## Research Article

# Full Static Output Feedback Equivalence 

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We present a constructive solution to the problem of full output feedback equivalence, of linear, minimal, time-invariant systems. The equivalence relation on the set of systems is transformed to another on the set of invertible block Bezout/Hankel matrices using the isotropy subgroups of the full state feedback group and the full output injection group. The transformation achieving equivalence is calculated solving linear systems of equations. We give a polynomial version of the results proving that two systems are full output feedback equivalent, if and only if they have the same family of generalized Bezoutians. We present a new set of output feedback invariant polynomials that generalize the breakaway polynomial of scalar systems.

## 1. Introduction

1.1. Problem Statement. This paper addresses the problem of full static output feedback equivalence on the set $\Sigma$ of linear, minimal, time-invariant systems, described by the following equations:

$$
\begin{align*}
& \dot{x}=A x+B u, \quad y=C x \text { with } \\
& x(t) \in \mathbb{R}^{n}, \quad u(t) \in \mathbb{R}^{m}, \quad y(t) \in \mathbb{R}^{r}, \tag{1}
\end{align*}
$$

where $x(t)$ is the state vector, $u(t)$ is the input vector, $y(t)$ is the output vector, and $A, B, C$, are real matrices of appropriate dimensions. Systems $\sigma \in \Sigma$ are supposed to be controllable and observable with lists of controllability and observability indices, $E=\left(p_{1}, \ldots, p_{m}\right)$ and $\Pi=\left(q_{1}, \ldots, q_{r}\right)$, respectively, arranged in decreasing order $\left(p_{1} \geq \cdots \geq p_{m}, q_{1} \geq \cdots \geq q_{r}\right)$. The matrices $B, C$ are supposed to have full rank. Any $\sigma \in \Sigma$ is uniquely determined by the 3-tuples of matrices ( $C, A, B$ ) and it will be denoted by it.

The systems $\sigma \in \Sigma$ are liable to static control transformations implying the following system transformations:
(ia) change of basis of the state space $x \longmapsto P x$,

$$
P \in G L_{n}(\mathbb{R}) \Longrightarrow(C, A, B) \longmapsto\left(C P, P^{-1} A P, P^{-1} B\right)
$$

(ib) change of basis of the state space $x \longmapsto Q^{-1} x$,

$$
Q \in G L_{n}(\mathbb{R}) \Longrightarrow(C, A, B) \longmapsto\left(C Q^{-1}, Q A Q^{-1}, Q B\right)
$$

(ii) change of basis of the input space $u \longmapsto F u$,

$$
F \in G L_{m}(\mathbb{R}) \Longrightarrow(C, A, B) \longmapsto(C, A, B F)
$$

(iii) change of basis of the output space $y \longmapsto G^{-1} y$,

$$
G \in G L_{r}(\mathbb{R}) \Longrightarrow(C, A, B) \longmapsto(G C, A, B)
$$

(iv) static output feedback $u \longmapsto u+H y$,

$$
H \in \mathbb{R}^{m \times r} \Longrightarrow(C, A, B) \longmapsto(C, A+B H C, B)
$$

(v) state feedback $u \longmapsto u+K x$,

$$
K \in \mathbb{R}^{m \times n} \Longrightarrow(C, A, B) \longmapsto(C, A+B K, B)
$$

(vi) output injection $\dot{x} \longmapsto \dot{x}+J y$,

$$
\begin{equation*}
J \in \mathbb{R}^{n \times r} \Longrightarrow(C, A, B) \longmapsto(C, A+J C, B) \tag{2}
\end{equation*}
$$

Each subset of the set of transformations (2) induces an equivalence relation on $\Sigma$ regardless of the order of their application, as one can verify by straightforward calculations.

In this paper we are interested in the full output feedback equivalence relation, induced by the subset of transformations ((2), (ia), (ii), (iii), (iv)).

We present explicit and checkable necessary and sufficient conditions for full static output feedback equivalence, leading to the construction of the transformation matrices ( $F, G, P, H$ ) achieving equivalence.

The conditions we present are expressed in terms of
(A) full state feedback equivalence, that is, the equivalence relation induced by the subset of transformations ((2), (ia), (ii), (v));
(B) full output injection equivalence, that is, the equivalence relation induced by the subset of transformations ((2) (ib), (iii), (vi)).

For a more compact, coherent, and comprehensive presentation of the results of this paper we consider the group structures underlying ordered subsets of transformations (2).

They are built in the following way.
(1) First we fix the action transformation of the group under construction to verify the ordered sequence of control transformations. For instance the ordered set of transformations ((2), (v), (ia), (ii)) gives

$$
\begin{align*}
& (((C, A, B)(K))(P))(F) \\
& \quad=\left(C P, P^{-1}(A+B K) P, P^{-1} B F\right)  \tag{3}\\
& \quad=(C, A, B)(P, K, F) .
\end{align*}
$$

(2) Second we fix the composition law of the group to satisfy action axioms. The inverse element is calculated using the composition law. The unit of the group is the ordered set of units of the groups apart. The previous ordered set of transformations gives

$$
\begin{align*}
& \left((C, A, B)\left(P_{1}, K_{1}, F_{1}\right)\right)\left(P_{2}, K_{2}, F_{2}\right) \\
& \quad=(C, A, B)\left(\left(P_{1}, K_{1}, F_{1}\right)\left(P_{2}, K_{2}, F_{2}\right)\right) \\
& \quad \Longrightarrow\left(P_{1}, K_{1}, F_{1}\right)\left(P_{2}, K_{2}, F_{2}\right) \\
& \quad=\left(P_{1} P_{2}, K_{1}+F_{1} K_{2} P_{1}^{-1}, F_{1} F_{2}\right)  \tag{4}\\
& (P, K, F)(P, K, F)^{-1}=\left(I_{n}, O_{m \times n}, I_{m}\right) \\
& \quad \Longrightarrow(P, K, F)^{-1}=\left(P^{-1},-F^{-1} K P, F^{-1}\right) .
\end{align*}
$$

The groups generated through the permutations of a subset of control transformations (2) are isomorphic and induce the same equivalence relation on the set of systems. The order of application of control transformations is not crucial. The choice of order used in this paper for the definition of various groups reflects our point of view.

Definition 1. Full output feedback, full state feedback, and full output injection are the groups generated by the ordered set of transformations ((2), (ia), (iv), (ii), (iii)), ((2), (v), (ia), (ii)), ((2), (vi), (ib), (iii)), respectively, and they are denoted by $\mathscr{Z}$, $X, \mathscr{Y}$.

In the appendix are listed the composition laws and inverse elements of the just defined groups as calculated applying rules (1) and (2).

The problem of full static output feedback equivalence on $\Sigma$ is formulated now in the following way. Given two systems $\sigma=(C, A, B), \widehat{\sigma}=(\widehat{C}, \widehat{A}, \widehat{B}) \in \Sigma$, find necessary and sufficient conditions for the existence of a $z=(G, T, H, F) \in \mathscr{Z}$ with $\hat{\sigma}=\sigma z$ or

$$
\begin{align*}
(\widehat{C}, \widehat{A}, \widehat{B}) & =(C, A, B)(G, T, H, F)  \tag{5}\\
& =\left(G C T, T^{-1}(A+B H C) T, T^{-1} B F\right)
\end{align*}
$$

To develop our results we need to consider subsets of $\Sigma$. Let $\Sigma_{i}, \Sigma_{o}$ be the sets of subsystems of $\Sigma$ described by the equations $\dot{x}=A x+B u$ and $\dot{x}=A x, y=C x$, respectively. The subsystems $\sigma_{i} \in \Sigma_{i}, \sigma_{o} \in \Sigma_{o}$ are uniquely determined by the pairs $(A, B),(C, A)$, respectively, and they will be denoted by them. We consider the restriction of the action transformation of the group $\mathcal{X}$, on $\Sigma_{i}$ :

$$
\begin{equation*}
(A, B) \longmapsto(A, B)(P, K, F)=\left(P^{-1}(A+B K) P, P^{-1} B F\right) . \tag{6}
\end{equation*}
$$

The restriction of the action transformation of the group $\mathscr{Y}$, on $\Sigma_{o}$ is

$$
\begin{equation*}
(C, A) \longmapsto(G, J, Q)(C, A)=\left(G C Q^{-1}, Q(A+J C) Q^{-1}\right) \tag{7}
\end{equation*}
$$

For the sequel of this paper the equivalence relation induced on a set $\Sigma$ by the action of a group $\mathscr{G}$ is referred to as $\mathscr{G}$-equivalence on $\Sigma$. For the elements $\sigma, \bar{\sigma} \in \Sigma$ meeting $\mathscr{G}$ equivalence we write $\sigma \mathscr{G} \widehat{\sigma}$. In this paper we present necessary and sufficient conditions for $\mathscr{Z}$-equivalence on $\Sigma$, using necessary and sufficient conditions for $\mathscr{X}$-equivalence on $\Sigma_{i}$ and $\mathscr{Y}$-equivalence on $\Sigma_{o}$.
1.2. Background. The classical solution to the problem of the equivalence relation $E$ on a set $\Sigma$ [1, page 254] is by means of a complete system of $E$-invariants. A function $\varphi: \Sigma \rightarrow \Phi$ is said to be complete $E$-invariant if $\sigma E \bar{\sigma} \Leftrightarrow \varphi(\sigma)=\varphi(\bar{\sigma})$. If the complete $E$-invariant function $\varphi$ is a list of functions $\varphi=\left(\varphi_{1}, \varphi_{2}, \ldots, \varphi_{k}\right): \Sigma \rightarrow \Phi_{1} \times \Phi_{2} \times \cdots \times \Phi_{k}$, the list $\left(\varphi_{1}(\sigma), \varphi_{2}(\sigma), \ldots, \varphi_{k}(\sigma)\right)$ is said to be a complete system of $E$ invariants. If the set $\Phi$ is a subset of $\Sigma$, then $\varphi(\sigma)$ is said to be a $E$-canonical form of $\sigma$.

A well-known complete $E$-invariant function is the canonical projection $p_{E}: \Sigma \rightarrow \Sigma / E$. But the canonical projection $p_{E}$ is neither explicit nor computable. We usually search for an explicit set $\Phi$ and a computable bijection $\varphi^{\prime}$ : $\Sigma / E \leftrightarrow \Phi$. Under these circumstances $\varphi=\varphi^{\prime} \circ p_{E}$ is a complete $E$-invariant function of $\Sigma$. The problem of finding $\Phi, \varphi$ is universal [1] and no appropriate method for its solution is known.

As far as I am informed, the first results on equivalence on $\Sigma$, under subsets of transformations (2), are obtained
applying Kronecker's theory for equivalence of singular pencils of matrices (Gantmacher [2]). The system ( $C, A, B$ ) is considered as a singular pencil of matrices $\left[\begin{array}{cc}A-\lambda I_{n} & B \\ C & O_{r \times m}\end{array}\right]$ and the system transformations (2) as left and right operations on it:

$$
\begin{align*}
& {\left[\begin{array}{cc}
A-\lambda I_{n} & B \\
C & O_{r \times m}
\end{array}\right]} \\
& \quad \longmapsto\left[\begin{array}{cc}
P^{-1} & J \\
O_{r \times n} & G
\end{array}\right]\left[\begin{array}{cc}
A-\lambda I_{n} & B \\
C & O_{r \times m}
\end{array}\right]\left[\begin{array}{cc}
P & O_{n \times m} \\
K & F
\end{array}\right] . \tag{8}
\end{align*}
$$

The problem of the equivalence relation induced by the transformation (9) is addressed by Morse [3]. Kronecker's theory finds a beautiful application in the case of $\mathscr{X}$-equivalence on $\Sigma_{i}$ :

$$
\left[\begin{array}{ll}
A-\lambda I_{n} & B
\end{array}\right] \longmapsto P^{-1}\left[\begin{array}{ll}
A-\lambda I_{n} & B
\end{array}\right]\left[\begin{array}{cc}
P & O_{n \times m}  \tag{9}\\
K & F
\end{array}\right] .
$$

In this case the list of Kronecker's indices, well known now as list of controllability indices, forms a complete system of $\mathcal{X}$-invariants as it is proved almost simultaneously by various authors and (Brunovsky [4], Kalman [5], Rosenbrock [6]).

The problem of $\mathscr{X}$-equivalence on $\Sigma$ is addressed by Wang and Davison [7]. As far as we are informed no other complete system of invariants of the set of linear minimal time invariant systems $\Sigma$, for an equivalence relation induced by subsets of transformations (2), is known, unless scalar systems $m=r=$ 1 are considered or only changes of basis of the state space are allowed. The techniques of [8] can apply to $\mathscr{X}$-equivalence on $\Sigma$ for single input or single output systems ( $m=1$ or $r=1$ ).
1.3. Another Point of View. We can imagine several distinct ways to affront an equivalence problem. A routine approach is to transform the initial universal problem to another universal problem which may be simpler. One searches for a function $\vartheta: \Sigma \rightarrow \Theta$ and an equivalence relation $R$ on $\Theta$, not necessarily equality, with $\sigma E \hat{\sigma} \Leftrightarrow \quad \vartheta(\sigma) R \vartheta(\hat{\sigma})$. Apparently if the function $\varphi: \Theta \rightarrow \Phi$ is complete $R-$ invariant of $\Theta$, the function $\varphi \circ \vartheta$ is complete $E$-invariant of $\Sigma$. An admissible solution is also to find a pair of explicit and computable functions $\phi, \widehat{\phi}$ on $\Sigma$ to an appropriate set $\Phi$, with $\sigma E \hat{\sigma} \Leftrightarrow \phi(\sigma)=\hat{\phi}(\hat{\sigma})$. The problem of $\mathcal{X}$-equivalence on $\Sigma$ is addressed by Wang and Davison [7] transforming the initial equivalence relation to another equivalence relation which is simpler. We present their approach in our context. The authors construct a list of functions, $\phi=\left(\phi_{1}, \phi_{2}\right)$. The first function $\phi_{1}$ assigns to each system the list of controllability indices $\phi_{1}: \Sigma \mapsto \mathbb{N}^{m}, \sigma \mapsto E=\left(p_{1}, \ldots, p_{m}\right)$. The second function $\phi_{2}$ assigns to each system $\sigma$ an $r \times n$ real matrix. The authors consider a new equivalence relation $R$ on $\mathbb{R}^{r \times n}$ and prove that

$$
\begin{equation*}
\sigma \mathscr{X} \hat{\sigma} \Longleftrightarrow\left(\phi_{1}(\sigma)=\phi_{1}(\hat{\sigma})\right) \wedge\left(\phi_{2}(\sigma) R \phi_{2}(\hat{\sigma})\right) \tag{10}
\end{equation*}
$$

The new equivalence relation $R$ is induced on $\mathbb{R}^{r \times n}$, by the action of a group $\mathscr{P} \subset G L_{n}(\mathbb{R})$ of block Toeplitz matrices
depending on the list of controllability indices. As this action is linear, one can find necessary and sufficient conditions for the existence of a $P \in \mathscr{P}$ with $\phi_{2}(\sigma) P=\phi_{2}(\bar{\sigma})$.

