

## Research Article

# Improved Results on $H_\infty$ State Estimation of Static Neural Networks with Time Delay

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This paper studies the problem of  $H_\infty$  state estimation for a class of delayed static neural networks. The purpose of the problem is to design a delay-dependent state estimator such that the dynamics of the error system is globally exponentially stable and a prescribed  $H_\infty$  performance is guaranteed. Some improved delay-dependent conditions are established by constructing augmented Lyapunov-Krasovskii functionals (LKFs). The desired estimator gain matrix can be characterized in terms of the solution to LMIs (linear matrix inequalities). Numerical examples are provided to illustrate the effectiveness of the proposed method compared with some existing results.

## 1. Introduction

Neural networks (NNs) have drawn a great deal of attention due to their extensive applications in various fields such as associative memory, pattern recognition, signal processing, combinatorial optimization, and adaptive control [1–3]. In the real world, time delays are unavoidably encountered in electronic implementations of neural networks because of the finite switching speed of the amplifiers. The presence of time delay may cause instability or deteriorate the performance of neural networks. Thus many recent literatures [1–9] have been working on the stability problem of delayed NNs.

We mainly focus on static neural networks (SNNs) in this paper, which is one type of recurrent neural networks (RNNs). Another type is local field neural networks, which has been fully studied while relatively less effort has been paid to the delayed SNNs. The main difference between SNNs and local field neural networks is whether the neuron states or the local field states of neurons are taken as basic variables. As mentioned in [10, 11], local field neural networks models and SNNs models are not always equivalent. Thus, it is necessary to study SNNs separately. Recently, many interesting results

on the stability analysis of SNNs have been addressed in the literature [2, 12–16].

Meanwhile, the state estimation of neural networks is an important issue. Generally, a neural network is a highly interconnected network with a great number of neurons. As a result, it would be very difficult to completely acquire the state information of all neurons. On the other hand, one needs to know the information of the neuron states and then make use of the neural networks in practice. Some results on the state estimation problem for the neural networks have been investigated in [17–30]. Among them  $H_\infty$  state estimation of static neural networks with time delay was studied in [17–19, 28, 30, 31]. In [28], a delay partition approach was proposed to deal with the state estimation problem for a class of static neural networks with time-varying delay. In [30], the state estimation problem of the guaranteed  $H_\infty$  and  $H_2$  performance of static neural networks was considered. Further improved results were obtained in [17, 18, 31] by using the convex combination approach. The exponential state estimation of time-varying delayed neural networks was studied in [19]. However, the information of neuron activation functions has not been

adequately taken into account. Moreover, the inequalities used may lead to conservatism to some extent. Therefore, the guaranteed performance state estimation problem has not yet been fully studied and remains a space for improvement.

This paper investigates the problem of  $H_\infty$  state estimation for a class of delayed static neural networks. The delay-dependent criteria are proposed such that the resulting filtering error system is globally exponentially stable with a guaranteed  $H_\infty$  performance. Different from time-varying delays considered in many papers such as [17, 19, 28], we consider the range of delay varies in an interval for which the lower bound is nonzero and fully take into account the information of lower bound of the delay. By using delay equal-partitioning method, augmented Lyapunov-Krasovskii functionals (LKFs) are properly constructed which is different from the existing relevant results. Then the free-weighting matrix technique is used to get a tighter upper bound on the derivatives of the LKFs. As mentioned in Remark 10, we also reduce conservatism by taking advantage of the information on activation function. Therefore, slack variables were introduced in our results, and it will increase the computational burden. To reduce decision variables so as to reduce computational burden, integral inequalities together with reciprocally convex approach are considered. Compared with existing results in [17–19], the criteria in this paper not only lead to less conservatism, but also have a balance between computational burden and conservatism.

The main contributions of this paper are as follows:

- (1) Augmented LKFs are properly constructed based on equal-partitioning method.
- (2) We also make use of integral inequalities to reduce computational burden.
- (3) Time delay is discussed under two different conditions: time-invariant delay and time-varying delay. In time-varying delay case, we consider the range of delay varies in an interval for which the lower bound is nonzero. Information of the lower bound is fully taken into account in LKFs.
- (4) We reduce conservatism by taking advantage of the information on activation function.

The remainder of this paper is organized as follows. The state estimation problem is formulated in Section 2. Section 3 is dedicated to deal with the designs of the state estimators for delayed static neural networks under two different conditions. In Section 4, two numerical examples with simulation results are provided to show the effectiveness of the results. Finally, some conclusions are made in Section 5.

*Notations.* The notations used throughout the paper are fairly standard.  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space;  $\mathbb{R}^{n \times m}$  is the set of all  $n \times m$  real matrices; the notation  $A > 0$  ( $< 0$ ) means  $A$  is a symmetric positive (negative) definite matrix;  $A^{-1}$  and  $A^T$  denote the inverse of matrix  $A$  and the transpose of matrix  $A$ ;  $I$  represents the identity matrix with proper dimensions, respectively; a symmetric term in a symmetric matrix is denoted by  $(*)$ ;  $\text{sym}(A)$  represents  $(A + A^T)$ ;  $\text{diag}\{\cdot\}$  stands for a block-diagonal matrix. Matrices,

if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Problem Formulation

Consider the delayed static neural network subject to noise disturbances described by

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + f(Wx(t - h(t)) + J) + B_1w(t), \\ y(t) &= Cx(t) + Dx(t - h(t)) + B_2w(t), \\ z(t) &= Hx(t), \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned} \quad (1)$$

where  $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$  is the state vector of the neural network,  $y(t) \in \mathbb{R}^m$  is the neural network output measurement,  $z(t) \in \mathbb{R}^r$  to be estimated is a linear combination of the state,  $w(t) \in \mathbb{R}^q$  is the noise input belonging to  $L_2[0, \infty)$ ,  $f(\cdot)$  denotes the neuron activation function,  $A = \text{diag}\{a_1, a_2, \dots, a_n\}$  with  $a_i > 0, i = 1, 2, \dots, n$  is the positive diagonal matrix,  $B_1, B_2, C, D$ , and  $H$  are real known matrices with appropriate dimensions,  $W \in \mathbb{R}^{n \times n}$  denote the connection weights,  $h(t)$  represent the time-varying delays,  $J$  represents the exogenous input vector, the function  $\phi(t)$  is the initial condition, and  $\tau = \sup_{t \geq 0}\{h(t)\}$ .

In this paper, time delay is discussed under two different conditions described as follows:

- (c1) time-invariant delay:  $0 \leq h(t) \equiv d$ ,
- (c2) time-varying delay:  $0 \leq d_1 \leq h(t) \leq d_2, \dot{h}(t) \leq \mu$ ,

where  $d_1, d_2$ , and  $\mu$  are constants.

In order to conduct the analysis, the following assumptions are necessary.

