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Research Article

Higher-Order Generalized Invexity in Control Problems

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We introduce a higher-order duality (Mangasarian type and Mond-Weir type) for the control problem. Under the higher-order generalized invexity assumptions on the functions that compose the primal problems, higher-order duality results (weak duality, strong duality, and converse duality) are derived for these pair of problems. Also, we establish few examples in support of our investigation.

1. Introduction

We consider the control problem

$$\text{(CP)}\ \min \quad \int_a^b f(t, x(t), \dot{x}(t), u(t))dt, \tag{1}$$

$$\text{subject to} \quad g(t, x(t), \dot{x}(t), u(t)) \leq 0, \tag{2}$$

$$G(t, x(t), \dot{x}(t), u(t)) = 0, \tag{3}$$

$$\begin{aligned} x(a) &= \gamma_1, \qquad x(b) = \gamma_2; \\ u(a) &= \delta_1, \qquad u(b) = \delta_2, \end{aligned} \tag{4}$$

where f, g, and G are twice continuously differentiable functions from $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}$, $\mathbb{R}^r$, and $\mathbb{R}^s$, respectively, $I = [a, b] \subseteq \mathbb{R}$.

Mangasarian [1] formulated a class of higher-order dual problems for a nonlinear programming problems involving twice continuously differentiable functions. He did not prove the weak duality and hence gave a limited strong duality theorem. Mond and Zhang [2] introduced invexity type conditions under which duality holds between Mangasarian [1] primal problem and various higher-order dual programming problems.

One practical advantage of higher-order duality is that it provides tighter bounds for the value of the objective function of the primal problem when approximations are used, because there are more parameters involved. Higher-order duality in nonlinear programming has been studied by several researchers like Mond and Zhang [2], Chen [3], Mishra and Rueda [4], and Yang et al. [5]. Recently, Gulati and Gupta [6] studied the higher-order symmetric duality over arbitrary cones for Wolfe and Mond-Weir type models. Obtained duality results for various higher-order dual problems under higher-order and type higher-order duality to higher-order type I functions.

Bhatia and Kumar [7] have studied the multiobjective control problems under ρ-pseudoinvexity, ρ-strictly pseudoinvexity, ρ-quasi-invexity, and ρ-strictly quasi-invexity assumptions. Nahak and Nanda [8] have studied the efficiency and duality for multiobjective control problems under $(F - \rho)$ convexity. Again Nahak and Nanda [9] proposed a sufficient condition for solutions and duality for the multiobjective variational control problems under V-invexity. Recently, Padhan and Nahak [10] considered a class of constrained nonlinear control primal problem and formulated the second-order dual. He also gave some duality results (weak duality, strong duality, and converse duality) under generalized invexity assumptions. But in our knowledge, no one has talked about higher-order duality for the control problem. In this paper, we study both Mangasarian and Mond-Weir type higher-order duality of the control primal problem (CP). We give more general type conditions that is higher-order generalized invexity under which duality holds between (CP) and (MHCD), and

(CP) and (MWHVD). Our approach is similar to that of Mangasarian [1]. Again, we discuss many counterexamples to justify our work.

2. Notations and Preliminaries

Let $S(I,\mathbb{R}^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Longleftrightarrow x(t) = \kappa + \int_a^t u(s)ds, \tag{5}$$

where κ is a given boundary value; thus, $d/dt = D$ except at discontinuities.

The higher-order generalized invexity functions are defined as follows.

Definition 2.1. The scalar functional $F(x,u) = \int_a^b f(t,x(t),\dot{x}(t),u(t))dt$ is said to be higher-order $\rho-(\eta,\xi,\theta)$-invex in x, $\dot{x}$, and u if there exist $\eta,\theta : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^n$, $\xi : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^m$, $h : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}$, $h_1 : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ and $\rho\in\mathbb{R}$, such that

$$\begin{aligned}
&F(x,u)-F(y,v)\\
&\quad\geq \int_a^b\Big[\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\qquad\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\qquad\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad+h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad-p^T\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad-q^T\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad+\rho\|\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\|^2\Big]dt.
\end{aligned} \tag{6}$$

Definition 2.2. The scalar functional $F(x,u) = \int_a^b f(t,x(t),\dot{x}(t),u(t))dt$ is said to be higher-order $\rho-(\eta,\xi,\theta)$-pseudoinvex in x, $\dot{x}$, and u if there exist $\eta,\theta : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^n$, $\xi : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^m$, $h : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}$, $h_1 : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ and $\rho\in\mathbb{R}$, such that

$$\begin{aligned}
&\int_a^b\Big[\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\qquad\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\qquad\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad+\rho\|\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\|^2\Big]dt\geq 0\\
&\Longrightarrow F(x,u)-F(y,v)\\
&\quad-\int_a^b\Big[h(t,y(t),\dot{y}(t),v(t),p)+h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad-p^T\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad-q^T\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\Big]dt\geq 0.
\end{aligned} \tag{7}$$

Definition 2.3. The scalar functional $F(x,u) = \int_a^b f(t,x(t),\dot{x}(t),u(t))dt$ is said to be higher-order $\rho-(\eta,\xi,\theta)$-quasi-invex in x, $\dot{x}$ and u, if there exist $\eta,\theta : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^n$, $\xi : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^m$, $h : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}$, $h_1 : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}$ and $\rho\in\mathbb{R}$, such that

$$\begin{aligned}
&F(x,u)-F(y,v)\\
&\quad-\int_a^b\Big[h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad-p^T\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad-q^T\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\Big]dt\leq 0\\
&\Longrightarrow \int_a^b\Big[\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\qquad\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\qquad\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad+\rho\|\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\|^2\Big]dt\\
&\quad\leq 0.
\end{aligned} \tag{8}$$

Remark 2.4. If

$$\begin{aligned}
&h(t,y(t),\dot{y}(t),v(t),p)\\
&\quad= p^T\Big[f_x(t,y(t),\dot{y}(t),v(t))-\frac{d}{dt}f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))\Big],\\
&h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\quad= q^T f_u(t,y(t),\dot{y}(t),v(t)),
\end{aligned} \tag{9}$$

with $\eta = 0$ at $t = a$ and $t = b$, then the above definitions becomes the definitions of invexity defined by Padhan and Nahak [10].

