\
\sum_B &= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in V(H_1), \\ s_1, s_2 \in V(H_1)}} [d^2(s_1, t) + d^2(s_2, t)] + \sum_{t \in V(H_2)} \sum_{\substack{s_1 \in V(H_1), \\ s_2 \in V(R_k(H_1) - H_1)}} [d^2(s_1, t) + d^2(s_2, t)] \\
&+ \sum_{t \in V(H_2)} \sum_{s_1, s_2 \in V(R_k(H_1) - H_1)} [d^2(s_1, t) + d^2(s_2, t)] = \sum_{B_1} + \sum_{B_2} + \sum_{B_3}. \\
\sum_{B_1} &= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(R_k(H_1)), \\ s_2 \in V(H_1)}} [(d(s_1) + d(t) + d(s_1)d(t))^2 + (d(s_2) + d(t) + d(s_2)d(t))^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(R_k(H_1)), \\ s_2 \in V(H_1)}} [4(d^2(s_1) + d^2(s_2)) + 2d^2(t) + 4d^2(t)(d^2(s_1) + d^2(s_2))] \\
&+ 4d(t)(d(s_1) + d(s_2)) + 4d^2(t)d(s_1) + d(s_2) + 8d(t)(d^2(s_1) + d^2(s_2))] \\
&= 4|V(H_2)|F(H_1) + 2|E(H_1)||M_1(H_2)| + 4M_1(H_2)F(H_1) + 8|E(H_2)||M_1(H_1)| \\
&+ 4M_1(H_1)M_1(H_2) + 16|E(H_2)|F(H_1), \\
\sum_{B_2} &= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(R_k(H_1)), \\ s_2 \in V(R_k(H_1) - H_1)}} [(d(s_1) + d(t) + d(s_1)d(t))^2 + (d(s_2) + d(s_2)d(t))^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(R_k(H_1)), \\ s_2 \in V(R_k(H_1) - H_1)}} [4d(s_1)^2 + d(s_2)^2 + d(t)^2(4d(s_1)^2 + d(s_2)^2)] \\
&+ 8d(t)(d(s_1)^2 + 4d(t)d(s_1) + 4d^2(t)(d(s_1) + d^2(t) + 2d(s_2)^2d(t))) \\
&= 4|V(H_2)|F(H_1) + 10|E(H_1)||M_1(H_2)| + 4M_1(H_2)F(H_1) + 8|E(H_2)||M_1(H_1)| \\
&+ 4M_1(H_1)M_1(H_2) + 16|E(H_2)|F(H_1) + 8|V(H_2)||E(H_1)| + 32|E(H_1)||E(H_2)|, \\
\sum_{B_3} &= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(R_k(H_1)), \\ s_2 \in V(R_k(H_1) - H_1)}} [(d(s_1) + d(s_1)d(t))^2 + (d(s_2) + d(s_2)d(t))^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(R_k(H_1)), \\ s_2 \in V(R_k(H_1) - H_1)}} [(d(s_1)^2 + d(s_2)^2) + d(t)^2(d(s_1)^2 + d(s_2)^2) + 2d(t)(d^2(s_1) + d^2(s_2))]
\end{aligned}$$

