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Research Article

Design of a Nonlinear Finite-Time Converging Observer for a Class of Nonlinear Systems

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This paper proposes a nonlinear finite-time converging observer for a class of nonlinear systems. The estimate is recovered from the present and delayed estimates provided by two independent dynamical systems converging to a function of the state with linear error dynamics. The estimation is carried out using only the Jacobian matrix of both transformations determined by solving two systems of partial derivative equations. The results are illustrated on a bioreactor model.

1. INTRODUCTION

The estimation of the state of a dynamical system from output measurements is a key issue in control theory, and in particular in process control, when some key variables are not accessible by online measurements. For instance, in chemical or biochemical industry, measuring the concentration of the process components often requires very specific and possibly highly expensive measuring devices whose use may be limited in practice. In such a context, the online estimation of component concentrations using a state observer is an attractive option.

Since the observers developed by Kalman [1] and Luenberger [2] more than four decades ago for linear systems, several different state observation techniques have been proposed to handle the systems nonlinearities [3]. A first category of techniques consists in applying linear algorithms to the system linearized around the estimated trajectory. These are known as the extended Kalman and Luenberger observers. Alternatively, the nonlinear dynamics are splitted into a linear part and a nonlinear one. The observer gains are then chosen large enough so that the linear part dominates the nonlinear one. Such observers are known as high-gain observers (e.g., [4, 5]). In a third approach the nonlinear system is transformed into a linear one by an appropriate change of coordinates [6–8]. The estimate is computed in these new coordinates and the original coordinates are retrieved through the inverse transformation. The nonlinear observer developed in [9] is based on a similar concept; however, it has the advantage that the inverse transformation does not need to be computed. Indeed the observer is a dynamical system that makes use of the Jacobian of a linearizing change of coordinates providing an estimate in the original coordinates in one step only.

All the above techniques have the common disadvantage to provide an asymptotically converging estimate. The authors of [10] have proposed an observer that converges exactly to the state within a finite-time interval. This finite-time converging observer is designed using a nonsmooth function of the reconstruction error and can be seen as a generalization of the common asymptotically converging observers which are linear in the reconstruction error. Although the nonsmooth observer reaches the state within a finite time interval, the convergence time interval depends on the initial conditions and is therefore a priori unknown.

More recently, Engel and Kreisselmeier have designed a state observer that converges exactly to the state after a predefined time delay [11]. The finite-time converging estimate is computed from the present and delayed estimates provided by two distinct state observers. The observer design arises from solving the set of four equations formed by the estimates definitions at present and delayed time instances. In [11] the finite time converging observer performance relies on the linearity of the reconstruction error dynamics and its use is therefore restricted to linear time-invariant systems.

The field of application of this technique has been extended to linear time-varying systems [12]. The use of the transition matrix of the system is introduced to compare the delayed and present estimates. The same authors have also extended the technique to nonlinear systems that can be transformed into the observer canonical form [13]. Once the nonlinear system is transformed into its normal form, two observers with linear error dynamics can be developed and the finite time estimation can be carried out in these coordinates. The estimate in the original coordinates is then retrieved by the inverse transformation.

The authors of [14] propose an observer for autonomous systems that uses an injective observation mapping instead of an invertible change of variable. The observation mapping is obtained by applying an integral operator to the system output so that the obtained variable is governed by a linear dynamics. A finite time estimate can be computed from both present and delayed estimates if the observation mapping is computed by integrating the system output between both present and delayed time instances. The pseudoinverse transformation has to be computed to retrieve the original coordinates.

In this paper, we design a nonlinear finite time converging observer that proceeds in one step only. This observer relies on the existence of coordinates in which two observers with linear error dynamics can be designed as in [13]. However the estimate in the original coordinates is provided by a dynamical system using only the Jacobian matrix of the linearizing change of coordinates and not the inverse state transformation.

The main contribution of this paper is the generalization of the finite time converging observer concept by considering Kazantzis' observers that allow to deal with the system in noncanonical forms. These observers are dynamical systems converging each to a function of the state with a linear error dynamics that can be arbitrarily assigned. The main advantages are that the output map does not have to be linear in the new coordinates and that the transformations can be computed by solving a system of partial differential equations.

The paper is organized as follows. Section 2 recalls two important state observer definitions. Section 3 develops the design of an observer converging in finite time for systems that can be written in a linear canonical observability form via an appropriate state transformation. Section 4 deals with the generalized version of the nonlinear finite-time converging observer which is based on two state transformations that linearize the system dynamics. Section 5 presents the application of the nonlinear finite-time converging observer to a bioreactor model.

2. DEFINITIONS

Let us first recall the observer definition introduced by Kazantzis and Kravaris [9], which is a generalization of the original Luenberger observer for linear systems [2].