3. Mangasarian Type Higher-Order Duality

In this section, we propose the following Mangasarian type higher-order dual (MHCD) to (CP):

$$\begin{aligned}(\text{MHCD})\max \quad & \int_a^b \Big[f(t,y(t),\dot{y}(t),v(t)) \\ & + \alpha(t)^T g(t,y(t),\dot{y}(t),v(t)) \\ & + \beta(t)^T G(t,y(t),\dot{y}(t),v(t)) \\ & + h(t,y(t),\dot{y}(t),v(t),p) \\ & + \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p) \\ & + \beta(t)^T l(t,y(t),\dot{y}(t),v(t),p) \\ & + h_1(t,y(t),\dot{y}(t),v(t),p) \\ & + \alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q) \\ & + \beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q) \Big] dt, \end{aligned} \tag{10}$$

$$\begin{aligned}\text{subject to} \quad & \nabla_p h(t,y(t),\dot{y}(t),v(t),p) \\ & + \nabla_p \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p) \\ & + \nabla_p \beta(t)^T l(t,y(t),\dot{y}(t),v(t),p) = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} & \nabla_q h_1(t,y(t),\dot{y}(t),v(t),q) \\ & + \nabla_q \alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q) \\ & + \nabla_q \beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} & y(a) = \gamma_1, \qquad y(b) = \gamma_2; \\ & v(a) = \delta_1, \qquad v(b) = \delta_2, \end{aligned} \tag{13}$$

$$\begin{aligned} & \alpha(t) \in \mathbb{R}_+^r, \qquad \beta(t) \in \mathbb{R}^s, \qquad \alpha_1(t) \in \mathbb{R}_+^{r_1}, \\ & \beta_1(t) \in \mathbb{R}^{s_1}, \qquad p \in \mathbb{R}^n, \qquad q \in \mathbb{R}^m. \end{aligned} \tag{14}$$

Remark 3.1. If

$$\begin{aligned} & h(t,y(t),\dot{y}(t),v(t),p) \\ & = p\left[f_x(t,y(t),\dot{y}(t),v(t)) - \frac{d}{dt} f_{\dot{x}}(t,y(t),\dot{y}(t),v(t)) \right] \\ & + \frac{1}{2} p \Big[f_{xx}(t,y(t),\dot{y}(t),v(t)) \\ & \quad - 2\frac{d}{dt} f_{x\dot{x}}(t,y(t),\dot{y}(t),v(t)) \\ & \quad + \frac{d^2}{dt^2} f_{\dot{x}\dot{x}}(t,y(t),\dot{y}(t),v(t)) \Big] p, \\ & h_1(t,y(t),\dot{y}(t),v(t),q) \\ & = q[f_u(t,y(t),\dot{y}(t),v(t))] \\ & + \frac{1}{2} q[f_{uu}(t,y(t),\dot{y}(t),v(t)) + 2 f_{xu}(t,y(t),\dot{y}(t),v(t))] q, \\ & k^j(t,y(t),\dot{y}(t),v(t),p) \\ & = p\left[g_x^j(t,y(t),\dot{y}(t),v(t)) - \frac{d}{dt} g_{\dot{x}}^j(t,y(t),\dot{y}(t),v(t)) \right] \\ & + \frac{1}{2} p \Big[g_{xx}^j(t,y(t),\dot{y}(t),v(t)) \\ & \quad - 2\frac{d}{dt} g_{x\dot{x}}^j(t,y(t),\dot{y}(t),v(t)) \\ & \quad + \frac{d^2}{dt^2} g_{\dot{x}\dot{x}}^j(t,y(t),\dot{y}(t),v(t)) \Big] p, \\ & \qquad j = 1,2,3,\ldots,r, \\ & k_1^j(t,y(t),\dot{y}(t),v(t),q) \\ & = q g_u^j(t,y(t),\dot{y}(t),v(t)) \\ & + \frac{1}{2} q \Big[g_{uu}^j(t,y(t),\dot{y}(t),v(t)) \\ & \quad + 2 g_{xu}^j(t,y(t),\dot{y}(t),v(t)) \Big] q, \\ & \qquad j = 1,2,\ldots,r_1, \\ & l^i(t,y(t),\dot{y}(t),v(t),p) \\ & = p\left[G_x^i(t,y(t),\dot{y}(t),v(t)) - \frac{d}{dt} G_{\dot{x}}^i(t,y(t),\dot{y}(t),v(t)) \right] \\ & + \frac{1}{2} p \Big[G_{xx}^i(t,y(t),\dot{y}(t),v(t)) \\ & \quad - 2\frac{d}{dt} G_{x\dot{x}}^i(t,y(t),\dot{y}(t),v(t)) \\ & \quad + \frac{d^2}{dt^2} G_{\dot{x}\dot{x}}^i(t,y(t),\dot{y}(t),v(t)) \Big] p, \\ & \qquad i = 1,2,3,\ldots,s, \\ & l_1^i(t,y(t),\dot{y}(t),v(t),q) \\ & = q G_u^i(t,y(t),\dot{y}(t),v(t)) \\ & + \frac{1}{2} q \Big[G_{uu}^i(t,y(t),\dot{y}(t),v(t)) \\ & \quad + 2 G_{xu}^i(t,y(t),\dot{y}(t),v(t)) \Big] q, \quad i = 1,2,3,\ldots,s_1, \end{aligned} \tag{15}$$

then (MHCD) is similar to the second-order duality given by Padhan and Nahak [10].