The element $(T, K, F) \in \mathscr{X}$ achieving equivalence is then calculated upon the entries of the block Toeplitz matrix $P$, achieving $R$-equivalence ( $[7$, Proposition 2.2]). As the action of the group $\mathscr{P}$ on $\mathbb{R}^{r \times n}$ is linear, the authors arrive to construct a complete $R$-invariant function (Algorithm 8).
1.4. $\mathscr{Z}$-Equivalence on $\Sigma$. The solution to the problem of $\mathscr{L}$ equivalence on $\Sigma$ we present in this paper draws inspiration from the solution to the problem of $\mathscr{X}$-equivalence on $\Sigma$ given at [7]. We consider both $\mathscr{X}$-equivalence on $\Sigma_{i}$ and $\mathscr{Y}$-equivalence on $\Sigma_{o}$ and we construct a list of functions $\theta=\left(\theta_{1}, \theta_{2}, \theta_{3}\right)$. The first assigns to each system the list of controllability indices $\theta_{1}: \Sigma_{i} \mapsto \mathbb{N}^{m}, \sigma \mapsto E=\left(p_{1}, \ldots, p_{m}\right)$, the second assigns to each system the list of observability indices $\theta_{2}: \Sigma_{o} \mapsto \mathbb{N}^{r}, \sigma \mapsto \Pi=\left(q_{1}, \ldots, q_{r}\right)$, and the third assigns to each system $\sigma$ a matrix $\mathbf{B} \in G L_{n}(\mathbb{R})$ of a particular structure. Then we consider the group $\mathscr{P}$ of [7] and its dual $Q \subset G L_{n}(\mathbb{R})$ of block Toeplitz matrices depending on the list of observability indices. We prove that the group $Q \times \mathscr{P}$ acts on the set $\mathscr{B}$ of matrices B inducing an equivalence relation $R$. After that we prove that if $\phi_{3}(\sigma)=\mathbf{B}, \phi_{3}(\widehat{\sigma})=\widehat{\mathbf{B}}$,

$$
\begin{equation*}
\sigma \mathscr{Z} \widehat{\sigma} \Longleftrightarrow(E=\widehat{E}) \wedge(\Pi=\hat{\Pi}) \wedge(\mathbf{B} R \widehat{\mathbf{B}}) . \tag{11}
\end{equation*}
$$

The action transformation of $Q \times \mathscr{P}$ on $\mathscr{B}$ is bilinear, $\mathbf{B} \mapsto$ $Q B P$, and it seems quite difficult to construct complete $R$ invariant functions based on it. However, thanks to the group structure, we can consider the actions of the groups opposite (Q), $\mathscr{P}$ on $\mathscr{B}$ which are both linear, to find necessary and sufficient conditions for the existence of a $\widehat{Q} \in$ opposite $(Q)$ and a $P \in \mathscr{P}$ with $\widehat{Q} \widehat{\mathbf{B}}=\mathbf{B} P$. The element achieving $\mathscr{Z}$-equivalence is then constructed upon the elements $\widehat{Q}, P$ achieving $R$-equivalence.

An effort is made to link the results on $\mathscr{Z}$-equivalence of this paper with known material of control theory. We prove that the matrix $\mathbf{B}$ has a close relation with the generalized polynomial Bezoutians and that $R$-equivalence amounts to an equivalence relation on the set of generalized Bezoutians. The problem of construction of a complete system of $\mathscr{Z}$ invariants remains open.

Helmke and Fuhrmann [9] prove that the matrix B is a Bezoutian for scalar systems and correlate it with the breakaway polynomial and other $\mathscr{E}$-invariants. We have no doubt that the matrix $\mathbf{B}$ has a very important role to play for multivariable systems. In Yannakoudakis [10] it is proved that the matrix $\mathbf{B}$ is related to the multivariate polynomial Bezoutian introduced by Anderson and Jury [11]. The result is reproduced in this paper. The notion of the breakaway polynomial is generalized for multivariable systems via $\mathbf{B}$.

What is very strange in relation (11) is the number of $n^{2}$ equations involved in $R$. For the scalar case the number is reduced to $2 n-1$ equations of invariants [9, 12]. For the multivariable case it is proved in [13] (the result is reproduced
in this paper) that the matrix $\mathbf{H}=\mathbf{B}^{-1}$ has an $m \times r$ blockHankel structure. So, only $n(m+r)-m r$ entries of $\mathbf{H}$ are independent.

In this paper we generalize the previous result. We prove that the matrix $\mathbf{H}$ conserves its block-Hankel structure when it is left-multiplied by matrices $P_{N} \in \mathscr{P}$ and right-multiplied by matrices $Q_{N} \in \mathbb{Q}$. Consequently only $n(m+r)-m r$ equations involved in (11) are independent.

To summarize, the $\mathscr{X}$-equivalence problem on $\Sigma$ is transformed to another equivalence problem on the set of generalized Bezoutian matrices or block Hankel matrices. The new equivalence relation involves $n(m+r)-m r$ bilinear equations and it is not easy to construct complete invariant functions based on it. However, thanks to the group structure of the block Toeplitz matrices involved, we can make a decision on $\mathscr{X}$-equivalence solving a linear system of $n(m+r)-m r$ equations with a number of unknowns depending on the distribution of controllability and observability indices. The equivalence relation on the generalized Bezoutian matrices has its analogue on the set of polynomial generalized Bezoutian matrices. The polynomial version of the results of this paper seems to open a path for a deeper understanding of the structure of the closed loop by an output feedback state space.
1.5. Paper Structure. This paper is organized in four sections. After this introduction we present in the second section the preliminary results. First of all we give our fundamental theorem. We prove that $\mathscr{X}$-equivalence on $\Sigma$ amounts to $\mathscr{X}$ equivalence on $\Sigma_{i}, \mathscr{Y}$-equivalence on $\Sigma_{o}$ and a condition on the basis of the state space.

To take advantage of this theorem we need an explicit formula of the elements of $X$ and $\mathscr{y}$ achieving equivalence on $\Sigma_{i}$ and $\Sigma_{o}$, respectively.

We present this explicit formula in terms of the isotropy subgroup in the general case of a group $\mathscr{G}$ acting on a set $\Sigma$.

Then we explain that the pioneer work of [7] amounts to the parameterization of the isotropy subgroups of $\mathscr{X}$.

By dualization of the result of [7] we parameterize the isotropy subgroups of $\mathscr{Y}$.

Finally we present algorithms that parameterize the elements of $\mathscr{X}$ and $\mathscr{Y}$ achieving equivalence on $\Sigma_{i}$ and $\Sigma_{o}$, respectively.

In the third section we present the main result on $\mathscr{Z}$ equivalence on $\Sigma$. The third condition of the fundamental theorem, after the parameterizations of the previous section, drives to another equivalence relation on a set of matrices of particular structure (block Bezout/Hankel). We present necessary and sufficient conditions for full output feedback equivalence and an application example. We give also a polynomial version of the main result.

In the fourth section, among the $n^{2}$ equations involved in the equivalence relation, we carry out the linearly independent ones.

## 2. Preliminary Results

In this section we develop the preliminary results necessary for the solution to the $\mathscr{E}$-equivalence problem. Theorem 2 expresses $\mathscr{X}$-equivalence on $\Sigma$, in terms of $\mathscr{X}$-equivalence on $\Sigma_{i}, \mathscr{Y}$-equivalence on $\Sigma_{o}$, and a third condition on the changes of bases of the state space.

Theorem 2. The systems $\sigma=(C, A, B), \hat{\sigma}=(\hat{C}, \hat{A}, \hat{B}) \in \Sigma$ are $\mathscr{Z}$-equivalent; that is, $\exists(G, T, H, F) \in \mathscr{E}$ with $(\widehat{C}, \widehat{A}, \widehat{B})=$ $(C, A, B)(G, T, H, F)=\left(G C T, T^{-1}(A+B H C) T, T^{-1} B F\right)$, if and only if
(I) the pairs $(A, B),(\widehat{A}, \widehat{B}) \in \Sigma_{i}$ are $\mathscr{X}$-equivalent; that is, $\exists(P, K, F) \in \mathcal{X}$ with

$$
\begin{equation*}
(\widehat{A}, \widehat{B})=(A, B)(P, K, F)=\left(P^{-1}(A+B K) P, P^{-1} B F\right), \tag{12}
\end{equation*}
$$

(II) the pairs $(C, A),(\hat{C}, \hat{A},) \in \Sigma_{o}$ are $\mathscr{Y}$-equivalent; that is, $\exists(G, J, Q) \in \mathscr{Y}$ with

$$
\begin{equation*}
(\widehat{C}, \widehat{A},)=(G, J, Q)(C, A)=\left(G C Q^{-1}, Q(A+J C) Q^{-1}\right) \tag{13}
\end{equation*}
$$

(III) there is $(P, K, F) \in \mathscr{X}$ satisfying (12) and $(G, J, Q) \in \mathscr{Y}$ satisfying (13) with

$$
\begin{equation*}
Q P=I_{n} . \tag{14}
\end{equation*}
$$

Proof. Necessity:

$$
\begin{align*}
(\widehat{C}, \widehat{A}, \widehat{B}) & =(C, A, B)(G, T, H, F)  \tag{15}\\
& =\left(G C T, T^{-1}(A+B H C) T, T^{-1} B F\right) .
\end{align*}
$$

Putting $H C=K$ and $T=P$ the equation for $\mathscr{Z}$-equivalence becomes

$$
\begin{align*}
(\widehat{C}, \widehat{A}, \widehat{B}) & =\left(G C P, P^{-1}(A+B K) P, P^{-1} B F\right) \Longrightarrow(\hat{A}, \widehat{B}) \\
& =\left(P^{-1}(A+B K) P, P^{-1} B F\right) . \tag{16}
\end{align*}
$$

Putting $B H=J$ and $T=Q^{-1}$ the equation for $\mathscr{Z}$-equivalence becomes

$$
\begin{align*}
(\widehat{C}, \widehat{A}, \widehat{B}) & =\left(G C Q^{-1}, Q(A+J C) Q^{-1}, Q B F\right) \Longrightarrow(\widehat{C}, \widehat{A}) \\
& =\left(G C Q^{-1}, Q(A+J C) Q^{-1}\right) \tag{17}
\end{align*}
$$

Obviously $Q P=I_{n}$.

Sufficiency:

$$
\left.\left.\begin{array}{l}
(\hat{A}, \widehat{B})=\left(P^{-1}(A+B K) P, P^{-1} B F\right) \\
(\widehat{C}, \widehat{A})=\left(G C Q^{-1}, Q(A+J C) Q^{-1}\right)  \tag{18}\\
Q P=I_{n}
\end{array}\right\} \Longrightarrow, ~ \begin{array}{c}
(\hat{A}, \widehat{B})=\left(P^{-1}(A+B K) P, P^{-1} B F\right) \\
(\hat{C}, \widehat{A})=\left(G C P, P^{-1}(A+J C) P\right)
\end{array}\right\} \Longrightarrow \quad \begin{gathered}
\begin{array}{c}
\widehat{B}=P^{-1} B F \\
A+B K=A+J C .
\end{array}
\end{gathered}
$$

But $B K=J C \Rightarrow K=B^{\dagger} J C \wedge J=B K C^{\dagger} \Rightarrow B^{\dagger} J=$ $K C^{\dagger}$ ( $\dagger$ denotes the Moore-Penrose inverse).

Putting $H=B^{\dagger} J=K C^{\dagger}$ conditions (I), (II), and (III) of Theorem 2 imply

$$
\begin{align*}
(\widehat{C}, \widehat{A}, \widehat{B}) & =(C, A, B)(G, P, H, F)  \tag{19}\\
& =\left(G C P, P^{-1}(A+B H C) P, P^{-1} B F\right)
\end{align*}
$$

To take advantage of Theorem 2 we need an explicit formula for the elements $(P, K, F)$ of $X$ achieving equivalence on $\Sigma_{i}$ as well as for the elements ( $G, J, Q$ ) of $\mathscr{y}$ achieving equivalence on $\Sigma_{o}$. This explicit formula is given in Proposition 5 in the general case of a set $\Sigma$ and a group $\mathscr{G}$, acting on it.

We recall from [1] that if $\Sigma$ is a set and $\mathscr{G}$ group acting on it, the set of elements $g \in \mathscr{G}$ with $\sigma g=\sigma$ is a group $\mathscr{G}_{I} \subset$ $\mathscr{G}$ called the stabilizer of $\sigma$ at $\mathscr{G}$ or the isotropy subgroup of $\mathscr{G}$ at $\sigma$. Given the particular weight of the term "stabilize" in control theory we prefer the term "isotropy." The following proposition uses the isotropy subgroup to parameterize the set of $g \in \mathscr{G}$ with $\bar{\sigma}=\sigma g$.

Proposition 3. The set of $g \in \mathscr{G}$ with $\hat{\sigma}=\sigma g$ is given by $g_{0} \mathscr{G}_{I}$, where $g_{0}$ is a particular solution $\hat{\sigma}=\sigma g_{0}$ and $\mathscr{G}_{I}$ the isotropy subgroup of $\mathscr{G}$ at $\bar{\sigma}$.

## Proof. Consider

$$
\begin{align*}
& \left(\exists g_{0} \in \mathscr{G} \text { with } \hat{\sigma}=\sigma g_{0}\right) \wedge\left(g_{I} \in \mathscr{G}_{I} \text { at } \hat{\sigma}\right) \Longrightarrow \hat{\sigma} g_{I}  \tag{20}\\
& \\
& =\sigma g_{0} g_{I} \Longrightarrow \hat{\sigma}=\sigma g_{0} g_{I}
\end{align*}
$$

In other words, if $g_{0}$ is a solution $g_{0} g_{I}$ is also a solution:

$$
\begin{gather*}
\left(\hat{\sigma}=\sigma g_{0}\right) \wedge\left(\hat{\sigma}=\sigma g_{1}\right) \Longrightarrow \hat{\sigma} g_{0}^{-1}=\hat{\sigma} g_{1}^{-1} \Longrightarrow \hat{\sigma} g_{0}^{-1} g_{1}  \tag{21}\\
=\hat{\sigma} \Longrightarrow g_{0}^{-1} g_{1}=g_{I} \in \mathscr{G}_{I} \Longrightarrow g_{1}=g_{0} g_{I}
\end{gather*}
$$

In other words, if $g_{0}, g_{1}$ are solutions, then $\exists g_{I} \in \mathscr{G}_{I}$ with $g_{1}=g_{0} g_{I}$.

Now having the isotropy subgroup $\mathscr{E}_{I}$ at a point $\sigma$, we can obtain the isotropy subgroup at any other point of the equivalence class $\sigma g$.

Proposition 4. If $\mathscr{G}_{I}$ is the isotropy subgroup of $\mathscr{G}$ at $\sigma$, $g^{-1} \mathscr{G}_{I} g$ is the isotropy subgroup of $\mathscr{G}$ at $\sigma g$.

Proof. Consider
$g_{I} \in \mathscr{G}_{I} \quad$ at $\sigma \Longleftrightarrow \sigma g_{I}=\sigma \Longleftrightarrow \sigma g\left(g^{-1} g_{I} g\right)=\sigma g_{I} g=\sigma g$.

Suppose now that $\sigma_{c}$ is a $\mathscr{G}$-canonical form of $\sigma$ and $\sigma \mathcal{g}_{c}=$ $\sigma_{c}$. Let $\mathscr{G}_{N}$ be the isotropy subgroup of $\mathscr{G}$ at $\sigma_{c}$.