Let us consider the following nonlinear dynamical system:

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}$, and $x(t_0) = x_0$. An observer for the above system is a dynamical system driven by the output y of the dynamical system (1) which is able to reconstruct an invertible function of the state.

Definition 1. The dynamical system

$$\dot{z} = \phi(z, y), \tag{2}$$

with $z \in \mathrm{R}^n$ and $\phi : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n$, is an observer for system (1) if there exists a locally invertible map $\Psi(x)$ such that if, for any time τ, $z(\tau) = \Psi(x(\tau))$, then $z(t) = \Psi(x(t))$ for all $t > \tau$. The observer is called an identity observer if the function Ψ is the identity map.

Remark 1. It is worth noting that in [15] the authors have shown that the function Ψ has to be only left invertible and thus that the dimension of z may be greater than n. In the following, we consider that the dimensions of x and z are identical so that the Jacobian matrices of Ψ are invertible square matrices.

Let us now define the nonlinear finite time observer as a particular case of this definition. A finite time observer is an observer which provides an estimate that reaches exactly the state after a certain time. This can be formalized as follows.

Definition 2. The dynamical system

$$\dot{z} = \phi(z, y), \tag{3}$$

starting at $z(t_0) = z_0 \neq x_0$, is a finite-time converging observer for system (1) if there exists a time instant D and a locally invertible map $\Psi(x)$ such that for all $t > t_0 + D$, $z(t) = \Psi(x(t))$.

A direct consequence of the above definition is that if z is a finite time observer for x, then the following system:

$$\dot{\hat{x}}(t) = \left(\frac{\partial \Psi}{\partial x}\right)^{-1} \dot{z}, \tag{4}$$

with initial conditions satisfying

$$\psi(\hat{x}(t_0)) = z_0, \tag{5}$$

is an identity finite-time converging observer for x.

3. FINITE-TIME OBSERVER FOR LINEARIZED SYSTEMS

Let us consider system (1) and assume that there exists a change of coordinates

$$z = \Psi(x) \tag{6}$$

that transforms system (1) into the following observable pseudolinear system [8, 16]:

$$\begin{aligned} \dot{z} &= Az + \beta(y, u), \\ y &= Cz, \end{aligned} \tag{7}$$

with $z(t_0) = z_0$ and where β is a known nonlinear function that only depends on the input and output of system (1). The system's observability implies that two gain matrices H_1 and H_2 can be computed so that both the following matrices have arbitrarily assigned eigenvalues with negative real parts:

$$\begin{aligned} F_1 &= A - H_1 C, \\ F_2 &= A - H_2 C. \end{aligned} \tag{8}$$

This implies that both the following systems are identity observers for system (7):

$$\begin{aligned} \dot{z}_1 &= Az_1 + \beta(y, u) + H_1(y - Cz_1), \\ \dot{z}_2 &= Az_2 + \beta(y, u) + H_2(y - Cz_2). \end{aligned} \tag{9}$$

Each of the reconstruction errors associated with the above observers is governed by the following linear dynamical systems ($i = 1, 2$):

$$\dot{\epsilon}_i = F_i \epsilon_i, \tag{10}$$

with

$$\epsilon_i = z - z_i. \tag{11}$$

Therefore, provided the matrices F_1 and F_2 are computed such that

$$e^{-F_1 D} - e^{-F_2 D} \tag{12}$$

is invertible, and by assuming that the following initial conditions of systems (9) are identical:

$$z_1(t_0) = z_2(t_0), \tag{13}$$

then the following combination of the estimates $z_1(t)$ and $z_2(t)$:

$$\begin{aligned} \hat{z}(t) &= \left(e^{-F_1 D} - e^{-F_2 D}\right)^{-1} \\ &\quad \times \left(e^{-F_1 D}\hat{z}_1(t) - \hat{z}_1(t - D) - e^{-F_2 D}\hat{z}_2(t) + \hat{z}_2(t - D)\right) \end{aligned} \tag{14}$$

provides an estimate that converges exactly to z within the predefined time D [11].

This approach has been adopted in [13]. The authors transform the system into its pseudolinear canonical form. Then they compute a finite-time estimate in these coordinates and the estimation in the original coordinates is finally retrieved by the inverse transformation $\hat{x} = \Psi^{-1}(z)$. The following theorem presents a dynamical system that provides a finite-time estimate that converges exactly to the state in the original coordinates.