$$\begin{aligned}
&= 8(k-1)(V(H_2)|E(H_1)| + 8(k-1)M_1(H_2)|E(H_1)| + 32(k-1)|E(H_2)||E(H_1)|), \\
\sum_C &= \sum_{t_1 t_2 \in E(H_2)} \sum_{s_1 s_2 \in V(H_1)} [d^2(s_1, t_1) + d^2(s_2, t_2)] + \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 \in V(H_1), \\ s_2 \in V(R_k(H_1)-H_1)}} [d^2(s_1, t_1) + d^2(s_2, t_2)] \\
&\quad + \sum_{t_1 t_2 \in E(H_2)} \sum_{s_1, s_2 \in V(R_k(H_1)-H_1)} [d^2(s_1, t_1) + d^2(s_2, t_2)] = \sum_{C_1} + \sum_{C_2} + \sum_{C_3}. \\
\sum_{C_1} &= \sum_{t_1 t_2 \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1))s_1, \\ s_2 \in V(H_1)}} [(d(s_1) + d(t_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(t_2) + d(s_2)d(t_2))^2] \\
&= \sum_{t_1 t_2 \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1))s_1, \\ s_2 \in V(H_1)}} [d(s_1)^2 + d(s_2)^2 + d^2(t_1) + d^2(t_2) + d(t_1)^2 d(s_1)^2 + d^2(t_2)d(s_2)^2 + 2d(t_1)d(s_1)] \\
&\quad + 2d(t_2)d(s_2) + 2d^2(t_1)d(s_1) + 2d^2(t_2)d(s_2) + 2d(t_1)d^2(s_1) + 2d^2(t_2)d(s_2)] \\
&= 8|E(H_2)|F(H_1) + 2|E(H_1)|F(H_2) + 4F(H_1)F(H_2) + 4M_1(H_1)M_1(H_2) \\
&\quad + 4M_1(H_1)F(H_2) + 8M_1(H_2)F(H_1), \\
\sum_{C_2} &= \sum_{t_1 t_2 \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1))s_1 \in V(H_1), \\ s_2 \in V(R_k(H_1)-H_1)}} [(d(s_1) + d(t_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \tag{9} \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1))s_1 \in V(H_1), \\ s_2 \in V(R_k(H_1)-H_1)}} [d(s_1)^2 + d(s_2)^2 + d^2(t_1) + d(t_1)^2 d(s_1)^2 + 2d(t_1)d(s_1) + 2d(t_1)d(s_1)^2] \\
&\quad + 2d^2(t_1)d(s_1) + d(t_2)^2 d(s_2)^2 + 2d(t_2)d(s_2)^2] \\
&= 8|E(H_2)|F(G_1) + 16|E(H_1)||E(H_2)| + 4F(G_1)F(G_2) + 4M_1(H_1)M_1(H_2) + 8F(G_1)M_1(H_2) \\
&\quad + 2|E(H_1)|F(G_2) + 4M_1(H_1)F(G_2) + 8|E(H_1)|F(G_2) + 16|E(H_1)|M_1(H_2), \\
\sum_{C_3} &= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1))s_1, \\ s_2 \in V(R_k(H_1)-H_1)}} [(d(s_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \\
&= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(R_k(H_1))s_1, \\ s_2 \in V(R_k(H_1)-H_1)}} [(2 + 2d(t_1))^2 + (2 + 2d(t_2))^2] \\
&= 16(k-1)|E(H_2)||E(H_1)| + 8(k-1)F(H_2)|E(H_1)| + 16(k-1)|E(H_1)|M_1(H_2).
\end{aligned}$$

We arrive at our desired result.

□ **Theorem 3.** Let H_1 and H_2 be two connected graphs, then

$$\begin{aligned}
F(H_{1_{Q_k}} \boxtimes H_2) &= 2(k-1)[F(H_1) + 2M_2(H_1)][3n(H_2) + 5M_1(H_2) + 14|E(H_2)| + F(H_2)] + k[|V(H_2)| \\
&\quad + 6|E(H_2)| + 3M_1(H_2) + F(H_2)][M_4(H_1) - 2F(H_1) + 2M_2(H_1) - 4M_2(H_1) \\
&\quad + \sum_{u \in V(H_1)} d^2(u) \sum_{v \in N(u)} d(v)] + 6|E(H_2)|M_1(H_1) + 10|E(H_2)|F(H_2) + 3F(H_1)F(H_2) \\
&\quad + 6M_1(H_1)M_1(H_2) + F(H_2)[|V(H_1)| + 3M_1(H_1) + 6|E(H_2)| + 4M_2(H_1)] \\
&\quad + F(H_1)[|V(H_2)| + 7M_1(H_2)] + 6|E(H_2)|M_2(H_1) + 8M_2(H_1)[|E(H_2)| + M_1(H_2)]. \tag{10}
\end{aligned}$$