Proposition 5. If $\sigma \mathscr{G} \hat{\sigma}$, the set of solutions $g \in \mathscr{G}$ with $\sigma \mathcal{G}=\hat{\sigma}$ is given by $g=g_{c} \mathscr{G}_{N} \widehat{g}_{c}^{-1}$ with $g_{c}, \widehat{g}_{c}$ the elements of $\mathscr{G}$ projecting $\sigma, \hat{\sigma}$ to their $\mathscr{G}$-canonical form $\sigma_{c}$ and $\mathscr{G}_{N}$ the isotropy subgroup of $\mathscr{G}$ at $\sigma_{c}$.

Proof. One has

$$
\begin{align*}
\sigma \mathscr{G} \hat{\sigma} & \Longleftrightarrow \exists g_{c}, \hat{g}_{c} \in \mathscr{G} \text { with }\left(\sigma g_{c}=\sigma_{c}\right) \wedge\left(\hat{\sigma} \hat{g}_{c}=\sigma_{c}\right) \\
& \Longleftrightarrow \sigma g_{c} \mathscr{G}_{N}=\sigma_{c} \wedge \hat{\sigma} \hat{g}_{c}=\sigma_{c} \Longrightarrow \sigma g_{c} \mathscr{G}_{N} \widehat{g}_{c}^{-1}=\hat{\sigma} \tag{23}
\end{align*}
$$

Let us come back to the problem of parameterization of the set of solutions $(P, K, F) \in X$ with $(\hat{A}, \bar{B})=$ $(A, B)(P, K, F)=\left(P^{-1}(A+B K) P, P^{-1} B F\right)$. According to Proposition 5 we need to find
an element of $\mathscr{X}$ projecting $(A, B)$ to its $\mathscr{X}$-canonical form:

$$
\begin{equation*}
\left(A_{\varepsilon}, B_{\varepsilon}\right)=(A, B)\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right) \tag{24a}
\end{equation*}
$$

an element of $\mathscr{X}$ projecting $(\widehat{A}, \widehat{B})$ to its $\mathscr{X}$-canonical form:

$$
\begin{equation*}
\left(\hat{A}_{\varepsilon}, \hat{B}_{\varepsilon}\right)=(\hat{A}, \hat{B})\left(\hat{P}_{\varepsilon}, \hat{K}_{\varepsilon}, \hat{F}_{\varepsilon}\right) . \tag{24b}
\end{equation*}
$$

If $\left(\hat{A}_{\varepsilon}, \widehat{B}_{\varepsilon}\right)=\left(A_{\varepsilon}, B_{\varepsilon}\right)$ we need to parameterize the elements $\left(P_{N}, K_{N}, F_{N}\right) \in \mathscr{X}_{N} \subset \mathscr{X}$ of the isotropy subgroup of the full state feedback group at $\left(A_{\varepsilon}, B_{\varepsilon}\right)$. Then the set of transformations achieving $\mathscr{X}$-equivalence is

$$
\begin{equation*}
(P, K, F)=\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(P_{N}, K_{N}, F_{N}\right)\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \widehat{F}_{\varepsilon}\right)^{-1} \tag{25}
\end{equation*}
$$

We can calculate $\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right),\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \widehat{F}_{\varepsilon}\right) \in X$ using the techniques of Brunovsky [4]. For the parameterization of
the isotropy subgroup of the full state feedback group at $\left(A_{\varepsilon}, B_{\varepsilon}\right)$, we use the results of Wang and Davison [7]. The authors found out all the elements $P_{N} \in G L_{n}(\mathbb{R})$, $K_{N} \in \mathbb{R}^{m \times n}, F_{N} \in G L_{m}(\mathbb{R})$ with $\left(A_{\varepsilon}, B_{\varepsilon}\right)=\left(P_{N}\left(A_{\varepsilon}+\right.\right.$ $\left.\left.B_{\varepsilon} K_{N}\right) P_{N}^{-1}, P_{N} B_{\varepsilon} F_{N}\right)$. In our terms they parameterize the isotropy subgroup of the group $X^{\prime}$ generated through the ordered set of transformations (IIiv, IIb, IIii) at the canonical form of Brunovsky $\left(A_{\varepsilon}, B_{\varepsilon}\right)$. The previous result is very deep as it parameterizes the state feedback transformations that do not alter the eigenvalues of the system matrix $A$, but only its eigenvectors. The authors exploit it at the canonical form, but thanks to the conjugation of Proposition 4 we use it in this paper at the current coordinates of the state space.

Proposition 6 (essentially Proposition 2.1 of [7]). The matri$\operatorname{ces} P_{N} \in G L_{n}(\mathbb{R}), K_{N} \in \mathbb{R}^{m \times n}, F_{N} \in G L_{m}(\mathbb{R})$ with $\left(A_{\varepsilon}, B_{\varepsilon}\right)=$ $\left(P_{N}\left(A_{\varepsilon}+B_{\varepsilon} K_{N}\right) P_{N}^{-1}, P_{N} B_{\varepsilon} F_{N}\right)$ are as follows.
(i) The matrices $P_{N}$ have an $m \times m$ block structure $P_{N}=$ $\left\{P_{\zeta \xi}\right\}, 1 \leq \zeta, \xi \leq m$. Each block $P_{\zeta \xi}$ has dimension $p_{\zeta} \times$ $p_{\xi}\left(p_{\zeta}, p_{\xi} \in E\right)$ and a Toeplitz structure with

$$
\begin{gather*}
P_{\zeta \xi}=\left[\begin{array}{ccccccc}
\gamma_{\zeta \xi}^{1} & \gamma_{\zeta \xi}^{2} & \cdots & \gamma_{\zeta \xi}^{\eta} & 0 & \cdots & 0 \\
0 & \gamma_{\zeta \xi}^{1} & \gamma_{\zeta \xi}^{2} & \cdots & \gamma_{\zeta \xi}^{\eta} & & 0 \\
\vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\
0 & \cdots & 0 & \gamma_{\zeta \xi}^{1} & \gamma_{\zeta \xi}^{2} & \cdots & \gamma_{\zeta \xi}^{\eta}
\end{array}\right]  \tag{26a}\\
\text { if } \quad p_{\xi} \geq p_{\zeta}\left(\eta=p_{\xi}-p_{\zeta}+1\right), \\
P_{\zeta \xi}=O_{p_{\zeta} \times p_{\xi}}
\end{gather*} \text { if } p_{\xi}<p_{\zeta} .
$$

(ii) The matrices $K_{N}$ are calculated substituting $P_{N}$ from (26a) in the equation

$$
\begin{equation*}
A_{\varepsilon}=P_{N}\left(A_{\varepsilon}+B_{\varepsilon} K_{N}\right) P_{N}^{-1} \tag{26b}
\end{equation*}
$$

(iii) The matrices $F_{N}$ are calculated substituting $P_{N}$ from (26a) in the equation

$$
\begin{equation*}
B_{\varepsilon}=P_{N} B_{\varepsilon} F_{N} . \tag{26c}
\end{equation*}
$$

The authors prove that the matrices $P_{N}$ form a group. As $\left(P_{N}\right)^{-1}$ has the structure (26a) of $P_{N}$, there are $\left(K_{N}, F_{N}\right)$ satisfying

$$
\begin{equation*}
A_{\varepsilon}=P_{N}^{-1}\left(A_{\varepsilon}+B_{\varepsilon} K_{N}\right) P_{N}, \quad B_{\varepsilon}=P_{N}^{-1} B_{\varepsilon} F_{N} \tag{26d}
\end{equation*}
$$

We give without proof the following.
Proposition 7. The 3-tuples of matrices $\left(P_{N}, K_{N}, F_{N}\right)$ satisfying (26a) and (26d) form a group which is the isotropy subgroup $\mathcal{X}_{N} \subset \mathcal{X}$ of the full state feedback group at the Brunovsky's canonical form $\left(A_{\varepsilon}, B_{\varepsilon}\right) \in \Sigma_{i}$.

The set of $(P, K, F)$ satisfying (12) is

$$
\begin{equation*}
(P, K, F)=\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(P_{N}, K_{N}, F_{N}\right)\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \hat{F}_{\varepsilon}\right)^{-1} \tag{27a}
\end{equation*}
$$

Applying the formulas of composition law and inverse element of $\mathscr{X}$ given in the appendix we obtain

$$
\begin{align*}
(P, K, F)=( & P_{\varepsilon} P_{N} \widetilde{P}_{\varepsilon}^{-1}, K_{\varepsilon}+F_{\varepsilon} K_{N} P_{\varepsilon}^{-1} \\
& \left.\quad-F_{\varepsilon} F_{N} \widehat{F}_{\varepsilon}^{-1} \widehat{K}_{\varepsilon} P_{\varepsilon} \widehat{P}_{\varepsilon}^{-1} P_{N}^{-1}, F_{\varepsilon} F_{N} \widehat{F}_{\varepsilon}^{-1}\right) \tag{27b}
\end{align*}
$$

The calculation of all $(P, K, F)$ satisfying (12) is summarized in the following.

Algorithm 8. Given $(A, B),(\hat{A}, \widehat{B}) \in \Sigma_{i}$, to find all the solutions $(P, K, F)$ with $(\widehat{A}, \widehat{B})=(A, B)(P, K, F)$ we have to do the following.
(1) We calculate the lists of controllability indices $E, \bar{E}$ of the subsystems $(A, B),(\hat{A}, \widehat{B})$.
(2) If $E \neq \widehat{E}$, there is no solution $(P, K, F)$.
(3) If $E=\hat{E}$, we calculate the elements $\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)$, $\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \hat{F}_{\varepsilon}\right) \in \mathscr{X}$ projecting $(A, B),(\hat{A}, \widehat{B})$ to their controllability canonical form of Brunovsky $\left(A_{\varepsilon}, B_{\varepsilon}\right)$.
(4) The general solution of $(\hat{A}, \widehat{B})=(A, B)(P, K, F)$ is given by (27b).

Example 9. $A=\left[\begin{array}{ccc}-1 & 2 & 8 \\ 0 & -2 & -1 \\ 0 & 0 & -3\end{array}\right], B=\left[\begin{array}{cc}-4 & -12 \\ 1 & 3 \\ 1 & 2\end{array}\right]$. The list of controllability indices is $E=(2,1)$. Let $P_{\varepsilon}=\left[\begin{array}{ccc}-2 & -4 & -8 \\ 1 & 1 & 2 \\ 1 & 1 & 1\end{array}\right]$.

The change of basis of the state space $x \mapsto P_{\varepsilon} x$ projects $(A, B)$ to its controllability canonical form of Popov: $(A, B)\left(P_{\varepsilon}, O_{m \times n}, I_{m}\right)=\left(A_{c}, B_{c}\right)=\left(\left[\begin{array}{ccc}0 & 1 & 0 \\ -3 & -4 & -1 \\ 0 & 0 & -2\end{array}\right],\left[\begin{array}{ll}0 & 0 \\ 1 & 1 \\ 0 & 1\end{array}\right]\right)$. With a state feedback $u \mapsto u+B_{c} K_{\varepsilon}^{\prime}$ and a change of basis of the input space $u \mapsto F_{\varepsilon} u$ we project the canonical form of Popov to the canonical form of Brunovsky $\left(A_{c}, B_{c}\right)\left(I_{n}, K_{\varepsilon}^{\prime}, F_{\varepsilon}\right)=$ $\left(A_{\varepsilon}, B_{\varepsilon}\right) \Rightarrow\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)=\left(P_{\varepsilon}, O_{m \times n}, I_{m}\right)\left(I_{n}, K_{\varepsilon}^{\prime}, F_{\varepsilon}\right)=\left(P_{\varepsilon}\right.$, $K_{\varepsilon}^{\prime} P_{\varepsilon}^{-1}, F_{\varepsilon}$.

The isotropy subgroup $X_{N} \subset \mathcal{X}$ at the $\mathscr{X}$-canonical form of Brunovsky $\left(A_{\varepsilon}, B_{\varepsilon}\right)$ is

$$
\begin{gather*}
P_{N}=\left[\begin{array}{lll}
a & 0 & 0 \\
0 & a & 0 \\
b & c & d
\end{array}\right], \\
K_{N}=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & \frac{b}{a} & 0
\end{array}\right],  \tag{28}\\
F_{N}=\left[\begin{array}{ll}
a & 0 \\
c & d
\end{array}\right] .
\end{gather*}
$$

The isotropy subgroup $\left(P_{I}, K_{I}, F_{I}\right)=\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(P_{N}, K_{N}\right.$, $\left.F_{N}\right)\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)^{-1}$ at $(A, B)$ is

$$
\begin{align*}
& P_{I}=\left[\begin{array}{ccc}
a-4 b+4 c & 8 a-16 b+24 c-8 d & -8 a-16 c+8 d \\
b-c & -a+4 b-6 c+2 d & 2 a+4 c-2 d \\
\frac{(b-c)}{2} & -a+2 b-3 c+d & 2 a+2 c-d
\end{array}\right], \\
& F_{I}=\left[\begin{array}{cc}
a-c & a-c-d \\
c & c+d
\end{array}\right], \\
& K_{I} \\
& =\frac{1}{a d} \\
& \times\left[\begin{array}{cc}
\frac{\left(-d b+d c+a b-a c-c b+c^{2}\right)}{2} & \frac{\left(d b-d c+c b-c^{2}\right)}{2} \\
a d-d b-a^{2}-2 a c+2 a b+3 c^{2}-2 c b & d b-a c-3 c^{2}+2 c b \\
-a d-2 d b+4 d c+a^{2}+a c-2 c^{2} & 2 d b-4 d c+a c+2 c^{2}
\end{array}\right]^{T}, \\
& K_{\varepsilon}^{\prime}=\left[\begin{array}{ccc}
-3 & -4 & 1 \\
0 & 0 & -2
\end{array}\right], \\
& F_{\varepsilon}=\left[\begin{array}{cc}
1 & -1 \\
0 & 1
\end{array}\right], \quad K_{\varepsilon}=\left[\begin{array}{ccc}
-\frac{1}{2} & -7 & 9 \\
0 & 2 & -2
\end{array}\right], \\
& \left(A_{\varepsilon}, B_{\varepsilon}\right)=\left(\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right],\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1
\end{array}\right]\right) . \tag{29}
\end{align*}
$$

Consider now a second system $(\hat{A}, \widehat{B})$
$\hat{A}=\left[\begin{array}{ccc}-20 & 7 & -30 \\ 7 & -4 & 12 \\ 3 & -1 & 3\end{array}\right], \hat{B}=\left[\begin{array}{cc}-24 & 9 \\ 9 & -3 \\ 4 & -1\end{array}\right], \widehat{E}=(2,1) x \mapsto \widehat{P}_{\varepsilon} x$ with $\widehat{P}_{\varepsilon}=\left[\begin{array}{ccc}-9 & -24 & 9 \\ 6 & 9 & -3 \\ 3 & 4 & -1\end{array}\right]$ being the change of basis of the state space projecting $(A, B)$ to its controllability canonical form of Popov $\left(\widehat{A}_{c}, \widehat{B}_{c}\right)$.

Putting $\widehat{K}_{\varepsilon}=\left[\begin{array}{ccc}-5 / 6 & -4 / 3 & 5 / 2 \\ 1 / 6 & -10 / 3 & 17 / 2\end{array}\right] \hat{F}_{\varepsilon}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ we have $(\hat{A}, \widehat{B})$ $\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \widehat{F}_{\varepsilon}\right)=\left(\left[\begin{array}{lll}0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right],\left[\begin{array}{lll}0 & 0 \\ 1 & 0 \\ 0 & 1\end{array}\right]\right)=\left(A_{\varepsilon}, B_{\varepsilon}\right)$.