Theorem 1. *Let D be a real positive constant and assume that matrices F_1 and F_2 are designed such that the following term is invertible:*

$$e^{-F_1 D} - e^{-F_2 D}. \tag{15}$$

Then the dynamical system

$$\begin{aligned} \dot{\hat{x}} &= \left(\frac{\partial \Psi}{\partial x}\right)^{-1}_{x=\hat{x}} \left(e^{-F_1 D} - e^{-F_2 D}\right)^{-1} \\ &\quad \times \left[e^{-F_1 D}\dot{\hat{z}}_1(t) - \dot{\hat{z}}_1(t - D) - e^{-F_2 D}\dot{\hat{z}}_2(t) + \dot{\hat{z}}_2(t - D)\right], \end{aligned} \tag{16}$$

with initial conditions satisfying

$$\Psi(\hat{x}(t_0)) = \hat{z}_1(t_0) = \hat{z}_2(t_0), \tag{17}$$

is an identity finite-time converging observer for system (1) *which converges exactly within the time delay D.*

Proof. Let us choose an arbitrary positive constant D and define the following variable:

$$\begin{aligned} \hat{z} &= \left(e^{-F_1 D} - e^{-F_2 D}\right)^{-1} \\ &\quad \times \left(e^{-F_1 D} z_1(t) - z_1(t - D) - e^{-F_2 D} z_2(t) + z_2(t - D)\right). \end{aligned} \tag{18}$$

Since the dynamical systems (9) are observers converging to z with linear error dynamics and starting from the same initial conditions, we have for any time $t \geq t_0 + D$,

$$\hat{z}(t) = z(t) = \Psi(x(t)). \tag{19}$$

Equation (18) is obtained by integrating the following equation:

$$\begin{aligned} \dot{\hat{z}} &= \left(e^{-F_1 D} - e^{-F_2 D}\right)^{-1} \\ &\quad \times \left(e^{-F_1 D}\dot{z}_1(t) - \dot{z}_1(t - D) - e^{-F_2 D}\dot{z}_2(t) - \dot{z}_2(t - D)\right) \end{aligned} \tag{20}$$

with the following initial conditions:

$$\hat{z}(t_0) = \hat{z}_1(t_0) = \hat{z}_2(t_0). \tag{21}$$

Therefore the above dynamical system is a finite-time observer for system (1) in the sense of Definition 2. It follows that the dynamical system

$$\dot{\hat{x}}(t) = \left(\frac{\partial \Psi}{\partial x}\right)^{-1}_{x=\hat{x}} \dot{\hat{z}}(t), \tag{22}$$

with the following initial conditions:

$$\Psi(\hat{x}(t_0)) = \hat{z}(t_0), \tag{23}$$

is an identity observer for system (1) that converges exactly to the state within the time delay D. ◻

It is worth emphasizing the difference between this approach and that of [13]. In [13] the system proceeds in two steps. Firstly, the system is transformed into its pseudolinear canonical form, and an estimate that converges in finite time is computed in these transformed coordinates. Secondly, the estimate in the original coordinates is retrieved by the inverse change of coordinates. The observer proposed here is a dynamical system that proceeds in one step only. The computation relies on the existence of the linearizing change of coordinates but it only requires to compute the inverse of its Jacobian matrix.

4. NONLINEAR FINITE-TIME OBSERVER

In the following, we consider the autonomous nonlinear observable system

$$\begin{aligned} \dot{x} &= f(x), \\ y &= h(x), \end{aligned} \tag{24}$$

with state $x \in \mathbb{R}^n$ and output $y \in \mathbb{R}$. Without loss of generality, let us assume that the origin is an equilibrium point of the system $f(0) = 0$, with $h(0) = 0$.

Furthermore, let us assume that there exist two changes of variables $\Psi_1(x)$ and $\Psi_2(x)$ that transform system (24) into the following pseudolinear systems:

$$\begin{aligned} \dot{\Psi}_1(x) &= A_1\Psi_1(x) + \beta_1(y), \\ \dot{\Psi}_2(x) &= A_2\Psi_2(x) + \beta_2(y), \end{aligned} \tag{25}$$

where matrices A_1 and A_2 are Hurwitz, and β_1 and β_2 are functions of the system output.

Then both the following systems are state observers for system (24) in the sense of Definition 1:

$$\dot{z}_1 = A_1 z_1 + \beta_1(y), \tag{26}$$

$$\dot{z}_2 = A_2 z_2 + \beta_2(y). \tag{27}$$

Remark 2. The existence of the functions Ψ_i $(i = 1, 2)$ in a neighborhood of the origin is guaranteed by the Lyapunov Auxiliary theorem provided $[A_i, \beta_i]$ form controllable pairs and provided the eigenvalues $\lambda_{A_{i,j}}$ of the matrices A_i are not related to the eigenvalue $\lambda_{f,l}$ of the Jacobian matrix of f evaluated at the origin through any relation of the type

$$\sum_{l=1}^{n} m_{i,l}\lambda_{f,l} = \lambda_{A_{i,j}} \tag{28}$$

$(j = 1, \ldots, n)$, where the $m_{i,l}$ are nonnegative integers [9].