*Assumption 1.* For any  $x, y \in \mathbb{R}, (x \neq y), i \in \{1, 2, \dots, n\}$ , the activation function satisfies

$$k_i^- \leq \frac{f_i(x) - f_i(y)}{x - y} \leq k_i^+, \quad (2)$$

where  $k_i^-, k_i^+$  are constants and we define  $K^- = \text{diag}\{k_1^-, k_2^-, \dots, k_n^-\}, K^+ = \text{diag}\{k_1^+, k_2^+, \dots, k_n^+\}$ .

We construct a state estimator for estimation of  $z(t)$ :

$$\begin{aligned} \dot{\hat{x}}(t) &= -A\hat{x}(t) + f(W\hat{x}(t - h(t)) + J) \\ &\quad + K(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C\hat{x}(t) + D\hat{x}(t - h(t)), \\ \hat{z}(t) &= H\hat{x}(t), \\ \hat{x}(t) &= 0, \quad t \in [-\tau, 0], \end{aligned} \quad (3)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimated state vector of the neural network,  $\hat{z}(t)$  and  $\hat{y}(t)$  denote the estimated measurements of  $z(t)$  and  $y(t)$ , and  $K$  is the gain matrix to be determined.

Define the error  $e(t) = x(t) - \hat{x}(t)$ ,  $\bar{z}(t) = z(t) - \hat{z}(t)$ ; we can easily obtain the error system:

$$\begin{aligned} \dot{e}(t) = & -(A + KC)e(t) - KDe(t - h(t)) \\ & + g(We(t - h(t))) + (B_1 - KB_2)w(t), \end{aligned} \quad (4)$$

$$\bar{z}(t) = He(t),$$

where  $g(We(t)) = f(Wx(t) + J) - f(W\hat{x}(t) + J)$ .

**Definition 2** (see [18]). For any finite initial condition  $\phi(t) \in \mathcal{C}^1([- \tau, 0]; \mathbb{R}^n)$ , the error system (4) with  $w(t) = 0$  is said to be globally exponentially stable with a decay rate  $\beta$ , if there exist constants  $\lambda > 0$  and  $\beta > 0$  such that

$$\|e(t)\|^2 \leq \lambda e^{-\beta t} \sup_{- \tau < s < 0} \left\{ \|\phi(s)\|^2, \|\dot{\phi}(s)\|^2 \right\}. \quad (5)$$

Given a prescribed level of disturbance attenuation level  $\gamma > 0$ , the error system is said to be globally exponentially stable with  $H_\infty$  performance, when the error system is globally exponentially stable and the response  $\bar{z}(t)$  under zero initial condition satisfies

$$\|\bar{z}\|_2^2 \leq \gamma^2 \|w\|_2^2, \quad (6)$$

for every nonzero  $w(t) \in L_2[0, \infty)$ , where  $\|\phi\|_2 = \sqrt{\int_0^\infty \phi^T(t)\phi(t)dt}$ .

**Lemma 3** (see [32]). For any constant matrix  $Z \in \mathbb{R}^{n \times n}$ ,  $Z = Z^T > 0$ , scalars  $h_2 > h_1 > 0$ , and vector function  $x : [h_1, h_2] \rightarrow \mathbb{R}^n$  such that the following integrations are well defined, then

$$\begin{aligned} & - (h_2 - h_1) \int_{t-h_2}^{t-h_1} x^T(s) Z x(s) ds \\ & \leq - \int_{t-h_2}^{t-h_1} x^T(s) ds Z \int_{t-h_2}^{t-h_1} x(s) ds. \end{aligned} \quad (7)$$

**Lemma 4** (Schur complement). Given constant symmetric matrices  $S_1, S_2$ , and  $S_3$ , where  $S_1 = S_1^T$  and  $S_2 = S_2^T > 0$ , then  $S_1 + S_3^T S_2^{-1} S_3 < 0$  if and only if

$$\begin{aligned} & \begin{bmatrix} S_1 & S_3^T \\ S_3 & -S_2 \end{bmatrix} < 0 \\ \text{or} & \begin{bmatrix} -S_2 & S_3 \\ S_3^T & S_1 \end{bmatrix} < 0. \end{aligned} \quad (8)$$

**Lemma 5** (see [33]). For  $k_i(t) \in [0, 1]$ ,  $\sum_{i=1}^N k_i(t) = 1$  and vector  $\eta_i$  which satisfy  $\eta_i = 0$  with  $k_i(t) = 0$ , and matrices  $R_i > 0$ , there exists matrix  $S_{ij}$  ( $i = 1, 2, \dots, N-1; j = i+1, \dots, N$ ), which satisfies

$$\begin{bmatrix} R_i & S_{ij} \\ * & R_j \end{bmatrix} \geq 0 \quad (9)$$

such that the following inequality holds:

$$\sum_{i=1}^N \frac{1}{k_i(t)} \eta_i^T R_i \eta_i \geq \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}^T \begin{bmatrix} R_1 & \cdots & S_{1,N} \\ * & \ddots & \vdots \\ * & * & R_N \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}. \quad (10)$$

### 3. State Estimator Design

In this section, the state estimation problem will be discussed under two different conditions: time-invariant delay and time-varying delay. We consider the constant time delay case first. For convenience of presentation, we denote

$$\begin{aligned} \eta_1(t) &= \begin{bmatrix} e(t) \\ e\left(t - \frac{1}{m}d\right) \\ e\left(t - \frac{2}{m}d\right) \\ \vdots \\ e\left(t - \frac{m-1}{m}d\right) \end{bmatrix}, \\ \eta_2(t) &= \begin{bmatrix} g(We(t)) \\ g\left(We\left(t - \frac{1}{m}d\right)\right) \\ g\left(We\left(t - \frac{2}{m}d\right)\right) \\ \vdots \\ g\left(We\left(t - \frac{m-1}{m}d\right)\right) \end{bmatrix}, \end{aligned} \quad (11)$$

$$\theta_1^T(t) = [\eta_1^T(t), e^T(t-d)],$$

$$\theta_2^T(t) = [\eta_2^T(t), g^T(We(t-d))],$$

$$\hat{\eta}_1(t) = \eta_1(t)|_{d=d_1},$$

$$\hat{\eta}_2(t) = \eta_2(t)|_{d=d_1},$$

$$\hat{\theta}_1(t) = \theta_1(t)|_{d=d_1},$$

$$\hat{\theta}_2(t) = \theta_2(t)|_{d=d_1};$$

here we use “|” to give the condition.