A particular solution $(\hat{A}, \widehat{B})=(A, B)\left(P_{0}, K_{0}, F_{0}\right)$ is then $\left(P_{0}, K_{0}, F_{0}\right)=\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \widehat{F}_{\varepsilon}\right)^{-1}:$

$$
\begin{align*}
& P_{0}=\left[\begin{array}{ccc}
\frac{5}{3} & \frac{56}{3} & -33 \\
-\frac{1}{3} & -\frac{13}{3} & 8 \\
-\frac{1}{6} & -\frac{7}{3} & \frac{9}{2}
\end{array}\right], \\
& K_{0}=\left[\begin{array}{ccc}
7 & 50 & -36 \\
-2 & -9 & 1
\end{array}\right], \\
& F_{0}=\left[\begin{array}{cc}
1 & -1 \\
0 & 1
\end{array}\right] . \tag{30}
\end{align*}
$$

The general solution $(P, K, F)$ of $(\hat{A}, \widehat{B})=(A, B)(P, K, F)$ is

$$
\begin{align*}
& (P, K, F)=\left(P_{0}, K_{0}, F_{0}\right)\left(P_{I}, K_{I}, F_{I}\right) \\
& =\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(P_{N}, K_{N}, F_{N}\right)\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \widehat{F}_{\varepsilon}\right)^{-1} \\
& =\left(P_{\varepsilon} P_{N} \widetilde{P}_{\varepsilon}^{-1}, K_{\varepsilon}+F_{\varepsilon} K_{N} P_{\varepsilon}^{-1}-F_{\varepsilon} F_{N} \widetilde{F}_{\varepsilon}^{-1} \widehat{K}_{\varepsilon} P_{\varepsilon} \widetilde{P}_{\varepsilon}^{-1} P_{N}^{-1}, F_{\varepsilon} F_{N} \widetilde{F}_{\varepsilon}^{-1}\right) \\
& P=\left[\begin{array}{ccc}
\frac{(a-4 b+4 c+4 d)}{3} & \frac{8(a-2 b)}{3}+8 c+16 d & -5 a+4 b-12 c-28 d \\
\frac{(b-c-d)}{3} & -\frac{a}{3}+\frac{4 b}{3}-2 c-4 d & a-b+3 c+7 d \\
\frac{(b-c-d)}{6} & -\frac{a}{3}+\frac{2 b}{3}-c-2 d & a+\frac{(3 c+7 d-b)}{2}
\end{array}\right],  \tag{31}\\
& F=\left[\begin{array}{cc}
a-c & -d \\
c & d
\end{array}\right], \\
& K=\frac{1}{a d}\left[\begin{array}{cc}
d(5 a-b-4 c+2 d)-3 a(b+c)+3 c(b-c) & d b+4 d c-2 d^{2}-3 b c+3 c^{2} \\
d(36 a-3 b-31 c+8 d)+6 a(a+2 c-2 b)-6 c(3 c+2 b) & d(3 b+31 c-8 d-a)+6 c(a+3 c-2 b) \\
d(-30 a-2 b+30 c)-6 a(a+c)+12 c^{2} & d a+2 d b-30 d c-6 a c-12 c^{2}
\end{array}\right]^{T} .
\end{align*}
$$

To take advantage of Theorem 2, we need also an explicit formula for the elements $(G, J, Q) \in \mathscr{y}$ with $(\hat{C}, \bar{A})=$
$\left(G C Q^{-1}, Q(A+J C) Q^{-1}\right)$. We provide it by dualization without further discussion.

Observability canonical forms of Popov ( $C_{o}, A_{o}$ ) and Brunovsky $\left(C_{\pi}, A_{\pi}\right)$ are the transposes of the controllability canonical form of Popov [14] and Brunovsky of the pair ( $A^{T}, C^{T}$ ), respectively.

Let us consider the changes of basis of the state space $x \mapsto$ $Q_{\pi}^{-1} x, x \mapsto \widehat{Q}_{\pi}^{-1} x$, projecting the pairs $(C, A),(\widehat{C}, \widehat{A})$ to their observability canonical forms of Popov $\left(C_{o}, A_{o}\right),\left(\bar{C}_{o}, \widehat{A}_{o}\right)$, respectively,

$$
\begin{gather*}
\left(C_{o}, A_{o}\right)=\left(C Q_{\pi}^{-1}, Q_{\pi} A Q_{\pi}^{-1}\right) \\
\left(\widehat{C}_{o}, \overparen{A}_{o}\right)=\left(\widehat{C} \widehat{Q}_{\pi}^{-1}, \widehat{Q}_{\pi} \widehat{A} \widehat{Q}_{\pi}^{-1}\right) \tag{32}
\end{gather*}
$$

We can always find changes of basis of the output space $y \mapsto$ $G_{\pi}^{-1} y, y \mapsto \widehat{G}_{\pi}^{-1} y$ and output injections $\dot{x} \mapsto \dot{x}+J_{\pi}^{\prime} y, \dot{x} \mapsto \dot{x}+$ $\widehat{J}_{\pi} y$ projecting the canonical form of Popov to the canonical form of Brunovsky $\left(C_{o}, A_{o}\right),\left(\widehat{C}_{o}, \widehat{A}_{o}\right)$.

One has $\left(C_{\pi}, A_{\pi}\right)=\left(G_{\pi}, J_{\pi}^{\prime}, I_{n}\right)\left(C_{o}, A_{o}\right),\left(\widehat{C}_{\pi}, \hat{A}_{\pi}\right)=$ $\left(\bar{G}_{\pi}, \widehat{J}_{\pi}^{\prime}, I_{n}\right)\left(\bar{C}_{o}, \hat{A}_{o}\right)$. Then $\left(C_{\pi}, A_{\pi}\right)=\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)(C, A)$, $\left(\widehat{C}_{\pi}, \widehat{A}_{\pi}\right)=\left(\widehat{G}_{\pi}, \widehat{J}_{\pi}, \widehat{Q}_{\pi}\right)(\widehat{C}, \hat{A})$ with $J_{\pi}=Q_{\pi}^{-1} J_{\pi}^{\prime}, \widehat{J}_{\pi}=\widehat{Q}_{\pi}^{-1} J_{\pi}^{\prime}$.

The equality of the lists of observability indices $\Pi=\bar{\Pi}$ implies $\left(C_{\pi}, A_{\pi}\right)=\left(\widehat{C}_{\pi}, \widehat{A}_{\pi}\right)$.

Then a particular solution achieving $\mathscr{Y}$-equivalence is given by the formula

$$
\begin{equation*}
\left(G_{0}, J_{0}, Q_{0}\right)=\left(\hat{G}_{\pi}, \widehat{J}_{\pi}, \hat{Q}_{\pi}\right)^{-1}\left(G_{\pi}, J_{\pi}, Q_{\pi}\right) \tag{33}
\end{equation*}
$$

The parameterization of the elements $\left(G_{N}, J_{N}, Q_{N}\right)$ of the isotropy subgroup of the full output injection group at the observability canonical form $\left(C_{\pi}, A_{\pi}\right)$ is given by Proposition 10 which is the dual of Proposition 7.

Proposition 10. The elements $\left(G_{N}, J_{N}, Q_{N}\right)$ of the isotropy subgroup $\mathscr{Y}_{N}$ of the full output injection group $\mathscr{y}$ at the observability canonical form of Brunovsky $\left(C_{\pi}, A_{\pi}\right)$ of the pair $(C, A)$ are as follows.

The matrices $Q_{N}$ have an $r \times r$ block structure $Q_{N}=$ $\left\{Q_{\zeta \xi}\right\}, 1 \leq \zeta, \xi \leq r$. Each block $Q_{\zeta \xi}$ has dimension $q_{\zeta} \times q_{\xi}\left(q_{\zeta}, q_{\xi} \in \Pi\right)$ and a Toeplitz structure with $Q_{\zeta \xi}=$ $O_{q_{\zeta} \times q_{\xi}}$ if $q_{\zeta}<q_{\xi}$ :

$$
Q_{\zeta \xi}=\left[\begin{array}{ccccccc}
\delta_{\zeta \xi}^{1} & \delta_{\zeta \xi}^{2} & \cdots & \delta_{\zeta \xi}^{\eta} & 0 & \cdots & 0  \tag{34a}\\
0 & \delta_{\zeta \xi}^{1} & \delta_{\zeta \xi}^{2} & \cdots & \delta_{\zeta \xi}^{\eta} & & 0 \\
\vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\
0 & \cdots & 0 & \delta_{\zeta \xi}^{1} & \delta_{\zeta \xi}^{2} & \cdots & \delta_{\zeta \xi}^{\eta}
\end{array}\right]^{T}
$$

The matrices $J_{N}$ are calculated substituting $Q_{N}$ from (34a) in the equation

$$
\begin{equation*}
A_{\pi}=Q_{N}\left(A_{\pi}+J_{N} C_{\pi}\right) Q_{N}{ }^{-1} \tag{34b}
\end{equation*}
$$

The matrices $G_{N}$ are calculated substituting $Q_{N}$ from (34a) in the equation

$$
\begin{equation*}
C_{\pi}=G_{N} C_{\pi} Q_{N}{ }^{-1} \tag{34c}
\end{equation*}
$$

The elements $\left(G_{I}, J_{I}, Q_{I}\right)$ of the isotropy subgroup $\mathscr{Y}_{I}$ of the full output injection group $\mathscr{Y}$ at $(C, A)$ are then $\left(G_{I}, J_{I}, Q_{I}\right)=\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)^{-1}\left(G_{N}, J_{N}, Q_{N}\right)\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)$.
The elements $(G, J, Q) \in \mathscr{Y}$ with $(\hat{C}, \hat{A})=\left(G C Q^{-1}\right.$, $\left.Q(A+J C) Q^{-1}\right)$ are
$(G, J, Q)=\left(\hat{G}_{\pi}, \hat{J}_{\pi}, \hat{Q}_{\pi}\right)^{-1}\left(G_{N}, J_{N}, Q_{N}\right)\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)$
$(G, J, Q)=\left(\widehat{Q}_{\pi}^{-1} Q_{N} Q_{\pi}, J_{\pi}+Q_{\pi}^{-1} J_{N} G_{\pi}\right.$

$$
\begin{equation*}
\left.-Q_{N}^{-1} \widehat{Q}_{\pi}^{-1} Q_{\pi} \widehat{J}_{\pi} \widehat{G}_{\pi}^{-1} G_{N} G_{\pi}, \widehat{G}_{\pi}^{-1} G_{N} G_{\pi}\right) \tag{35b}
\end{equation*}
$$

The calculation of all ( $G, J, Q$ ) satisfying (13) is summarized in the following.

Algorithm 11. Given $(C, A),(\hat{C}, \hat{A}) \in \Sigma_{0}$, to find all the solutions $(G, J, Q)$ with $(\widehat{C}, \widehat{A})=(G, J, Q)(C, A)$ we have to do the following.
(1) We calculate the lists of observability indices $\Pi, \overparen{\Pi}$.
(2) If $\Pi \neq \hat{\Pi}$, there is no solution $(G, J, Q)$.
(3) If $\Pi=\bar{\Pi}$, we calculate the elements $\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)$, $\left(\widehat{G}_{\pi}, \widehat{J}_{\pi}, \widehat{Q}_{\pi}\right) \in \mathscr{Y}$ projecting $(C, A),(\widehat{C}, \widehat{A})$ to their Brunovsky's canonical form ( $C_{\pi}, A_{\pi}$ ).
(4) The general solution of $(\hat{C}, \hat{A})=(G, J, Q)(C, A)$ is given by (35b).

Now we can take advantage of Theorem 2.

## 3. Full Output Feedback Equivalence

The main result of this paper on full output feedback equivalence is obtained substituting $P, Q$ in the third condition of Theorem 2 by the values given in (27b) and (35b), respectively.

Theorem 12. The systems $\sigma=(C, A, B), \widehat{\sigma}=(\widehat{C}, \widehat{A}, \widehat{B}) \in \Sigma$ are $\mathscr{Z}$-equivalent; that is, there is $(G, T, H, F) \in \mathscr{Z}$ with $(\hat{C}, \widehat{A}, \widehat{B})$ $=(C, A, B)(G, T, H, F)=\left(G C T, T^{-1}(A+B H C) T, T^{-1} B F\right)$, if and only if
(I) the pairs $(A, B),(\widehat{A}, \widehat{B}) \in \Sigma_{i}$ have the same lists of controllability indices:

$$
\begin{equation*}
E=\widehat{E} \tag{36}
\end{equation*}
$$

(II) the pairs $(C, A),(\bar{C}, \bar{A},) \in \Sigma_{0}$ have the same lists of observability indices:

$$
\begin{equation*}
\Pi=\bar{\Pi} \tag{37}
\end{equation*}
$$

(III) there are an element $\left(P_{N}, K_{N}, F_{N}\right)$ of the isotropy subgroup of the full state feedback group at the controllability canonical form $\left(A_{\varepsilon}, B_{\varepsilon}\right)$ and an element $\left(G_{N}, J_{N}, Q_{N}\right)$ of the isotropy subgroup of the full output injection group at the observability canonical form $\left(C_{\pi}, A_{\pi}\right)$ with

$$
\begin{equation*}
\widehat{Q}_{\pi} \widehat{P}_{\varepsilon}=Q_{N} Q_{\pi} P_{\varepsilon} P_{N} \tag{38}
\end{equation*}
$$

$x \mapsto P_{\varepsilon} x, x \mapsto \widehat{P}_{\varepsilon} x$ are the changes of bases of the state space, projecting $(A, B),(\hat{A}, \hat{B})$ to their controllability canonical forms of Popov $\left(A_{c}, B_{c}\right),\left(\hat{A}_{c}, \widehat{B}_{c}\right)$ and $x \mapsto Q_{\pi}^{-1} x, x \mapsto \bar{Q}_{\pi}^{-1} x$ are the changes of bases of the state space, projecting $(C, A),(\bar{C}, \bar{A})$ to their observability canonical forms of Popov $\left(C_{o}, A_{o}\right),\left(\bar{C}_{o}, \widehat{A}_{o}\right)$.