Let us introduce the following definitions:

$$\nabla\Psi_i = \left(\frac{\partial\Psi_i}{\partial x}\right)_{x(t)}, \tag{29}$$

$$\nabla^D\Psi_i = \left(\frac{\partial\Psi_i}{\partial x}\right)_{x(t-D)}, \tag{30}$$

$$\Delta(z_i) = e^{-A_i D} z_i(t) - z_i(t-D), \tag{31}$$

$$\Theta = \nabla^D\Psi_1\left(\nabla^D\Psi_2\right)^{-1}, \tag{32}$$

$$\Omega = e^{-A_1 D}\nabla\Psi_1 - \Theta e^{-A_2 D}\nabla\Psi_2. \tag{33}$$

The design of a nonlinear finite-time converging observer is now presented in Theorem 2.

Theorem 2. *Let D be an arbitrary positive real constant. Let $[A_1, \beta_1]$ and $[A_2, \beta_2]$ form controllable pairs and assume that the following term is invertible:*

$$e^{-A_1 D}\nabla\Psi_1 - \nabla^D\Psi_1\left(\nabla^D\Psi_2\right)^{-1} e^{-A_2 D}\nabla\Psi_2. \tag{34}$$

Then the dynamical system

$$\dot{\hat{x}}(t) = \Omega^{-1}\left(\Delta(\dot{z}_1) - \Theta\Delta(\dot{z}_2)\right), \tag{35}$$

with initial conditions satisfying

$$z_1(t_0) = \Psi_1(\hat{x}(t_0)), \tag{36}$$

$$z_2(t_0) = \Psi_2(\hat{x}(t_0)), \tag{37}$$

$$z_1(t_0) = \nabla^0\Psi_1\left(\nabla^0\Psi_2\right)^{-1} z_2(t_0), \tag{38}$$

is an identity finite-time observer for (24).

Proof. The proof proceeds in two steps. Firstly, we show that the estimate dynamics is identical to the state dynamics after the time delay D. Secondly, we show that the estimate reaches exactly the state at time $t = t_0 + D$.

Let us define the reconstruction errors associated to observers (26) and (27) as follows:

$$\begin{aligned} \epsilon_1(t) &= \Psi_1(x(t)) - z_1(t), \\ \epsilon_2(t) &= \Psi_2(x(t)) - z_2(t), \end{aligned} \tag{39}$$

which are governed by the following linear dynamics:

$$\begin{aligned} \dot{\epsilon}_1 &= A_1\epsilon_1, \\ \dot{\epsilon}_2 &= A_2\epsilon_2. \end{aligned} \tag{40}$$

Introducing the reconstruction errors definitions (39) into (35), the observer expression becomes

$$\dot{\hat{x}} = \Omega^{-1}\left(\Delta(\dot{\Psi}_1(x)) - \Theta\Delta(\dot{\Psi}_2(x)) - \Delta(\dot{\epsilon}_1) + \Theta\Delta(\dot{\epsilon}_2)\right). \tag{41}$$

By definition of Δ, Θ, and Ω ((31), (32), (33)), it can be seen that

$$\Delta(\dot{\Psi}_1(x)) - \Theta\Delta(\dot{\Psi}_2(x)) = \Omega\dot{x}. \tag{42}$$

Therefore, (41) can be rewritten as follows:

$$\dot{x}(t) - \dot{\hat{x}}(t) = \Omega^{-1}\left(\Delta(\dot{\epsilon}_1) - \Theta\Delta(\dot{\epsilon}_2)\right). \tag{43}$$

As the reconstruction error ϵ_1 and ϵ_2 have linear dynamics (40), it can be seen that for any time $t > t_0 + D$, we have

$$\begin{aligned} \Delta(\dot{\epsilon}_1) &= 0, \\ \Delta(\dot{\epsilon}_2) &= 0. \end{aligned} \tag{44}$$

Consequently, the right-hand side of (43) vanishes after the time delay D and we have for any time $t > t_0 + D$

$$\dot{\hat{x}}(t) = \dot{x}(t). \tag{45}$$

This implies that the estimate dynamics and the state dynamics are identical after the convergence time interval. It remains to show that the estimate reaches exactly the state at time $t = t_0 + D$.

Let us focus on the time interval $[t_0, t_0 + D]$. During this time interval, the delayed values for the different variables remain constant and equal to their initial values:

$$\begin{aligned} z_1(t - D) &= z_1(t_0), \\ z_2(t - D) &= z_2(t_0), \\ \Theta &= \Theta(t_0). \end{aligned} \tag{46}$$

The observer expression (35) becomes

$$\dot{\hat{x}}(t) = \Omega^{-1}\left(e^{-A_1D}\dot{z}_1 - \Theta(t_0)e^{-A_1D}\dot{z}_2\right). \tag{47}$$