**Theorem 6.** Under Assumption 1, for given scalars  $d \geq 0$ ,  $\gamma > 0$ , and  $\alpha \geq 0$  and an integer  $m \geq 1$ , system (4) is globally exponentially stable with  $H_\infty$  performance  $\gamma$  if there exist positive diagonal matrices  $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ ,  $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , matrices  $P > 0$ ,  $R_i > 0$  ( $i = 1, 2, \dots, m+1$ ),  $\begin{bmatrix} Q_1 & V \\ * & Q_1 \end{bmatrix} > 0$ , and any matrix with appropriate dimensions  $M$ , such that the following LMI holds:

$$\begin{bmatrix} \Xi & \hat{H}^T \\ * & -I \end{bmatrix} < 0, \quad (12)$$

where  $\Xi = (\Xi_{i,j})_{4 \times 4}$  is symmetric with

$$\begin{aligned}
\Xi_{1,1} &= \alpha E_1^T P E_1 + W_1^T Q_1 W_1 - e^{-\alpha d/m} W_2^T Q_1 W_2 \\
&\quad - \sum_{i=1}^m e^{-\alpha di/m} [E_i^T - E_{i+1}^T] R_i [E_i - E_{i+1}] \\
&\quad - E_1^T (MA + A^T M^T) E_1 \\
&\quad - E_1^T (GC + C^T G^T) E_1 - E_1^T G D E_{m+1} \\
&\quad - E_{m+1}^T D^T G^T E_1 - 2E_1^T W^T K^- \Gamma K^+ W E_1 \\
&\quad - 2E_{m+1}^T W^T K^- \Lambda K^+ W E_{m+1}, \\
\Xi_{1,2} &= W_1^T V W_1 - e^{-\alpha d/m} W_2^T V W_2 + E_1^T M E_{m+1} \\
&\quad + E_1^T W^T (K^- + K^+) \Gamma E_1 \\
&\quad + E_{m+1}^T W^T (K^- + K^+) \Lambda E_{m+1}, \\
\Xi_{1,3} &= E_1^T M B_1 - E_1^T G B_2, \\
\Xi_{1,4} &= E_1^T P - E_1^T M - E_1^T A^T M^T - E_1^T C^T G^T \\
&\quad - E_{m+1}^T (GD)^T, \\
\Xi_{2,2} &= W_1^T Q_2 W_1 - e^{-\alpha d/m} W_2^T Q_2 W_2 - 2E_1^T \Gamma E_1 \\
&\quad - 2E_{m+1}^T \Lambda E_{m+1}, \\
\Xi_{2,4} &= E_{m+1}^T M^T, \\
\Xi_{3,3} &= -\gamma^2 I, \\
\Xi_{3,4} &= B_1^T M^T - (GB_2)^T, \\
\Xi_{4,4} &= \left(\frac{d}{m}\right)^2 \left(\sum_{i=1}^m R_i\right) - M - M^T,
\end{aligned} \tag{13}$$

and other entries of  $\Xi$  are zeros:

$$\begin{aligned}
W_1 &= [I_{mn}, 0_{mn \times n}], \\
W_2 &= [0_{mn \times n}, I_{mn}], \\
E_i &= [0_{n \times (i-1)n}, I_n, 0_{n \times (m+1-i)n}], \quad i = 1, 2, \dots, m+1, \\
\widehat{H} &= [H \times E_1, 0, 0, 0].
\end{aligned} \tag{14}$$

The estimator gain matrix is given by  $K = M^{-1}G$ .

*Proof.* Construct a Lyapunov-Krasovskii functional candidate as follows:

$$V(t, e_t) = \sum_{i=1}^3 V_i(t, e_t), \tag{15}$$

where

$$\begin{aligned}
V_1(t, e_t) &= e^T(t) P e(t), \\
V_2(t, e_t) &= \int_{t-d/m}^t e^{-\alpha(t-s)} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \end{bmatrix} ds, \\
V_3(t, e_t) &= \frac{d}{m} \sum_{i=1}^m \int_{t-(i/m)d}^{t-((i-1)/m)d} \int_{t+\theta}^t e^{-\alpha(t-s)} \dot{e}^T(s) R_i \dot{e}(s) ds d\theta.
\end{aligned} \tag{16}$$

Calculating the derivative of  $V(t, e_t)$  along the trajectory of system, we obtain

$$\begin{aligned}
\dot{V}_1(t, e_t) &= 2e^T(t) P \dot{e}(t) = 2\theta_1^T(t) E_1^T P \dot{e}(t), \\
\dot{V}_2(t, e_t) &= -\alpha V_2(t, e_t) + \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \\
&\quad - e^{-\alpha d/m} \begin{bmatrix} \eta_1\left(t - \frac{d}{m}\right) \\ \eta_2\left(t - \frac{d}{m}\right) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \eta_1\left(t - \frac{d}{m}\right) \\ \eta_2\left(t - \frac{d}{m}\right) \end{bmatrix} = -\alpha V_2(t, e_t) \\
&\quad + \begin{bmatrix} W_1 \theta_1(t) \\ W_1 \theta_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} W_1 \theta_1(t) \\ W_1 \theta_2(t) \end{bmatrix} \\
&\quad - e^{-\alpha d/m} \begin{bmatrix} W_2 \theta_1(t) \\ W_2 \theta_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \begin{bmatrix} W_2 \theta_1(t) \\ W_2 \theta_2(t) \end{bmatrix}, \\
\dot{V}_3(t, e_t) &\leq -\alpha V_3(t, e_t) + \left(\frac{d}{m}\right)^2 \dot{e}^T(t) \left(\sum_{i=1}^m R_i\right) \dot{e}(t) \\
&\quad - \left(\frac{d}{m}\right) \sum_{i=1}^m \int_{t-(i/m)d}^{t-((i-1)/m)d} e^{-\alpha di/m} \dot{e}^T(s) R_i \dot{e}(s) ds,
\end{aligned} \tag{17}$$

and using Lemma 3, we can obtain

$$\begin{aligned}
\dot{V}_3(t, e_t) &\leq -\alpha V_3(t, e_t) + \left(\frac{d}{m}\right)^2 \dot{e}^T(t) \left(\sum_{i=1}^m R_i\right) \dot{e}(t) \\
&\quad - \sum_{i=1}^m e^{-\alpha di/m} \left[ e^T\left(t - \frac{i-1}{m}d\right) - e^T\left(t - \frac{i}{m}d\right) \right] \\
&\quad \cdot R_i \left[ e\left(t - \frac{i-1}{m}d\right) - e\left(t - \frac{i}{m}d\right) \right] = -\alpha V_3(t, e_t)
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{d}{m} \right)^2 \dot{e}^T(t) \left( \sum_{i=1}^m R_i \right) \dot{e}(t) \\
& - \sum_{i=1}^m e^{-\alpha di/m} \left[ \theta_1^T(t) E_i^T - \theta_1^T(t) E_{i+1}^T \right] \\
& \cdot R_i \left[ E_i \theta_1(t) - E_{i+1} \theta_1(t) \right].
\end{aligned} \tag{18}$$

According to Assumption 1, we have

$$\begin{aligned}
& 2 \sum_{i=1}^n \gamma_i \left( g_i(W_i e(t)) - k_i^- W_i e(t) \right) \\
& \cdot \left( g_i(W_i e(t)) - k_i^+ W_i e(t) \right) \leq 0, \quad i = 1, 2, \dots, n,
\end{aligned} \tag{19}$$

which is equivalent to

$$\begin{aligned}
& 2 \left( g(We(t)) - K^- We(t) \right) \Gamma \left( g(We(t)) - K^+ We(t) \right) \\
& \leq 0,
\end{aligned} \tag{20}$$

where  $\Gamma = \text{diag} \{ \gamma_1, \gamma_2, \dots, \gamma_n \}$ .