Proof. The list of controllability indices $E$ is a complete system of $\mathscr{X}$-invariants:

$$
\begin{equation*}
(A, B) \mathscr{X}(\hat{A}, \widehat{B}) \Longleftrightarrow E=\widehat{E} \tag{39}
\end{equation*}
$$

The variety of elements $(P, K, F) \in X$ with $(\hat{A}, \widehat{B})=$ $(A, B)(P, K, F)$ is given by (27b):

$$
\begin{align*}
&(P, K, F)=\left(P_{\varepsilon} P_{N} \widetilde{P}_{\varepsilon}^{-1}, K_{\varepsilon}+F_{\varepsilon} K_{N} P_{\varepsilon}^{-1}\right. \\
&\left.\quad-F_{\varepsilon} F_{N} \widetilde{F}_{\varepsilon}^{-1} \widehat{K}_{\varepsilon} P_{\varepsilon} \widetilde{P}_{\varepsilon}^{-1} P_{N}^{-1}, F_{\varepsilon} F_{N} \widetilde{F}_{\varepsilon}^{-1}\right) . \tag{40}
\end{align*}
$$

The list of observability indices $\Pi$ is a complete system of $\mathscr{y}$ invariants:

$$
\begin{equation*}
(C, A) \mathscr{Y}(\hat{C}, \hat{A}) \Longleftrightarrow \Pi=\hat{\Pi} . \tag{41}
\end{equation*}
$$

The variety of elements $(G, J, Q) \in \mathscr{Y}$ with $(\hat{C}, \hat{A})=$ $(G, J, Q)(C, A)$ is given by (35b):

$$
\begin{align*}
(G, J, Q)=( & \widehat{Q}_{\pi}^{-1} Q_{N} Q_{\pi}, J_{\pi}+Q_{\pi}^{-1} J_{N} G_{\pi} \\
& \left.\quad-Q_{N}^{-1} \widehat{Q}_{\pi}^{-1} Q_{\pi} \widehat{J}_{\pi} \widehat{G}_{\pi}^{-1} G_{N} G_{\pi}, \overparen{G}_{\pi}^{-1} G_{N} G_{\pi}\right) \tag{42}
\end{align*}
$$

As $P=P_{\varepsilon} P_{N} \widehat{P}_{\varepsilon}^{-1}$ and $Q=\widehat{Q}_{\pi}^{-1} Q_{N} Q_{\pi}$, the third condition of Theorem 2 is written as

$$
\begin{equation*}
Q P=I_{n} \Longleftrightarrow \widehat{Q}_{\pi} \widehat{P}_{\varepsilon}=Q_{N} Q_{\pi} P_{\varepsilon} P_{N} \tag{43}
\end{equation*}
$$

For the calculation of the element $(G, T, H, F) \in \mathscr{Z}$, achieving equivalence, one has to solve the matrix equation (43) for $Q_{N}, P_{N}$, derive $K_{N}, F_{N}$ from the isotropy subgroup of the full state feedback group (Proposition 7), derive $G_{N}$, $J_{N}$ from the isotropy subgroup of the full output injection group (Proposition 10), and substitute them in (27b), (35b). Then the changes of coordinates of the state space are direct $T=P=$ $Q^{-1}$. The output feedback gain is calculated by the formula $H=K C^{\dagger}$ or $H=B^{\dagger} J$. The change of basis of the coordinates of the input and output spaces is direct.

Equation (43) is not linear but as the inverses $\left(Q_{N}\right)^{-1},\left(P_{N}\right)^{-1}$ conserve the structure of $Q_{N}, P_{N}$ we can solve the linear system $\widehat{Q}_{N} \widehat{Q}_{\pi} \widehat{P}_{\varepsilon}=Q_{\pi} P_{\varepsilon} P_{N}$ with $\widehat{Q}_{N}=\left(Q_{N}\right)^{-1}$.

The solution to the $\mathscr{E}$-equivalence problem on $\Sigma$ is given through the following algorithm.

Algorithm 13. To check if the systems $\sigma=(C, A, B)$ and $\widehat{\sigma}=$ $(\widehat{C}, \widehat{A}, \widehat{B}) \in \Sigma$ are $\mathscr{Z}$-equivalent
(1) we calculate the general solution of $(\hat{A}, \hat{B})=$ $(A, B)(P, K, F)$ using Algorithm 8 :
$(P, K, F)=\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(P_{N}, K_{N}, F_{N}\right)\left(\hat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \hat{F}_{\varepsilon}\right)^{-1}$,
(2) we calculate the general solution of $(\widehat{C}, \widehat{A})=$ $(G, J, Q)(C, A)$ using Algorithm 11:
$(G, J, Q)=\left(\hat{G}_{\pi}, \hat{J}_{\pi}, \hat{Q}_{\pi}\right)^{-1}\left(G_{N}, J_{N}, Q_{N}\right)\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)$.
(3) If there is no solution for the equation $\widehat{Q}_{N} \widehat{Q}_{\pi} \widehat{P}_{\varepsilon}=$ $Q_{\pi} P_{\varepsilon} P_{N}$, we have no $\mathscr{Z}$-equivalence.
(4) If $\left(\widehat{Q}_{N}, P_{N}\right)$ is a solution, we calculate $Q_{N}=\left(\widehat{Q}_{N}\right)^{-1}$.
(5) We substitute the values of the entries of $P_{N}$ in $\left(K_{N}, F_{N}\right)$.
(6) We substitute the value of $\left(P_{N}, K_{N}, F_{N}\right)$ in the general solution $(P, K, F)$ to obtain $T, F$.
(7) As we have $\mathscr{E}$-equivalence we have also $K=H C$. So $H=K C^{\dagger}$.
(8) We substitute the value of $Q_{N}$ in $\left(G_{N}, J_{N}\right)$.
(9) We substitute the value of $\left(G_{N}, J_{N}, Q_{N}\right)$ in the general solution ( $G, J, Q$ ) to obtain $G$ and alternatively $T, H$ by the relations $T=Q^{-1}, H=B^{\dagger} J$.

Obviously the output feedback gains $H$ calculated in steps (7) and (9) of Algorithm 13 are identical.

Example 14. Systems: $\sigma=(C, A, B)$ and $\bar{\sigma}=(\widehat{C}, \hat{A}, \widehat{B}) \in \Sigma$

$$
\begin{align*}
{\left[\begin{array}{ll}
A & B \\
C &
\end{array}\right]=} & {\left[\begin{array}{cccccccccc}
-\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & 1 & -1 & 0 \\
\mathbf{0} & -\mathbf{2} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & -1 & -1 & 1 \\
\mathbf{0} & \mathbf{0} & -\mathbf{3} & \mathbf{0} & -\mathbf{4} & \mathbf{0} & \mathbf{3} & -1 & 0 & 1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & -2 & -3 & -1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & 1 & 1 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{2} & -\mathbf{2} & 0 & 0 & -1 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 4 & 0 & 2 & & & \\
0 & 0 & 0 & 0 & 1 & 1 & 2 & & &
\end{array}\right], } \\
{\left[\begin{array}{ll}
\hat{A} & \widehat{B} \\
\widehat{C} & ]
\end{array}\right] } & {\left[\begin{array}{cccccccccc}
-\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 2 & 2 & 1 \\
-\mathbf{1} & -\mathbf{3} & -\mathbf{1} & -\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1 & -1 \\
\mathbf{1} & \mathbf{1} & -\mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 2 & 3 & 3 \\
-\mathbf{2} & -\mathbf{2} & -\mathbf{2} & -\mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & -3 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & 2 & 3 & 2 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & -\mathbf{1} & \mathbf{1} & 1 & 2 & 0 \\
-\mathbf{1} & -\mathbf{1} & -\mathbf{1} & -\mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{2} & 2 & 3 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & &
\end{array}\right] . } \tag{46}
\end{align*}
$$

The lists of controllability indices are $E=\widehat{E}=(3,2,2)$. The changes of bases of the state space, $x \mapsto P_{\varepsilon} x, x \mapsto \widehat{P}_{\varepsilon} x$ projecting $(A, B),(\hat{A}, \widehat{B})$, to their controllability canonical forms of Popov $\left(A_{c}, B_{c}\right),\left(\hat{A}_{c}, \widehat{B}_{c}\right)$ respectively, are

$$
\begin{aligned}
P_{\varepsilon} & {\left[\begin{array}{ccccccc}
0 & 3 & 1 & -4 & -2 & 0 & 0 \\
0 & -3 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\
0 & -5 & -1 & 2 & 1 & 2 & 1 \\
-3 & -7 & -2 & -2 & -1 & -2 & -1 \\
0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{2} & 0 & -\frac{5}{2} & -1 \\
0 & 0 & 0 & 2 & 1 & 2 & 1
\end{array}\right], } \\
\left(A_{c}, B_{c}\right)= & {\left[\begin{array}{cccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & -4 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -\frac{5}{2} & 1 & \frac{1}{2} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{2} & -1 & -\frac{5}{2} & 0 & 0 & 1
\end{array}\right], }
\end{aligned}
$$

$$
\begin{align*}
& \widehat{P}_{\varepsilon}=\left[\begin{array}{cccccccc}
-10 & 1 & 2 & -2 & -1 & -2 & -1 \\
-\frac{5}{2} & -3 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -2 \\
0 & 2 & 2 & 0 & 0 & 0 & 0 \\
-8 & -5 & 0 & 0 & 0 & -6 & -3 \\
0 & 6 & 2 & 0 & 0 & 0 & 0 \\
\frac{5}{2} & 2 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\
-2 & 3 & 2 & 0 & 0 & -6 & -3
\end{array}\right], \\
& \left(\widehat{A}_{c}, \widehat{B}_{c}\right)=\left[\begin{array}{ccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 0 & 1 & -1 & -\frac{1}{2} & -7 & -5 & 1 & \frac{3}{2} & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{50}{3} & -\frac{31}{3} & 0 & -\frac{8}{3} & -\frac{10}{3} & -\frac{20}{3} & -\frac{13}{3} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-\frac{83}{6} & -\frac{29}{3} & 0 & -\frac{11}{6} & -\frac{7}{6} & -\frac{23}{6} & -\frac{14}{3} & 0 & 0 & 1
\end{array}\right] . \tag{47}
\end{align*}
$$

The state feedback transformations $u \mapsto u+K_{\varepsilon}^{\prime} x, u \mapsto$ $u+\widehat{K}_{\varepsilon}^{\prime} x$ and the change of bases of the input spaces $u \mapsto$ $F u, u \mapsto \tilde{F u}$ projecting the controllability canonical forms of $\operatorname{Popov}\left(A_{c}, B_{c}\right),\left(\widehat{A}_{c}, \widehat{B}_{c}\right)$ to the controllability canonical forms of Brunovsky $\left(A_{\varepsilon}, B_{\varepsilon}\right),\left(\widehat{A}_{\varepsilon}, \widehat{B}_{\varepsilon}\right)$ are

$$
\begin{gathered}
K_{\varepsilon}^{\prime}=\left[\begin{array}{ccccccc}
0 & 3 & 4 & -1 & -\frac{5}{2} & 1 & \frac{1}{2} \\
0 & 0 & 0 & 1 & \frac{5}{2} & -1 & -\frac{1}{2} \\
0 & 0 & 0 & -1 & -\frac{1}{2} & 1 & \frac{5}{2}
\end{array}\right], \\
\widehat{K}_{\varepsilon}^{\prime}=\left[\begin{array}{ccccccc}
-\frac{173}{6} & -\frac{151}{6} & -1 & -\frac{29}{6} & -\frac{17}{3} & -\frac{41}{6} & -\frac{37}{6} \\
\frac{50}{3} & \frac{31}{3} & 0 & \frac{8}{3} & \frac{10}{3} & \frac{20}{3} & \frac{13}{3} \\
\frac{83}{6} & \frac{29}{3} & 0 & \frac{11}{6} & \frac{7}{6} & \frac{23}{6} & \frac{14}{3}
\end{array}\right], \\
F_{\varepsilon}=\left[\begin{array}{ccc}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right], \quad \widehat{F}_{\varepsilon}=\left[\begin{array}{ccc}
1 & -\frac{3}{2} & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right],
\end{gathered}
$$

$$
\begin{gather*}
\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)=\left(P_{\varepsilon}, O_{m \times r}, I_{r}\right)\left(I_{n}, K_{\varepsilon}^{\prime}, F_{\varepsilon}\right)=\left(P_{\varepsilon}, K_{\varepsilon}^{\prime} P_{\varepsilon}^{-1}, F_{\varepsilon}\right), \\
\left(\hat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \hat{F}_{\varepsilon}\right)=\left(\widehat{P}_{\varepsilon}, O_{m \times r}, I_{r}\right)\left(I_{n}, \widehat{K}_{\varepsilon}^{\prime}, \hat{F}_{\varepsilon}\right) \\
=\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}^{\prime} \widehat{P}_{\varepsilon}^{-1}, \widehat{F}_{\varepsilon}\right), \\
K_{\varepsilon}=\left[\begin{array}{ccccccc}
-\frac{1}{2} & 1 & \frac{9}{2} & 0 & 10 & -3 & -8 \\
\frac{1}{2} & -1 & 0 & 0 & -\frac{3}{2} & 3 & \frac{7}{2} \\
-\frac{1}{2} & 3 & 0 & 0 & \frac{7}{2} & -1 & -\frac{3}{2}
\end{array}\right] \\
\hat{K}_{\varepsilon}=  \tag{48}\\
{\left[\begin{array}{ccccccc}
-\frac{5}{6} & \frac{19}{3} & -\frac{11}{6} & \frac{2}{3} & \frac{8}{3} & -\frac{20}{3} & -\frac{1}{3} \\
\frac{2}{3} & -\frac{11}{3} & \frac{5}{3} & -\frac{1}{3} & -\frac{7}{3} & \frac{13}{3} & -\frac{1}{3} \\
-\frac{2}{3} & -\frac{4}{3} & \frac{13}{12} & -\frac{2}{3} & \frac{1}{12} & -\frac{1}{3} & \frac{1}{3}
\end{array}\right]}
\end{gather*}
$$

$\left(P_{0}, K_{0}, F_{0}\right)=\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)\left(\widehat{P}_{\varepsilon}, \widehat{K}_{\varepsilon}, \widehat{F}_{\varepsilon}\right)^{-1}$ is a particular solution of $(\widehat{A}, \widehat{B})=(A, B)\left(P_{0}, K_{0}, F_{0}\right)$ :

$$
P_{0}=\left[\begin{array}{ccccccc}
2 & 0 & -\frac{4}{9} & -\frac{28}{9} & -\frac{7}{2} & 0 & \frac{22}{9}  \tag{49}\\
\frac{1}{2} & -\frac{2}{3} & \frac{43}{72} & -\frac{13}{36} & -\frac{13}{8} & \frac{1}{3} & \frac{7}{36} \\
-1 & 0 & \frac{11}{12} & \frac{5}{3} & \frac{5}{4} & 0 & -\frac{5}{3} \\
1 & 0 & -\frac{5}{12} & -\frac{7}{6} & -\frac{11}{4} & 0 & \frac{7}{6} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\frac{3}{2} & -\frac{4}{3} & \frac{17}{72} & -\frac{47}{36} & -\frac{27}{8} & \frac{5}{3} & \frac{53}{36} \\
-1 & 0 & \frac{5}{12} & \frac{5}{3} & \frac{9}{4} & 0 & -\frac{5}{3}
\end{array}\right],
$$

The lists of observability indices are $\Pi=\bar{\Pi}=(4,3)$.