The above expression can be rewritten as follows:

$$\Omega\dot{\hat{x}}(t) = e^{-A_1D}\dot{z}_1 - \Theta(t_0)e^{-A_1D}\dot{z}_2 \tag{48}$$

and, by definition of Ω, it leads to the following equation:

$$\begin{aligned} &e^{-A_1D}\dot{\Psi}_1(\hat{x}(t)) - \Theta(t_0)e^{-A_2D}\dot{\Psi}_2(\hat{x}(t)) \\ &\quad = e^{-A_1D}\dot{z}_1 - \Theta(t_0)e^{-A_2D}\dot{z}_2 \end{aligned} \tag{49}$$

which can be integrated between t_0 and t using the initial conditions described by (36) and (38). This leads to

$$\begin{aligned} &e^{-A_1D}\Psi_1(\hat{x}(t)) - \Theta(t_0)e^{-A_2D}\Psi_2(\hat{x}(t)) \\ &\quad = e^{-A_1D}z_1(t) - \Theta(t_0)^0 e^{-A_2D}z_2(t). \end{aligned} \tag{50}$$

This equation can be rewritten as follows using the reconstruction errors expressions (39):

$$\begin{aligned} &e^{-A_1D}\Psi_1(\hat{x}(t)) - \Theta(t_0)e^{-A_2D}\Psi_2(\hat{x}(t)) \\ &\quad = e^{-A_1D}\Psi_1(x(t)) - \Theta(t_0)e^{-A_2D}\Psi_2(x(t)) \\ &\qquad - e^{-A_1D}\epsilon_1(t) + \Theta(t_0)e^{-A_2D}\epsilon_2(t). \end{aligned} \tag{51}$$

Evaluating of the above expression at time $t = t_0 + D$ leads to the following expression:

$$\begin{aligned} &e^{-A_1D}\Psi_1(\hat{x}(t_0 + D)) - \Theta(t_0)e^{-A_2D}\Psi_2(\hat{x}(t_0 + D)) \\ &\quad = e^{-A_1D}\Psi_1(x(t_0 + D)) - \Theta(t_0)e^{-A_2D}\Psi_2(x(t_0 + D)) \\ &\qquad - \epsilon_1(t_0) + \Theta(t_0)\epsilon_2(t_0). \end{aligned} \tag{52}$$

The assumption on the initial conditions (38) is such that

$$\hat{x}(t_0 + D) = x(t_0 + D). \tag{53}$$

This shows that the estimate reaches the state at time $t = t_0 + D$. This completes the proof. □

The implementation of the above observer does not require to compute any change of variable; however, it requires to compute the Jacobian matrices of two transformations. This can be achieved by solving both the following partial differential equations systems obtained from (25):

$$\frac{\partial\Psi_1}{\partial x}f(x) = A_1\Psi_1(x) + \beta_1(y), \tag{54}$$

$$\frac{\partial\Psi_2}{\partial x}f(x) = A_2\Psi_2(x) + \beta_2(y), \tag{55}$$

where the Hurwitz matrices A_1 and A_2 and the functions β_1 and β_2 have been chosen so that $[A_1, \beta_1]$ and $[A_2, \beta_2]$ form controllable pairs and so that there is no resonance between the eigenvalues of A and the function f. This ensures that each of the above systems has a unique solution around the origin [9, 15].

The algorithm proposed in [9] is a practical way to compute the Jacobian matrices $\partial\Psi_i/\partial x$ ($i = 1, 2$). The idea is to approximate the functions Ψ_i by their truncated Taylor series around the origin which is an equilibrium point. The different Taylor series coefficients are computed recursively by evaluating both sides of (54) and (55) and their successive partial derivatives with respect to the state at the origin.

The implementation of the finite time converging observer also requires to impose corresponding initial conditions for the different dynamical systems (36) and (37). This can easily be done by setting the initial conditions at the origin. Furthermore, this also involves that condition (38) is satisfied.

The nonlinear finite-time observer presented in this section is the generalization of the finite-time converging observer for linear systems proposed in [11]. This latter takes advantage of the linear reconstruction error dynamics of two state observers to link the present and delayed errors. The state of the system can then be expressed as a function of the estimates at present and delayed time instances by solving the following set of equations:

$$\begin{aligned} x(t) &= z_1(t) + \epsilon_1(t), \\ x(t) &= z_2(t) + \epsilon_2(t), \\ x(t - D) &= z_1(t - D) + e^{-A_1D}\epsilon_1(t), \\ x(t - D) &= z_2(t - D) + e^{-A_2D}\epsilon_2(t), \end{aligned} \tag{56}$$

where x is the state, and z_1 and z_2 are the estimates provided by two observers with linear error dynamics characterized by the matrices A_1 and A_2, respectively. The finite-time converging observer $\hat{x}$ is then written as the solution of the above set of equation and is therefore exactly equal to the state after the time delay D.