Similarly, we obtain

$$\begin{aligned}
& 2 \left( g(We(t-d)) - K^- We(t-d) \right) \\
& \cdot \Lambda \left( g(We(t-d)) - K^+ We(t-d) \right) \leq 0,
\end{aligned} \tag{21}$$

where  $\Lambda = \text{diag} \{ \lambda_1, \lambda_2, \dots, \lambda_n \}$ .

According to the system equation, the following equality holds:

$$\begin{aligned}
& 2 \left( e^T(t) + \dot{e}^T(t) \right) M \left\{ - (A + KC) e(t) - KDe(t-d) \right. \\
& \left. + g(We(t-d)) + (B_1 - KB_2) w(t) - \dot{e}(t) \right\} = 0.
\end{aligned} \tag{22}$$

Combining the qualities and inequalities from (17), (18), (20), (21), and (22), we can obtain

$$\begin{aligned}
& \bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, x_t) + \alpha V(t, x_t) \\
& \leq \zeta_t^T (\Xi) \zeta_t + \theta_1^T(t) E_1^T H^T H E_1 \theta_1(t),
\end{aligned} \tag{23}$$

where  $\zeta_t$  is defined as

$$\zeta_t^T = \left[ \theta_1^T(t), \theta_2^T(t), w^T(t), \dot{e}^T(t) \right]. \tag{24}$$

Based on Lemma 4, one can deduce that

$$\begin{aligned}
& \bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, x_t) + \alpha V(t, x_t) \\
& \leq \zeta_t^T \begin{bmatrix} \Xi & \widehat{H}^T \\ * & -I \end{bmatrix} \zeta_t,
\end{aligned} \tag{25}$$

where  $\widehat{H} = [H \times E_1, 0, 0, 0]$ .

If LMI (12) holds, then

$$\begin{aligned}
& \bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, x_t) + \alpha V(t, x_t) \\
& \leq 0;
\end{aligned} \tag{26}$$

$\alpha V(t, x_t) \geq 0$ , so we can obtain

$$\int_0^\infty \left[ \bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, e(t)) \right] dt \leq 0, \tag{27}$$

and since  $V(t, e(t)) \geq 0$ , under the zero initial condition, we have

$$\int_0^\infty \bar{z}^T(t) \bar{z}(t) dt \leq \gamma^2 \int_0^\infty w^T(t) w(t) dt, \tag{28}$$

which is equal to

$$\|\bar{z}\|_2^2 \leq \gamma^2 \|w\|_2^2; \tag{29}$$

therefore, the error system (4) guaranteed  $H_\infty$  performance according to Definition 2. In sequel, we show the globally exponential stability of the estimation error system with  $w(t) = 0$ . When  $w(t) = 0$ , the error system (4) becomes

$$\begin{aligned}
\dot{e}(t) & = - (A + KC) e(t) - KDe(t-h(t)) \\
& + g(We(t-h(t))),
\end{aligned} \tag{30}$$

$$\bar{z}(t) = He(t).$$

Equation (22) becomes

$$\begin{aligned}
& 2 \left( e^T(t) + \dot{e}^T(t) \right) M \left\{ - (A + KC) e(t) - KDe(t-d) \right. \\
& \left. + g(We(t-d)) - \dot{e}(t) \right\} = 0;
\end{aligned} \tag{31}$$

considering the same Lyapunov-Krasovskii functional candidate and calculating its time-derivative along the solution of (30), we can derive

$$\dot{V}(t, e_t) + \alpha V(t, e_t) \leq \bar{\zeta}_t^T (\Omega) \bar{\zeta}_t, \tag{32}$$

where

$$\Omega = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \Omega_{1,3} \\ * & \Omega_{2,2} & \Omega_{2,3} \\ * & * & \Omega_{3,3} \end{bmatrix},$$

$$\Omega_{1,1} = \alpha E_1^T P E_1 + W_1^T Q_1 W_1 - e^{-\alpha d/m} W_2^T Q_1 W_2$$

$$- \sum_{i=1}^m e^{-\alpha di/m} \left[ E_i^T - E_{i+1}^T \right] R_i \left[ E_i - E_{i+1} \right]$$

$$- E_1^T \left( MA + A^T M^T \right) E_1$$

$$- E_1^T \left( GC + C^T G^T \right) E_1 - E_1^T G D E_{m+1}$$

$$- E_{m+1}^T D^T G^T E_1 - 2 E_1^T W^T K^- \Gamma K^+ W E_1$$

$$- 2 E_{m+1}^T W^T K^- \Lambda K^+ W E_{m+1},$$

$$\Omega_{1,2} = W_1^T V W_1 - e^{-\alpha d/m} W_2^T V W_2 + E_1^T M E_{m+1}$$

$$+ E_1^T W^T \left( K^- + K^+ \right) \Gamma E_1$$

$$+ E_{m+1}^T W^T \left( K^- + K^+ \right) \Lambda E_{m+1},$$

$$\begin{aligned}
\Omega_{1,3} &= E_1^T P - E_1^T M - E_1^T A^T M^T - E_1^T C^T G^T \\
&\quad - E_{m+1}^T (GD)^T, \\
\Omega_{2,2} &= W_1^T Q_2 W_1 - e^{-\alpha d/m} W_2^T Q_2 W_2 - 2E_1^T \Gamma E_1 \\
&\quad - 2E_{m+1}^T \Lambda E_{m+1}, \\
\Omega_{2,3} &= E_{m+1}^T M^T, \\
\Omega_{3,3} &= \left(\frac{d}{m}\right)^2 \left(\sum_{i=1}^m R_i\right) - M - M^T, \\
\bar{\zeta}_t^T &= [\theta_1^T(t), \theta_2^T(t), e^T(t)].
\end{aligned} \tag{33}$$

Let  $G = MK$ , and it is obvious that if  $\Xi < 0$ , then  $\Omega < 0$ , so we get

$$\dot{V}(t, e_t) + \alpha V(t, e_t) \leq 0. \tag{34}$$

Integrating inequality (34), so we obtain

$$V(t, e_t) \leq e^{-\alpha t} V(0, e_0). \tag{35}$$

From (15), we have

$$\begin{aligned}
V(t, e_t) &\geq \lambda_{\min}(P) \|e(t)\|^2, \\
V(0, e_0) &\leq b \sup_{-\tau < s < 0} \{\|\phi(s)\|^2, \|\dot{\phi}(s)\|^2\},
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
b &= \lambda_{\max}(P) + (1 + \rho^2) d \lambda_{\max} \left( \begin{bmatrix} Q_1 & V \\ * & Q_2 \end{bmatrix} \right) \\
&\quad + \frac{d}{m} \sum_{i=1}^m \frac{((i/m)d)^2 - (((i-1)/m)d)^2}{2} \lambda_{\max}(R_i), \\
\rho &= \max_{1 \leq i \leq 6} (|k_i^-|, |k_i^+|).
\end{aligned} \tag{37}$$