The change of bases of the state space $x \mapsto Q_{\pi}^{-1} x, x \mapsto$ $\widehat{Q}_{\pi}^{-1} x$ projects $(C, A),(\widehat{C}, \hat{A})$ to their observability canonical form of Popov $\left(C_{o}, A_{o}\right),\left(\bar{C}_{o}, \bar{A}_{o}\right)$ :

$$
\begin{aligned}
& Q_{\pi}=\frac{1}{4}\left[\begin{array}{ccccccc}
0 & 0 & 0 & 6 & 6 & 0 & 12 \\
6 & 3 & 2 & 11 & 18 & 0 & 26 \\
5 & 4 & 2 & 6 & 16 & 0 & 14 \\
1 & 1 & 1 & 1 & 4 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 1 & 4 \\
0 & 0 & 0 & 0 & 1 & 1 & 2
\end{array}\right], \\
& \hat{Q}_{\pi}=\left[\begin{array}{ccccccc}
0 & 0 & 0 & 6 & 0 & 0 & 26 \\
6 & 3 & 2 & 11 & 26 & 13 & 41 \\
5 & 4 & 3 & 6 & 15 & 14 & 16 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 4 & 2 & 6 \\
0 & 0 & 0 & 0 & 2 & 2 & 2
\end{array}\right],
\end{aligned}
$$

$$
\left[\begin{array}{l}
A_{o}  \tag{50}\\
C_{o}
\end{array}\right]=\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -6 & 0 & 0 & 0 \\
0 & 1 & 0 & -11 & 0 & 0 & 0 \\
0 & 0 & 1 & -6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right],
$$

$$
\left[\begin{array}{c}
\hat{A}_{o} \\
\widehat{C}_{o}
\end{array}\right]=\left[\begin{array}{ccccccc}
0 & 0 & 0 & -38 & 0 & 0 & 48 \\
1 & 0 & 0 & -44 & 0 & 0 & \frac{161}{2} \\
0 & 1 & 0 & -25 & 0 & 0 & \frac{63}{2} \\
0 & 0 & 1 & -8 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 0 & 6 \\
0 & 0 & 0 & -2 & 1 & 0 & 8 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1
\end{array}\right] .
$$

The output injection transformations $\dot{x} \mapsto \dot{x}+J_{\pi}^{\prime} y, \dot{x} \mapsto$ $\dot{x}+\bar{J}_{\pi}^{\prime} y$ and the change of bases of the output spaces $y \mapsto$ $G_{\pi}^{-1} y, y \mapsto \widehat{G}_{\pi}^{-1} y$ projecting systems $\left(C_{o}, A_{o}\right),\left(\widehat{C}_{o}, \widehat{A}_{o}\right)$ to the
observability canonical form of Brunovsky $\left(C_{\pi}, A_{\pi}\right),\left(\widehat{C}_{\pi}, \hat{A}_{\pi}\right)$ are

$$
\begin{align*}
& \left(C_{\pi}, A_{\pi}\right)=\left(G_{\pi}, J_{\pi}^{\prime}, I_{n}\right)\left(C_{o}, A_{o}\right)=\left(\widehat{G}_{\pi}, \widehat{J}_{\pi}^{\prime}, I_{n}\right)\left(\widehat{C}_{o}, \widehat{A}_{o}\right), \\
& G_{\pi}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right], \\
& J_{\pi}^{\prime T}=\left[\begin{array}{ccccccc}
0 & 6 & 11 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 3
\end{array}\right] \quad \widehat{G}_{\pi}=\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right], \\
& \hat{J}_{\pi}^{\prime T}=\left[\begin{array}{ccccccc}
-10 & -\frac{73}{2} & -\frac{13}{2} & 8 & -2 & -6 & -1 \\
-48 & -\frac{161}{2} & -\frac{63}{2} & 0 & -6 & -8 & -1
\end{array}\right], \\
& \left(C_{\pi}, A_{\pi}\right)=\left(G_{\pi}, J_{\pi}^{\prime}, I_{n}\right)\left(I_{r}, O_{n \times r}, Q_{\pi}\right)(C, A) \\
& =\left(\hat{G}_{\pi}, \widehat{J}_{\pi}^{\prime}, I_{n}\right)\left(I_{r}, O_{n \times r}, \hat{Q}_{\pi}\right)(\hat{C}, \widehat{A}), \\
& \left(C_{\pi}, A_{\pi}\right)=\left(G_{\pi}, J_{\pi}, Q_{\pi}\right)(C, A)=\left(\widehat{G}_{\pi}, J_{\pi}, \widehat{Q}_{\pi}\right)(\widehat{C}, \widehat{A}) \\
& J_{\pi}^{T}=\left[\begin{array}{ccccccc}
\frac{1}{2} & -8 & \frac{27}{2} & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & -1 & 4 & 0
\end{array}\right], \\
& \tilde{J}_{\pi}^{T}=\left[\begin{array}{ccccccc}
3 & -30 & 35 & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} \\
\frac{5}{2} & -23 & \frac{45}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2}
\end{array}\right] . \tag{51}
\end{align*}
$$

A particular solution of $(\hat{C}, \widehat{A})=\left(G_{0}, J_{0}, Q_{0}\right)(C, A)$ is $\left(G_{0}, J_{0}, Q_{0}\right)=\left(\widehat{G}_{\pi}, J_{\pi}, \widehat{Q}_{\pi}\right)^{-1}\left(G_{\pi}, J_{\pi}, Q_{\pi}\right):$

$$
G_{0}=\left[\begin{array}{cc}
1 & 0 \\
-1 & 1
\end{array}\right], \quad J_{0}=\left[\begin{array}{cc}
8 & -\frac{5}{2} \\
-3 & \frac{13}{2} \\
-\frac{20}{3} & -12 \\
-\frac{13}{3} & 2 \\
2 & 0 \\
2 & -2 \\
-2 & 3
\end{array}\right],
$$

$$
Q_{0}=\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & -3 & 0 & 1  \tag{52}\\
0 & 1 & 0 & 0 & 1 & -\frac{11}{2} & -\frac{11}{2} \\
0 & 0 & 1 & 0 & \frac{9}{2} & 5 & \frac{17}{3} \\
0 & 0 & 0 & 1 & 1 & 0 & -\frac{1}{6} \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{array}\right] .
$$

For the elements $\left(P_{N}, K_{N}, F_{N}\right),\left(G_{N}, J_{N}, Q_{N}\right)$ of the isotropy subgroups we have

$$
\begin{align*}
P_{N} & =\left[\begin{array}{lllllll}
a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 \\
b & c & 0 & d & 0 & e & 0 \\
0 & b & c & 0 & d & 0 & e \\
f & g & 0 & h & 0 & k & 0 \\
0 & f & g & 0 & h & 0 & k
\end{array}\right], \\
Q_{N} & =\left[\begin{array}{lllllll}
v & 0 & 0 & 0 & w & 0 & 0 \\
0 & v & 0 & 0 & x & w & 0 \\
0 & 0 & v & 0 & 0 & x & w \\
0 & 0 & 0 & v & 0 & 0 & x \\
0 & 0 & 0 & 0 & y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & y
\end{array}\right],  \tag{53}\\
\widehat{Q}_{N} & =\left[\begin{array}{lllllll}
\widehat{v} & 0 & 0 & 0 & \widehat{w} & 0 & 0 \\
0 & \widehat{v} & 0 & 0 & \widehat{x} & \widehat{w} & 0 \\
0 & 0 & \widehat{v} & 0 & 0 & \widehat{x} & \widehat{w} \\
0 & 0 & 0 & \widehat{v} & 0 & 0 & \widehat{x} \\
0 & 0 & 0 & 0 & \widehat{y} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \widehat{y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \widehat{y}
\end{array}\right] .
\end{align*}
$$

The equation $\widehat{Q}_{N} \widehat{Q}_{\pi} \widehat{P}_{\varepsilon}=Q_{\pi} P_{\varepsilon} P_{N}$ has infinitely many solutions for $\widehat{Q}_{N}, P_{N}$. One of them is $a=-4, b=-4, c=$ $-2, d=-1, e=2, f=6, g=-2, h=1, k=4, \widehat{v}=-2, \widehat{w}=$ 13, $\widehat{x}=1, \widehat{y}=-1$

We conclude that $(C, A, B)$ and $(\widehat{C}, \widehat{A}, \widehat{B})$ are full output feedback equivalent.

To calculate the transformation achieving equivalence, we substitute the values of $a, b, c, d, e, f, g, h, k$, in $K_{N}, F_{N}$, we
calculate $Q_{N}=\left(\widehat{Q}_{N}\right)^{-1}$, and we substitute the values of $v, w, x, y$ in $G_{N}, J_{N}$ :

$$
\begin{array}{cc}
K_{N}=\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0
\end{array}\right], \quad F_{N}=\left[\begin{array}{ccc}
-4 & 0 & 0 \\
-2 & -1 & 2 \\
-2 & 1 & 4
\end{array}\right], \\
G_{N}=\left[\begin{array}{cc}
-\frac{1}{2} & -\frac{1}{2} \\
0 & -1
\end{array}\right], & J_{N}^{T}=\left[\begin{array}{lllllll}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 13 & 0 & 0 & 0
\end{array}\right] . \tag{54}
\end{array}
$$

For the $(G, T, H, F) \in \mathscr{Z}$ achieving equivalence we have the following.
(a) From full state feedback equivalence $(P, K, F)$

$$
\begin{align*}
& (P, K, F)=\left(P_{\varepsilon} P_{N} \widetilde{P}_{\varepsilon}^{-1}, K_{\varepsilon}+F_{\varepsilon} K_{N} P_{\varepsilon}^{-1}\right.  \tag{55}\\
& \left.-F_{\varepsilon} F_{N} \widehat{F}_{\varepsilon}^{-1} \widehat{K}_{\varepsilon} P_{\varepsilon} \mathcal{P}_{\varepsilon}^{-1} P_{N}^{-1}, F_{\varepsilon} F_{N} \widetilde{F}_{\varepsilon}^{-1}\right), \\
& T=P=\left[\begin{array}{ccccccc}
-2 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & -2 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 & 4 & 0 & -2 \\
0 & 0 & 0 & -2 & 2 & 0 & 4 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -2
\end{array}\right], \tag{56}
\end{align*}
$$

$$
K=\left[\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 4 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 2 \\
-1 & -1 & -1 & -1 & 3 & 1 & 0
\end{array}\right]
$$

$$
F=\left[\begin{array}{lll}
-2 & -4 & 2 \\
-2 & -4 & 0 \\
-2 & -2 & 2
\end{array}\right]
$$

As the systems are full output feedback equivalent, there is no doubt that the state feedback gain $K$ is of the form $K=H C$. However, we check it. The rows of the matrix $N_{C}$ form a basis for the space $\operatorname{Kernel}(C)$ :

$$
N_{C}=\left[\begin{array}{rrrrrrr}
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & -1 & 0 & 1 \\
6 & 0 & 0 & 0 & -2 & 0 & 1
\end{array}\right],
$$

$$
\begin{align*}
K N_{C}^{T}= & O_{3 \times 5} \\
& \Longrightarrow \exists H \quad \text { with } K=H C \\
& \Longrightarrow H=K C^{\dagger}=\left[\begin{array}{rr}
1 & 0 \\
0 & 1 \\
-1 & 1
\end{array}\right]\left[\begin{array}{rr}
1 & 0 \\
0 & 1 \\
-1 & 1
\end{array}\right] . \tag{57}
\end{align*}
$$

(b) From full output injection equivalence $G=\widehat{G}_{\pi}^{-1} G_{N} G_{\pi}$ $=\left[\begin{array}{cc}-1 / 2 & -1 / 2 \\ 1 / 2 & -1 / 2\end{array}\right]$.

We presented explicit and computable necessary and sufficient conditions for full output feedback equivalence on the set of linear, time invariant, minimal systems driving to the construction of the full output feedback transformation achieving equivalence. The initial equivalence relation is described by $n(n+m+r)$ nonlinear equations with $n^{2}+m^{2}+r^{2}+m r$ unknowns and 3 constraints, $(\operatorname{det}(T) \neq 0, \operatorname{det}(F) \neq 0, \operatorname{det}(G) \neq 0)$. It is transformed to another equivalence relation, described by $n^{2}+m+r$ equations (including equalities of invariant indices) with a number of unknowns depending on the distribution of controllability and observability indices and 2 constraints, ( $\left.\operatorname{det}\left(P_{N}\right) \neq 0, \operatorname{det}\left(Q_{N}\right) \neq 0\right)$. This is not palatable for the control engineer as the balance equations-unknowns is harder in the second case. In Example 14 the difference equations minus unknowns for the initial problem is 16 and for the final 41.

The removal of the impasse is given through the study of the structure of the matrix $Q_{\pi} P_{\varepsilon}$. The entries of this matrix are not independent. Indeed,

$$
\begin{align*}
A_{c} & =P_{\varepsilon}^{-1} A P_{\varepsilon} \wedge A_{o} \\
& =Q_{\pi} A Q_{\pi}^{-1} \Longrightarrow P_{\varepsilon} A_{c} P_{\varepsilon}^{-1}=A \wedge Q_{\pi}^{-1} A_{o} Q_{\pi} \\
& =A \Longrightarrow P_{\varepsilon} A_{c} P_{\varepsilon}^{-1}=Q_{\pi}^{-1} A_{o} Q_{\pi}  \tag{58}\\
& \Longrightarrow Q_{\pi} P_{\varepsilon} A_{c}=A_{o} Q_{\pi} P_{\varepsilon}
\end{align*}
$$

It is proved in Yannakoudakis [12] (as it is referred to by Helmke and Fuhrmann [9] and Byrnes and Crouch [15]) that the function $\sigma \mapsto Q_{\pi} P_{\varepsilon}$ is complete static output invariant for scalar systems. Furthermore, the first column and the last row of $Q_{\pi} P_{\varepsilon}$ are a complete and independent (as defined in Popov [14]) system of static output invariants for scalar systems.

Helmke and Fuhrmann [9] prove that the matrix $Q_{\pi} P_{\varepsilon}$ is a Bezoutian. It is related to the polynomial Bezoutian $b(\lambda, \mu)=$ $(a(\lambda) z(\mu)-a(\mu) z(\lambda)) /(\lambda-\mu)$ of the characteristic $a(s)$ and the zero $z(s)$ polynomials of the system, by the relation $b(\lambda, \mu)=$ $\left[\begin{array}{lll}1 & \mu \cdots \mu^{n-1}\end{array}\right] Q_{\pi} P_{\varepsilon}\left[\begin{array}{ll}1 & \lambda \cdots \lambda^{n-1}\end{array}\right]^{T}$.

Let us now expand the Bezoutian $b(\lambda, \mu)=\sum_{k=0}^{n-1} z_{k}(\lambda) \mu^{k}$. The coefficients of the polynomials $z_{0}(s), z_{n-1}(s)$ are the entries of the first and last rows of $Q_{\pi} P_{\varepsilon}$. We conclude that the set of roots of the pair of polynomials $\left(z_{0}(s), z_{n-1}(s)\right)$ is a complete system of independent $\mathscr{Z}$-invariants.

Anderson and Jury [11] generalize the polynomial Bezoutian $b(\lambda, \mu)$ for scalar systems to the generalized Bezoutian, for multivariable systems.

Let $X(s)=C\left(s I_{n}-A\right)^{-1} B$ be the transfer function matrix of a system $(C, A, B) \in \Sigma$. Consider a left and a right coprime factorization of the transfer function matrix $X(s)=$ $D_{L}^{-1}(s) Z_{L}(s)=Z_{R}(s) D_{R}^{-1}(s)$. The generalized Bezoutian associated with the pair of coprime factorizations is

$$
\begin{align*}
B(\lambda, \mu) & =\frac{Z_{L}(\mu) D_{R}(\lambda)-D_{L}(\mu) Z_{R}(\lambda)}{\lambda-\mu} \\
& =D_{L}(\mu) \frac{X(\mu)-X(\lambda)}{\lambda-\mu} D_{R}(\lambda) . \tag{59}
\end{align*}
$$

Apparently there is an infinity of generalized Bezoutians associated with each system $\sigma \in \Sigma$. For any two of them $B(\lambda, \mu), \widehat{B}(\lambda, \mu)$ there are unimodular matrices $V(\lambda), U(\mu)$ with $U(\mu) B(\lambda, \mu) V(\lambda)=\widehat{B} \quad(\lambda, \mu)$. Let $\mathscr{B}$ be the family of the generalized Bezoutians associated with a system $\sigma \in \Sigma$. Theorem 12 is equivalent to the following.

Theorem 15. The family of generalized Bezoutians is a complete system of $\mathscr{X}$-invariants: $\sigma \mathscr{Z} \hat{\sigma} \Leftrightarrow \mathscr{B}=\widehat{\mathscr{B}}$.