Proof. Let $d(s, t) = d_{H_{1_{Q_k}} \boxtimes H_2}(s, t)$ is the degree of the vertex (s, t) in $H_{1_{Q_k}} \boxtimes H_2$, then

$$\begin{aligned}
F(H_{1Q_k} \boxtimes H_2) &= \sum_{(s,t) \in V(H_{1Q_k} \boxtimes H_2)} d(s,t)^3 = \sum_{(s_1,t_1)(s_2,t_2) \in E(H_{1Q_k} \boxtimes H_2)} [d(s_1,t_1)^2 + d(s_2,t_2)^2] \\
&= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [d(s,t_1)^2 + d(s,t_2)^2] + \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(Q_k(H_1))} [d(s_1,t)^2 + d(s_2,t)^2] \\
&\quad + \sum_{t_1 t_2 \in E(H_2)} \sum_{s_1 s_2 \in E(Q_k(H_1))} [d(s_1,t_1)^2 + d(s_2,t_2)^2] = \sum_A + \sum_B + \sum_C. \\
\sum_A &= \sum_{s \in V(H_1)} \sum_{t_1 t_2 \in E(H_2)} [d(s,t_1)^2 + d(s,t_2)^2] = 2|E(H_2)|M_1(H_1) + |V(H_1)|F(H_2) \\
&\quad + M_1(H_1)F(H_2) + 4|E(H_1)|M_1(H_2) + 2M_1(H_1)M_1(H_2) + 4|E(H_1)|F(H_2). \\
\sum_B &= \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in E(Q_k(H_1))} [d^2(s_1,t) + d^2(s_2,t)] = \sum_{t \in V(H_2)} \sum_{\substack{s_1 \in V(H_1), \\ s_2 \in V(Q_k(H_1)-H_1)}} [d^2(s_1,t) + d^2(s_2,t)] \\
&\quad + \sum_{t \in V(H_2)} \sum_{s_1 s_2 \in V(Q_k(H_1)-H_1)} [d^2(s_1,t) + d^2(s_2,t)] = \sum_{B_1} + \sum_{B_2}. \\
\sum_C &= \sum_{B_1} \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1), s_2 \in V(Q_k(H_1)-H_1)}} [(d(s_1) + d(t) + d(s_1)d(t))^2 + (d(s_2) + d(s_2)d(t))^2] \\
&= \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1), s_2 \in V(Q_k(H_1)-H_1)}} [d(s_1)^2 + d(s_2)^2 + d(t)^2(d(s_1)^2 + d(s_2)^2) + 2d(t)d(s_1)^2 + 2d(t)d^2(s_2)^2] \\
&\quad + 2d(t)d^2(s_2)^2 + 2d(t)d(s_1) + d^2(t)].
\end{aligned} \tag{11}$$

Consider $s_1 \in V(H_1)$ and $d^2(s_1)$ occurs $d(s_1)$ times. Thus,

$$D_1 = \sum_{\substack{s_1 s_2 \in E(Q(H_1)), \\ s_1 \in V(H_1), s_2 \in V(Q(H_1)-V(H_1))}} d^3(s_1) = F(H_1). \tag{12}$$

Let

$$D_2 = \sum_{\substack{s_1 s_2 \in E(Q(H_1)), \\ s_1 \in V(H_1), s_2 \in V(Q(H_1)-V(H_1))}} d^2(s_2). \tag{13}$$

as $s_2 = uv \in E(H_1)$ and $d^2(s_2)$ occurs two times. Therefore,

$$\begin{aligned}
D_2 &= 2 \sum_{s_2 = uv \in V(Q(H_1)) - V(H_1)} [d(u) + d(v)]^2 = 2 \sum_{uv \in E(H_1)} [d^2(u) + d^2(v) + 2d(u)d(v)] = 2[F(H_1) + 2M_2(H_1)] \\
&= |V(H_2)|F(H_1) + 2|E(H_1)|M_1(H_2) + M_1(H_2)F(H_1) + 4|E(H_2)|[M_1(H_1) + F(H_1)] + 2M_1(H_1)M_1(H_2) \\
&\quad + 2|V(H_2)|[F(H_1) + 2M_2(H_1)] + 2M_1(H_2)[F(H_1) + 2M_2(H_1)] + 8|E(H_2)|[F(H_1) + 2M_2(H_1)]. \\
\sum &= \sum_{B_2} \sum_{t \in V(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1)) \\ s_1 \in V(H_1), s_2 \in V(Q_k(H_1)-H_1)}} [(d(s_1) + d(s_1)d(t))^2 + (d(s_2) + d(s_2)d(t))^2].
\end{aligned} \tag{14}$$