Combining (35) and (36) it yields

$$\begin{aligned}
\|e(t)\|^2 &\leq \frac{1}{\lambda_{\min}(P)} V(t, e_t) \\
&\leq \frac{b}{\lambda_{\min}(P)} e^{-\alpha t} \sup_{-\tau < s < 0} \{\|\phi(s)\|^2, \|\dot{\phi}(s)\|^2\},
\end{aligned} \tag{38}$$

and hence the error system (4) is globally exponentially stable. Above all, if  $\Xi < 0$ , then the state estimator for the static neural network has the prescribed  $H_\infty$  performance and guarantees the globally exponential stability of the error system. This completes the proof.  $\square$

*Remark 7.* Based on the method of delay partitioning together with the free-weighting matrix approach, a new delay-dependent condition is proposed in Theorem 6 for the  $H_\infty$  state estimation of static neural networks (1) with

time-invariant delay. Delay partitioning method reduces the conservatism by employing more detailed information of time delay. The simulation results in Numerical Examples reveal the effectiveness of the delay partitioning approach to the design of the state estimator for static neural networks.

In the following, we will study the time-varying delay case; the result is as follows.

**Theorem 8.** Under Assumption 1, for given scalars  $\mu$ ,  $0 \leq d_1 \leq d_2$ ,  $\gamma > 0$ ,  $\alpha \geq 0$  and an integer  $m \geq 1$ , system (4) is globally exponentially stable with  $H_\infty$  performance  $\gamma$  if there exist matrices  $P > 0$ ,  $R_i > 0$  ( $i = 1, 2, \dots, m+1$ ),  $\begin{bmatrix} Q_1 & V_1 \\ * & Q_2 \end{bmatrix} > 0$ ,  $\begin{bmatrix} Q_3 & V_2 \\ * & Q_4 \end{bmatrix} > 0$ ,  $\begin{bmatrix} Q_5 & V_3 \\ * & Q_6 \end{bmatrix} > 0$ , positive diagonal matrices  $\Gamma$ ,  $\Lambda_1$ ,  $\Lambda_2$ ,  $\Lambda_3$ , and any matrices with appropriate dimensions  $M$ ,  $U_1$ , such that the following LMIs hold:

$$\begin{bmatrix} \hat{\Xi} & \bar{H}^T \\ * & -I \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} R_{m+1} & U_1 \\ * & R_{m+1} \end{bmatrix} \geq 0, \tag{40}$$

where  $\hat{\Xi} = (\Xi_{i,j})_{8 \times 8}$  is symmetric with

$$\begin{aligned}
\hat{\Xi}_{1,1} &= \alpha E_1^T P E_1 + W_1^T Q_1 W_1 - e^{-\alpha d_1/m} W_2^T Q_1 W_2 \\
&\quad - \sum_{i=1}^m e^{-\alpha d_1 i/m} [E_i^T - E_{i+1}^T] R_i [E_i - E_{i+1}] \\
&\quad + E_1^T Q_3 E_1 + E_1^T Q_5 E_1 \\
&\quad - \frac{1}{d_2 - d_1} e^{-\alpha d_2} E_{m+1}^T R_{m+1} E_{m+1} \\
&\quad - E_1^T (MA + A^T M^T) E_1 \\
&\quad - E_1^T (GC + C^T G^T) E_1 \\
&\quad - 2E_{m+1}^T W^T K^- \Lambda_1 K^+ W E_{m+1} \\
&\quad - 2E_1^T W^T K^- \Gamma K^+ W E_1,
\end{aligned}$$

$$\begin{aligned}
\hat{\Xi}_{1,2} &= W_1^T V_1 W_1 - e^{-\alpha d_1/m} W_2^T V_1 W_2 + E_1^T V_2 E_1 \\
&\quad + E_1^T V_3 E_1 + E_{m+1}^T W^T (K^- + K^+) \Lambda_1 E_{m+1} \\
&\quad + E_1^T W^T (K^- + K^+) \Gamma E_1,
\end{aligned}$$

$$\begin{aligned}
\hat{\Xi}_{1,3} &= \frac{1}{d_2 - d_1} e^{-\alpha d_2} E_{m+1}^T R_{m+1} - E_1^T G D \\
&\quad - \frac{1}{d_2 - d_1} e^{-\alpha d_2} E_{m+1}^T U_1,
\end{aligned}$$

$$\hat{\Xi}_{1,4} = \frac{1}{d_2 - d_1} e^{-\alpha d_2} E_{m+1}^T U_1,$$

$$\hat{\Xi}_{1,5} = E_1^T M,$$

$$\begin{aligned}
\hat{\Xi}_{1,7} &= E_1^T M B_1 - E_1^T G B_2, \\
\hat{\Xi}_{1,8} &= E_1^T P - E_1^T M - E_1^T A^T M^T - E_1^T C^T G^T, \\
\hat{\Xi}_{2,2} &= W_1^T Q_2 W_1 - e^{-\alpha d_1/m} W_2^T Q_2 W_2 + E_1^T Q_4 E_1 \\
&\quad + E_1^T Q_6 E_1 - 2E_{m+1}^T \Lambda_1 E_{m+1} - 2E_1^T \Gamma E_1, \\
\hat{\Xi}_{3,3} &= -(1-\mu) e^{-\alpha d_2} Q_3 - \frac{2}{d_2 - d_1} e^{-\alpha d_2} R_{m+1} \\
&\quad - W^T K^- \Lambda_3 K^+ W \\
&\quad + \frac{1}{d_2 - d_1} e^{-\alpha d_2} (U_1 + U_1^T), \\
\hat{\Xi}_{3,4} &= \frac{1}{d_2 - d_1} e^{-\alpha d_2} R_{m+1} - \frac{1}{d_2 - d_1} e^{-\alpha d_2} U_1, \\
\hat{\Xi}_{3,5} &= -(1-\mu) e^{-\alpha d_2} V_2 + W^T K^- \Lambda_3 + W^T K^+ \Lambda_3, \\
\hat{\Xi}_{3,8} &= -(GD)^T, \\
\hat{\Xi}_{4,4} &= -e^{-\alpha d_2} Q_5 - \frac{1}{d_2 - d_1} e^{-\alpha d_2} R_{m+1} \\
&\quad - W^T K^- \Lambda_2 K^+ W, \\
\hat{\Xi}_{4,6} &= -e^{-\alpha d_2} V_3 + W^T K^- \Lambda_2 + W^T K^+ \Lambda_2, \\
\hat{\Xi}_{5,5} &= -(1-\mu) e^{-\alpha d_2} Q_4 - 2\Lambda_3, \\
\hat{\Xi}_{5,8} &= -M^T, \\
\hat{\Xi}_{6,6} &= -e^{-\alpha d_2} Q_6 - 2\Lambda_2, \\
\hat{\Xi}_{7,7} &= -\gamma^2 I, \\
\hat{\Xi}_{7,8} &= B_1^T M^T - (GB_2)^T, \\
\hat{\Xi}_{8,8} &= \left(\frac{d_1}{m}\right)^2 \left(\sum_{i=1}^m R_i\right) + (d_2 - d_1) R_{m+1} - M - M^T,
\end{aligned} \tag{41}$$

and other entries of  $\Xi$  are zeros:

$$\begin{aligned}
W_1 &= [I_{mn}, 0_{mn \times n}], \\
W_2 &= [0_{mn \times n}, I_{mn}], \\
E_i &= [0_{n \times (i-1)n}, I_n, 0_{n \times (m+1-i)n}], \quad i = 1, 2, \dots, m+1, \\
\bar{H} &= [H \times E_1, 0, 0, 0, 0, 0, 0].
\end{aligned} \tag{42}$$