Proof. Let $\mathbb{R}_{r}^{p \times q}[s]$ be the set of $p \times q$ column proper real polynomial matrices of degree $r$ and let $\mathbb{R}_{r}^{p \times q}\{s\}$ be the set of $p \times q$ row proper real polynomial matrices of degree $r$.

Consider a left coprime factorization of the transfer matrix state output and a right coprime factorization of the transfer matrix input state:

$$
\begin{align*}
& C\left(s I_{n}-A\right)^{-1} \\
& \qquad \begin{array}{l}
=D_{L}^{-1}(s) N_{L}(s),\left(D_{L}(s), N_{L}(s)\right) \in \mathbb{R}_{n}^{r \times r}\{s\} \times \mathbb{R}_{n-r}^{r \times n}\{s\} \\
\left(s I_{n}-A\right)^{-1} B= \\
\quad N_{R}(s) D_{R}^{-1}(s),\left(N_{R}(s), D_{R}(s)\right) \\
\quad \in \mathbb{R}_{n-m}^{n \times m}[s] \times \mathbb{R}_{n}^{m \times m}[s] .
\end{array}
\end{align*}
$$

Let $X(s)=C\left(s I_{n}-A\right)^{-1} B$ be the transfer function matrix of the system $\sigma \in \Sigma$. The following trivial calculation is due to Kimura [16]:

$$
\begin{aligned}
& \frac{X(\mu)-X(\lambda)}{\lambda-\mu} \\
& \quad=\frac{C\left(\mu I_{n}-A\right)^{-1} B-C\left(\lambda I_{n}-A\right)^{-1} B}{\lambda-\mu} \\
& \quad=\frac{C\left(\left(\mu I_{n}-A\right)^{-1}-\left(\lambda I_{n}-A\right)^{-1}\right) B}{\lambda-\mu} \\
& \quad \Longleftrightarrow \frac{X(\mu)-X(\lambda)}{\lambda-\mu}=C\left(\mu I_{n}-A\right)^{-1}\left(\lambda I_{n}-A\right)^{-1} B .
\end{aligned}
$$

Substituting (58) in (61) we obtain

$$
\begin{align*}
& \frac{X(\mu)-X(\lambda)}{\lambda-\mu} \\
& \quad=D_{L}^{-1}(\mu) N_{L}(\mu) N_{R}(\lambda) D_{R}^{-1}(\lambda) \\
& \quad \Longleftrightarrow N_{L}(\mu) N_{R}(\lambda)=\frac{N_{L}(\mu) B D_{R}(\lambda)-D_{L}(\mu) C N_{R}(\lambda)}{\lambda-\mu} \\
& \quad \Longleftrightarrow N_{L}(\mu) N_{R}(\lambda)=\frac{Z_{L}(\mu) D_{R}(\lambda)-D_{L}(\mu) Z_{R}(\lambda)}{\lambda-\mu} \\
& \Longleftrightarrow N_{L}(\mu) N_{R}(\lambda) \in \mathscr{B} . \tag{62}
\end{align*}
$$

In other words, the product $N_{L}(\mu) N_{R}(\lambda)$ is a generalized Bezoutian. We can assign to each system exactly one generalized Bezoutian. Notice that the group $\mathscr{X}$ acts on $\mathbb{R}_{n-m}^{n \times m}[s] \times$ $\mathbb{R}_{n}^{m \times m}[s]$. Let $(N(s), D(s))$ be a right coprime factorization of $(s I-A)^{-1} B$, and let $(\widetilde{N}(s), \widetilde{D}(s))$ be a right coprime factorization of $P^{-1}(s I-A-B K)^{-1} B F$ and $(P, K, F) \in \mathscr{X}$ :

$$
\begin{align*}
&(s I-A)^{-1} B \\
&=N(s) D^{-1}(s) \Longrightarrow P^{-1}(s I-A-B K)^{-1} P P^{-1} B F \\
&=\widetilde{N}(s) \widetilde{D}^{-1}(s) \Longrightarrow(s I-A-B K)^{-1} B F \widetilde{D}(s)=P \widetilde{N}(s) \\
& \Longrightarrow B F \widetilde{D}(s)=(s I-A-B K) P \widetilde{N}(s) \\
& \Longrightarrow B F \widetilde{D}(s)+B K P \widetilde{N}(s) \\
&=(s I-A) P \widetilde{N}(s) \Longrightarrow B(F \widetilde{D}(s)+K P \widetilde{N}(s)) \\
&=(s I-A) P \widetilde{N}(s) \Longrightarrow F \widetilde{D}(s)+K P \widetilde{N}(s)=D(s), \\
& P \widetilde{N}(s)=N(s) \Longrightarrow(P, K, F)(N(s), D(s)) \\
& \quad=\left(P^{-1} N(s), F^{-1}(D(s)+K N(s))\right) . \tag{63}
\end{align*}
$$

Let now $\left(N_{\varepsilon}(s), D_{\varepsilon}(s)\right)$ be a particular right coprime factorization of $\left(s I_{n}-A_{\varepsilon}\right)^{-1} B_{\varepsilon}$ :

$$
\begin{gather*}
N_{\varepsilon}(s)=\left[\begin{array}{cccc}
1 s \cdots s^{p_{1}-1} & 00 \cdots 0 & \cdots & 00 \cdots 0 \\
00 \cdots 0 & 1 s \cdots s^{p_{2}-1} & \cdots & 00 \cdots 0 \\
\vdots & \vdots & \ddots & \vdots \\
00 \cdots 0 & 00 \cdots 0 & \cdots & 1 s \cdots s^{p_{m}-1}
\end{array}\right]^{T},  \tag{64}\\
D_{\varepsilon}(s)=\left[\begin{array}{cccc}
s^{p_{1}} & 0 & \cdots & 0 \\
0 & s^{p_{2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s^{p_{m}}
\end{array}\right] .
\end{gather*}
$$

Obviously $\left(N_{\varepsilon}(s), D_{\varepsilon}(s)\right)$ is a $\mathscr{X}$-canonical form for the equivalence relation induced by (63).

As $(A, B)\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)=\left(A_{\varepsilon}, B_{\varepsilon}\right) \Rightarrow\left(N_{R_{0}}(s), D_{R_{0}}(s)\right)=$ $\left(P_{\varepsilon}, K_{\varepsilon}, F_{\varepsilon}\right)^{-1}\left(N_{\varepsilon}(s), D_{\varepsilon}(s)\right)$ is a uniquely determined right coprime factorization of $(s I-A)^{-1} B$ :

$$
\begin{equation*}
N_{R 0}(s)=P_{\varepsilon} N_{\varepsilon}(s) \tag{65}
\end{equation*}
$$

By duality we have that $N_{L 0}(s)=N_{\pi}(s) Q_{\pi}$

$$
N_{\pi}(s)=\left[\begin{array}{cccc}
1 s \cdots s^{q_{1}-1} & 00 \cdots 0 & \cdots & 00 \cdots 0  \tag{66}\\
00 \cdots 0 & 1 s \cdots s^{q_{2}-1} & \cdots & 00 \cdots 0 \\
\vdots & \vdots & \ddots & \vdots \\
00 \cdots 0 & 00 \cdots 0 & \cdots & 1 s \cdots s^{q_{r}-1}
\end{array}\right]
$$

We conclude that to each system we can assign exactly one generalized Bezoutian:

$$
\begin{equation*}
B_{0}(\lambda, \mu)=N_{L 0}(s) N_{R 0}(s)=N_{\pi}(s) Q_{\pi} P_{\varepsilon} N_{\varepsilon}(s) \tag{67}
\end{equation*}
$$

To any block-Toeplitz matrix $P_{N}$ we assign a unimodular matrix $V_{N}(s)=\left\{v_{\zeta \xi}(s)\right\}, v_{\zeta \xi}(s)=\sum_{k=1}^{p_{\zeta}-p_{\xi}+1} \gamma_{\zeta \xi}^{k} s^{k-1}$ Then $P_{N} N_{\varepsilon}(s)=N_{\varepsilon}(s) V_{N}(s)$.

To any block-Toeplitz matrix $Q_{N}$ we assign a unimodular matrix $U_{N}(s)=\left\{v_{\zeta \xi}(s)\right\}, v_{\zeta \xi}(s)=\sum_{k=1}^{q_{\zeta}-q_{\xi}+1} \delta_{\zeta \xi}^{k} s^{k-1}$. Then $N_{\pi}(s) Q_{N}=U_{N}(s) N_{\pi}(s)$

$$
\begin{align*}
\hat{Q}_{\pi} \widehat{P}_{\varepsilon}= & Q_{N} Q_{\pi} P_{\varepsilon} P_{N} \Longleftrightarrow N_{\pi}(\mu) \hat{Q}_{\pi} \widehat{P}_{\varepsilon} N_{\varepsilon}(\lambda) \\
= & N_{\pi}(\mu) Q_{N} Q_{\pi} P_{\varepsilon} P_{N} N_{\varepsilon}(\lambda) \Longleftrightarrow \hat{N}_{L_{0}}(\mu) \hat{N}_{R_{0}}(\lambda) \\
= & U_{N}(\mu) N_{\varepsilon}(\mu) Q_{\pi} P_{\varepsilon} N_{\varepsilon}(\lambda) V_{N}(\lambda) \\
& \Longleftrightarrow \hat{N}_{L_{0}}(\mu) \hat{N}_{R_{0}}(\lambda) \\
= & U_{N}(\mu) N_{L_{0}}(\mu) N_{R_{0}}(\lambda) V_{N}(\lambda) \Longleftrightarrow \widehat{B}_{0}(\mu, \lambda) \\
= & U_{N}(\mu) B_{0}(\mu, \lambda) V_{N}(\lambda) \Longleftrightarrow \widehat{\mathscr{B}}=\mathscr{B} . \tag{68}
\end{align*}
$$

Theorem 15 is a polynomial version of Theorem 12. It does not add something important to the equivalence problem. The family of generalized Bezoutians is a complete system of $\mathscr{Z}$-invariants but it is infinite. It has however a huge importance considering the solution of control problems involving output feedback. The solution of such problems is obligated to have an expression in terms of the generalized Bezoutian. We give a simple example of generalization. The breakaway polynomial for scalar systems is $w(s)=$ $b(s, s)$ [9]. The invariant factors of the polynomial matrix $W(s)=B(s, s)$ are $\mathscr{Z}$-invariant and seem to have the same geometric interpretation with scalar breakaway polynomial. Rank deficiency of $W\left(s_{0}\right)$, means that $s_{0}$ is a double closed loop (by an output feedback) pole.

## 4. Minimal Number of Equations

In this section we explain why among the $n^{2}$ equations involved in equivalence relation (43) only $n(m+r)-m r$ are independent. First of all we reproduce a result of [13].

Proposition 16. The matrix $\mathbf{H}=\left(Q_{\pi} P_{\varepsilon}\right)^{-1}$ has an $m \times r$ block structure. Block $H_{\zeta \xi}$ has dimension $p_{\zeta} \times q_{\xi}$, entries $H_{\zeta \xi}^{\nu}, 1 \leq$ $\nu \leq p_{\zeta}, 1 \leq \kappa \leq q_{\xi}$, and a Hankel structure, that is, $H_{\zeta \xi}^{(\nu+1) \kappa}=$ $H_{\zeta \xi}^{\nu(\kappa+1)}$.

Proof. From the equations of controllability and observability canonical forms $A_{c}=P_{\varepsilon}^{-1} A P_{\varepsilon}$ and $A_{o}=Q_{\pi} A Q_{\pi}^{-1}$ we conclude that $A_{c} \mathbf{H}=\mathbf{H} A_{o}$. Let us now write the matrix $A_{c}$ as a sum of two matrices $A_{c}=A_{\varepsilon}+A_{c}^{\prime}$. The matrix $A_{\varepsilon}^{\prime}$ is zero except its rows $\mathbf{p}_{i}=\sum_{k=1}^{i} p_{i}$ that are those of the matrix $A_{c}$, and the matrix $A_{o}$ as a sum of two matrices $A_{o}=A_{\pi}+A_{o}^{\prime}$ with $A_{o}^{\prime}$ zero, except its columns $\mathbf{q}_{i}=\sum_{k=1}^{i} q_{i}$ that are those of the matrix $A_{c}$. Then,

$$
\begin{align*}
\left(A_{\varepsilon}+A_{c}^{\prime}\right) \mathbf{H} & =\mathbf{H}\left(A_{\pi}+A_{o}^{\prime}\right) \Longrightarrow A_{\varepsilon} \mathbf{H}-\mathbf{H} A_{\pi} \\
& =\mathbf{H} A_{o}^{\prime}-A_{c}^{\prime} \mathbf{H} . \tag{69}
\end{align*}
$$

The matrix $\mathbf{H} A_{o}^{\prime}$ has only its entries on the columns $\mathbf{q}_{1}, \mathbf{q}_{2}, \ldots, \mathbf{q}_{r}$ different than zero. The matrix $A_{c}^{\prime} \mathbf{H}$ has only its entries on the rows $\mathbf{p}_{1}, \mathbf{p}_{2}, \ldots, \mathbf{p}_{m}$ different than zero. So the matrix $Z=H A_{o}^{\prime}-A_{c}^{\prime} H$ has an $m \times r$ block structure. $Z=\left\{Z_{\zeta \xi}\right\}, 1 \leq \zeta \leq m, 1 \leq \xi \leq r$ with blocks:

$$
Z_{\zeta \xi}=\left[\begin{array}{cccc}
0 & \cdots & 0 & \beta_{\zeta \zeta}^{1}  \tag{70}\\
\vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \beta_{\zeta \zeta}^{p_{\zeta}-1} \\
\alpha_{\zeta \xi}^{1} & \cdots & \alpha_{\zeta \xi}^{q_{\xi}-1} & z_{\zeta \zeta}
\end{array}\right]
$$

The matrix $A_{\varepsilon},\left(A_{\pi}\right)$ has an $m \times m,(r \times r)$ block structure: its block with coordinates $\zeta, \xi$ the $\left(A_{\varepsilon}\right)_{\zeta \xi},\left(\left(A_{\pi}\right)_{\zeta \xi}\right)$ has dimension $p_{\zeta} \times p_{\xi},\left(q_{\zeta} \times q_{\xi}\right)$ and verifies the relations

$$
\begin{align*}
\left(A_{\varepsilon}\right)_{\zeta \zeta}=\left[\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{array}\right], \quad\left(A_{\pi}\right)_{\xi \xi}=\left[\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{array}\right], \\
\left(A_{\varepsilon}\right)_{\zeta \xi}=O_{p_{\zeta} \times p_{\xi}} \text { if } \zeta \neq \xi, \\
\left(A_{\pi}\right)_{\zeta \xi}=O_{q \zeta \times q_{\xi}} \text { if } \zeta \neq \xi .
\end{align*}
$$

Obviously $H$ has an $m \times r$ block structure of dimension $p_{\zeta} \times q_{\xi}$. The matrix $A_{\varepsilon} \mathbf{H}-\mathbf{H} A_{\pi}$ has its entries in the intersection of the rows $\mathbf{p}_{1}, \mathbf{p}_{2}, \ldots, \mathbf{p}_{m}$ and the columns $\mathbf{q}_{1}, \mathbf{q}_{2}, \ldots, \mathbf{q}_{r}$ zero. So in (70) we must have $z_{\zeta \xi}=0$. The equality of the blocks with coordinates $\zeta, \xi$ of both sides of (36) gives

$$
\begin{equation*}
\left(A_{\varepsilon}\right)_{\zeta \zeta} H_{\zeta \xi}-H_{\zeta \xi}\left(A_{\pi}\right)_{\xi \xi}=Z_{\zeta \zeta} \tag{72}
\end{equation*}
$$

The entry with coordinates 1,1 of the right part of (72) is $H_{\zeta \xi}^{21}-H_{\zeta \xi}^{12}$ and it must be zero so $H_{\zeta \xi}^{21}=H_{\zeta \xi}^{12}$. Notice that in general the entry with coordinates $\kappa$, $v$ verifying $1 \leq \kappa<$ $p_{\zeta}, 1 \leq \nu<q_{\xi}$ of the left part of (72) is $H_{\zeta \xi}^{(\kappa+1) v}-H_{\zeta \xi}^{\kappa(\nu+1)}$. As it must be zero one has that $H_{\zeta \xi}^{(\kappa+1) v}=H_{\zeta \xi}^{\kappa(\nu+1)}$.