Now, we split sum in two parts, s_1 and s_2 , where $s_1 s_2 \in V(Q_k(H_1)) - V(H_1)$. Suppose that $\sum_{B_2} = \sum_{B_{21}} + \sum_{B_{22}}$, where $\sum_{B_{21}}$ covers the edges of $Q_k(H_1)$ which are in the

same edges of H_1 and $\sum_{B_{22}}$ of $Q_k(H_1)$ in two different adjacent edges of H_1 .

$$\begin{aligned}
\sum_{B_{21}} &= \sum_{teV(H_1)} \sum_{s_1, s_2 \in E(Q(H_1))} [\{d(s_1) + d(s_1)d(t)\}^2 + \{d(s_2) + d(s_2)d(t)\}^2] \\
&= \sum_{teV(H_2)} \sum_{s_1, s_2 \in E(Q(H_1))} [[d^2(s_1) + d^2(s_2)] + d^2(t)[d^2(s_1) + d^2(s_2)] + 2d(t)[d^2(s_1) + d^2(s_2)]] \\
&= \sum_{teV(H_2)} 2(k-1)[F(H_1) + 2M_2(H_1)] + d^2(t)[F(H_1) + 2M_2(H_1)] + 2d(t)[F(H_1) + 2M_2(H_1)] \\
&= 2(k-1)[F(H_1) + 2M_2(H_1)][|V(H_2)| + M_1(H_2) + 4|E(H_2)|], \\
\sum_{B_{22}} &= \sum_{teV(H_2)} \sum_{\substack{s_1, s_2 \in E(Q_k(H_1)), \\ s_2 \in V(Q_k(H_1)-H_1)}} [d(s_1)^2 + d(s_2)^2 + d(t)^2(d(s_1)^2 + d(s_2)^2) + 2d(t)(d(s_1)^2 + d(s_2)^2)] \\
D_3 &= \sum_{\substack{s_1, s_2 \in E(Q(H_1)), \\ s_2 \in V(Q(H_1)-V(H_1))}} [d^2(s_1) + d^2(s_2)].
\end{aligned} \tag{15}$$

In D_3 , the coefficient of

$$\begin{aligned}
d^2(u) &= 2 \binom{2}{d(H_1)(u)} + \sum_{v \in N(u)} d(v) - d(u) \\
&= d^2(u) - 2d(u) + \sum_{v \in N(u)} d(v).
\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}
\sum_{u \in V(H_1)} d^2(u) &= M_4(H_1) - 2F(H_1) + \sum_{u \in V(H_1)} d^2(u) \\
&\quad \cdot \sum_{v \in N(u)} d(v).
\end{aligned} \tag{17}$$

For the coefficient of $dudv$, let $s_1, s_2 \in E(Q(H_1))$ with $s_1 = uv$ and $s_2 = wz$. As $s_1, s_2 \in E(Q(H_1))$, we have either $v = w$ or $z = u$ or $u = w$ or z . So, uv is adjacent to all those vertices in H_1 which are adjacent to u and v . So, the number of such $dudv$ is $(du + dv - 2)$. Therefore,

$$\begin{aligned}
2 \sum_{uv \in E(H_1)} dudv &= 2 \sum_{uv \in E(H_1)} (du + dv - 2)dudv \\
&= 2 \sum_{uv \in E(H_1)} (du + dv)dudv - 4 \sum_{uv \in E(H_1)} dudv \\
&= 2M_2^1(H_1) - 4M_2(H_1).
\end{aligned} \tag{18}$$