The estimator gain matrix is given by  $K = M^{-1}G$ .

*Proof.* Construct a Lyapunov-Krasovskii functional candidate as follows:

$$V(t, e_t) = \sum_{i=1}^5 V_i(t, e_t), \tag{43}$$

where

$$\begin{aligned}
V_1(t, e_t) &= e^T(t) P e(t), \\
V_2(t, e_t) &= \int_{t-(d_1/m)}^t e^{-\alpha(t-s)} \begin{bmatrix} \hat{\eta}_1(s) \\ \hat{\eta}_2(s) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_1 & V_1 \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1(s) \\ \hat{\eta}_2(s) \end{bmatrix} ds \\
V_3(t, e_t) &= \int_{t-h(t)}^t e^{-\alpha(t-s)} \begin{bmatrix} e(s) \\ g(We(s)) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_3 & V_2 \\ * & Q_4 \end{bmatrix} \begin{bmatrix} e(s) \\ g(We(s)) \end{bmatrix} ds \\
V_4(t, e_t) &= \int_{t-d_2}^t e^{-\alpha(t-s)} \begin{bmatrix} e(s) \\ g(We(s)) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_5 & V_3 \\ * & Q_6 \end{bmatrix} \begin{bmatrix} e(s) \\ g(We(s)) \end{bmatrix} ds, \\
V_5(t, e_t) &= \frac{d_1}{m} \\
&\quad \cdot \sum_{i=1}^m \int_{-(i/m)d_1}^{-(i-1/m)d_1} \int_{t+\theta}^t e^{-\alpha(t-s)} \dot{e}^T(s) R_i \dot{e}(s) ds d\theta \\
&\quad + \int_{-d_2}^{-d_1} \int_{t+\theta}^t e^{-\alpha(t-s)} \dot{e}^T(s) R_{m+1} \dot{e}(s) ds d\theta,
\end{aligned} \tag{44}$$

and calculating the derivative of  $V(t, e_t)$  along the trajectory of system, we obtain

$$\begin{aligned}
\dot{V}_1(t, e_t) &= 2e^T(t) P \dot{e}(t) = 2\hat{\theta}_1^T(t) E_1^T P \dot{e}(t), \\
\dot{V}_2(t, e_t) &= -\alpha V_2(t, e_t) + \begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_1 & V_1 \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix} - e^{-\alpha d_1/m} \begin{bmatrix} \hat{\eta}_1\left(t - \frac{d_1}{m}\right) \\ \hat{\eta}_2\left(t - \frac{d_1}{m}\right) \end{bmatrix}^T \\
&\quad \cdot \begin{bmatrix} Q_1 & V_1 \\ * & Q_2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1\left(t - \frac{d_1}{m}\right) \\ \hat{\eta}_2\left(t - \frac{d_1}{m}\right) \end{bmatrix} = -\alpha V_2(t, e_t) \\
&\quad + \begin{bmatrix} W_1 \hat{\theta}_1(t) \\ W_1 \hat{\theta}_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V_1 \\ * & Q_2 \end{bmatrix} \begin{bmatrix} W_1 \hat{\theta}_1(t) \\ W_1 \hat{\theta}_2(t) \end{bmatrix} \\
&\quad - e^{-\alpha d_1/m} \begin{bmatrix} W_2 \hat{\theta}_1(t) \\ W_2 \hat{\theta}_2(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & V_1 \\ * & Q_2 \end{bmatrix} \begin{bmatrix} W_2 \hat{\theta}_1(t) \\ W_2 \hat{\theta}_2(t) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\dot{V}_3(t, e_t) &= -\alpha V_3(t, e_t) + \begin{bmatrix} e(t) \\ g(We(t)) \end{bmatrix}^T \\
&\cdot \begin{bmatrix} Q_3 & V_2 \\ * & Q_4 \end{bmatrix} \begin{bmatrix} e(t) \\ g(We(t)) \end{bmatrix} - (1 - \mu) \\
&\cdot e^{-\alpha h(t)} \begin{bmatrix} e(t-h(t)) \\ g(We(t-h(t))) \end{bmatrix}^T \\
&\cdot \begin{bmatrix} Q_3 & V_2 \\ * & Q_4 \end{bmatrix} \begin{bmatrix} e(t-h(t)) \\ g(We(t-h(t))) \end{bmatrix} \leq -\alpha V_3(t, e_t) \\
&+ \begin{bmatrix} E_1 \hat{\theta}_1(t) \\ E_1 \hat{\theta}_2(t) \end{bmatrix}^T \begin{bmatrix} Q_3 & V_2 \\ * & Q_4 \end{bmatrix} \begin{bmatrix} E_1 \hat{\theta}_1(t) \\ E_1 \hat{\theta}_2(t) \end{bmatrix} - (1 - \mu) \\
&\cdot e^{-\alpha d_2} \begin{bmatrix} e(t-h(t)) \\ g(We(t-h(t))) \end{bmatrix}^T \\
&\cdot \begin{bmatrix} Q_3 & V_2 \\ * & Q_4 \end{bmatrix} \begin{bmatrix} e(t-h(t)) \\ g(We(t-h(t))) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\dot{V}_4(t, e_t) &= -\alpha V_4(t, e_t) + \begin{bmatrix} E_1 \hat{\theta}_1(t) \\ E_1 \hat{\theta}_2(t) \end{bmatrix}^T \\
&\cdot \begin{bmatrix} Q_5 & V_3 \\ * & Q_6 \end{bmatrix} \begin{bmatrix} E_1 \hat{\theta}_1(t) \\ E_1 \hat{\theta}_2(t) \end{bmatrix} \\
&- e^{-\alpha d_2} \begin{bmatrix} e(t-d_2) \\ g(We(t-d_2)) \end{bmatrix}^T \\
&\cdot \begin{bmatrix} Q_5 & V_3 \\ * & Q_6 \end{bmatrix} \begin{bmatrix} e(t-d_2) \\ g(We(t-d_2)) \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\dot{V}_5(t, e_t) &\leq -\alpha V_5(t, e_t) + \left(\frac{d_1}{m}\right)^2 \dot{e}^T(t) \left(\sum_{i=1}^m R_i\right) \dot{e}(t) \\
&- \left(\frac{d_1}{m}\right) \sum_{i=1}^m \int_{t-(i/m)d_1}^{t-(i-1)/m d_1} e^{-\alpha d_1 i/m} e^T(s) R_i \dot{e}(s) ds \\
&+ (d_2 - d_1) \dot{e}^T(t) R_{m+1} \dot{e}(t) \\
&- \int_{t-h(t)}^{t-d_1} e^{-\alpha d_2} \dot{e}^T(s) R_{m+1} \dot{e}(s) ds \\
&- \int_{t-d_2}^{t-h(t)} e^{-\alpha d_2} \dot{e}^T(s) R_{m+1} \dot{e}(s) ds,
\end{aligned} \tag{45}$$