Theorem 3.2 (weak duality). *Let $(x(t), u(t))$ and $(y(t), v(t), \alpha(t), \beta(t), p, q)$ be the feasible solutions of (CP) and (MHCD), respectively. Let $\int_a^b f(t,\cdot,\cdot,\cdot)dt$, $\int_a^b \alpha(t)^T g(t,\cdot,\cdot,\cdot)dt$ and $\int_a^b \beta(t)^T G(t,\cdot,\cdot,\cdot)dt$ be higher-order $\rho_0-(\eta,\xi,\theta)$-invex, higher-order $\rho_1-(\eta,\xi,\theta)$-invex, and higher-order $\rho_2-(\eta,\xi,\theta)$-invex functions in x, $\dot{x}$, and u on I with respect to the same functions η,ξ,θ, with $\rho_0+\rho_1+\rho_2 \geq 0$, then the following inequality holds between the primal (CP) and the dual (MHCD):*

$$\begin{aligned}
&\int_a^b f(t,x(t),\dot{x}(t),u(t))dt \\
&\quad \geq \int_a^b \Big[f(t,y(t),\dot{y}(t),v(t)) + \alpha(t)^T g(t,y(t),\dot{y}(t),v(t)) \\
&\qquad + \beta(t)^T G(t,y(t),\dot{y}(t),v(t)) \\
&\qquad + h(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad + h_1(t,y(t),\dot{y}(t),v(t),q) \\
&\qquad + \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad + \alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q) \\
&\qquad + \beta(t)^T l(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad + \beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q) \Big] dt.
\end{aligned} \tag{16}$$

Proof. Since $(x(t), u(t))$ and $(y(t), v(t), \alpha(t), \beta(t), p, q)$ be the feasible solutions of (CP) and (MHCD), respectively, we have

$$\begin{aligned}
&\int_a^b f(t,x(t),\dot{x}(t),u(t))dt \\
&\quad - \int_a^b \Big[f(t,y(t),\dot{y}(t),v(t)) + \alpha(t)^T g(t,y(t),\dot{y}(t),v(t)) \\
&\qquad + \beta(t)^T G(t,y(t),\dot{y}(t),v(t)) \\
&\qquad + h(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad + h_1(t,y(t),\dot{y}(t),v(t),q) \\
&\qquad + \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad + \alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q) \\
&\qquad + \beta(t)^T l(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad + \beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q) \Big] dt \\
&\quad \geq \left[\int_a^b f(t,x(t),\dot{x}(t),u(t))dt \right. \\
&\qquad \left. - \int_a^b f(t,y(t),\dot{y}(t),v(t))dt \right] \\
&\qquad + \left[\int_a^b \alpha(t)^T g(t,x(t),\dot{x}(t),u(t))dt \right. \\
&\qquad\quad \left. - \int_a^b \alpha(t)^T g(t,y(t),\dot{y}(t),v(t))dt \right] \\
&\qquad + \left[\int_a^b \beta(t)^T G(t,x(t),\dot{x}(t),u(t))dt \right. \\
&\qquad\quad \left. - \int_a^b \beta(t)^T G(t,y(t),\dot{y}(t),v(t))dt \right] \\
&\qquad - \int_a^b \Big[h(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad\quad + h_1(t,y(t),\dot{y}(t),v(t),q) \\
&\qquad\quad + \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad\quad + \alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q) \\
&\qquad\quad + \beta(t)^T l(t,y(t),\dot{y}(t),v(t),p) \\
&\qquad\quad + \beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q) \Big] dt \\
&\quad \geq 0.
\end{aligned} \tag{17}$$

(by the higher-order $\rho-(\eta,\xi,\theta)$-invexity of $\int_a^b f(t,\cdot,\cdot,\cdot)dt$, $\int_a^b \alpha(t)^T g(t,\cdot,\cdot,\cdot)dt$) and $\int_a^b \beta(t)^T G(t,\cdot,\cdot,\cdot)dt, \rho_0+\rho_1+\rho_2 \geq 0$ and equations (11)–(14)). This completes the proof. □

We construct the following example which verifies Theorem 3.2 above, in which the objective and the constraints functions are higher-order $\rho-(\eta,\xi,\theta)$-invex.