The finite-time converging observer for nonlinear systems presented in this paper is also based on the existence of two observers with linear error dynamics. However, these ones are state observers in the sense of Definition 1, that is, they are dynamical systems that are able to reconstruct a function of the state with a linear reconstruction error dynamics. The corresponding set of equations then is

$$\begin{aligned}
\Psi_1(x(t)) &= z_1(t) + \epsilon_1(t),\\
\Psi_2(x(t)) &= z_2(t) + \epsilon_2(t),\\
\Psi_1(x(t-D)) &= z_1(t-D) + e^{-A_1 D}\epsilon_1(t),\\
\Psi_2(x(t-D)) &= z_2(t-D) + e^{-A_2 D}\epsilon_2(t),
\end{aligned} \tag{57}$$

where x is the state, and z_1 and z_2 are two state observers reconstructing the functions $\Psi_1(x)$ and $\Psi_2(x)$ with linear errors dynamics characterized by the matrices A_1 and A_2, respectively. The above set of equations can be differentiated to make $\dot{x}$ appear explicitly:

$$\begin{aligned}
\nabla\Psi_1\dot{x}(t) &= \dot{z}_1(t) + \dot{\epsilon}_1(t),\\
\nabla\Psi_2\dot{x}(t) &= \dot{z}_2(t) + \dot{\epsilon}_2(t),\\
\nabla^D\Psi_1\dot{x}(t-D) &= \dot{z}_1(t-D) + \dot{\epsilon}_1(t),\\
\nabla^D\Psi_2\dot{x}(t-D) &= \dot{z}_2(t-D) + \dot{\epsilon}_2(t).
\end{aligned} \tag{58}$$

The finite-time converging observer is then written as the solution of the above system for $\dot{x}$. It is thus written as a dynamical system $\hat{\dot{x}}$ which is exactly equal to $\dot{x}$ after the time delay D.

It is worth noting that the finite-time converging observer developed in Section 3 is a particular case of this nonlinear finite time converging observer. If Ψ_1 is the solution of (54) and if the output map of the system is a linear function of the new coordinates

$$y = C_1\Psi_1, \tag{59}$$

then Ψ_1 is also the solution of (55) if the matrix A_2 and the function β_2 satisfy the following conditions:

$$\begin{aligned}
A_2 &= A_1 - aC_1,\\
\beta_2 &= \beta_1 + aC_1,
\end{aligned} \tag{60}$$

where a is a real number. In this case, $\Psi_1 = \Psi_2$, and it follows from (32) that

$$\begin{aligned}
\Theta(t_0) &= I_n,\\
\Omega &= e^{-A_1 D} - e^{-A_2 D}\frac{\partial\Psi_1}{\partial x},
\end{aligned} \tag{61}$$

which leads to exactly the same dynamical system as in Section 3. This shows that the changes of variables Ψ_1 and Ψ_2 can be the same but the error dynamics of observers z_1 and z_2 have to be different.

Table 1: Numerical values used for the simulation.

Parameter	Value	Unit	Variable	Initial value	Unit
$\frac{Q}{V}$	0.1	h^{-1}	$x_1(0)$	0.01	g/l
$\mu_{\max}$	0.5	h^{-1}	$x_2(0)$	0.1	g/l
K_S	10	g/l	$\hat{x}_1(0)$	0	g/l
S_{in}	25	g/l	$\hat{x}_2(0)$	0	g/l
$\overline{X}$	56.25	g/l	—	—	—
$\overline{S}$	2.5	g/l	—	—	—
k	0.4	—	—	—	—

5. APPLICATION TO A BIOREACTOR

Let us consider a continuous stirred tank reactor with a simple microbial growth reaction

$$S \longrightarrow X, \tag{62}$$

where S and X are the substrate and the biomass, respectively. The dynamical model of such a process is given by the following mass balance equations [17]:

$$\begin{aligned}
\frac{dX}{dt} &= (\mu - D)X,\\
\frac{dS}{dt}7 &= D(S_{in} - S) + k\mu X,
\end{aligned} \tag{63}$$

where X, S, S_{in}, D, k, and μ are the biomass concentration (g/l), the substrate concentration in the reactor (g/l), the inlet substrate concentration (g/l), the dilution rate (h^{-1}), the yield coefficient, and the specific growth rate (h^{-1}), respectively. Furthermore, we assume that the specific growth rate μ obeys to the common Monod kinetics:

$$\mu = \frac{\mu_{\max}S}{K_S + S}, \tag{64}$$

where $\mu_{\max}$ (h^{-1}) and K_S (g/l) are the maximum specific growth rate and the half saturation constant, respectively. The numerical values used for the system parameters are listed in Table 1.