So,

$$\begin{aligned}
D_3 &= M_4(H_1) - 2F(H_1) + \sum_{u \in V(H_1)} d^2(u) \sum_{v \in N(u)} d(v) + 2M_2^1(H_1) - 4M_2(H_1) \\
&= (k)[|V(H_2)| + 4|E(H_2)| + M_1(H_2)][M_4(H_1) - 2F(H_1) + 2M_2^1(H_1) - 4M_2(H_1) \\
&\quad + \sum_{u \in V(H_1)} d^2(u) \sum_{v \in N(u)} d(v)], \\
\sum_C &= \sum_{t_1, t_2 \in E(H_2)} \sum_{s_1, s_2 \in E(Q_k(H_1))} [d^2(s_1, t_1) + d^2(s_2, t_2)] = \sum_{t_1, t_2 \in E(H_2)} \sum_{\substack{s_1 \in V(H_1), \\ s_2 \in V(Q_k(H_1)-H_1)}} [d^2(s_1, t_1) + d^2(s_2, t_2)] \\
&\quad + \sum_{t_1, t_2 \in E(H_2)} \sum_{s_1, s_2 \in V(Q_k(H_1)-H_1)} [d^2(s_1, t_1) + d^2(s_2, t_2)] = \sum_{C_1} + \sum_{C_2}. \\
\sum_{C_1} &= \sum_{t_1, t_2 \in V(H_2)} \sum_{\substack{s_1, s_2 \in E(Q_k(H_1)), \\ s_2 \in V(Q_k(H_1)-H_1)}} [(d(s_1) + d(t_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \\
&= \sum_{teV(H_2)} \sum_{\substack{s_1, s_2 \in E(Q_k(H_1)), \\ s_2 \in V(Q_k(H_1)-H_1)}} [d(s_1)^2 + d(t_1)^2 d(s_1)^2 + 2d(t_1)d(s_1) + 2d^2(t_1)d(s_1) + 2d(t_1)d(s_1)^2] \\
&\quad + d^2(t_1) + d(s_2)^2 + d(t_2)^2 d(s_2)^2 + 2d(t_2)d(s_2)^2] \\
&= 6[|E(H_2)| + M_1(H_2)]F(H_1) + 3F(H_1)F(H_2) + 2M_1(H_1)M_1(H_2) + 2[|E(H_1)| + M_1(H_1) \\
&\quad + 2M_2(H_1)]F(H_2) + 8M_2(H_1)[|E(H_2)| + M_1(H_2)]. \\
\sum_{C_2} &= \sum_{t_1, t_2 \in E(H_2)} \sum_{\substack{s_1, s_2 \in E(Q_k(H_1)), \\ s_2 \in V(Q_k(H_1)-H_1)}} [(d(s_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2].
\end{aligned} \tag{19}$$

TABLE 1: F -index of F_1 -sum path graphs.

$[n_1, n_2]$	$F(P_{n_{1S_1}} \boxtimes P_{n_2})$	$F(P_{n_{1R_1}} \boxtimes P_{n_2})$	$F(P_{n_{1Q_1}} \boxtimes P_{n_2})$	$F(P_{n_{1T_1}} \boxtimes P_{n_2})$
(3, 3)	1808	6414	3442	8048
(4, 4)	4836	18120	8826	22,110
(5, 5)	9320	35,746	16,692	43,118
(6, 6)	15,260	59,292	27,040	71,072
(7, 7)	22,656	88,758	39,870	105,972

TABLE 2: F -index of F_2 -sum path graphs.