Example 3.3. Let us consider the following control problem:

$$\min \quad \int_a^b (x^2(t) - x(t) - u(t))dt, \tag{18}$$

$$\begin{aligned}
\text{subject to} \quad & u(t) - x^2(t) \leq 0, \\
& x(t) - u^2(t) = 0, \\
& x(a) = \gamma_1, \qquad x(b) = \gamma_2, \\
& u(a) = \delta_1, \qquad u(b) = \delta_2,
\end{aligned} \tag{19}$$

where $I = [a,b]$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t,x(t),\dot{x}(t),u(t)) = x^2(t) - x(t) - u(t)$, $g(t,x(t),\dot{x}(t),u(t)) = u(t) - x^2(t)$ and $G(t,x(t),\dot{x}(t),u(t)) = x(t) - u^2(t)$. It is clear that the objective function f, the inequality constraint function g and the equality constraint function G are not $1-(\eta,\xi,\theta)$-invex, $-(1/2)-(\eta,\xi,\theta)$-invex and $-(1/2)-(\eta,\xi,\theta)$-invex, respectively, as defined by Padhan and Nahak [10]. But for $u(t)v(t) > 1$ and $y(t) > v(t)$, they are higher-order $1-(\eta,\xi,\theta)$-invex, higher-order

$-(1/2)-(\eta,\xi,\theta)$-invex and higher-order $-(1/2)-(\eta,\xi,\theta)$-invex, respectively, with respect to the same η,ξ and θ. The functions $\eta,\xi,\theta : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $h : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $k : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $l : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $h_1 : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $k_1 : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $l_1 : I\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are defined as follows:

$$\begin{aligned}
&\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\\
&\quad=-(x^2(t)+x(t)+u(t)+y(t)),\\
&\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\\
&\quad=-(2x^2(t)y(t)+2x(t)y(t)+2y(t)u(t)-u(t)+v(t)),\\
&\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\\
&\quad=\sqrt{x^2(t)+y^2(t)},\\
&h(t,y(t),\dot{y}(t),v(t),p)\\
&\quad=(2y(t)-1)p-y^2(t)-v^2(t),\\
&k(t,y(t),\dot{y}(t),v(t),p)\\
&\quad=-2y(t)p-v^2(t),\\
&l(t,y(t),\dot{y}(t),v(t),p)\\
&\quad=p-y^2(t)-v^2(t),\\
&h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\quad=-q-y^2(t),\\
&k_1(t,y(t),\dot{y}(t),v(t),q)\\
&\quad=q-v^2(t),\\
&l_1(t,y(t),\dot{y}(t),v(t),q)\\
&\quad=-y^2(t)-v^2(t).
\end{aligned}\tag{20}$$

The Mangasarian higher-order dual of the control problem is

$$\max\quad\int_a^b(-4y^2(t)-6v^2(t))\,dt,\tag{21}$$

$$\begin{aligned}
\text{subject to}\quad & y(a)=\gamma_1,\qquad y(b)=\gamma_2;\\
& v(a)=\delta_1,\qquad v(b)=\delta_2,\\
& \alpha(t)=\beta(t)=\alpha_1(t)=\beta_1(t)=1.
\end{aligned}\tag{22}$$

The above problem satisfies weak duality Theorem 3.2 for $x^2(t)+4y^2(t)+6v^2(t)\geq x(t)+u(t)$.

Necessary conditions for the existence of an extremal solution for a variational problem subject to the both equality and inequality constraints were given by Valentine [11]. Using Valentine's results, Berkovitz [12] obtained the corresponding necessary conditions for the control problem (CP). These may be stated in the following way. If $(y(t),v(t))$ is an optimal solution for (CP), then

$$\begin{aligned}
&\mu_0 f_x(t,y(t),\dot{y}(t),v(t))+g_x(t,y(t),\dot{y}(t),v(t))^T\alpha(t)\\
&\quad+G_x(t,y(t),\dot{y}(t),v(t))^T\beta(t)\\
&\quad=\frac{d}{dt}\Big(\mu_0 f_{\dot{x}}(t,y(t),\dot{y}(t),v(t))\\
&\qquad+g_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T\alpha(t)\\
&\qquad+G_{\dot{x}}(t,y(t),\dot{y}(t),v(t))^T\beta(t)\Big),
\end{aligned}\tag{23}$$

$$\begin{aligned}
&\mu_0 f_u(t,y(t),\dot{y}(t),v(t))+g_u(t,y(t),\dot{y}(t),v(t))^T\alpha(t)\\
&\quad+h_u(t,y(t),\dot{y}(t),v(t))^T\beta(t)=0,
\end{aligned}\tag{24}$$

$$\alpha(t)^T g(t,y(t),\dot{y}(t),v(t))=0,\tag{25}$$

$$\beta(t)^T G(t,y(t),\dot{y}(t),v(t))=0,\tag{26}$$

$$\alpha(t)\in\mathbb{R}^r_+\tag{27}$$

hold throughout $a\leq t\leq b$ (except for the values of t corresponding to points of discontinuity of $u(t)$, (23) holds for right and left hand limits). Here, μ_0 is nonnegative constant, $\beta(t)$ is continuous in $a\leq t\leq b$, and $\mu_0,\alpha(t)$, and $\beta(t)$ cannot vanish simultaneously for any $a\leq t\leq b$. It will be assumed that the minimizing arc determined by $y(t),v(t)$ is normal, that is, that μ_0 can be taken equal to 1.