Let us define the following state variables in order the origin to be an equilibrium point:

$$\begin{aligned}
x_1 &= X - \overline{X},\\
x_2 &= S - \overline{S},
\end{aligned} \tag{65}$$

where $\overline{X}$ and $\overline{S}$ are the equilibrium values for X and S (see Table 1). Let us consider that the biomass concentration is measured online; we have the following state-space model:

$$\dot{x}_1 = f_1(x_1, x_2), \tag{66}$$

$$\begin{aligned}
\dot{x}_2 &= f_2(x_1, x_2),\\
y &= h(x_1, x_2),
\end{aligned} \tag{67}$$

TABLE 2: Computed values for the Taylors series coefficients.

Coeff.	Value	Coeff.	Value	Coeff.	Value	Coeff.	Value
P_0^1	0	Q_0^1	0	P_0^2	0	Q_0^2	0
P_1^1	5.25E-02	Q_1^1	5.00E-02	P_1^2	4.37E-02	Q_1^2	1.87E-02
P_2^1	−5.17E-03	Q_2^1	−4.70E-03	P_2^2	−1.72E-03	Q_2^2	−5.68E-04
P_{11}^1	−1.96E-07	Q_{11}^1	−1.70E-07	P_{11}^2	−2.56E-08	Q_{11}^2	−5.00E-09
P_{12}^1	−9.35E-05	Q_{12}^1	−8.48E-05	P_{12}^2	−3.07E-05	Q_{12}^2	−1.01E-05
P_{22}^1	4.60E-04	Q_{22}^1	4.15E-04	P_{22}^2	1.43E-04	Q_{22}^2	4.65E-05
P_{111}^1	−5.42E-11	Q_{111}^1	−4.55E-11	P_{111}^2	−3.09E-13	Q_{111}^2	−4.45E-14
P_{112}^1	−2.59E-08	Q_{112}^1	−2.27E-08	P_{112}^2	−3.90E-09	Q_{112}^2	−8.00E-10
P_{122}^1	8.86E-06	Q_{122}^1	7.98E-06	P_{122}^2	2.63E-06	Q_{122}^2	8.44E-07
P_{222}^1	−4.27E-05	Q_{222}^1	−3.83E-05	P_{222}^2	−1.21E-05	Q_{222}^2	−3.84E-06
P_{1111}^1	7.47E-14	Q_{1111}^1	5.90E-14	P_{1111}^2	1.10E-16	Q_{1111}^2	−4.60E-16
P_{1112}^1	3.59E-11	Q_{1112}^1	2.95E-11	P_{1112}^2	1.48E-13	Q_{1112}^2	−7.10E-14
P_{1122}^1	1.23E-08	Q_{1122}^1	1.07E-08	P_{1122}^2	1.50E-09	Q_{1122}^2	3.00E-10
P_{1222}^1	−9.08E-07	Q_{1222}^1	−8.08E-07	P_{1222}^2	−2.32E-07	Q_{1222}^2	−7.11E-08
P_{2222}^1	4.16E-06	Q_{2222}^1	3.70E-06	P_{2222}^2	1.04E-06	Q_{2222}^2	3.19E-07

with

$$f_1 = (x_1 + \overline{X})\left(\frac{\mu_{\max}(x_2 + \overline{S})}{K_S + x_2 + \overline{S} - D}\right),$$
$$f_2 = D(S_{in} - (x_2 + \overline{S})) + k\frac{\mu_{\max}(x_2 + \overline{S})}{K_S + x_2 + \overline{S}}(x_1 + \overline{X}),$$
$$h = x_1, \tag{68}$$

and where the following conditions are satisfied:

$$f(0,0) = 0, \qquad h(0,0) = 0. \tag{69}$$

We now have to choose the matrices A_1 and A_2 and the functions β_1 and β_2 in order to design both state observers $\dot{z}_1$ (26) and $\dot{z}_2$ (27). The matrices A_1 and A_2 and the functions β_1 and β_2 can be arbitrarily defined provided there is no resonance between the eigenvalues of A_i and the function f and provided $[A_i, \beta_i]$ form controllable pairs. Using the numerical values of Table 1, we can compute the eigenvalues of the Jacobian matrix of f at the origin:

$$\lambda_{f,1} = -0.1, \quad \lambda_{f,2} = -0.72. \tag{70}$$

We arbitrarily assign double eigenvalues to the matrices A_1 and A_2 so that there is no resonance with f:

$$\lambda_{A_1} = -20, \quad \lambda_{A_2} = -40. \tag{71}$$

Finally, we choose the following matrices:

$$A_1 = \begin{pmatrix} -20 & 1 \\ 0 & -20 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -50 & 10 \\ -10 & -30 \end{pmatrix}, \tag{72}$$

and impose the following functions $\beta_i(y)$ to form controllable pairs:

$$\beta_1 = \begin{pmatrix} y \\ y \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 2y \\ y \end{pmatrix}. \tag{73}$$