Proposition 17. The structure of the block-Hankel matrix $\mathbf{H}$ is not altered by right multiplication with matrices $Q_{N}$ or by left multiplication with matrices $P_{N}$.

Proof. The block with coordinates $\zeta, \xi$ of the matrix $P_{N} \mathbf{H}$ is $\left(P_{N} H\right)_{\zeta \xi}=\sum_{\rho=1}^{m} P_{\zeta \rho} H_{\rho \xi}$. We will prove that the block

$$
W_{\zeta \rho \xi}=P_{\zeta \rho} H_{\rho \xi}=\left[\begin{array}{ccccccc}
\gamma_{\zeta \rho}^{1} & \gamma_{\zeta \rho}^{2} & \cdots & \gamma_{\zeta \rho}^{v} & 0 & \cdots & 0  \tag{73}\\
0 & \gamma_{\zeta \rho}^{1} & \gamma_{\zeta \rho}^{2} & \cdots & \gamma_{\zeta \rho}^{v} & & 0 \\
\vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\
0 & \cdots & 0 & \gamma_{\zeta \rho}^{1} & \gamma_{\zeta \rho}^{2} & \cdots & \gamma_{\zeta \rho}^{v}
\end{array}\right] H_{\rho \xi}
$$

has a Hankel structure since its entry with coordinates $\kappa, \lambda$ equals its entry with coordinates $\kappa-1, \lambda+1$. Let $W_{\zeta \rho \xi}^{\kappa \lambda}$ be the entry with coordinates $\kappa, \lambda$ of the block $W_{\zeta \rho \xi}$

$$
\begin{gather*}
W_{\zeta \rho \xi}^{\kappa \lambda}=\sum_{\mu=1}^{v} \gamma_{\zeta \rho}^{\mu} H_{\rho \xi}^{(\kappa+\mu-1) \lambda} \\
W_{\zeta \rho \xi}^{(\kappa-1)(\lambda+1)}=\sum_{\mu=1}^{v} \gamma_{\zeta \rho}^{\mu} H_{\rho \xi}^{(\kappa+\mu-2)(\lambda+1)} . \tag{74}
\end{gather*}
$$

As $H_{\rho \xi}^{(\kappa+\mu) \lambda}=H_{\rho \xi}^{(\kappa+\mu-1)(\lambda+1)}$ we have $W_{\zeta \rho \xi}^{\kappa \lambda}=W_{\zeta \rho \xi}^{(\kappa-1)(\lambda+1)}$.
The sum of Hankel matrices is a Hankel matrix and Proposition 17 is proved.

Example 18. For the systems of Example 14

$$
\begin{aligned}
\left(Q_{\pi} P_{\varepsilon}\right)^{-1} & =\mathbf{H}=\left[\begin{array}{lll}
H_{11} & H_{12} \\
H_{21} & H_{22} \\
H_{32} & H_{33}
\end{array}\right] \\
& =\left[\begin{array}{ccccccc}
-\frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & \frac{3}{4} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{12} & -\frac{1}{4} & \frac{3}{4} & -\frac{9}{4} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{4} & \frac{3}{4} & -\frac{9}{4} & \frac{27}{4} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{3}{8} & -\frac{1}{2} & \frac{3}{4} & -\frac{5}{4} & -\frac{3}{4} & \frac{5}{4} & -2 \\
-\frac{1}{2} & \frac{3}{4} & -\frac{5}{4} & \frac{9}{4} & \frac{5}{4} & -2 & \frac{7}{2} \\
-\frac{5}{8} & 1 & -\frac{7}{4} & \frac{13}{4} & \frac{3}{4} & -\frac{3}{4} & 1 \\
1 & -\frac{7}{4} & \frac{13}{4} & -\frac{25}{4} & -\frac{3}{4} & 1 & -\frac{3}{2}
\end{array}\right]
\end{aligned}
$$

$$
\begin{align*}
& \left(\widehat{Q}_{\pi} \widehat{P}_{\varepsilon}\right)^{-1}=\hat{\mathbf{H}}=\left[\begin{array}{ll}
\hat{H}_{11} & \hat{H}_{12} \\
\widehat{H}_{21} & \hat{H}_{22} \\
H_{32} & H_{33}
\end{array}\right] \\
& =\left[\begin{array}{ccccccc}
-\frac{1}{24} & \frac{1}{24} & -\frac{1}{8} & \frac{3}{8} & \frac{5}{12} & -\frac{1}{3} & \frac{3}{4} \\
\frac{1}{24} & -\frac{1}{8} & \frac{3}{8} & -\frac{9}{8} & -\frac{1}{3} & \frac{3}{4} & -2 \\
-\frac{1}{8} & \frac{3}{8} & -\frac{9}{8} & \frac{27}{8} & \frac{3}{4} & -2 & \frac{23}{4} \\
\frac{13}{1} & -\frac{13}{9} & \frac{5}{2} & -\frac{29}{6} & -\frac{559}{72} & \frac{175}{18} & -16 \\
-\frac{13}{9} & \frac{5}{2} & -\frac{29}{6} & \frac{61}{6} & \frac{175}{18} & -16 & \frac{89}{3} \\
\frac{1}{8} & -\frac{19}{72} & \frac{5}{8} & -\frac{37}{24} & -\frac{59}{72} & \frac{13}{9} & -\frac{13}{4} \\
-\frac{19}{72} & \frac{5}{8} & -\frac{37}{24} & \frac{95}{24} & \frac{13}{9} & -\frac{13}{4} & \frac{47}{6}
\end{array}\right], \\
& P_{N}=\left[\begin{array}{lllllll}
a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 \\
b & c & 0 & d & 0 & e & 0 \\
0 & b & c & 0 & d & 0 & e \\
f & g & 0 & h & 0 & k & 0 \\
0 & f & g & 0 & h & 0 & k
\end{array}\right], \\
& \widehat{Q}_{N}=\left[\begin{array}{ccccccc}
\widehat{v} & 0 & 0 & 0 & \widehat{w} & 0 & 0 \\
0 & \widehat{v} & 0 & 0 & \widehat{x} & \widehat{w} & 0 \\
0 & 0 & \widehat{v} & 0 & 0 & \widehat{x} & \widehat{w} \\
0 & 0 & 0 & \widehat{v} & 0 & 0 & \widehat{x} \\
0 & 0 & 0 & 0 & \widehat{y} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \widehat{y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \widehat{y}
\end{array}\right] \Longrightarrow \\
& S=P_{N}\left[\begin{array}{ll}
H_{11} & H_{12} \\
H_{21} & H_{22} \\
H_{32} & H_{33}
\end{array}\right]-\left[\begin{array}{ll}
\hat{H}_{11} & \hat{H}_{12} \\
\hat{H}_{21} & \hat{H}_{22} \\
H_{32} & H_{33}
\end{array}\right], \\
& \widehat{Q}_{N}=\left[\begin{array}{ll}
S_{11} & S_{12} \\
S_{21} & S_{22} \\
S_{32} & S_{33}
\end{array}\right]=O_{n \times n}, \\
& S_{11}=a H_{11}-\hat{H}_{11} \widehat{v}=O_{3 \times 4}, \\
& S_{12}=a H_{12}-\hat{H}_{11}\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right] \widehat{w}+\hat{H}_{11}\left[\begin{array}{lll}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \widehat{x}+\hat{H}_{12} \hat{y} \\
& =\mathrm{O}_{3 \times 3} \\
& \text { (entry } 2,1 \text { of } S_{12} \text { ) } a H_{12}^{21}-\widehat{H}_{11}^{21} \widehat{w}+\widehat{H}_{11}^{22} \widehat{x}+\widehat{H}_{12}^{21} \hat{y}=0 \\
& \text { (entry } 1,2 \text { of } S_{12} \text { ) } a H_{12}^{12}-\widehat{H}_{11}^{12} \widehat{w}+\widehat{H}_{11}^{13} \widehat{x}+\widehat{H}_{12}^{12} \widehat{y}=0 . \tag{75}
\end{align*}
$$

The matrix $S$ has a block-Hankel structure because the blocks $H_{\zeta \xi}, \widehat{H}_{\zeta \xi}$ have a Hankel structure.

Among the $n^{2}$ equations $P_{N} \widehat{\mathbf{H}}=\mathbf{H} \widehat{Q}_{N}$ only $n(m+r)-m r$ are (in the general case of systems with $n$ states $m$ inputs and $r$ outputs) independent. With the distribution of controllability and observability indices $E=\langle 3,2,2\rangle, \Pi=\langle 4,3\rangle$ of Example 14 the linearly independent equations are those of the first and fifth columns and those of the third, fifth, and seventh rows of the matrix equation.

## 5. Conclusions

In this paper we presented the solution of a problem of equivalence, open for several decades, illustrated with didactic examples. We exploit the fact that an output feedback is simultaneously a state feedback and an output injection. We use the isotropy subgroups to parameterize the solutions of two separate problems, the state feedback and the output injection equivalence. The group structure allows the "linearization" of the resulting bilinear system of equations.

The results of this paper are obtained using the state space representation of the systems. We presented also a bivariate polynomial variant of the problem of full output feedback equivalence involving generalized Bezoutians. Even though it is not clear how the equivalence of generalized Bezoutians can drive to the transformations achieving output feedback equivalence of systems without consideration of a state space representation, we believe that they have a very important role to play in the comprehension of the output feedback closed loop structure of the state space. The generalization of the breakaway polynomial for multivariable systems is only one step.

## Appendix

Here we present binary operations and inverse elements for the groups we use in this paper.

For the group $\mathscr{X}$

$$
\begin{align*}
&(((C, A, B)(K))(P))(F) \\
&=((C, A+B K, B)(P))(F) \\
&=\left(C P, P^{-1}(A+B K) P, P^{-1} B F\right) \\
&=(C, A, B)(P, K, F) \\
& \Longrightarrow\left((C, A, B)\left(P_{1}, K_{1}, F_{1}\right)\right)\left(P_{2}, K_{2}, F_{2}\right) \\
&=\left(C P_{1} P_{2}, P_{2}^{-1}\left(P_{1}^{-1}\left(A+B K_{1}\right) P_{1}+P_{1}^{-1} B F_{1} K_{2}\right) P_{2},\right. \\
&\left.P_{2}^{-1} P_{1}^{-1} B F_{1} F_{2}\right) \\
& \Longrightarrow\left(P_{1}, K_{1}, F_{1}\right)\left(P_{2}, K_{2}, F_{2}\right) \\
&=\left(P_{1} P_{2}, K_{1}+F_{1} K_{2} P_{1}^{-1}, F_{1} F_{2}\right) \\
& \Longrightarrow(P, K, F)^{-1}=\left(P^{-1},-F^{-1} K P, F^{-1}\right) . \tag{A.1}
\end{align*}
$$

For the group $\mathscr{y}$ we obtain by duality

$$
\begin{align*}
& \left(G_{1}, J_{1}, Q_{1}\right)\left(G_{2}, J_{2}, Q_{2}\right) \\
& \quad=\left(G_{2} G_{1}, J_{1}+Q_{1}^{-1} J_{2} G_{1}, Q_{2} Q_{1}\right)  \tag{A.2}\\
& \quad \Longrightarrow(G, J, Q)^{-1}=\left(G_{1}^{-1},-Q J G_{1}^{-1}, Q^{-1}\right)
\end{align*}
$$

For the group $\mathscr{Z}$,

$$
\begin{align*}
& \left(G_{1}, P_{1}, H_{1}, F_{1}\right)\left(G_{2}, P_{2}, H_{2}, F_{2}\right) \\
& \quad=\left(G_{2} G_{1}, P_{1} P_{2}, H_{1}+F_{1} H_{2} G_{1}, F_{1} F_{2}\right) \\
& \quad \Longrightarrow(G, P, H, F)^{-1}=\left(G^{-1}, P^{-1},-F^{-1} H G^{-1}, F^{-1}\right) \tag{A.3}
\end{align*}
$$

## References

[1] S. MacLane and G. Birkoff, Algebra, American Mathematical Society Chelsea Publishing, 3rd edition.
[2] F. R. Gantmacher, The Theory of Matrices, AMS, Chelsea, Mich, USA, 2000.
[3] A. S. Morse, "Structural invariants of linear multivariable systems," SIAM Journal on Control, vol. 11, pp. 446-465, 1973.
[4] P. Brunovsky, "A classification of linear controllable systems," Kybernetika Cislo, vol. 6, no. 3, pp. 173-188, 1970.
[5] R. E. Kalman, "Kronecker invariants and feedback," in Proceedings of the Conference on Ordinary Differential Equations (Mathematics Research Center, Madison, Wis. 1971), Naval Research Laboratory, Washington, DC, USA, 1971.
[6] H. H. Rosenbrock, State-Space and Multivariable Theory, John Wiley-Interscience, New York, NY, USA, 1970.
[7] S. H. Wang and E. J. Davison, "Canonical forms of linear multivariable systems," SIAM Journal on Control and Optimization, vol. 14, no. 2, pp. 236-250, 1976.
[8] M. S. Ravi, J. Rosenthal, and U. Helmke, "Output feedback invariants," Linear Algebra and Its Applications, vol. 351-352, pp. 623-637, 2002.
[9] U. Helmke and P. A. Fuhrmann, "Bezoutians," Linear Algebra and Its Applications, vol. 122-124, pp. 1039-1097, 1989.
[10] A. Yannakoudakis, "Bezoutians and output feedback stabilizability," in Proceedings of the 12th Mediterranean Conference on Control and Automation, Kuşadası, Turkey, June 2004.
[11] B. D. O. Anderson and E. I. Jury, "Generalized bezoutian and sylvester matrices in multivariable linear control," IEEE Transactions on Automatic Control, vol. AC-21, no. 4, pp. 551556, 1976.
[12] A. Yannakoudakis, "Invariant algebraic structures in multivariable control theory," Laboratoire Automatique de Grenoble. In press.
[13] A. Yannakoudakis, "Output feedback equivalence," in Proceedings of the European Control Conference, Cos Island, Greece, July 2007.
[14] V. M. Popov, "Invariant description of linear time-invariant controllable systems," SIAM Journal on Control, vol. 10, no. 2, pp. 252-264, 1972.
[15] C. I. Byrnes and P. E. Crouch, "Geometric methods for the classification of linear feedback systems," Systems and Control Letters, vol. 6, no. 4, pp. 239-246, 1985.
[16] H. Kimura, "On pole assignment by output feedback," International Journal of Control, vol. 28, no. 1, pp. 11-22, 1978.


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