Theorem 3.4 (strong duality). *Let $(\overline{x}(t),\overline{u}(t))$ be a local or global optimal solution of (CP) at which the constraint qualification (23)–(27) are satisfied, and for the piecewise smooth functions $\overline{\alpha}:I\to\mathbb{R}^r$, $\overline{\beta}:I\to\mathbb{R}^s$, let*

(i) $h(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=h_1(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=0$,

(ii) $k(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=k_1(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=0$,

(iii) $l(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=l_1(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=0$,

(iv) $\nabla_p h(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=f_x(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))-(d/dt)\ f_{\dot{x}}(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))$,

(v) $\nabla_p\overline{\alpha}(t)^T k(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=g_x(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))^T\overline{\alpha}(t)-(d/dt)g_{\dot{x}}(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))^T\overline{\alpha}(t)$,

(vi) $\nabla_p\overline{\beta}(t)^T l(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=G_x(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))^T\overline{\beta}(t)-(d/dt)G_{\dot{x}}(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))^T\overline{\beta}(t)$,

(vii) $\nabla_q h_1(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=f_u(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))$,

(viii) $\nabla_q\overline{\alpha}(t)^T k_1(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=g_u(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))^T\overline{\alpha}(t)$,

(ix) $\nabla_q\overline{\beta}(t)^T l_1(t,\overline{x}(t),\overline{x}(t),\overline{u}(t),0)=G_u(t,\overline{x}(t),\overline{x}(t),\overline{u}(t))^T\overline{\beta}(t)$.

Then, $(\overline{x}(t),\overline{u}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p}=0,\overline{q}=0)$ is feasible for (MHCD). Moreover, if the weak duality Theorem 3.2 holds between the control primal (CP) and the Mangasarian higher-order dual (MHCD). Then, $(\overline{x}(t),\overline{u}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p}=0,\overline{q}=0)$ is an optimal solution of (MHCD), and the optimal values of (CP) and (MHCD) are equal.

Proof. Since $(\overline{x}(t),\overline{u}(t))$ is an optimal solution of (CP), from (23)–(27), we can easily conclude that $(\overline{x}(t),\overline{u}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p}=0,\overline{q}=0)$ satisfies the constraints of (MHCD) and objective values of (CP) and (MHCD) are equal. Hence, the result follows. □

Theorem 3.5 (converse duality). *Let $(\overline{x}(t),\overline{u}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p},\overline{q})$ be an optimal solution of (MWHCD). Suppose that $\int_a^b f(t,\cdot,\cdot,\cdot)dt$, $\int_a^b \overline{\alpha}(t)^T g(t,\cdot,\cdot,\cdot)dt$ and $\int_a^b \overline{\beta}(t)^T G(t,\cdot,\cdot,\cdot)dt$ are higher-order $\rho_0-(\eta,\xi,\theta)$-invex, higher-order $\rho_1-(\eta,\xi,\theta)$-invex, and higher-order $\rho_2-(\eta,\xi,\theta)$-invex functions in $\overline{x}$, $\dot{\overline{x}}$, and $\overline{u}$ on I with respect to the same functions η,ξ,θ, with $\rho_0+\rho_1+\rho_2\geq 0$. Moreover, if*

$$\begin{aligned}
&\overline{\alpha}(t)^T g(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t)) + \overline{\beta}(t)^T G(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t)) \\
&\quad\times h(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\
&\quad+ h_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) + \overline{\alpha}k(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\
&\quad+ \overline{\alpha}_1 k_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) + \overline{\beta}l(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\
&\quad+ \overline{\beta}_1 l_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) \\
&\geq 0,
\end{aligned} \tag{28}$$

then $(\overline{x}(t),\overline{u}(t))$ is an optimal solution of (CP).

Proof. Suppose that $(\overline{x}(t),\overline{u}(t))$ is not an optimal solution of (CP). Then, there exists a feasible solution $(x(t),u(t))$ of the primal (CP) such that

$$\int_a^b f(t,x(t),\dot{x}(t),u(t))dt < \int_a^b f(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))dt. \tag{29}$$

Since $\int_a^b f(t,\cdot,\cdot,\cdot)dt$, $\int_a^b \overline{\alpha}(t)^T g(t,\cdot,\cdot,\cdot)dt$ and $\int_a^b \overline{\beta}(t)^T G(t,\cdot,\cdot,\cdot)dt$ are higher-order $\rho_0-(\eta,\xi,\theta)$-invex, higher-order $\rho_1-(\eta,\xi,\theta)$-invex, and higher-order $\rho_2-(\eta,\xi,\theta)$-invex functions with respect to same η, ξ and θ, we have

$$\begin{aligned}
&\int_a^b f(t,x(t),\dot{x}(t),u(t))dt - \int_a^b f(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))dt \\
&\geq \int_a^b \Big[\overline{\alpha}(t)^T g(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t)) \\
&\qquad+ \overline{\beta}(t)^T G(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t)) \\
&\qquad+ h(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\
&\qquad+ h_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) \\
&\qquad+ \overline{\alpha}k(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\
&\qquad+ \overline{\alpha}_1 k_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) \\
&\qquad+ \overline{\beta}l(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\
&\qquad+ \overline{\beta}_1 l_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q})\Big]dt \\
&\quad\text{(by equations (2), (3), (11), (12), and (14))} \\
&\geq 0 \\
&\quad\text{(by equations (28)).}
\end{aligned} \tag{30}$$

This completes the proof. □

Mond-Weir type higher-order duality is established to weaken the higher-order invexity requirements, that is, higher-order pseudoinvexiy and higher-order quasi-invexity.