$[n_1, n_2]$	$F(P_{n_{1S_2}} \boxtimes P_{n_2})$	$F(P_{n_{1R_2}} \boxtimes P_{n_2})$	$F(P_{n_{1Q_2}} \boxtimes P_{n_2})$	$F(P_{n_{1T_2}} \boxtimes P_{n_2})$
(3, 3)	2496	7102	5764	10,370
(4, 4)	6516	19,800	14,496	27,780
(5, 5)	12,424	38,850	27,168	53,594
(6, 6)	20,220	64,252	43,780	87,812
(7, 7)	29,904	96,006	64,332	130,434

Now, we split this sum into two parts for the vertices, t_1 and t_2 , where $s_1 s_2 \in V(Q_k(G) - V(G))$. Assume that (Tex translation failed), where (Tex translation failed) cover

the edges of $Q_k(H_1)$ which are in the same edges of H_1 and (Tex translation failed) of $Q_k(H_1)$ in two different adjacent edges of H_1 .

$$\begin{aligned}
\sum_{C_{21}} &= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1))_{s_1}, \\ s_2 \in V(Q_k(H_1) - H_1)}} [(d(s_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \\
&= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1))_{s_1}, \\ s_2 \in V(Q_k(H_1) - H_1)}} [d^2(s_1) + d^2(s_2) + d^2(s_1)d^2(t_1) + d^2(s_2)d^2(t_2) + 2[d^2(s_1)d(t_1) + d^2(s_2)d(t_2)]] \\
&= 2(k-1)[F(H_1) + 2M_2(H_1)][2|E(H_2)| + F(H_2) + 2M_1(H_2)], \\
\sum_{C_{22}} &= \sum_{t_1 t_2 \in E(H_2)} \sum_{\substack{s_1 s_2 \in E(Q_k(H_1))_{s_1}, \\ s_2 \in V(Q_k(H_1) - H_1)}} [(d(s_1) + d(s_1)d(t_1))^2 + (d(s_2) + d(s_2)d(t_2))^2] \\
&= (k)[2|E(H_2)| + F(H_2) + 2M_1(H_2)]([M_4(H_1) - 2F(H_1) + 2M_2(H_1) - 4M_2(H_1) \\
&\quad + \sum_{u \in V(H_1)} d^2(u) \sum_{v \in N(u)} d(v)].
\end{aligned} \tag{20}$$

We arrive at our desired result.

□ **Theorem 4.** Let H_1 and H_2 be two connected graphs, then

$$\begin{aligned}
F(H_{1T_k} \boxtimes H_2) &= 2(k-1)[F(H_1) + 2M_2(H_1)][n(H_2) + 3M_1(H_2) + 6|E(H_2)| + F(H_2)] + k[|V(H_2)| \\
&\quad + 6|E(H_2)| + 3M_1(H_2) + F(H_2)][M_4(H_1) - 2F(H_1) + 2M_2(H_1) - 4M_2(H_1) \\
&\quad + \sum_{u \in V(H_1)} d^2(u) \sum_{v \in N(u)} d(v)] + [F(H_1) + 2M_2(H_1)][2n(H_2) + 6M_1(H_2) + 12|E(H_2)| + 2F(H_2)] \\
&\quad + 4F(H_1)[2n(H_2) + 6M_1(H_2) + 12|E(H_2)| + 2F(H_2)] + F(H_2)[n(H_1) + 12M_1(H_1) \\
&\quad + 12|E(H_2)|] + 12|E(H_1)|M_1(H_2) + 16|E(H_2)|M_1(H_1) + 20M_1(H_1)M_1(H_2).
\end{aligned} \tag{21}$$

Proof. It follows from Theorems 2 and 3.

□

FIGURE 8: (a) $F(P_{n_{S_1}} \boxtimes P_{n_2})$, $F(P_{n_{Q_1}} \boxtimes P_{n_2})$, $F(P_{n_{R_1}} \boxtimes P_{n_2})$, and $F(P_{n_{T_1}} \boxtimes P_{n_2})$ denoted by red, purple, green, and pink colours, respectively. (b) $F(P_{n_{S_2}} \boxtimes P_{n_2})$, $F(P_{n_{Q_2}} \boxtimes P_{n_2})$, $F(P_{n_{R_2}} \boxtimes P_{n_2})$, and $F(P_{n_{T_2}} \boxtimes P_{n_2})$ denoted by purple, green, pink, and red colours, respectively.