and using Lemmas 3 and 5, we can obtain

$$\begin{aligned}
\dot{V}_5(t, e_t) &\leq -\alpha V_5(t, e_t) + \left(\frac{d_1}{m}\right)^2 \dot{e}^T(t) \left(\sum_{i=1}^m R_i\right) \dot{e}(t) \\
&+ (d_2 - d_1) \dot{e}^T(t) R_{m+1} \dot{e}(t)
\end{aligned}$$

$$\begin{aligned}
&- \sum_{i=1}^m e^{-\alpha d_1 i/m} \left[ \hat{\theta}_1^T(t) E_i^T - \theta_1^T(t) E_{i+1}^T \right] \\
&\cdot R_i \left[ E_i \theta_1(t) - E_{i+1} \theta_1(t) \right] - \frac{1}{d_2 - d_1} \\
&\cdot e^{-\alpha d_2} \begin{bmatrix} e(t-d_1) - e(t-h(t)) \\ e(t-h(t)) - e(t-d_2) \end{bmatrix}^T \\
&\cdot \begin{bmatrix} R_{m+1} & U_1 \\ * & R_{m+1} \end{bmatrix} \begin{bmatrix} e(t-d_1) - e(t-h(t)) \\ e(t-h(t)) - e(t-d_2) \end{bmatrix}.
\end{aligned} \tag{46}$$

According to Assumption 1, Similarly to (20), we obtain

$$\begin{aligned}
&2(g(We(t)) - K^- We(t))^T \Gamma (g(We(t)) - K^+ We(t)) \\
&\leq 0, \\
&2(g(We(t-d_1)) - K^- We(t-d_1))^T \\
&\cdot \Lambda_1 (g(We(t-d_1)) - K^+ We(t-d_1)) \leq 0, \\
&2(g(We(t-d_2)) - K^- We(t-d_2))^T \\
&\cdot \Lambda_2 (g(We(t-d_2)) - K^+ We(t-d_2)) \leq 0, \\
&2(g(We(t-h(t))) - K^- We(t-h(t)))^T \\
&\cdot \Lambda_3 (g(We(t-h(t))) - K^+ We(t-h(t))) \leq 0,
\end{aligned} \tag{47}$$

and according to the system equation, the following equality holds:

$$\begin{aligned}
&2(e^T(t) + \dot{e}^T(t)) M \{-(A + KC) e(t) \\
&- KDe(t-h(t)) + g(We(t-h(t))) \\
&+ (B_1 - KB_2) w(t) - \dot{e}(t)\} = 0.
\end{aligned} \tag{48}$$

Combining the qualities and inequalities from (45) to (48), we can obtain

$$\begin{aligned}
&\bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, x_t) + \alpha V(t, x_t) \\
&\leq \tilde{\zeta}_t^T (\hat{\Xi}) \tilde{\zeta}_t + \hat{\theta}_1^T E_1^T H^T H E_1 \hat{\theta}_1,
\end{aligned} \tag{49}$$

where  $\tilde{\zeta}_t$  is defined as

$$\begin{aligned}
\tilde{\zeta}_t^T &= \left[ \hat{\theta}_1^T(t), \hat{\theta}_2^T(t), e^T(t-h(t)), e^T(t-d_2), \right. \\
&\left. g(We(t-h(t)))^T, g(We(t-d_2))^T, w^T(t), \dot{e}^T(t) \right].
\end{aligned} \tag{50}$$

Based on Lemma 4, one can deduce that

$$\begin{aligned}
&\bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, x_t) + \alpha V(t, x_t) \\
&\leq \tilde{\zeta}_t^T \begin{bmatrix} \hat{\Xi} & \bar{H}^T \\ * & -I \end{bmatrix} \tilde{\zeta}_t,
\end{aligned} \tag{51}$$

where  $\bar{H} = [H \times E_1, 0, 0, 0, 0, 0, 0, 0]$ .

If LMI (39) holds, then

$$\begin{aligned} & \bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, x_t) + \alpha V(t, x_t) \\ & < 0; \end{aligned} \quad (52)$$

$\alpha V(t, x_t) > 0$ , so we can obtain

$$\int_0^\infty \left[ \bar{z}^T(t) \bar{z}(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t, e(t)) \right] dt < 0, \quad (53)$$

and since  $V(t, e(t)) > 0$ , under the zero initial condition, we have

$$\|\bar{z}(t)\|^2 \leq \gamma^2 \|w(t)\|^2. \quad (54)$$

Therefore, the error system (4) guaranteed  $H_\infty$  performance  $\gamma$  according to Definition 2. The remainder of proof is similar to the proof of Theorem 6. This completes the proof.  $\square$

*Remark 9.* If we only use the free-weighting matrix method together with the delay partitioning method to deal with the  $H_\infty$  state estimation problem of static neural networks (1), a great many free-weighting matrices will be introduced with the increasing number of partitions. That will lead to complexity and computational burden. So in this paper we also make use of integral inequalities to reduce decision variables so as to reduce computational burden, because only one matrix is introduced no matter how large the number of partitions is. Moreover, reciprocally convex approach was used with integral inequalities to reduce the conservatism.

*Remark 10.* In some previous literatures [18, 19, 30],  $k_i^- \leq f_i(x)/x \leq k_i^+$ , which is a special case of  $k_i^- \leq (f_i(x) - f_i(y))/(x - y) \leq k_i^+$ , was used to reduce the conservatism. In our proof, not only  $k_i^- \leq g_i(W_i e(t))/W_i e(t) \leq k_i^+$ , but also  $k_i^- \leq g_i(W_i e(t - h(t)))/W_i e(t - h(t)) \leq k_i^+$ ,  $k_i^- \leq g_i(W_i e(t - d_1))/W_i e(t - d_1) \leq k_i^+$ , and  $k_i^- \leq g_i(W_i e(t - d_2))/W_i e(t - d_2) \leq k_i^+$  have been used, which play an important role in reducing the conservatism.

#### 4. Numerical Examples

In this section, numerical examples are provided to illustrate effectiveness of the developed method for the state estimation of static neural networks.