4. Mond-Weir Type Higher-Order Duality

In this section, we propose the following Mond-Weir type higher-order dual (MWHCD) to (CP):

$$\begin{aligned}
\text{(MWHCD)}\ \max\ & \int_a^b \Big[f(t,y(t),\dot{y}(t),v(t)) \\
&\quad+ h(t,y(t),\dot{y}(t),v(t),p) \\
&\quad+ h_1(t,y(t),\dot{y}(t),v(t),q) \\
&\quad- p^T\nabla_p h(t,y(t),\dot{y}(t),v(t),p) \\
&\quad- q^T\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\Big]dt,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\text{subject to}\quad & \nabla_p h(t,y(t),\dot{y}(t),v(t),p) \\
&\quad+ \nabla_p \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p) \\
&\quad+ \nabla_p \beta(t)^T l(t,y(t),\dot{y}(t),v(t),p) \\
&= 0,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q) \\
&\quad+ \nabla_q \alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q) \\
&\quad+ \nabla_q \beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q) \\
&= 0,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\alpha(t)^T g(t,y(t),\dot{y}(t),v(t)) \\
&\quad+ \alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p)
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q)\\
&-p^T\nabla_p\alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p)\\
&-q^T\nabla_q\alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q)\\
&\geq 0,
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&\beta_1(t)^T G(t,y(t),\dot{y}(t),v(t))\\
&+\beta(t)^T l(t,y(t),\dot{y}(t),v(t),p)\\
&+\beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q)\\
&-p^T\nabla_p\beta(t)^T l(t,y(t),\dot{y}(t),v(t),p)\\
&-q^T\nabla_q\beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q)\\
&\geq 0,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&y(a)=\gamma_1, \qquad y(b)=\gamma_2,\\
&v(a)=\delta_1, \qquad v(b)=\delta_2,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&\alpha(t)\in\mathbb{R}_+^r, \qquad \beta(t)\in\mathbb{R}^s, \qquad \alpha_1(t)\in\mathbb{R}_+^{r_1},\\
&\beta_1(t)\in\mathbb{R}^{s_1}, \qquad p\in\mathbb{R}^n, \qquad q\in\mathbb{R}^m.
\end{aligned}
\tag{37}
$$

Theorem 4.1 (weak duality). *Let $(x(t),u(t))$ and $(y(t),v(t),\alpha(t),\beta(t),p,q)$ be the feasible solutions of (CP) and (MWHCD), respectively. Let $\int_a^b f(t,\cdot,\cdot,\cdot)dt$, $\int_a^b \alpha(t)^T g(t,\cdot,\cdot,\cdot)dt$ and $\int_a^b \beta(t)^T G(t,\cdot,\cdot,\cdot)dt$ be higher-order $\rho_0-(\eta,\xi,\theta)$-pseudoinvex, higher-order $\rho_1-(\eta,\xi,\theta)$-quasi-invex, and higher-order $\rho_2-(\eta,\xi,\theta)$-quasi-invex functions in x, $\dot{x}$, and u on I with respect to the same functions η,ξ, and θ, with $\rho_0+\rho_1+\rho_2\geq 0$, then the following inequality holds between the primal (CP) and the dual (MHCD),*

$$
\begin{aligned}
&\int_a^b f(t,x(t),\dot{x}(t),u(t))dt\\
&\quad\geq \int_a^b\Big[f(t,y(t),\dot{y}(t),v(t))+h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+h_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad-p^T\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad-q^T\nabla_q h_1(t,y(t),\dot{y}(t),v(t),q)\Big]dt.
\end{aligned}
\tag{38}
$$

Proof. Since $(x(t),u(t))$ and $(y(t),v(t),\alpha(t),\beta(t),p,q)$ are the feasible solutions of (CP) and (MWHCD), respectively, from (46), (3), (34), (35) and (37), we have

$$
\begin{aligned}
&\alpha(t)^T g(t,x(t),\dot{x}(t),u(t))-\alpha(t)^T g(t,y(t),\dot{y}(t),v(t))\\
&\quad-\alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p)\\
&\quad-\alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q)\\
&\quad+p^T\nabla_p\alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p)\\
&\quad+q^T\nabla_q\alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q)\leq 0,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&\beta_1(t)^T G(t,x(t),\dot{x}(t),u(t))-\beta_1(t)^T G(t,y(t),\dot{y}(t),v(t))\\
&\quad-\beta(t)^T l(t,y(t),\dot{y}(t),v(t),p)\\
&\quad-\beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q)\\
&\quad+p^T\nabla_p\beta(t)^T l(t,y(t),\dot{y}(t),v(t),p)\\
&\quad+q^T\nabla_q\beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q)\leq 0.
\end{aligned}
\tag{40}
$$

Since $\int_a^b \alpha(t)^T g(t,\cdot,\cdot,\cdot)dt$ and $\int_a^b \beta(t)^T G(t,\cdot,\cdot,\cdot)dt$ are higher-order $\rho_1-(\eta,\xi,\theta)$-quasi-invex and higher-order $\rho_2-(\eta,\xi,\theta)$-quasi-invex functions, (39) and (40) gives

$$
\begin{aligned}
&\int_a^b\Big[\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\quad\Big(\nabla_p\alpha(t)^T k(t,y(t),\dot{y}(t),v(t),p)\\
&\qquad+\nabla_p\beta(t)^T l(t,y(t),\dot{y}(t),v(t),p)\Big)\\
&\quad+\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\quad+\Big(\nabla_q\alpha_1(t)^T k_1(t,y(t),\dot{y}(t),v(t),q)\\
&\qquad+\nabla_q\beta_1(t)^T l_1(t,y(t),\dot{y}(t),v(t),q)\Big)\\
&\quad+(\rho_1+\rho_2)\|\theta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))\|^2\Big]dt\\
&\leq 0\\
&\Longrightarrow \int_a^b\Big[\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T\\
&\quad\nabla_p h(t,y(t),\dot{y}(t),v(t),p)\\
&\quad+\xi(t,x(t),y(t),\dot{x}(t),\dot{y}(t),u(t),v(t))^T
\end{aligned}
$$

$$\nabla_q h_1(t, y(t), \dot{y}(t), v(t), q)$$

$$+\rho\|\theta(t, x(t), y(t), \dot{x}(t), \dot{y}(t), u(t), v(t))\|^2\Big]dt \geq 0$$