FIGURE 9: Compression between F -index of F_1 -sum and F_2 -sum graphs. (a) $F(P_{n_{S_1}} \boxtimes P_{n_2})$ and $F(P_{n_{S_2}} \boxtimes P_{n_2})$ are denoted by light green dotted line and dark green solid line, and $F(P_{n_{R_1}} \boxtimes P_{n_2})$ and $F(P_{n_{R_2}} \boxtimes P_{n_2})$ are denoted by sky blue dotted line and blue solid line, respectively. (b) $F(P_{n_{Q_1}} \boxtimes P_{n_2})$ and $F(P_{n_{Q_2}} \boxtimes P_{n_2})$ are denoted by pink dotted line and red solid line, and $F(P_{n_{T_1}} \boxtimes P_{n_2})$ and $F(P_{n_{T_2}} \boxtimes P_{n_2})$ by grey dotted line and black solid line, respectively.

4. Applications and Discussion

In this paper, we computed the F -index of the generalized sum graphs. Assume that $H_1 = P_{n_1}$ and $H_2 = P_{n_2}$ are particular alkane called as paths of orders $n_1, n_2 \geq 3$, and $k \geq 1$, respectively. Then, the following outcomes are the direct consequences of the achieved results:

$$(i) \quad F(P_{n_{S_1}} \boxtimes P_{n_2}) = 128k(n_1 - 1) + 250(n_2 + n_1) + 512(n_2 - 2)(n_1 - 2) + 216k(n_1 - 1)(n_2 - 2) - 892$$

$$(ii) \quad F(P_{n_{R_1}} \boxtimes P_{n_2}) = 128k(n_1 - 1) + 216k(n_1 - 1)(n_2 - 2) + 2744(n_2 - 2)(n_1 - 2) + 1458n_1 + 1024n_2 - 4464$$

$$(iii) \quad F(P_{n_{Q_1}} \boxtimes P_{n_2}) = 432k(n_1 - 1) + 729k(n_1 - 1)(n_2 - 2) + 512(n_2 - 2)(n_1 - 2) + 250(n_1 + n_2) - 892$$

$$(iv) \quad F(P_{n_{T_1}} \boxtimes P_{n_2}) = 432k(n_1 - 1) + 729k(n_1 - 1)(n_2 - 2) + 2744(n_2 - 2)(n_1 - 2) + 1458n_1 + 1024n_2 - 4464$$

(1) Four subdivision operations ($S_1(G)$, $R_1(G)$, $Q_1(G)$, $T_1(G)$) are restricted in case of newly

addition of vertices upto $k = 1$, while four subdivision operations $(S_k(G), R_k(G), Q_k(G), T_k(G))$ are not restricted in case of newly addition of vertices as k can be any natural number.

- (2) For a certain graph G , newly constructed graphs $(S_1(G), R_1(G), Q_1(G), T_1(G))$ will have specific number of edges. While, in $(S_k(G), R_k(G), Q_k(G), T_k(G))$, the number of edges will be varying according to k .
- (3) Molecular structure of $(S_k(G), R_k(G), Q_k(G), T_k(G))$ is more general as compared to the $(S_1(G), R_1(G), Q_1(G), T_1(G))$.
- (4) Molecular structure of F_k -sum is more complex and general as compared to the molecular structure of F_1 -sum.
- (5) In this paper, main results $F(H_{1_{Sk}} \boxtimes H_2)$, $F(H_{1_{Ok}} \boxtimes H_2)$, $F(H_{1_{Rk}} \boxtimes H_2)$, and $F(H_{1_{Tk}} \boxtimes H_2)$ based on strong product are generalization of $F(H_{1_{Si}} \boxtimes H_2)$, $F(H_{1_{Qi}} \boxtimes H_2)$, $F(H_{1_{Ri}} \boxtimes H_2)$, and $F(H_{1_{Ti}} \boxtimes H_2)$ based on strong product.

Now, we present its tabular form, Table 1 and Table 2 and graphical representations in Figures 8 and 9 for $k = 1$ and $k = 2$.

In this paper, we computed the F -index of the subdivision-related generalized F -sum graphs based on strong product. However, the problem is still open for other topological indices on the generalized F -sum graphs.

Data Availability

The data used to support the findings of this study are included within the article. However, additional data may be provided by the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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