Once A_1, A_2, β_1, and β_2 have been defined, it remains to compute the Jacobian matrices of the functions $\Psi_1(x)$ and $\Psi_2(x)$ that transform the system into the pseudolinear forms (25). This is achieved by solving the partial differential equation systems (54) and (55). According to the algorithm described in [9], we approximate both functions $\Psi_1(x)$ and $\Psi_2(x)$ by their Taylor series development around the origin (with $i = 1, 2$):

$$\Psi_i(x) = \begin{pmatrix} P_0^i + P_1^i x_1 + P_2^i x_2 + P_{11}^i x_1^2 + P_{12}^i x_1 x_2 + P_{22}^i x_2^2 + \cdots \\ Q_0^i + Q_1^i x_1 + Q_2^i x_2 + Q_{11}^i x_1^2 + Q_{12}^i x_1 x_2 + Q_{22}^i x_2^2 + \cdots \end{pmatrix}, \tag{74}$$

which leads to the following Jacobian matrices:

$$\frac{\partial \Psi_i}{\partial x} = \begin{pmatrix} P_1^i + 2P_{11}^i x_1 + P_{12}^i x_2 + \cdots & P_2^i + 2P_{22}^i x_2 + P_{12}^i x_1 + \cdots \\ Q_1^i + 2Q_{11}^i x_1 + Q_{12}^i x_2 + \cdots & Q_2^i + 2Q_{22}^i x_2 + Q_{12}^i x_1 + \cdots \end{pmatrix}, \tag{75}$$

then, by estimating both sides of (54) and (55) at the origin and taking advantage that $f(0) = 0$, we find that P_0 and Q_0 are both equal to zero. The other coefficients are computed by evaluating the successive partial derivatives of both sides

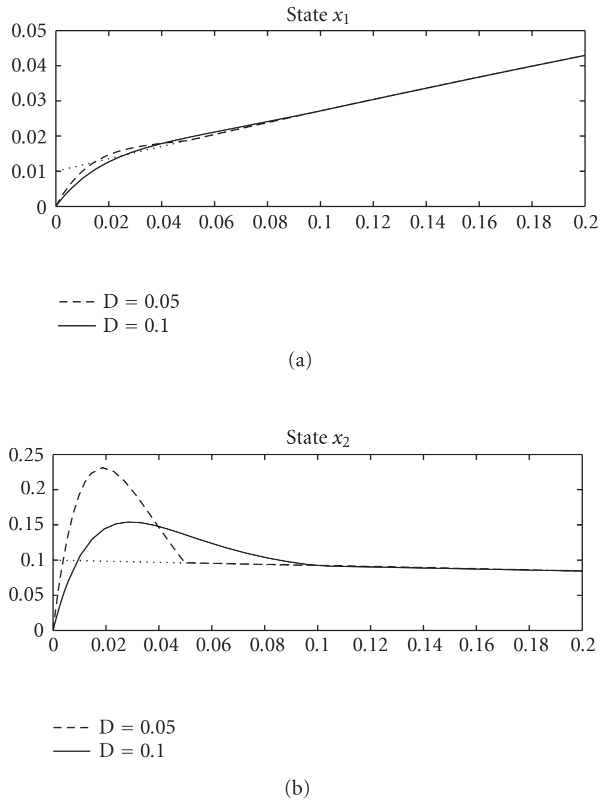

Figure 1: Finite-time estimation with different convergence time delays, $A : D = 0.05$, $B : D = 0.1$.

of (54) and (55) with respect to the state variables at the origin. The computed values for the coefficients can be found in Table 2. It can be seen that the coefficients' values become rapidly negligible; therefore, we have stopped the computing procedure to the 4th order.

The matrices A_1 and A_2, the functions β_1 and β_2, and the Jacobian matrices $\partial\Psi_1/\partial x$ and $\partial\Psi_2/\partial x$ being defined, the finite-time converging observer (35) can be implemented to reconstruct the state of system (66). The results of two simulations with different arbitrary convergence time interval are shown on Figure 1. The numerical values used for this simulation can be found in Table 1.

6. CONCLUSION

In this paper, we have presented the design of a nonlinear observer that converges in finite time.

This nonlinear observer is the generalization of the linear finite-time converging observer. It is written as a dynamical system that exploits the present and delayed estimates of two state observers converging each estimate to a function of the state with a linear error dynamics. Considering the dynamics instead of the states' variables allows to use the transformation Jacobian matrices rather than computing the change of coordinate. The nonlinear observer is therefore based on the existence of two transformations that linearize the system dynamics. However, the estimate is computed in one step only without computing any change of coordinate.

The Jacobian matrices of the linearizing transformations can be computed by solving a partial differential equations system. A practical way to compute them is to approximate the transformations by their truncated Taylor series development around an equilibrium point. Finally, we have illustrated the application of the designed nonlinear finite-time converging observer to a bioreactor model.

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