(by equations (32), (33) and $\rho_0 + \rho_1 + \rho_2 \geq 0$)

$$\Longrightarrow \int_a^b f(t, x(t), \dot{x}(t), u(t))dt$$

$$-\int_a^b \Big[f(t, y(t), \dot{y}(t), v(t)) + h(t, y(t), \dot{y}(t), v(t), p)$$

$$+ h_1(t, y(t), \dot{y}(t), v(t), q)$$

$$- p^T \nabla_p h(t, y(t), \dot{y}(t), v(t), p)$$

$$-q^T \nabla_q h_1(t, y(t), \dot{y}(t), v(t), q)\Big]dt \geq 0. \tag{41}$$

by the higher-order $\rho_0 - (\eta, \xi, \theta)$-pseudoinvexity of $\int_a^b f(t, \cdot, \cdot, \cdot)dt$. This completes the proof. □

We construct the following example which verifies Theorem 4.1 above, in which the objective function is higher-order $\rho - (\eta, \xi, \theta)$-pseudoinvex and the constraints functions are higher-order $\rho - (\eta, \xi, \theta)$-quasi-invex.

Example 4.2. Let us consider the following control problem:

$$\min \quad \int_a^b (x^2(t) + u^2(t) + x(t) + u(t))dt, \tag{42}$$

$$\begin{aligned} \text{subject to} \quad & x^2(t) - x(t)u(t) \leq 0, \\ & u^2(t) - x(t) = 0, \\ & x(a) = \gamma_1, \qquad x(b) = \gamma_2, \\ & u(a) = \delta_1, \qquad u(b) = \delta_2, \end{aligned} \tag{43}$$

where $I = [a, b], f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, G : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(t, x(t), \dot{x}(t), u(t)) = x^2(t) + u^2(t) + x(t) + u(t)$, $g(t, x(t), \dot{x}(t), u(t)) = x^2(t) - x(t)u(t)$, and $G(t, x(t), \dot{x}(t), u(t)) = u^2(t) - x(t)$. It is clear that the objective function f, the inequality constraint function g and the equality constraint function G are not higher-order $-(1/2)$-(η, ξ, θ)-invex, higher-order $1 - (\eta, \xi, \theta)-$invex and higher-order $-(1/2)$-(η, ξ, θ)-invex, respectively. But they are higher-order $-(1/2)$-(η, ξ, θ)-pseudoinvex, higher-order 1-(η, ξ, θ)-quasi-invex and higher-order $-(1/2)$-(η, ξ, θ)-quasi-invex, respectively, with respect to the same η, ξ and θ. The functions $\eta, \xi, \theta : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, h : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $l : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h_1 : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, k_1 : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $l_1 : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$are defined as follows:

$$\begin{aligned} &\eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t), u(t), v(t)) \\ &\quad = -(x^2(t) + u(t) + 1), \\ &\xi(t, x(t), y(t), \dot{x}(t), \dot{y}(t), u(t), v(t)) \\ &\quad = -(u^2(t) + x(t) + 1), \\ &\theta(t, x(t), y(t), \dot{x}(t), \dot{y}(t), u(t), v(t)) \\ &\quad = \sqrt{x^2(t)y^2(t) + u(t)y^2(t) + u^2(t)v^2(t) + x(t)v^2(t)}, \\ &h(t, y(t), \dot{y}(t), v(t), p) \\ &\quad = -(y^2(t) + v^2(t) - (y^2(t) + v^2(t) + 1)p), \\ &k(t, y(t), \dot{y}(t), v(t), p) \\ &\quad = y(t)v(t) + v^2(t) + y^2(t)p, \\ &h_1(t, y(t), \dot{y}(t), v(t), q) \\ &\quad = -(2v^2(t) + y(t) + (y^2(t) + v^2(t) + 1)q), \\ &h_1(t, y(t), \dot{y}(t), v(t), q) \\ &\quad = -(2v^2(t) + y(t) + (y^2(t) + v^2(t) + 1)q), \\ &k_1(t, y(t), \dot{y}(t), v(t), q) \\ &\quad = y^2(t) + v^2(t)q, \\ &l_1(t, y(t), \dot{y}(t), v(t), q) \\ &\quad = 4y^2(t) + 5v^2(t) + (y^2(t) + 1)q. \end{aligned} \tag{44}$$

The Mond-Weir higher-order dual of the control problem is

$$\max \quad \int_a^b (y^2(t) + v^2(t))dt, \tag{45}$$

$$\text{subject to} \quad 2y^2(t) + v^2(t) \geq 0, \tag{46}$$

$$6y^2(t) + 9v^2(t) \geq 0, \tag{47}$$

$$\begin{aligned} & y(a) = \gamma_1, \qquad y(b) = \gamma_2, \\ & v(a) = \delta_1, \qquad v(b) = \delta_2, \end{aligned} \tag{48}$$

$$\alpha(t) = \beta(t) = \alpha_1(t) = \beta_1(t) = 1. \tag{49}$$

The above problem satisfies weak duality Theorem 4.1.