*Example 1.* Consider the neural networks (1) with the following parameters:

$$A = \begin{pmatrix} 0.96 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 1.48 \end{pmatrix},$$

$$W = \begin{pmatrix} 0.5 & 0.3 & -0.36 \\ 0.1 & 0.12 & 0.5 \\ -0.42 & 0.78 & 0.9 \end{pmatrix},$$

TABLE 1: The  $H_\infty$  performance index  $\gamma$  with different  $(d_2, \mu)$ .

Methods	(0.8, 0.4)	(0.9, 0.7)	(1.1, 0.5)	(1, 1)	(1.2, 1)
[17]	1.2197	1.2719	1.8944	1.3720	Infeasible
[18]	1.2989	1.3164	1.6441	1.3720	Infeasible
Theorem 6 [19]	1.1391	1.1489	1.1538	1.1626	1.1731
Theorem 8, $m = 5$	0.4201	0.4602	0.5144	0.4969	0.5687
Theorem 8, $m = 10$	0.2883	0.3220	0.3890	0.3530	0.4422

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$B1 = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.1 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C = (1 \ 0 \ -2),$$

$$D = (0.5 \ 0 \ -1),$$

$$B2 = -0.1.$$

(55)

To compare with the existing results, we let  $\alpha = 0$ ,  $d_1 = 0$ ,  $K^- = 0$ , and  $K^+ = I$ . And we obtain the optimal  $H_\infty$  performance index  $\gamma$  for different values of delay  $d_2$  and  $\mu$ . It is summarized in Table 1.

From Table 1, it is clear that our results achieve better performance. In addition, the optimal  $H_\infty$  performance index  $\gamma$  becomes smaller as the partitioning number is increasing. It shows that delay partitioning method can reduce the conservatism effectively.

*Example 2.* Consider the neural networks (1) with the following parameters:

$$A = \begin{pmatrix} 1.56 & 0 & 0 \\ 0 & 2.42 & 0 \\ 0 & 0 & 1.88 \end{pmatrix},$$

$$W = \begin{pmatrix} -0.3 & 0.8 & -1.36 \\ 1.1 & 0.4 & -0.5 \\ 0.42 & 0 & -0.95 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix},$$

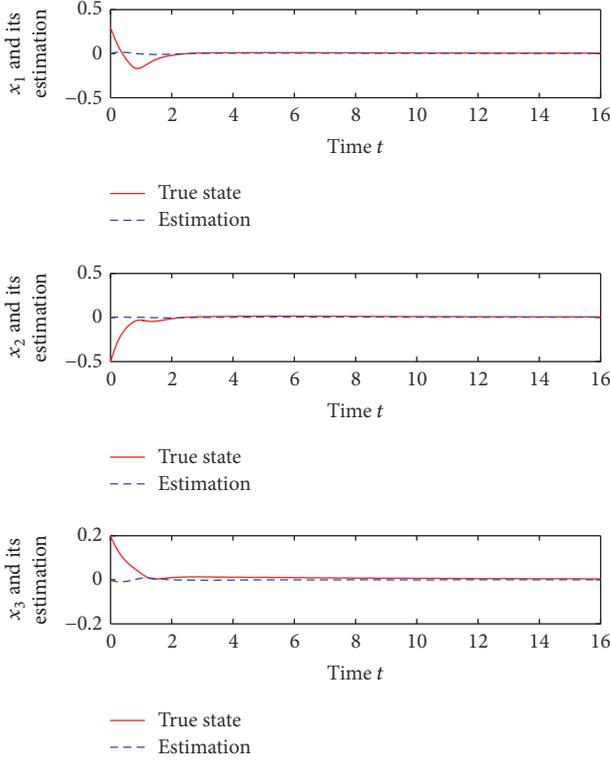


FIGURE 1: The state variables and their estimation.

$$B1 = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C = (1 \ 0 \ 0),$$

$$D = (2 \ 0 \ 0),$$

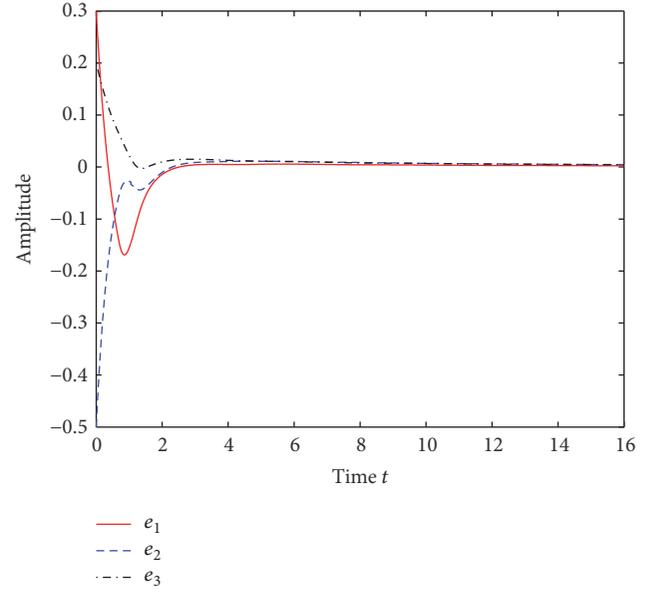
$$B2 = 0.4.$$

(56)

The activation function is  $f(x) = \tanh(x)$ , and it is easy to get that  $K^- = 0$ ,  $K^+ = I$ . And we set  $\gamma = 1$ ,  $\alpha = 0$ , and  $h(t) = 0.5 + 0.5 \sin(0.8t)$ , so the bound of time delay  $d_1 = 0$ ,  $d_2 = 1$ , and  $\mu = 0.4$ . The noise disturbance is assumed to be  $w(t) = 1/(0.8 + 1.2t)$ . By solving through the Matlab LMI toolbox, we obtain the gain matrix of the estimator:

$$K = \begin{pmatrix} 0.1078 \\ 0.0371 \\ -0.0655 \end{pmatrix}. \quad (57)$$

Figure 1 presents the state variables and their estimation of the neural network (1) from initial values  $[0.3, -0.5, 0.2]^T$ . Figure 2 shows the state response of the error system (4)

FIGURE 2: The response of the error  $e(t)$ .

under the initial condition  $e(0) = [0.3, -0.5, 0.2]^T$ . It is clear that  $e(t)$  converge rapidly to zero. The simulation results reveal the effectiveness of the proposed approach to the design of the state estimator for static neural networks.

## 5. Conclusions

In this paper, we investigated the  $H_\infty$  state estimation problem for a class of delayed static neural networks. By constructing augmented Lyapunov-Krasovskii functionals, new delay-dependent conditions were established. The designs of the desired estimator are achieved by solving a set of linear matrix inequalities, which can be facilitated efficiently by resorting to standard numerical algorithms. In the end, numerical examples were provided to illustrate the effectiveness of the proposed method compared with some existing results.

## Competing Interests

The authors declare that they have no competing interests.

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