Theorem 4.3 (strong duality). *Let $(\overline{x}(t), \overline{u}(t))$ be a local or global optimal solution of (CP), and for the piecewise smooth functions $\overline{\alpha} : I \to \mathbb{R}^r$, $\overline{\beta} : I \to \mathbb{R}^s$, let*

(i) $h(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), 0) = h_1(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), 0) = 0$,

(ii) $k(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), 0) = k_1(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), 0) = 0$,

(iii) $l(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), 0) = l_1(t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), 0) = 0$,

(iv) $\nabla_p h(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),0) = f_x(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t)) - (d/dt) f_{\dot{x}}(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))$,

(v) $\nabla_p \overline{\alpha}(t)^T k(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),0) = g_x(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))^T\overline{\alpha}(t) - (d/dt) g_{\dot{x}}(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))^T\overline{\alpha}(t)$,

(vi) $\nabla_p \overline{\beta}(t)^T l(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),0) = G_x(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))^T\overline{\beta}(t) - (d/dt) G_{\dot{x}}(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))^T\overline{\beta}(t)$,

(vii) $\nabla_q h_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),0) = f_u(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))$,

(viii) $\nabla_q \overline{\alpha}(t)^T k_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),0) = g_u(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))^T\overline{\alpha}(t)$,

(ix) $\nabla_q \overline{\beta}(t)^T l_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),0) = G_u(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t))^T\overline{\beta}(t)$.

Then, $(\overline{x}(t),\overline{u}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p} = 0,\overline{q} = 0)$ *is feasible for* (*MWHCD*). *Moreover, if the weak duality Theorem 4.1 holds between the control primal (CP) and the Mond-Weir higher-order dual* (*MWHCD*). *Then,* $(\overline{x}(t),\overline{u}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p} = 0,\overline{q} = 0)$ *is an optimal solution of* (*MWHCD*), *and the optimal values of* (*CP*) *and* (*MHCD*) *are equal.*

Proof. The proof is similar to that of Theorem 3.4. □

Theorem 4.4 (converse duality). *Let* $(\overline{y}(t),\overline{v}(t),\overline{\alpha}(t),\overline{\beta}(t),\overline{p},\overline{q})$ *be an optimal solution of (MWHCD). Suppose that* $\int_a^b f(t,\cdot,\cdot,\cdot)dt$, $\int_a^b \overline{\alpha}(t)^T g(t,\cdot,\cdot,\cdot)dt$ *and* $\int_a^b \overline{\beta}(t)^T G(t,\cdot,\cdot,\cdot)dt$ *are higher-order* $\rho_0-(\eta,\xi,\theta)$*-pseudoinvex, higher-order* $\rho_1-(\eta,\xi,\theta)$*-quasi-invex, and higher-order* $\rho_2-(\eta,\xi,\theta)$*-quasi-invex functions in* $\overline{x}$, $\overline{x}$ *and* $\overline{u}$ *on* I *with respect to the same functions* η,ξ,θ, *with* $\rho_0+\rho_1+\rho_2 \geq 0$. *Moreover, if*

$$\begin{aligned} &h(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) + h_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) \\ &\quad - \overline{p}^T\nabla_{\overline{p}} h(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{p}) \\ &\quad - \overline{q}^T\nabla_{\overline{q}} h_1(t,\overline{x}(t),\dot{\overline{x}}(t),\overline{u}(t),\overline{q}) \geq 0, \end{aligned} \tag{50}$$

then $(\overline{x}(t),\overline{u}(t))$ *is an optimal solution of (CP).*

Proof. The proof is similar to that of Theorem 3.5. □

5. Concluding Remarks

In this paper, we have studied both Mangasarian type and Mond-Weir type higher-order duality of the control problems. By taking different examples, it is verified that our higher-order generalized invexity is more general than the existing definitions of invexity in the literature. Example 3.3 shows that the objective and constraint functions of the (CP) are not invex as defined by Padhan and Nahak [10], but they are higher-order $\rho-(\eta,\xi,\theta)$-invex and also satisfy weak duality relation with Mangasarian type higher-order duality. In Example 4.2, the objective and constraint functions are not higher-order $\rho-(\eta,\xi,\theta)$-invex but the objective function is higher-order $\rho-(\eta,\xi,\theta)$-pseudoinvex and the constraint functions are higher-order $\rho-(\eta,\xi,\theta)$-quasi-invex and also satisfy weak duality relation with Mond-Weir higher-order duality.

In this paper, the objective function and the constraint functions are twice continuously differentiable. Relaxing this assumptions to include nonsmooth higher-order generalized invex functions for control problems is immediately a topic of further research.

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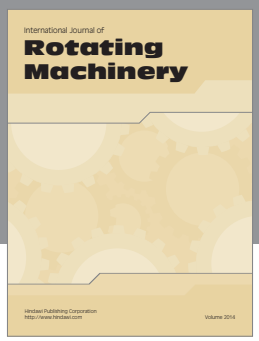

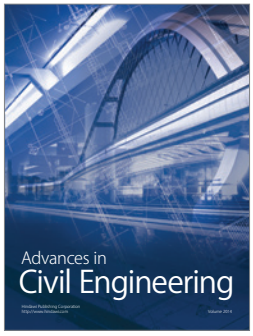

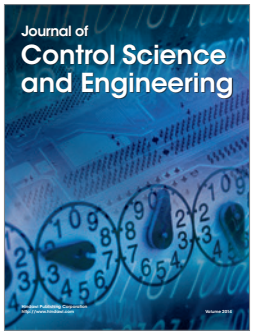

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